I.I.T. GUIDE

FOR

MATHEMATICS

CONTAINING

Detailed study of all the topics in more than 1500 Pages, with Problem sets and their solutions, followed by objective type questions,

and

I I T, Roorkee and M N R Papers Solved upto date

By

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Note —*A supplement to this guide containing additional topics on Mathematics for* Roorkee, Moti Lal Nehru *etc Engineering entrance Examinations is also available (See contents P vii) It contains 600 pages and its price is* Rs 50 00

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PREFACE

This book has been written for the help of those students who wish to seek admission through entrance examination to various I I Ts and also other Engineering Institutes like those of Roorkee, Moti Lal Nehru (Allahabad) etc

There was no dearth of various guides available in the market for the above entrance examinations but mostly they were covering all the subjects like Physics, Chemistry, Mathematics and English in one volume We felt that no justice could be done to any of the subjects within a limited number of pages alloted to each of the subjects Hence we felt that we should bring out an exhaustive guide covering our subject Mathematics We have discussed the various topics prescribed in greater details which has resulted in the volume of the book exceeding 1100 pages but we are sure that the students will draw the full benefit by studying the same This is going to be a boon for all those students who do not have the facility of extra guidance for the preparation

All the necessary results and theorems have been given in the beginning of each chapter and followed by problem set on these theorems Solutions to these problems have been given as well In the end we have given Objective Type Questions Our sincere advice to the students is that they should attempt the questions given in problems set independently If they fail to do so them they should look to the given solution

You should feel your preparation of the subject complete only when you are able to reproduce the solutions of the problems without consulting the solutions given in the book

We are grateful to the publishers M/s Jai Prakash Nath Publications and various presses for publishing and printing this voluminous book

We are confident that the book will meet the requirement of all those for whom it is meant

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PREFACE TO TWENTIETH EDITION

In this edition the book has again been thoroughly revised and enlarged Many new questions have been added resulting further increase in the volume of the book but thereby making it exhaustive and complete in all respects Objective type questions have been given after each chapter Latest papers of I I T, Roorkee & M N R examinations have been added in respective chapters We hope that this new edition of the book will be found very useful for various engineering competitive examinations

We shall very much appreciate suggestions from our readers for the improvement of the book

J N Sharma **M L Khanna**
Meerut

PREFACE TO SECOND EDITION

The subject matter has been thoroughly revised and printing mistakes have been removed as far as possible Rearrangement of problem sets and their solutions has been done so that the solution set follows the problem set Objective type questions have been given after each topic Questions from latest papers of I I T, Roorkee and M N R have been added Chapter on set Theory has been withdrawn as now it is not in the syllabi for I I T and a new chapter on Inequalities has been added This chapter on Set Theory has now been added in supplement for Roorkee and M N R All this has resulted in the increase of the volume of the book by about 100 pages Consequently the price of the book has also been increased by a little amount The printing of the book is in continuation except the last about one handered and fifty pages thus eliminating the error of missing pages

Any suggestions for improvement of the book from any quarter will be highly appreciated

J N Sharma **M L Khanna**
Meerut

WHERE IS WHAT ?

(*Part II*)

(*Part III*)

Probability

New Edition 1989

SUPPLEMENT OF I I T GUIDE

For

Roorkee and M N R Entrance Examinations

Containing

PART I

A. Co-ordinate Geometry

Parabola, Ellipse, Pair of Straight Lines, Hyperbola

B Algebra

Binomial Theorem for any rational index except zero, Exponential and Logarithmic Series

C Set Theory

PART II

G *Statics*

Composition and resolution of forces, Parallel forces, Moments and Couples, Coplanar, forces, Friction and Centre of Gravity

H Dynamics

Velocity, Composition of Velocity and relative velocity, Acceleration Composition of acceleration and motion under gravity, Projectiles, Direct impact of smooth spheres Principle of conservation of energy and momentum

PART I

TRIGONOMETRY

1

Trigonometrical Ratios and Identities

§ 1 **Measurement of angles** There are two common systems of measuring angles

(1) Sexagesimal Measure (2) Circular Measure

According to sexagesimal system of measurement, we divide a right angle into 90 equal parts called degrees Each degree is divided into sixty equal parts called minutes, and each minute is divided into sixty equal parts called seconds We denote one degree one minute and one second by the symboles $1°$, $1'$ and $1''$ respectively

In second system, the unit of measurement is called a *radian* whose definition is as follows

Take any circle whose centre is O In this, measure an arc AB equal in length to the radius Join O to A and B Then $\angle AOB$ is the measurement of one radian This is the unit of circular measure We denote the ratio of the circumference to the diameter of a circle by the Greek letter π

Thus $\dfrac{\text{Circumference}}{\text{Diameter}} = \pi$

It can be proved that π is an irrational number Its approximate value is often taken as 22/7 This value is correct only to places of decimals Its value correct to 5 decimal places is 3.14159 Sometimes, we require the value of $1/\pi$ which is given by

$$\frac{1}{\pi} = 0.3183098862$$

Relation between degree and radian Students must remember that

$$1 \text{ Radian} = 180° \times \frac{1}{\pi} = 180° \times 0.3183098862$$

$$= 57.2957795...° = 57° \ 17' \ 44.8''$$

Again $1^\circ = \frac{\pi}{180}$ radian

§ 2 Basic Formula

1 $\sin^2 A + \cos^2 A = 1$ or $\sin^2 A = 1 - \cos^2 A$ or $\cos^2 A = 1 - \sin^2 A$

2 $1 + \tan^2 A = \sec^2 A$ or $\sec^2 A - \tan^2 A = 1$
or $(\sec A - \tan A)(\sec A + \tan A) = 1$
or $\sec A + \tan A = \frac{1}{\sec A - \tan A}$, $(A \neq n\pi + \frac{1}{2}\pi, n \in I)$

3 $1 + \cot^2 A = \operatorname{cosec}^2 A$ or $\operatorname{cosec}^2 A - \cot^2 A = 1$
or $(\operatorname{cosec} A - \cot A)(\operatorname{cosec} A + \cot A) = 1$
or $\operatorname{cosec} A + \cot A = \frac{1}{\operatorname{cosec} A - \cot A}$ $(A \neq n\pi, n \in I)$

4 Since $\sin^2 A + \cos^2 A = 1$ hence each of $\sin A$ and $\cos A$ is numerically less than or equal to unity that is
$|\sin A| \leq 1$ and $|\cos A| \leq 1$ or $-1 \leq \sin A \leq 1$
and $-1 \leq \cos A \leq 1$

5 Since $\sec A$ and $\operatorname{cosec} A$ are respectively reciprocals of $\cos A$ and $\sin A$ therefore the value of $\sec A$ and $\operatorname{cosec} A$ are always numerically greater than or equal to unity In other words
$\sec A \geq 1$, $\sec A \leq -1$ and $\operatorname{cosec} A \geq 1$, $\operatorname{cosec} A \leq -1$

6 To determine the values of other trigonometrical ratios in terms of a given ratio (The following method is valid only for acute angles)

Let $\sin \theta = s = \frac{s}{1} = \frac{\text{Perp}}{\text{Hyp}} = \frac{p}{h}$

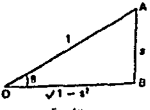


Fig 5v

Construct a triangle whose $p = s$ and $h = 1$ and hence the other side is $\sqrt{(1-s^2)}$
Now from the above triangle we can find the values of all the remaining ratios
Thus $\cos \theta = \sqrt{(1-s^2)} = \sqrt{(1-\sin^2 \theta)}$

$\tan \theta = \frac{s}{\sqrt{(1-s^2)}} = \frac{\sin \theta}{\sqrt{(1-\sin^2 \theta)}}$

The remaining three ratios are reciprocals of the above three ratios

Similarly if
$\cos \theta = c = \frac{c}{1} = \frac{\text{Base}}{\text{Hyp}} = \frac{b}{h}$, we construct a triangle where $b = c$

and $h=1$ so that $p=\sqrt{(1-c^2)}$

Now from the triangle we can find the values of all the remaining ratios Thus

$$\sin\theta=\sqrt{(1-c^2)}=\sqrt{(1-\cos^2\theta)},$$

$$\tan\theta=\frac{\sqrt{(1-c^2)}}{c}=\frac{\sqrt{(1-\cos^2\theta)}}{\cos\theta}$$

The remaining there ratios are reciprocals of the above three ratios

In a similar manner if

$$\tan\theta=t=\frac{t}{1}=\frac{\text{Perp}}{\text{Base}}=\frac{p}{b},$$ we

construct a triangle where $p=t$, $b=1$ so that $h=\sqrt{(1+t^2)}$

$$\sin\theta=\frac{t}{\sqrt{(1+t^2)}}=\frac{\tan\theta}{\sqrt{(1+\tan^2\theta)}}$$

$$\cos\theta=\frac{1}{\sqrt{(1+t^2)}}=\frac{1}{\sqrt{(1+\tan^2\theta)}}$$ etc

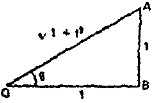


Fig 6)

In particular if $\sin\theta=5/13$ then taking $p=5$ and $h=13$, the base is $\sqrt{(13^2-5^2)}=12$

Hence $\cos\theta=\frac{12}{13}$, $\tan\theta=\frac{5}{12}$ etc

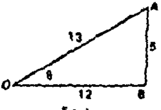


Fig 6

7 The following formulae of Algebra must be noted

$a^2+b^2=(a+b)^2-2ab$, $a^3+b^3=(a+b)^3-3ab\,(a+b)$

$a^3-b^3=(a-b)^3+3ab\,(a-b)$ $(a+b)\,(a-b)=a^2-b^2$

Method of componendo and dividendo

If $p/q=a/b$ then by componendo and dividendo we can write

$$\frac{p-q}{p+q}=\frac{a-b}{a+b} \text{ or } \frac{q-p}{q+p}=\frac{b-a}{b+a}$$

or

$$\frac{p+q}{p-q}=\frac{a+b}{a-b} \text{ or } \frac{q+p}{q-p}=\frac{b+a}{b-a}$$

Note Reference of the above formulae will be given in the solutions of problems

Problem Set (A)

Prove the following identities

1 (*a*) $2(\sin^6 A+\cos^6 A)-3(\sin^4 A+\cos^4 A)+1=0$

(*b*) $\sin^6 A+\cos^6 A+3\sin^2 A\cos^2 A=1$

(e) The expression

$$3\left[\sin^4\left(\frac{3\pi}{2}-\alpha\right)+\sin^4(3\pi+\alpha)\right]$$
$$-2\left[\sin^6\left(\frac{\pi}{2}+\alpha\right)+\sin^6(5\pi-\alpha)\right]$$

is equal to (A) 0, (B) 1, (C) 3, (D) $\sin 4\alpha+\cos 6\alpha$
(E) None of these (IIT 86)

2 The perimeter of a certain sector of a circle is equal to the length of the arc of a semi-circle having the same radius, express the angle of the sector in degrees, minutes and seconds

3 $3(\sin x-\cos x)^4+6(\sin x+\cos x)^2+4(\sin^6 x+\cos^6 x)=13$

4 $(\sin^8 A-\cos^8 A)=(\sin^2 A-\cos^2 A)(1-2\sin^2 A\cos^2 A)$

5 $(\tan A+\cot A)^2=\sec^2 A+\operatorname{cosec}^2 A=\sec^2 A\operatorname{cosec}^2 A$

6 $(1-\tan\alpha\tan\beta)^2+(\tan\alpha+\tan\beta)^2=\sec^2\alpha\sec^2\beta$

7 $\frac{\sec A+\tan A}{\sec A-\tan A}=1+2\sec A\tan A+2\tan^2 A$

8 $\frac{1}{\sec x-\tan x}-\frac{1}{\cos x}=\frac{1}{\cos x}-\frac{1}{\sec x+\tan x}$

9 (a) If $(\sec A+\tan A)(\sec B+\tan B)(\sec C+\tan C)$
$=(\sec A-\tan A)(\sec B-\tan B)(\sec C-\tan C)$
prove that each of the sides is equal to ± 1

(a) If $(1+\sin A)(1+\sin B)(1+\sin C)=(1-\sin A)(1-\sin B)\times(1-\sin C)$, show that each side is equal to $\pm\cos A\cos B\cos C$

10 $\sqrt{\left(\frac{1-\sin A}{1+\sin A}\right)}=\sec A-\tan A$, $\sqrt{\left(\frac{1+\cos\theta}{1-\cos\theta}\right)}=\operatorname{cosec}\theta+\cot\theta$

11 $\sec^4 A(1-\sin^4 A)-2\tan^2 A=1$

12 $\tan^2 A-\sin^2 A=\sin^4 A\sec^2 A=\tan^2 A\sin^2 A$

13 $\frac{\cot A+\tan B}{\cot B+\tan A}=\cot A\tan B$

14 $(1+\cot A-\operatorname{cosec} A)(1+\tan A+\sec A)=2$

15 $(\sin A+\cos A)(\tan A+\cot A)=\sec A+\tan A$

16 $(\operatorname{cosec}\theta-\sin\theta)(\sec\theta-\cos\theta)(\tan\theta+\cot\theta)=1$

17 $\frac{\cos A\operatorname{cosec} A-\sin A\sec A}{\cos A+\sin A}=\operatorname{cosec} A-\sec A$

18 $\frac{\tan A+\sec A-1}{\tan A-\sec A+1}=\frac{1+\sin A}{\cos A}$

19 $\frac{\sec x-1-\tan x}{\tan x-\sec x+1}=\frac{1+\cos x}{\sin x}$

20 $\dfrac{\cot^2\theta\,(\sec\theta-1)}{1+\sin\theta}=\sec^2\theta\,\dfrac{1-\sin\theta}{1+\sec\theta}$

21 $\dfrac{\cos A}{1-\tan A}+\dfrac{\sin A}{1-\cot A}=\sin A+\cos A$

22 $\dfrac{\tan A}{1-\cot A}+\dfrac{\cot A}{1-\tan A}=\sec A\operatorname{cosec} A+1$

23 $(\sin\alpha+\operatorname{cosec}\alpha)^2+(\cos\alpha+\sec\alpha)^2=\tan^2\alpha+\cot^2\alpha+7$

24 $(1+\cot A+\tan A)(\sin A-\cos A)=\dfrac{\sec A}{\operatorname{cosec}^2 A}-\dfrac{\operatorname{cosec} A}{\sec^2 A}$

25 $(\tan\theta+\operatorname{cosec}\phi)^2-(\cot\phi-\sec\theta)^2=2\tan\theta\cot\phi\,(\operatorname{cosec}\theta+\sec\phi)$

26 $\left(\dfrac{1+\sin\theta-\cos\theta}{1+\sin\theta+\cos\theta}\right)^2=\dfrac{1-\cos\theta}{1+\cos\theta}$

27 $\left(\dfrac{1}{\sec^2\alpha-\cos^2\alpha}+\dfrac{1}{\operatorname{cosec}^2\alpha-\sin^2\alpha}\right)\sin^2\alpha\cos^2\alpha$

$$=\frac{1-\sin^2\alpha\cos^2\alpha}{2+\sin^2\alpha\cos^2\alpha}$$

28 $(\operatorname{cosec}\theta-\sec\theta)(\cot\theta-\tan\theta)$

$$=(\operatorname{cosec}\theta+\sec\theta)(\sec\theta\operatorname{cosec}\theta-2)$$

29 If $\tan A+\sin A=m$ and $\tan A-\sin A=n$, then show that

$$m^2-n^2=4\sqrt{(mn)}$$

30 If $\operatorname{cosec}\theta-\sin\theta=m$, $\sec\theta-\cos\theta=n$, eliminate θ (I I T 76)

31 If $\cot\theta+\tan\theta=x$, $\sec\theta-\cos\theta=y$, eliminate θ

32 If $\cos x+\sin x=\sqrt{2}\cos x$, prove that $\cos x-\sin x=\sqrt{2}\sin x$

33 If $3\sin\theta+5\cos\theta=5$, show that $5\sin\theta-3\cos\theta=3$

34 If $a\cos\theta+b\sin\theta=p$, $a\sin\theta-b\cos\theta=q$, prove that $a^2+b^2=p^2+q^2$

35 If $a\cos\theta+b\sin\theta=c$, show that

$$a\sin\theta-b\cos\theta=\pm\sqrt{(a^2+b^2-c^2)}$$

36 If $\tan\theta=(1-e^2)$, prove that $\sec\theta+\tan^3\theta\operatorname{cosec}\theta=(2-e^2)^{3/2}$

37 If $\dfrac{ax}{\cos\theta}+\dfrac{by}{\sin\theta}=(a^2-b^2)$, $\dfrac{ax\sin\theta}{\cos^2\theta}-\dfrac{by\cos\theta}{\sin^2\theta}=0$, show that

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$$

38 If $\sin\theta=\dfrac{m^2-n^2}{m^2+n^2}$, determine the values of $\tan\theta$, $\sec\theta$ and $\operatorname{cosec}\theta$

39 If $\tan\theta=\dfrac{2x(x+1)}{2x+1}$, determine $\sin\theta$ and $\cos\theta$

40 If $\cos\theta = \frac{2x}{1+x^2}$, find the value of $\tan\theta$ and $\operatorname{cosec}\theta$

41 If $\tan\theta = p/q$, show that
$$\frac{p\sin\theta - q\cos\theta}{p\sin\theta + q\cos\theta} = \frac{p^2 - q^2}{p^2 + q^2}$$

42 Is the equation $\sec^2\theta = \frac{4xy}{(x+y)^2}$ possible for real values of x and y? If not, then find out a relation between x and y so that it may be possible

43 If $\sec\theta + \tan\theta = p$, obtain the values of $\sec\theta$, $\tan\theta$, $\sin\theta$ in terms of p

44 If $m^2 + m'^2 + 2mm' \cos\theta = 1$,
$n^2 + n'^2 + 2nn' \cos\theta = 1$,
and $mn + m'n' + (mn' + m'n)\cos\theta = 0$,
prove that $m^2 + n^2 = \operatorname{cosec}^2\theta$

45 If $C\cos^3\theta + 3C\cos\theta\sin^2\theta = m$,
$C\sin^3\theta + 3C\cos^2\theta\sin\theta = n$, then prove that
$(m+n)^{2/3} + (m-n)^{2/3} = 2C^{2/3}$

Solutions to Problem Set (A)

1 (a) We know that $\sin^2\theta + \cos^2\theta = 1$,
$a^3 + b^3 = (a+b)^3 - 3ab(a+b)$, $a^2 + b^2 = (a+b)^2 - 2ab$,
$\text{L H S} = 2\{(\sin^2 A + \cos^2 A)^3$
$- 3\sin^2 A\cos^2 A(\sin^2 A + \cos^2 A)\}$
$- 3\{(\sin^2 A + \cos^2 A)^2 - 2\sin^2 A\cos^2 A\} + 1$
$= 2 - 6\sin^2 A\cos^2 A - 3 + 6\sin^2 A\cos^2 A + 1 = 3 - 3 = 0$

(b) Proceed as in (a)

(c) Exp is $3(\cos^4\alpha + \sin^4\alpha) - 2(\cos^6\alpha + \sin^6\alpha) = 1$ by part (a)
(B) is correct

2 Let r be the radius of the circle and θ the number of radians in the angle. Then
$r\theta + 2r = \pi r$ or $\theta = (\pi - 2)$ radians
or $\theta = (\pi - 2) \times \frac{180^\circ}{\pi} = 180^\circ - 2 \times 57^\circ 17' 44.8''$
or $\theta = 180^\circ - 114^\circ 35' 29.6'' = 65^\circ 24' 30.4''$

(3 to 7) Do yourself

8 We have to prove that
$$\frac{1}{\sec x - \tan x} + \frac{1}{\sec x + \tan x} = \frac{2}{\cos x}$$
$$\text{L H S} = \frac{2\sec x}{\sec^2 x - \tan^2 x} = 2\sec x = \frac{2}{\cos x} = \text{R H S}$$

9 (*a*) Denoting the left hand and right hand sides by x and y, we have $x=y \Rightarrow x^2=xy$

or $x^2=(\sec^2 A-\tan^2 A)(\sec^2 B-\tan^2 B)(\sec^2 C-\tan^2 C)=1$

$x=\pm 1=y$

(*b*) If $x=y$, then $x^2=xy$

$x^2=(1-\sin^2 A)(1-\sin^2 B)(1-\sin^2 C)$

$=\cos^2 A\cos^2 B\cos^2 C$

$x=\pm\cos A\cos B\cos C=y$

10 L H S $=\sqrt{\left[\frac{(1-\sin A)^2}{(1-\sin^2 A)}\right]}=\frac{1-\sin A}{\cos A}=\frac{1}{\cos A}-\frac{\sin A}{\cos A}$

$=\sec A-\tan A$ etc

(11 to 17) Do yourself

18 We know that $\sec^2 A-\tan^2 A=1$, hence

$L\ H\ S=\frac{(\tan A+\sec A)-(\sec^2 A-\tan^2 A)}{\tan A-\sec A+1}$

$=\frac{(\tan A+\sec A)[1-(\sec A-\tan A)]}{\tan A-\sec A+1}$

$=\tan A+\sec A=\frac{\sin A+1}{\cos A}$

19 Change in terms of sin and cos

20 L H S $=\cot^2\theta\ \frac{\sec\theta-1}{1+\sin\theta}\ \frac{\sec\theta+1}{\sec\theta+1}$

$=\frac{\cot^2\theta(\sec^2\theta-1)}{(\sec\theta+1)(1+\sin\theta)}=\frac{\cot^2\theta\tan^2\theta}{(\sec\theta+1)}\ \frac{1-\sin\theta}{1-\sin^2\theta}$

$=\frac{1}{1+\sec\theta}\ \frac{1-\sin\theta}{\cos^2\theta}=\sec^2\theta\ \frac{1-\sin\theta}{1+\sec\theta}$

21 Do yourself

22 L H S $=\frac{\sin^2 A}{\cos A(\sin A-\cos A)}\ \frac{\cos^2 A}{\sin A(\sin A-\cos A)}$

$=\frac{\sin^3 A-\cos^3 A}{\sin A\cos A(\sin A-\cos A)}$

$=\frac{\sin^2 A+\cos^2 A+\sin A\cos A}{\sin A\cos A}=\frac{1+\sin A\cos A}{\sin A\cos A}$

$=\frac{1}{\sin A\cos A}+1=\sec A\operatorname{cosec} A+1$

23 Do yourself

24 L H S $=(\sin A-\cos A)\left(1+\frac{\cos A}{\sin A}+\frac{\sin A}{\cos A}\right)$

$=\frac{1}{\sin A\cos A}(\sin A-\cos A)(\sin^2 A+\cos^2 A+\sin A\cos A)$

$$=\frac{(\cos\theta-\sin\theta)^2}{\sin\theta\cos\theta}\cdot\frac{\cos\theta+\sin\theta}{\sin\theta\cos\theta}$$

$$=\frac{1-2\sin\theta\cos\theta}{\sin\theta\cos\theta}\left[\frac{1}{\sin\theta}+\frac{1}{\cos\theta}\right]$$

$=(\operatorname{cosec}\theta\sec\theta-2)(\operatorname{cosec}\theta-\sec\theta)=$ R H S

29 $m^2-n^2=(m+n)(m-n)=2\tan A\ 2\sin A=4\tan A\sin A$,

$4\sqrt{(mn)}=4\sqrt{(\tan^2 A-\sin^2 A)}$

$=4\sin A\sqrt{(\sec^2 A-1)}=4\sin A\tan A$

$\therefore m^2-n^2=4\sqrt{(mn)}$

30 $\operatorname{cosec}\theta-\sin\theta=m$ and $\sec\theta-\cos\theta=n$

$\frac{1}{\sin\theta}-\sin\theta=m$ and $\frac{1}{\cos\theta}-\cos\theta=n$

or $\frac{\cos^2\theta}{\sin\theta}=m$ and $\frac{\sin^2\theta}{\cos\theta}=n$

Multiplying, we get $\sin\theta\cos\theta=mn$ (1)

Again from $\cos^2\theta=m\sin\theta$ we get

$\cos^3\theta=m\sin\theta\cos\theta=m(mn)=m^2n$ by (1) (2)

Also from $\sin^2\theta=n\cos\theta$ we get

$\sin^3\theta=n(\cos\theta\sin\theta)=n(mn)=mn^2$ by (3) (3)

But $\cos^2\theta+\sin^2\theta=1$

or $(\cos^3\theta)^{2/3}+(\sin^3\theta)^{2/3}=1$ Put for $\cos^3\theta$ and $\sin^3\theta$ from (2) and (3), $(m^2n)^{2/3}+(mn^2)^{2/3}=1$ Proved

31 $\frac{1}{\tan\theta}+\tan\theta=x$, $\frac{1}{\cos\theta}-\cos\theta=y$

or $\frac{\sec^2\theta}{\tan\theta}=x$, $\frac{\sin^2\theta}{\cos\theta}=y$ or $\tan\theta\sin\theta=y$

Multiplying we get

$\sec^2\theta\sin\theta=xy$ or $\sec\theta\tan\theta=xy$ (1)

Again from $\sec^2\theta=x\tan\theta$, we get

$\sec^3\theta=x(\sec\theta\tan\theta)=x(xy)=x^2y$ (2)

Cubing both sides of (1) we get $\sec^3\theta\tan^3\theta=x^3y^3$

or $(x^2y)\tan^3\theta=x^3y^3$ by (2), $\tan^3\theta=xy^2$ (3)

But $\sec^2\theta-\tan^2\theta=1$ Hence from (2) and (3) we get

$(x^2y)^{2/3}-(xy^2)^{2/3}=1$

32 Squaring, $1+2\sin x\cos x=2(1-\sin^2 x)$

or $2\sin^2 x=1-2\sin x\cos x=(\cos x-\sin x)^2$

$\sqrt{2}\sin x=\cos x-\sin x$

33 Squaring $3\sin\theta+5\cos\theta=5$ we get

$9\sin^2\theta+25\cos^2\theta+30\sin\theta\cos\theta=25$

or $9(1-\cos^2\theta)+25(1-\sin^2\theta)+30\sin\theta\cos\theta=25$

$$9 = 9\cos^2\theta + 25\sin^2\theta - 30\sin\theta\cos\theta = (5\sin\theta - 3\cos\theta)^2$$

or $5\sin\theta - 3\cos\theta = 3$

34 Square and add and put $\sin^2\theta + \cos^2\theta = 1$

35 Square $a\cos\theta - b\sin\theta = c$ Then

$$a^2\cos^2\theta + b^2\sin^2\theta - 2ab\sin\theta\cos\theta = c^2$$

or $a^2(1-\sin^2\theta) + b^2(1-\cos^2\theta) - 2ab\sin\theta\cos\theta = c^2$

$$a^2 + b^2 - c^2 = a^2\sin^2\theta + b^2\cos^2\theta + 2ab\sin\theta\cos\theta$$

or $(a\sin\theta + b\cos\theta)^2 = a^2 + b^2 - c^2$

$$a\sin\theta + b\cos\theta = \pm\sqrt{(a^2+b^2-c^2)}$$

36 $\tan^2\theta = 1 - e^2$ (given)

Now $\sec\theta + \tan^3\theta\,\text{cosec}\,\theta$

$$= \sec\theta\left(1 + \tan^3\theta\,\frac{\text{cosec}\,\theta}{\sec\theta}\right) = \sec\theta\,(1 + \tan^3\theta\cot\theta)$$

$$= \sec\theta\,(1 + \tan^2\theta) = \sqrt{(1+\tan^2\theta)}\,(1 + \tan^2\theta)$$

$$= (1+\tan^2\theta)^{3/2} = (1 + 1 - e^2)^{3/2} = (2 - e^2)^{3/2}$$

37 From the second relation, we get $\dfrac{\sin^2\theta\sin\theta}{\cos^2\theta\cos\theta} = \dfrac{by}{ax}$

or $\tan^3\theta = \dfrac{by}{ax}$, $\tan\theta = \dfrac{(by)^{1/3}}{(ax)^{1/3}}$

$$\sin\theta = \frac{(by)^{1/3}}{[(ax)^{2/3} + (by)^{2/3}]^{1/2}},\quad \cos\theta = \frac{(ax)^{1/3}}{[(ax)^{2/3} + (by)^{2/3}]^{1/2}}$$

Putting for $\sin\theta$ and $\cos\theta$ in

$$\frac{ax}{\cos\theta} + \frac{by}{\sin\theta} = a^2 - b^2 \text{ we get}$$

$$[(ax)^{2/3} + (by)^{2/3}]^{1/2}\left[\frac{ax}{(ax)^{1/3}} + \frac{by}{(by)^{1/3}}\right] = a^2 - b^2$$

or $[(ax)^{2/3} + (by)^{2/3}]^{1/2}\,[(ax)^{2/3} + (by)^{2/3}] = a^2 - b^2$

or $[(ax)^{2/3} + (by)^{2/3}]^{3/2} = a^2 - b^2$

or $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$ Proved

38 We know that $(m^2+n^2)^2 - (m^2-n^2)^2 = 4m^2n^2$

$$\tan\theta = \frac{m^2-n^2}{2mn},\ \sec\theta = \frac{m^2+n^2}{2mn},\ \text{cosec}\,\theta = \frac{m^2+n^2}{m^2-n^2}$$

You have to make a triangle whose height is m^2-n^2 and hypotenuse m^2+n^2 so that its base is $2mn$ as explained in § 1 [illegible]

39 $\tan\theta = \dfrac{2x(x+1)}{2x+1}$

Make a triangle whose [illegible]

$PM = 2x(x+1)$, [illegible]

[illegible]

or $OP=[4x^4+8x^3+8x^2+4x+1]^{1/2}$

or $OP=[(2x^2)^2+(2x)^2+1+2(2x)(2x^2)+2(2x)1+2\cdot 1(2x^2)]^{1/2}$

$=[(2x^2+2x+1)^2]^{1/2}=2x^2+2x+1$

$\sin\theta=\dfrac{PM}{OP}$ and $\cos\theta=\dfrac{OM}{OP}$ gives

$\sin\theta=\dfrac{2x(x+1)}{2x^2+2x+1}$, $\cos\theta=\dfrac{2x+1}{2x^2+2x+1}$

40 $\tan\theta=\dfrac{1-x^2}{2x}$, $\operatorname{cosec}\theta=\dfrac{1+x^2}{1-x^2}$

41 $\dfrac{p\sin\theta-q\cos\theta}{p\sin\theta+q\cos\theta}=\dfrac{p\tan\theta-q}{p\tan\theta+q}$ on dividing by $\cos\theta$

$=\dfrac{p(p/q)-q}{p(p/q)+q}=\dfrac{p^2-q^2}{p^2+q^2}$, $\tan\theta=p/q$

42 Since $\sec^2\theta\geqslant 1$ we have from the given relation,

$\dfrac{4xy}{(x+y)^2}\geqslant 1$ or $(x+y)^2\leqslant 4xy$

or $(x+y)^2-4xy\leqslant 0$ or $(x-y)^2\leqslant 0$,

which implies that $x=y$

Thus the given equality can hold only when $x=y$

43 We know that $\sec^2\theta-\tan^2\theta=1$. Also $\sec\theta+\tan\theta=p$ (given)

$\sec\theta-\tan\theta=\dfrac{1}{\sec\theta+\tan\theta}=\dfrac{1}{p}$

Adding we get

$\sec\theta=\frac{1}{2}\left(p+\dfrac{1}{p}\right)$, subtracting we get $\tan\theta=\frac{1}{2}\left(p-\dfrac{1}{p}\right)$

Dividing we get $\sin\theta=\dfrac{p^2-1}{p^2+1}$

44 The given relations can be written as

$(m'+m\cos\theta)^2+m^2-m^2\cos^2\theta=1$

or $(m'+m\cos\theta)^2=1-m^2\sin^2\theta$ (1)

Similarly $(n'+n\cos\theta)^2=1-n^2\sin^2\theta$ (2)

and $(m'+m\cos\theta)(n'+n\cos\theta)+mn\sin^2\theta=0$

or $(m'+m\cos\theta)^2(n'+n\cos\theta)^2=m^2n^2\sin^4\theta$ (3)

Hence substituting from (1) and (2) in (3) we get

$(1-m^2\sin^2\theta)(1-n^2\sin^2\theta)=m^2n^2\sin^4\theta$

or $(m^2+n^2)\sin^2\theta=1$ i.e. $m^2+n^2=\operatorname{cosec}^2\theta$

45 Adding the given relations, we get

$C(\sin\theta+\cos\theta)^3=m+n$

or $C^{2/3}(\sin\theta+\cos\theta)^2=(m+n)^{2/3}$ (1)

Similarly, by subtracting, we shall get

$C^{2/3}(\sin\theta-\cos\theta)^2=(m-n)^{2/3}$ (2)

Now add (1) and (2)

§ 3 **Trigonometrical ratios for sum and difference**

1 $\sin(A+B)=\sin A\cos B+\cos A\sin B$

2 $\sin(A-B)=\sin A\cos B-\cos A\sin B$

3 $\cos(A+B)=\cos A\cos B-\sin A\sin B$

4 $\cos(A-B)=\cos A\cos B+\sin A\sin B$

5 $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

6 $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

($A\neq n\pi+\pi/2$, $B\neq m\pi+\pi/2$, $A\pm B\neq k\pi+\pi/2$)

7 $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A+\cot B}$

8 $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

($A\neq n\pi$, $B\neq m\pi$, $A\pm B\neq k\pi$)

*9 $\sin(A+B)\sin(A-B)=\sin^2 A-\sin^2 B=\cos^2 B-\cos^2 A$

*10 $\cos(A+B)\cos(A-B)=\cos^2 A-\sin^2 B=\cos^2 B-\sin^2 A$

11 $\sin 2\theta=2\sin\theta\cos\theta=\dfrac{2\tan\theta}{1+\tan^2\theta}$

*12 $\cos 2\theta=\cos^2\theta-\sin^2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}$

$=1-2\sin^2\theta=2\cos^2\theta-1$

*13 $1+\cos 2\theta=2\cos^2\theta$, $1-\cos 2\theta=2\sin^2\theta$

or $\cos^2\theta=\frac{1}{2}(1+\cos 2\theta)$ $\sin^2\theta=\frac{1}{2}(1-\cos 2\theta)$

14 (a) $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$ $\left(\theta\neq(2n+1)\dfrac{\pi}{4}\right)$

(b) $\dfrac{1-\cos\theta}{\sin\theta}=\tan\theta/2$ $(\theta\neq(2n+1)\pi)$

(c) $\dfrac{1+\cos\theta}{\sin\theta}=\cot\theta/2$ $(\theta\neq 2n\pi)$

(d) $\dfrac{1-\cos\theta}{1+\cos\theta}=\tan^2\theta/2$ $(\theta\neq(2n+1)\pi)$

(e) $\dfrac{1+\cos\theta}{1-\cos\theta}=\cot^2\theta/2$ $(\theta\neq 2n\pi)$

Triple angles

15 $\sin 3x=3\sin x-4\sin^3 x$ or $\sin^3 x=\frac{1}{4}(3\sin x-\sin 3x)$

16 $\cos 3x=4\cos^3 x-3\cos x$ or $\cos^3 x=\frac{1}{4}(3\cos x+\cos 3x)$

17 $\tan 3x = \dfrac{3\tan x - \tan^3 x}{1 - 3\tan^2 x} \left(x \neq n\pi + \dfrac{\pi}{6} \right)$

Sum and difference into products

18 $\sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

19 $\sin C - \sin D = 2 \cos \dfrac{C+D}{2} \sin \dfrac{C-D}{2}$

20 $\cos C + \cos D = 2 \cos \dfrac{C+D}{2} \cos \dfrac{C-D}{2}$

21 $\cos C - \cos D = 2 \sin \dfrac{C+D}{2} \sin \dfrac{D-C}{2}$ $\left(\text{Note } \dfrac{D-C}{2}\right)$

22 $\tan A \pm \tan B = \dfrac{\sin A}{\cos A} \pm \dfrac{\sin B}{\cos B} = \dfrac{\sin A \cos B \pm \cos A \sin B}{\cos A \cos B}$

$$= \frac{\sin (A \pm B)}{\cos A \cos B} \left(A \neq n\pi + \frac{\pi}{2},\ B \neq m\pi + \frac{\pi}{2} \right)$$

23 $\cot A \pm \cot B = \dfrac{\sin (B \pm A)}{\sin A \sin B} \ (A \neq n\pi,\ B \neq m\pi)$

24 $\cos A \pm \sin A = \sqrt{2} \sin \left(\dfrac{\pi}{4} \pm A \right) = \sqrt{2} \cos \left(\dfrac{\pi}{4} \mp A \right)$

Product into sum or difference

25 $2 \sin A \cos B = \sin (A+B) + \sin (A-B)$

26 $2 \cos A \sin B = \sin (A+B) - \sin (A-B)$

27 $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$

28 $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$ (Note)

29 **Values of Trigonometrical ratios for particular angles**

$\theta =$	0	30	45	60	90
$\sin \theta =$	$\sqrt{\frac{0}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{4}{4}}$
$\cos \theta =$	$\sqrt{\frac{4}{4}}$	$\sqrt{\frac{3}{4}}$	$\sqrt{\frac{2}{4}}$	$\sqrt{\frac{1}{4}}$	$\sqrt{\frac{0}{4}}$
$\tan \theta =$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined

The following table of values is useful to remember

θ	0	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	$5\pi/6$	π	$3\pi/2$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\frac{1}{\sqrt{2}}$	$-\frac{\sqrt{3}}{2}$	-1	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	undefined	$-\sqrt{3}$	-1	$-\frac{1}{\sqrt{3}}$	0	undefined
$\cot\theta$	un defined	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0	$-\frac{1}{\sqrt{3}}$	-1	$\sqrt{3}$	undefined	0
$\sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined	-2	$-\sqrt{2}$	$-\frac{2}{\sqrt{3}}$	-1	undefined
$\operatorname{cosec}\theta$	un defined	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	undefined	1

Note Students must note carefully that $\tan \pi/2$ is not defined. It can be easily seen that as $\theta \to \pi/2-$, $\tan\theta \to +\infty$ and as $\theta \to \pi/2+$, $\tan\theta \to -\infty$. Similarly it can be shown that $\tan 3\pi/2$, $\cot 0$, $\cot\pi$, $\sec \pi/2$, $\sec 3\pi/2$, $\operatorname{cosec} 0$ and $\operatorname{cosec}\pi$ are undefined.

30 Sign of Trigonometrical Ratios

Quadrant	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\operatorname{cosec}\theta$
I	+	+	+	+	+	+
II	+	−	−	−	−	+
III	−	−	+	+	−	−
IV	−	+	−	−	+	−

Aid to Memory First of all Remember the following sentence **Add Sugar To Coffee** This sentence consists of four words Write down the first letter of each word in the same order as

A, S, T, C

Taking the meaning of *A* as "All ratios are +ve" Second letter *S* means "Sine and cosec +ve and rest −ve" Third letter *T* means "Tan and cot +ve and rest −ve and the fourth letter *C* means "Cos and sec +ve and rest −ve"

31 Table for Reduction Formulae

θ	$\frac{\pi}{2}-\theta$	$\frac{\pi}{2}+\theta$	$\pi-\theta$	$\pi+\theta$	$\frac{3\pi}{2}-\theta$	$\frac{3\pi}{2}+\theta$	$2\pi-\theta$
$\sin\theta$	$\cos\theta$	$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$
$\cos\theta$	$\sin\theta$	$-\sin\theta$	$-\cos\theta$	$-\cos\theta$	$-\sin\theta$	$\sin\theta$	$\cos\theta$
$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$	$\tan\theta$	$\cot\theta$	$-\cot\theta$	$-\tan\theta$
$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$	$\cot\theta$	$\tan\theta$	$-\tan\theta$	$-\cot\theta$

Note How to remember the above Reduction Formulae Suppose $X'OX$ and $Y'OY$ divide the plane in four quadrants Then

(1) First determine the name of the trigonometrical ratio by the following rule, If θ is measured from $X'OX$ (that is, $\pi \pm \theta, 2\pi - \theta$), then retain the original name of the ratio and if θ is meanued from $Y'OY$ that is $\left(\frac{\pi}{2}+\theta, \frac{3\pi}{\theta} \pm \theta\right)$ then change sine, cosine, tangent, cotangent into cosine, sine, cotangent and tangent respectively

(2) Determine the sign of the trigonometrical ratio as follows Regarding θ as lying in the first quadrant find the quadrant in which $\frac{n\pi}{2} \pm \theta$ is situated and then determine the sign of the given ratio in this quadrant For example

$$\cot \frac{19\pi}{6} = \cot\left(3\pi+\frac{\pi}{6}\right) = \cot\left(\pi+\frac{\pi}{6}\right)$$

$$= \cot \pi/6, = \sqrt{3}$$

Note that by rule (1), the ratio in cot $(\pi+\pi/6)$ will remain cot and by rule (2), the sign will be positive since $\pi+\pi/6$, lies in the third quadrant Hence

$$\cot\left(\pi+\frac{\pi}{6}\right) = \cot \frac{\pi}{6}$$

Again $\tan\left(\frac{3\pi}{2}-\frac{\pi}{3}\right) = \cot\frac{\pi}{3} = \frac{1}{\sqrt{3}}$ since by rule (1) the ratio tan is changed to cot and by rule (2), sign is +ve Hence

$$\tan\left(\frac{3\pi}{2}-\frac{\pi}{3}\right) = \cot\frac{\pi}{3}$$

33 Values of trigonometrical ratios of some important angles

(i) $\sin 15° = \frac{1}{2\sqrt{2}}(\sqrt{3}-1)$, (ii) $\cos 15° = \frac{1}{2\sqrt{2}}(\sqrt{3}+1)$

(iii) $\tan 15° = 2-\sqrt{3}$ (iv) $\cot 15° = 2+\sqrt{3}$

(v) $\sin 22\frac{1}{2}° = \frac{1}{2}\sqrt{(2-\sqrt{2})}$ (vi) $\cos 22\frac{1}{2}° = \frac{1}{2}\sqrt{(2+\sqrt{2})}$

(vii) $\tan 22\frac{1}{2} = \sqrt{2}-1$ (viii) $\cot 22\frac{1}{2}° = \sqrt{2}+1$

(ix) $\sin 18° = \frac{1}{4}(\sqrt{5}-1) = \cos 72°$

(x) $\cos 18° = \frac{1}{4}\sqrt{(10+2\sqrt{5})} = \sin 72°$

(xi) $\sin 36° = \frac{1}{4}\sqrt{(10-2\sqrt{5})} = \cos 54°$

(xii) $\cos 36 = \frac{1}{4}(\sqrt{5}+1) = \sin 54$

34 Expression of sin A/2 and cos A/2 in terms of sin A

It is easy to see that

$$\left(\sin\frac{A}{2}+\cos\frac{A}{2}\right)^2 = 1+\sin A$$

$$\left(\sin\frac{A}{2}-\cos\frac{A}{2}\right)^2=1-\sin A$$

so that $\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{(1+\sin A)}$. (1)

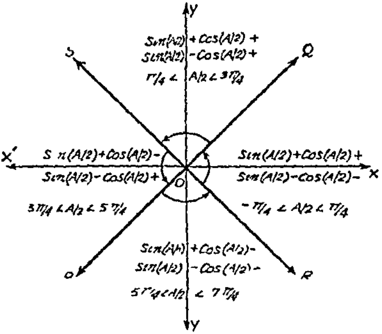
and $\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{(1-\sin A)}$. (2)

By addition and subtraction, we have

$$2\sin\frac{A}{2}=\pm\sqrt{(1+\sin A)}\pm\sqrt{(1-\sin A)} \quad (3)$$

and $2\cos\frac{A}{2}=\pm\sqrt{(1+\sin A)}\mp\sqrt{(1-\sin A)}$ (4)

The ambiguities of sign in relations (1) and (2) is determined by the following diagram



Problem Set (B)

1 Prove that

(*a*) $\cot 7\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)=\sqrt{2}+\sqrt{3}+\sqrt{4}+\sqrt{6}$ (**I I T 66**)

or $\tan 82\frac{1}{2}^\circ=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$ (**I I T 75**)

(*b*) $\cot 22\frac{1}{2}^\circ=\sqrt{\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right)}=\sqrt{\left(\frac{\sqrt{2}+1}{\sqrt{2}-1}\right)}=(\sqrt{2}+1)$

(*c*) $\tan(142^\circ 30')=2+\sqrt{2}-\sqrt{3}-\sqrt{6}$

(*d*) $\tan(11\frac{1}{4}^\circ)=\sqrt{\{4+2\sqrt{2}\}}-(\sqrt{2}+1)$

2 The efficiency E, of a certain screw jack is given by

$$E=\frac{1-\mu \tan \alpha}{\mu+\tan \alpha}$$

If $\mu=\tan \lambda$ prove that $E=\cot(\lambda+\alpha)$

Find λ if $E=1/\sqrt{3}$ and $\alpha=33°$

3 prove $\cot\theta-\cot 2\theta=\text{cosec}\, 2\theta$

4 If $m\tan(\theta-30°)=n\tan(\theta+120°)$, show that

$$\cos 2\theta=\frac{m+n}{2(m-n)}$$ (I I T 66)

Prove the following —

5 $\dfrac{1}{\tan 3A-\tan A}-\dfrac{1}{\cot 3A-\cot A}=\cot 2A$

6. $\dfrac{\sin(A-B)}{\sin A\sin B}+\dfrac{\sin(B-C)}{\sin B\sin C}+\dfrac{\sin(C-A)}{\sin C\sin A}=0$

7 $\dfrac{\sin(A-B)}{\cos A\cos B}+\dfrac{\sin(B-C)}{\cos B\cos C}+\dfrac{\sin(C-A)}{\cos C\cos A}=0$

8 $\dfrac{\tan(A+B)}{\cot(A-B)}=\dfrac{\sin^2 A-\sin^2 B}{\cos^2 A-\sin^2 B}$

9 If $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$
prove that $\cot\alpha\cot\beta\cot\gamma=\cot\delta$

10 $\sin^2(\pi/8+A/2)-\sin^2(\pi/8-A/2)=\dfrac{1}{\sqrt{2}}\sin A$

11 $\sin^2 24°-\sin^2 6°=\dfrac{\sqrt{5}-1}{8}$

12 $\sin^2(n+1)A-\sin^2 nA=\sin(2n+1)A\,\sin A$

13 $\sin^2 42°-\cos^2 78°=\dfrac{\sqrt{5}+1}{7}$

14 $\cos 2\theta\cos 2\phi+\sin^2(\theta-\phi)-\sin^2(\theta+\phi)=\cos 2(\theta+\phi)$

15 $\cos 2\alpha=2\sin^2\beta+4\cos(\alpha+\beta)\sin\alpha\sin\beta+\cos 2(\alpha+\beta)$ (I I T 77)

16 $\sin^2 A=\cos^2(A-B)+\cos^2 B-2\cos(A-B)\cos A\cos B$

17 $\sin^2 B=\sin^2 A+\sin^2(A-B)-2\sin A\cos B\sin(A-B)$

18 $\cos^2 A+\cos^2 B-2\cos A\cos B\cos(A+B)=\sin^2(A+B)$

19 (i) $\tan 3A\tan 2A\tan A=\tan 3A-\tan 2A-\tan A$ (M N R 82)

(ii) $\tan 15°+\tan 30°+\tan 15°\tan 30°=1$

20 $\tan 70°-\tan 20°=2\tan 50$

21 If $A+B=45°$ prove that

(i) $(1+\tan A)(1+\tan B)=2$, (I I T 73)

(ii) $(\cot A-1)(\cot B-1)=2$

22 (*i*) $\tan\alpha=\dfrac{m}{m+1}$, $\tan\beta=\dfrac{1}{2m+1}$, show that $\alpha+\beta=\pi/4$ **(I I T 78)**

(*ii*) If $\tan\alpha=\frac{1}{2}$, $\tan\beta=\frac{1}{3}$, show that $\alpha+\beta=\pi/4$ **(*I I T* 67)**

23 If $A+B=225°$, prove that

$$\frac{\cot A}{1+\cot A}\,\frac{\cot B}{1+\cot B}=\frac{1}{2}$$

24 Prove that

(*i*) $\tan\left(\dfrac{\pi}{4}+\theta\right)-\tan\left(\dfrac{\pi}{4}-\theta\right)=2\tan 2\theta$

(*ii*) $\tan\left(\dfrac{\pi}{4}+\theta\right)+\tan\left(\dfrac{\pi}{4}-\theta\right)=2\sec 2\theta$

(*iii*) $\tan\left(\dfrac{\pi}{4}+\theta\right)\tan\left(\dfrac{\pi}{4}-\theta\right)=1$

(*iv*) $\operatorname{cosec}\left(\dfrac{\pi}{4}+\theta\right)\operatorname{cosec}\left(\dfrac{\pi}{4}-\theta\right)=\sec\left(\dfrac{\pi}{4}+\theta\right)\sec\left(\dfrac{\pi}{4}-\theta\right)$
$=2\sec 2\theta,$

(*v*) $\tan(\pi/4+\theta/2)=\sqrt{\left(\dfrac{1+\sin\theta}{1-\sin\theta}\right)}=\tan\theta+\sec\theta$

(*vi*) $\dfrac{\tan^2 2\theta-\tan^2\theta}{1-\tan^2 2\theta\tan^2\theta}=\tan 3\theta\tan\theta$

25 $\dfrac{\cos 9°+\sin 9°}{\cos 9°-\sin 9°}=\tan 54°$

26 (*i*) $\tan A+\cot A=2\operatorname{cosec} 2A$

(*ii*) $\cot A-\tan A=2\cot 2A$

(*iii*) $\tan A+2\tan 2A+4\tan 4A+8\cot 8A=\cot A$ **(I I T 88)**

(*iv*) $\operatorname{cosec} A-2\cot 2A\cos A=2\sin A$

27 (*i*) $\operatorname{cosec} 2A-\cot 2A=\tan A$

(*ii*) $\dfrac{1-\cos A}{\sin A}=\tan\dfrac{A}{2}$ (*iii*) $\dfrac{1+\cos A}{\sin A}=\cot\dfrac{A}{2}$

(*iv*) $2\sin A\cos^3 A-2\sin^3 A\cos A=\frac{1}{2}\sin 4A$, **(*Roorkee* 75)**

28 (*i*) $\dfrac{1+\sin 2\theta-\cos 2\theta}{1+\sin 2\theta+\cos 2\theta}=\tan\theta$

(*ii*) $\dfrac{\sin\theta+\sin 2\theta}{1+\cos\theta+\cos 2\theta}=\tan\theta$

(*iii*) $2\cos\theta=\sqrt{\{2+\sqrt{(2+2\cos 4\theta)}\}}$

(*iv*) $1+\cos 56°+\cos 58°-\cos 66°=4\cos 28°\cos 29°\sin 33°$ **(I I T 64)**

29 (*a*) $\frac{\sec 8\theta-1}{\sec 4\theta-1}=\frac{\tan 8\theta}{\tan 2\theta}$

(*b*) $\tan A \sec 4A+\tan 4A=\tan A+\tan 4A \sec 2A$

30 If $2 \tan \beta+\cot \beta=\tan \alpha$, then $\cot \beta=2 \tan (\alpha-\beta)$

31 If $\cos 2\alpha=\frac{3 \cos 2\beta-1}{3-\cos 2\beta}$, then $\tan \alpha=\sqrt{2} \tan \beta$

32 If $\tan \theta=b/a$, then prove that $a \cos 2\theta+b \sin 2\theta=a$

33 Prove (*i*) $\sin A \sin (60°-A) \sin (60°+A)=\frac{1}{4} \sin 3A$

(*ii*) $\cos A \cos (60°-A) \cos (60°+A)=\frac{1}{4} \cos 3A$

(*iii*) If $\cos \theta=\frac{1}{2}(a+1/a)$, then prove that $\cos 3\theta=\frac{1}{2}(a^3+1/a^3)$

34 Prove (*i*) $\cos 5\theta=16 \cos^5 \theta-20 \cos^3 \theta+5 \cos \theta$,

(*ii*) $\sin 5\theta=5 \cos^4 \theta \sin \theta-10 \cos^2 \theta \sin^3 \theta+\sin^5 \theta$

(*iii*) *Express* $\sin 5\theta$ *in terms of* $\sin \theta$ *and hence find the value of* $\sin 36°$ (Roorkee 82)

Prove the following —

35 (*i*) $\cos^3 \theta+\cos^3 (120°+\theta)+\cos^3 (240°+\theta)=\frac{3}{4} \cos 3\theta$

(*ii*) $\cos^2 A+\cos^2 (A+120°)+\cos^2 (A-120°)=\frac{3}{2}$

(*iii*) $\cos^4 \frac{\pi}{8}+\cos^4 \frac{3\pi}{8}+\cos^4 \frac{5\pi}{8}+\cos^4 \frac{7\pi}{8}=\frac{3}{2}$

(*iv*) $\sin^4 \frac{\pi}{8}+\sin^4 \frac{3\pi}{8}+\sin^4 \frac{5\pi}{8}+\sin^4 \frac{7\pi}{8}=\frac{3}{2}$

(*Roorkee 80*)

36 (*a*) $\tan \theta \tan (\theta+60°)+\tan \theta \tan (\theta-60°)$
$+\tan (\theta+60°) \tan (\theta-60°)=-3$

(*b*) $\cot 16° \cot 44°+\cot 44° \cot 76°-\cot 76° \cot 16°=3$

37 (*i*) $\tan A+\tan (60°+A)-\tan (60°-A)=3 \tan 3A$

(*ii*) $\cot \alpha+\cot (60°+\alpha)-\cot (60°-\alpha)=3 \cot 3\alpha$

38 $\frac{\sin^2 A-\sin^2 B}{\sin A \cos A-\sin B \cos B}=\tan (A+B)$

39 (*i*) $\frac{\sin A+\sin 3A+\sin 5A+\sin 7A}{\cos A+\cos 3A+\cos 5A+\cos 7A}=\tan 4A$

(*ii*) $\frac{\sin 3\theta+\sin 5\theta+\sin 7\theta+\sin 9\theta}{\cos 3\theta+\cos 5\theta+\cos 7\theta+\cos 9\theta}=\tan 6\theta$ (Roorkee 73)

(*iii*) $\frac{\cos 3A+2 \cos 5A+\cos 7A}{\cos A+2 \cos 3A+\cos 5A}=\frac{\cos 5A}{\cos 3A}$
$=\cos 2A-\sin 2A \tan 3A$

(*iv*) $\frac{\cos 6\theta+6 \cos 4\theta+15 \cos 2\theta+10}{\cos 5\theta+5 \cos 3\theta+10 \cos \theta}=2 \cos \theta$

(*v*) $2 \cos x-\cos 3x-\cos 5x=16 \cos^3 x \sin^2 x$ (Roorkee 74) (*vi*) $1+\cos 2x+\cos 4x+\cos 6x=4\cos x\cos 2x\cos 3x$ (Roorkee 74)

40 (*i*) $(\cos\alpha+\cos\beta)^2+(\sin\alpha+\sin\beta)^2=4\cos^2\dfrac{\alpha-\beta}{2}$

(*ii*) $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=4\sin^2\dfrac{\alpha-\beta}{2}$

(*iii*) $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$$

(*iv*) $\left(\dfrac{\cos A+\cos B}{\sin A-\sin B}\right)^n+\left(\dfrac{\sin A+\sin B}{\cos A-\cos B}\right)^n=2\cot^n\left(\dfrac{A-B}{2}\right)$ or 0

according as n is an even or an odd positive integer

(*v*) If three angles A, B, C are in A P, prove that

$$\cot B=\frac{\sin A-\sin C}{\cos C-\cos A}$$

(*vi*) If A, B, C, D are angles of a cyclic quadrilateral, prove that $\cos A+\cos B+\cos C+\cos D=0$ (I I T 70)

41 (*i*) $\cos 20°+\cos 100°+\cos 140°=0$

(*ii*) $\sin 50°-\sin 70°+\sin 10°=0$ (M N R 79)

(*iii*) $2\cos\dfrac{\pi}{13}\cos\dfrac{9\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{5\pi}{13}=0$

(*iv*) $\cos 20°\cos 100°+\cos 100°\cos 140°-\cos 140°\cos 200°$

$$=-3/4$$

(*v*) $\cos\dfrac{2\pi}{7}+\cos\dfrac{4\pi}{7}+\cos\dfrac{6\pi}{7}=-\dfrac{1}{2}$

(*vi*) $\tan 10°+\tan 70°-\tan 50°=\sqrt{3}$

(*vii*) $\cos^2 73°+\cos^2 47°+\cos 73°\cos 47°=\dfrac{3}{4}$

(*viii*) $\cos 0+\cos\dfrac{\pi}{7}+\cos\dfrac{2\pi}{7}+\cos\dfrac{3\pi}{7}+\cos\dfrac{4\pi}{7}+\cos\dfrac{5\pi}{7}$

$$+\cos\frac{6\pi}{7}=1$$

42 (*i*) $\dfrac{\cos 2A\cos 3A-\cos 2A\cos 7A+\cos A\cos 10A}{\sin 4A\sin 3A-\sin 2A\sin 5A+\sin 4A\sin 7A}$

$$=\cot 6A\cot 5A$$

(*ii*) $\dfrac{\cos 8A\cos 5A-\cos 12A\cos 9A}{\sin 8A\cos 5A+\cos 12A\sin 9A}=\tan 4A$

(*iii*) If in a $\triangle ABC$, $\cos A=\dfrac{\sin B}{2\sin C}$

prove that it is an isosceles triangle

43 Prove that

$$\cos A \cos 2A \cos 4A \cos 8A = \frac{\sin 2^4 A}{2^4 \sin A}$$ **(Remember)**

and hence prove the following

(i) $\cos 20° \cos 40° \cos 60° \cos 80° = \frac{1}{16}$

Another form

$\sin 10° \sin 30° \sin 50° \sin 70° = \frac{1}{16}$

(ii) $\cos 7° \cos 14° \cos 28° \cos 56° = \frac{\sin 68°}{16 \cos 83°}$

(iii) $\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{16\pi}{15} = \frac{1}{16}$ **(I I T 83)**

Another form

$\cos \frac{2\pi}{15} \cos \frac{4\pi}{15} \cos \frac{8\pi}{15} \cos \frac{14\pi}{15} = \frac{1}{16}$

(iv) $\cos \frac{\pi}{7} \cos \frac{2\pi}{7} \cos \frac{4\pi}{7} = -\frac{1}{8}$

(v) $\cos \frac{\pi}{5} \cos \frac{2\pi}{5} = \frac{1}{4}$ and $\cos \frac{\pi}{5} + \cos \frac{3\pi}{5} = \frac{1}{2}$

without using standard values

(vi) $\sin 20° \sin 40° \sin 60° \sin 80° = \frac{3}{16}$ **(M N R 81)**

(vii) $\tan 20° \tan 40° \tan 60° \tan 80° = 3$ **(I I T 74)**

(viii) $\sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14} = \frac{1}{8}$

44 (a) $\sin 12° \sin 48° \sin 54° = \frac{1}{8}$ **(I I T 82)**

(b) $\sin \frac{\pi}{10} \sin \frac{13\pi}{10}$ is equal to

(i) $\frac{1}{2}$ (ii) $-\frac{1}{2}$ (iii) $-\frac{1}{4}$ (iv) 1 **(M N R 84)**

(c) $\cos 60° \cos 36° \cos 42° \cos 78° = 1/16$

(d) $\sin 6° \sin 42° \sin 66° \sin 78 = 1/16$

and $\cos 6° \cos 42° \cos 66° \cos 78° = 1/16$

(e) $\tan 6° \tan 42° \tan 66° \tan 78° = 1$

(f) $\sin 36° \sin 72° \sin 108° \sin 144° = 5/16$

(g) $\left(1+\cos \frac{\pi}{8}\right)\left(1+\cos \frac{3\pi}{8}\right)\left(1+\cos \frac{5\pi}{8}\right)\left(1+\cos \frac{7\pi}{8}\right) = \frac{1}{8}$

(h) $\cos \frac{\pi}{15} \cos \frac{2\pi}{15} \cos \frac{3\pi}{15} \cos \frac{4\pi}{15} \cos \frac{5\pi}{15} \cos \frac{6\pi}{15} \cos \frac{7\pi}{15} = \frac{1}{128}$

45 (*i*) If $\cos(\alpha+\beta)=4/5$ and $\sin(\alpha-\beta)=5/13$ and α, β, lie between 0 and $\pi/4$, find $\tan 2\alpha$ (I I T 79)

(*ii*) If $\cos\alpha=3/5$, $\cos\beta=5/13$, prove that

$$\sin^2\frac{\alpha-\beta}{2}=\frac{1}{65}, \cos^2\frac{\alpha-\beta}{2}=\frac{64}{65}$$

46 (*a*) If $\sin x+\sin y=a$ and $\cos x+\cos y=b$, show that

(*i*) $\sin(x+y)=\dfrac{2ab}{a^2+b^2}$

(*ii*) $\cos(x-y)=\dfrac{a^2+b^2-2}{2}$,

(*iii*) $\tan\dfrac{x-y}{2}=\pm\sqrt{\left(\dfrac{4-a^2-b^2}{a^2+b^2}\right)}$

(*b*) If $\sin x+\cos x=a$, evaluate $\sin^6 x+\cos^6 x$

47 (*i*) If $\tan A-\tan B=x$ and $\cot B-\cot A=y$, prove that

$$\cot(A-B)=\frac{1}{x}+\frac{1}{y}$$

(*ii*) If $2\cos A=x+1/x$ $2\cos B=y+1/y$, show that $2\cos(A-B)=x/y+y/x$

48 If α and β are the solutions of $a\cos\theta+b\sin\theta=c$, then show that

(*i*) $\sin\alpha+\sin\beta=\dfrac{2bc}{a^2+b^2}$, $\sin\alpha\sin\beta=\dfrac{c^2-a^2}{a^2+b^2}$

(*ii*) $\cos\alpha+\cos\beta=\dfrac{2ac}{a^2+b^2}$, $\cos\alpha\cos\beta=\dfrac{c^2-b^2}{a^2+b^2}$

49 (*i*) If α and β are the solutions of the equation $a\tan\theta+b\sec\theta=c$, then show that

$$\tan(\alpha+\beta)=\frac{2ac}{a^2-c^2}$$ (I I T 64, 73)

(*ii*) If $a\cos 2\theta+b\sin 2\theta=c$ has α and β as its solutions then prove that

$$\tan\alpha+\tan\beta=\frac{2b}{c+a}, \tan\alpha\tan\beta=\frac{c-a}{c+a}$$ (I I T 70)

(*iii*) If α and β are the two different values of θ lying between 0 and 2π which satisfy $3\cos\theta+4\sin\theta=6$ Find the value of $\sin(\alpha+\beta)$ In case the given equation be wrong then try it with 9/2 in place of 6 in R H S (I I T 71)

50 (*i*) Find the max and min value of $7\cos\theta+24\sin\theta$

(*ii*) Show that the max and min, values of $8\cos\theta-15\sin\theta$ are 17 and -17 respectively

(iii) Prove that $5\cos\theta+3\cos(\theta+\pi/3)+3$ lies between -4 and 10

(iv) Max value of $\sin x+\cos x$ is

(i) 1 (ii) 2 (iii) $\sqrt{2}$ (iv) $1/\sqrt{2}$ **(IIT 79)**

(v) The value of $\sqrt{3}\sin x+\cos x$ is max when x is equal to

(i) 30°, (ii) 90° (iii) 60° (iv) 45°, **(MNR 83)**

(vi) Prove that $\frac{1}{\sin 10^\circ}-\frac{\sqrt{3}}{\cos 10^\circ}=4$ **(MNR 78)**

(vii) Prove that $\sqrt{3}\,\text{cosec}\,20^\circ-\sec 20^\circ=4$ **(IIT 74)**

51 If $\cos^2\theta=\frac{1}{3}(a^2-1)$ and $\tan^2\frac{\theta}{2}=\tan\alpha$, **(IIT 88)**

Prove that $\cos^{2/3}\alpha+\sin^{2/3}\alpha=\left(\frac{2}{a}\right)^{2/3}$

52 (i) If $\tan\theta/2=\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\phi/2$, prove that

$$\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}$$

(ii) If $\cos\theta=\frac{a\cos\phi+b}{a+b\cos\phi}$, prove that

$$\tan\theta/2=\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\phi/2$$ **(Roorkee 78)**

(iii) If $\cos\theta=\frac{2\cos\phi-1}{2-\cos\phi}$ prove that $\tan\theta/2=\sqrt{3}\tan\phi/2$

and hence show that $\sin\phi=\frac{\sqrt{3}\sin\theta}{2+\cos\theta}$

53 (i) If $\cos\theta=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$ then prove that

$\tan\theta/2=\pm\tan\alpha/2\cot\beta/2$

(ii) If $\tan\beta=\cos\theta\tan\alpha$ then prove that

$\sin(\alpha-\beta)=\tan^2(\theta/2)\sin(\alpha+\beta)$

54 If $\tan^2\theta=2\tan^2\phi+1$, than $\cos 2\theta+\sin^2\phi=0$

55 If an angle θ be divided into two parts such that the tangent of one part is m times the tangent of the other then prove that their difference ϕ is obtained from the equation

$$\sin\phi=\frac{m-1}{m+1}\sin\theta$$

(ii) If $m\cos(\theta+\alpha)=n\cos(\theta-\alpha)$, show that

$(m-n)\cot\theta=(m+n)\tan\alpha$

(iii) If $\cos x=k\cos(x-2y)$ show that

$$\tan (x-y) \tan y = \frac{1-k}{1+k}$$

(iv) If $\sin \theta = n \sin (\theta+2\alpha)$, show that

$$\tan (\theta+\alpha) = \frac{1+n}{1-n} \tan \alpha$$

(v) If $\cot^2 \theta = \cot (\theta-\alpha) \cot (\theta-\beta)$, show that $\cot 2\theta = \frac{1}{2} (\cot \alpha + \cot \beta)$

(vi) If $\tan \beta = \frac{\tan \alpha + \tan \gamma}{1+\tan \alpha \tan \gamma}$, prove that

$$\sin 2\beta = \frac{\sin 2\alpha + \sin 2\gamma}{1+\sin 2\alpha \sin 2\gamma}$$

56 If $\theta = \frac{\pi}{2^n+1}$, prove that

$2^n \cos \theta \cos 2\theta \cos 2^2 \theta \ldots \cos 2^{n-1} \theta = 1$

57 If $\sin (y+z-x)$, $\sin (z+x-y)$, $\sin (x+y-z)$ be in A P, Prove that $\tan x$, $\tan y$, $\tan z$ are also in A P

58 If $\sec (\phi-\alpha)$, $\sec \phi$, $\sec (\phi+\alpha)$ are in A P, prove that $\cos \phi = \sqrt{2} \cos \alpha/2$

59 prove $\frac{1+\sin A}{\cos A} + \frac{\cos B}{1-\sin B} = \frac{2 \sin A - 2 \sin B}{\sin (A-B) + \cos A - \cos B}$

60 If $\tan \left(\frac{\pi}{4} + \frac{y}{2}\right) = \tan^3 \left(\frac{\pi}{4} + \frac{x}{2}\right)$, prove that

$$\sin y = \sin x \frac{3+\sin^2 x}{1+3 \sin^2 x}$$

61 $\sin 3x \sin^3 x + \cos 3x \cos^3 x = \cos^3 2x$

62 Let $\cos \alpha = \cos \beta \cos \phi = \cos \gamma \cos \theta$, $\sin \alpha = 2 \sin \phi/2 \sin \theta/2$, prove that $\tan^2 \alpha/2 = \tan^2 \beta/2 \tan^2 \gamma/2$

63 Show that for all real values of θ the expression $a \sin \theta + b \sin \theta \cos \theta + a \cos^2 \theta$ lies between $\frac{1}{2}(a+c) - \frac{1}{2}\sqrt{\{b^2+(a-c)^2\}}$ and $\frac{1}{2}(a+c) + \frac{1}{2}\sqrt{\{b^2+(a-c)^2\}}$

64 (a) Prove that

$$\tan \alpha + \tan \beta + \tan \gamma - \tan \alpha \tan \beta \tan \gamma = \frac{\sin (\alpha+\beta+\gamma)}{\cos \alpha \cos \beta \cos \gamma}$$

(b) If $0 < \alpha, \beta, \gamma < \pi/2$ prove that

$\sin \alpha + \sin \beta + \sin \gamma > \sin (\alpha+\beta+\gamma)$

65 If $\frac{\sin (\theta+A)}{\sin (\theta+B)} = \sqrt{\left(\frac{\sin 2A}{\sin 2B}\right)}$ prove that $\tan^2 \theta = \tan A \tan B$

66 If $\cos x = \tan y$, $\cos y = \tan z$, $\cos z = \tan x$, prove $\sin x = \sin y = \sin z = 2 \sin 18°$

67 If $\sin x+\sin y=3(\cos y-\cos x)$, prove that

$\sin 3x+\sin 3y=0$

68 Given that the angles α, β, γ are connected by the relation

$$2\tan^2\alpha\tan^2\beta\tan^2\gamma+\tan^2\alpha\tan^2\beta+\tan^2\beta\tan^2\gamma+\tan^2\gamma\tan^2\alpha=1$$

find the value of $\sin^2\alpha+\sin^2\beta+\sin^2\gamma$

69 (a) If $\tan\theta\tan\phi=\sqrt{\left[\dfrac{a-b}{a+b}\right]}$ prove that

$(a-b\cos 2\theta)(a-b\cos 2\phi)$ is independent of θ and ϕ

(b) If $\dfrac{\cos^4 x}{\cos^2 y}+\dfrac{\sin^4 x}{\sin^2 y}=1$, prove that $\dfrac{\cos^4 y}{\cos^2 x}+\dfrac{\sin^4 y}{\sin^2 x}=1$

70 (a) Show that

$$\cos^3(\alpha+\theta)\sin^3(\beta-\gamma)+\cos^3(\beta+\theta)\sin^3(\gamma-\alpha)+\cos^3(\gamma+\theta)\sin^3(\alpha-\beta)$$

$$=3\cos(\alpha+\theta)\cos(\beta+\theta)\cos(\gamma+\theta),\ \sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$$

(b) If $\sqrt{2}\cos A=\cos B+\cos^3 B$ and $\sqrt{2}\sin A=\sin B-\sin^3 B$

show that $\sin(A-B)=\pm\frac{1}{3}$

Solution to Problem Set (B)

1 $\tan 82\frac{1}{2}^\circ=\tan(90^\circ-7\frac{1}{2}^\circ)=\cot 7\frac{1}{2}^\circ=\cot A$, say, where $A=7\frac{1}{2}^\circ$

Now $\cot A=\dfrac{\cos A}{\sin A}=\dfrac{2\cos^2 A}{2\sin A\cos A}=\dfrac{1+\cos 2A}{\sin 2A}=\dfrac{1+\cos 15^\circ}{\sin 15^\circ}$

or $\cot 7\frac{1}{2}^\circ=\dfrac{1+\cos(45^\circ-30^\circ)}{\sin(45^\circ-30^\circ)}=\dfrac{1+\left(\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}+\frac{1}{\sqrt{2}}\frac{1}{2}\right)}{\frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2}-\frac{1}{\sqrt{2}}\frac{1}{2}}$

$=\dfrac{2\sqrt{2}+(\sqrt{3}+1)}{\sqrt{3}-1}\times\dfrac{\sqrt{3}+1}{\sqrt{3}+1}$

$=\dfrac{2\sqrt{2}(\sqrt{3}+1)+(\sqrt{3}+1)^2}{3-1}$

$=\dfrac{2\sqrt{6}+2\sqrt{2}+4+2\sqrt{3}}{2}=\sqrt{6}+\sqrt{2}+2+\sqrt{3}$

$=\sqrt{2}(\sqrt{2}+1)+\sqrt{3}(\sqrt{2}+1)=(\sqrt{2}+1)(\sqrt{3}+\sqrt{2})$

(b) Here choose $A=22\frac{1}{2}^\circ$ $\quad 2A=45^\circ$ etc

(c) $\tan(142^\circ 30')=\tan(90^\circ+45^\circ+7\frac{1}{2}^\circ)=-\cot(45^\circ+7\frac{1}{2}^\circ)$

$=\dfrac{-1}{\tan(45^\circ+7\frac{1}{2}^\circ)}=-\dfrac{1-\tan 7\frac{1}{2}^\circ}{1+\tan 7\frac{1}{2}^\circ}=\dfrac{\tan 7\frac{1}{2}^\circ-1}{\tan 7\frac{1}{2}^\circ+1}$

If $A=7\frac{1}{2}°$ then $\tan A=\frac{\sin A}{\cos A}=\frac{2\sin^2 A}{2\sin A\cos A}=\frac{1-\cos 2A}{\sin 2A}$

$$\frac{1-\cos 15°}{\sin 15°}$$

or $\tan 7\frac{1}{2}°=\sqrt{6}-\sqrt{3}+\sqrt{2}-2$ As in part (a)

Hence from (1) we get

$$\tan(142°\ 30')=\frac{\sqrt{6}-\sqrt{3}+\sqrt{2}-3}{\sqrt{6}-\sqrt{3}+\sqrt{2}-1}=\frac{(\sqrt{3}+1)(\sqrt{2}-\sqrt{3})}{(\sqrt{3}+1)(\sqrt{2}-1)}$$

$$=\frac{(\sqrt{2}-\sqrt{3})(\sqrt{2}+1)}{2-1}$$

$$=2+\sqrt{2}-\sqrt{3}-\sqrt{6}$$

$$\tan 22\tfrac{1}{2}°=\frac{1-\cos 45°}{\sin 45°}=\sqrt{2}-1=\frac{1}{\sqrt{2}+1}$$

$$\tan 11\tfrac{1}{4}°=\frac{1-\cos 22\frac{1}{2}°}{\sin 22\frac{1}{2}°}=\operatorname{cosec} 22\tfrac{1}{2}°-\cot 22\tfrac{1}{2}°,$$

$$=\sqrt{(1+\cot^2 22\tfrac{1}{2}°)}-\cot 22\tfrac{1}{2}=\sqrt{\{1+(\sqrt{2}+1)^2\}}-(\sqrt{2}+1)$$

$$=\sqrt{(4+2\sqrt{2})}-(\sqrt{2}+1)$$

2 Putting $\mu=\tan\lambda$ in the given relation we get

$$E=\frac{1-\mu\tan\alpha}{\mu+\tan\alpha}=\frac{1-\tan\lambda\tan\alpha}{\tan\lambda+\tan\alpha}=\frac{1}{\tan(\lambda+\alpha)}=\cot(\lambda+\alpha)$$

Now $E=\frac{1}{\sqrt{3}}$ and $\alpha=33°$ (given)

$$\frac{1}{\sqrt{3}}=\cot(\lambda+33°), \qquad \lambda+33°=\cot^{-1}\left(\frac{1}{\sqrt{3}}\right)=60°$$

$$\lambda=60°-33°=27°$$

3 Changing to $\sin\theta$ and $\cos\theta$, we get

$$\cot\theta-\cot 2\theta=\frac{\cos\theta}{\sin\theta}-\frac{\cos 2\theta}{\sin 2\theta}$$

$$=\frac{\sin 2\theta\cos\theta-\cos 2\theta\sin\theta}{\sin\theta\sin 2\theta}=\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}=\operatorname{cosec} 2\theta$$

4 $\frac{m}{n}=\frac{\tan(\theta+120°)}{\tan(\theta-30°)}=\frac{\tan A}{\tan B}=\frac{\sin A\cos B}{\cos A\sin B}$

where $A=\theta+120°$ and $B=\theta-30°$

Applying componendo and dividendo, we get

$$\frac{m+n}{m-n}=\frac{\sin(A+B)}{\sin(A-B)}=\frac{\sin(2\theta+90°)}{\sin 150°}$$

$$=\frac{\cos 2\theta}{\sin(180°-30°)}=\frac{\cos 2\theta}{\sin 30°}=2\cos 2\theta$$

$\cos 2\theta = \frac{m+n}{2(m-n)}$

5 Changing into sin and cos we get

$$\frac{\cos A \cos 3A}{\sin 3A \cos A - \cos 3A \sin A} - \frac{\sin 3A \sin A}{\cos 3A \sin A - \sin 3A \cos A}$$

$$= \frac{\cos A \cos 3A}{\sin (3A - A)} + \frac{\sin 3A \sin A}{\sin (3A - A)} = \frac{\cos (3A - A)}{\sin 2A} = \cot 2A$$

6 $\frac{\sin (A-B)}{\sin A \sin B} = \frac{\sin A \cos B}{\sin A \sin B} - \frac{\cos A \sin B}{\sin A \sin B} = \cot B - \cot A$

L H S $(\cot B - \cot A) + (\cot C - \cot B) + (\cot A - \cot C) = 0$

7 Proceed as above

8 On changing into sin and cos, we get

L H S $= \frac{\sin (A+B) \sin (A-B)}{\cos (A-B) \cos (A+B)} = \frac{\sin^2 A - \sin^2 B}{\cos^2 A - \sin^2 B}$

9 From the given relation, we have

$\frac{\sin (\gamma + \delta)}{\sin (\gamma - \delta)} = \frac{\cos (\alpha - \beta)}{\cos (\alpha + \beta)}$ Now apply comp and div

$$\frac{\sin (\gamma+\delta) + \sin (\gamma - \delta)}{\sin (\gamma + \delta) - \sin (\gamma - \delta)} = \frac{\cos (\alpha - \beta) + \cos (\alpha + \beta)}{\cos (\alpha - \beta) - \cos (\alpha + \beta)}$$

or $\frac{2 \sin \gamma \cos \delta}{2 \cos \gamma \sin \delta} = \frac{2 \cos \alpha \cos \beta}{2 \sin \alpha \sin \beta}$

$\cot \delta = \cot \alpha \cot \beta \cot \gamma$ **Proved**

10 Using $\sin^2 A - \sin^2 B = \sin (A+B) \sin (A-B)$ we get

L H S $= \sin (\pi/8 + A/2 + \pi/8 - A/2) \sin (\pi/8 + A/2 - \pi/8 + A/2)$

$= \sin \pi/4 \sin A = (1/\sqrt{2}) \sin A = $ R H S

11 L H S $= \sin (24° + 6°) \sin (24° - 6°) = \sin 30° \sin 18° = \frac{1}{2} \cdot \frac{\sqrt{5} - 1}{4}$

12 proceed as above

13 L H S $= -(\cos^2 78° - \sin^2 42°) = -\cos (78° + 42°) \cos (78° - 42°)$

$= -\cos 120° \cos 36° = -(-\frac{1}{2}) \frac{\sqrt{5}+1}{4} = \frac{\sqrt{5}+1}{8}$

14 L H S $= \cos 2\theta \cos 2\phi + [\sin (\theta - \phi + \theta + \phi) \sin (\theta - \phi - \theta - \phi)]$

$= \cos 2\theta \cos 2\phi + \sin 2\theta \sin (-2\phi)$

$= \cos 2\theta \cos 2\phi - \sin 2\theta \sin 2\phi = \cos 2 (\theta + \phi)$

15 R H S $= (1 - \cos 2\beta) + 4 \cos (\alpha + \beta) \sin \alpha \sin \beta$

$+ 2 \cos^2 (\alpha + \beta) - 1$

$= -\cos 2\beta + 2 \cos (\alpha + \beta) [2 \sin \alpha \sin \beta + \cos (\alpha + \beta)]$

$= -\cos 2\beta + 2 \cos (\alpha + \beta) [2 \sin \alpha \sin \beta + \cos \alpha \cos \beta$

$- \sin \alpha \sin \beta]$

$= -\cos 2\beta + 2\cos(\alpha+\beta)\cos(\alpha-\beta)$

$= -\cos 2\beta - (\cos 2\alpha + \cos 2\beta) = \cos 2\alpha = \text{L H S}$

16 R H S $= (A-B)\,[\cos(A-B) - 2\cos A\cos B] + \cos^2 B$

$= \cos(A-B\,[\cos A\cos B + \sin A\sin B - 2\cos A\cos B]$
$+\cos^2 B$

$= -\cos(A-B)\,[\cos A\cos B - \sin A\sin B] + \cos^2 B$

$= -\cos(A-B)\cos(A+B) + \cos^2 B$

$= -[\cos^2 A - \sin^2 B] + \cos^2 B$

$= -\cos^2 A + (\sin^2 B + \cos^2 B) = 1 - \cos^2 A = \sin^2 A$

17 Proceed as in Q 16

18 L H S $= \cos^2 A + \cos^2 B - 2\cos A\cos B\,(\cos A\cos B$
$-\sin A\sin B)$

$= (\cos^2 A - \cos^2 A\cos^2 B) + (\cos^2 B - \cos^2 A\cos^2 B)$
$+2\sin A\sin B\cos A\cos B$

$= \cos^2 A\,(1-\cos^2 B) + \cos^2 B\,(1-\cos^2 A)$
$+2\sin A\sin B\cos A\cos B$

$= \cos^2 A\sin^2 B + \cos^2 B\sin^2 A$
$+2\sin A\sin B\cos A\cos B$

$= (\sin A\cos B + \cos A\sin B)^2 = \sin^2(A+B)$

19 (i) Rearranging the given relation, we have to prove that
$\tan A + \tan 2A = \tan 3A\,(1 - \tan A\tan 2A)$

or $\dfrac{\tan A + \tan 2A}{1 - \tan A\tan 2A} = \tan 3A$

or $\tan(A+2A) = \tan 3A$ which is true

(ii) $\tan 15° + \tan 30° = 1 - \tan 15°\tan 30°$

or $\dfrac{\tan 15° + \tan 30°}{1 - \tan 15°\tan 30°} = 1$ or $\tan(15°+30°) = 1$

or $\tan 45° = 1$, which is true

20 $70° = 50° + 20°$, $\tan 70° = \dfrac{\tan 50° + \tan 20°}{1 - \tan 50\tan 20°}$

or $\tan 70° - \tan 70°\tan 50°\tan 20° = \tan 50° + \tan 20°$

or $\tan 70° - \tan 20° = \tan 50° + \tan 50° = 2\tan 50°$

[$\tan 70°\tan 20° = \tan(90°-20°)\tan 20° = \cot 20°\tan 20° = 1$]

Alit $\tan 70° - \tan 20° = \dfrac{\sin(70°-20°)}{\cos 70°\cos 20°} = \dfrac{2\sin 50°}{2\sin 20°\cos 20°}$

$= \dfrac{2\sin 50°}{\sin 40°} = \dfrac{2\sin 50°}{\cos 50°} = 2\tan 50$

21 $(1+\tan A)(1+\tan B) = 2$

$1 + \tan A + \tan B + \tan A\tan B = 2$

or $\tan A+\tan B=1-\tan A\tan B$

or $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=1$

or $\tan(A+B)=1$ This is true as $A+B=45°$

(ii) Change cot in terms of tan and proceed as above

22 (i) $\tan(\alpha+\beta)=\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\dfrac{\dfrac{m}{m+1}+\dfrac{1}{2m+1}}{1-\dfrac{m}{m+1}\dfrac{1}{2m+1}}$

$=\dfrac{2m^2+2m+1}{2m^2+3m+1-m}=1,$ $\alpha+\beta=\pi/4$

(ii) Proceed as above

23 Changing cot in terms of tan we have to prove that

$$\frac{1}{1+\tan A}\ \frac{1}{1+\tan B}=\frac{1}{2}$$

or $2=1+\tan A+\tan B+\tan A\tan B$

or $\tan A+\tan B=(1-\tan A\tan B)$

or $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=1$ or $\tan(A+B)=1$

or $\tan 225°=1$

But $\tan 225°=\tan(180°+45°)=\tan 45°=1$ **Proved**

24 (i) $\tan\left(\dfrac{\pi}{4}+\theta\right)=\dfrac{\tan\pi/4+\tan\theta}{1-\tan\pi/4\tan\theta}=\dfrac{1+\tan\theta}{1-\tan\theta}$

Similarly, $\tan\left(\dfrac{\pi}{4}-\theta\right)=\dfrac{1-\tan\theta}{1+\tan\theta}$

$\text{L H S}=\dfrac{1+\tan\theta}{1-\tan\theta}-\dfrac{1-\tan\theta}{1+\tan\theta}$

$=\dfrac{(1+\tan\theta)^2-(1-\tan\theta)^2}{1-\tan^2\theta}=\dfrac{4\tan\theta}{1-\tan^2\theta}$

$=2\dfrac{2\tan\theta}{1-\tan^2\theta}=2\tan 2\theta$

(ii) Proceed as above

$\text{L H S}=2\dfrac{(1+\tan^2\theta)}{1-\tan^2\theta}=2\dfrac{1}{\cos 2\theta}=2\sec 2\theta$

(iii) It is clear from part (i)

(iv) $\text{L H S}=\dfrac{1}{\sin(45+\theta)\sin(45°-\theta)}=\dfrac{1}{\sin^2 45°-\sin^2\theta}$

$=\dfrac{2}{1-2\sin^2\theta}=\dfrac{2}{\cos 2\theta}=2\sec 2\theta$

Similarly,

$$\sec\left(\frac{\pi}{4}+\theta\right)\sec\left(\frac{\pi}{4}-\theta\right)=\frac{1}{\cos^2 45^\circ-\sin^2\theta}=2\sec 2\theta \text{ as above}$$

(i) $\text{L H S}=\frac{1+\tan\theta/2}{1-\tan\theta/2}=\frac{\cos\theta/2+\sin\theta/2}{\cos\theta/2-\sin\theta/2}$

$$=\left[\frac{(\cos\theta/2+\sin\theta/2)^2}{(\cos\theta/2-\sin\theta/2)^2}\right]^{1/2}$$

$$=\left[\frac{\cos^2\theta/2+\sin^2\theta/2+2\sin\theta/2\cos\theta/2}{\cos^2\theta/2+\sin^2\theta/2-2\sin\theta/2\cos\theta/2}\right]^{1/2}=\sqrt{\left(\frac{1+\sin\theta}{1-\sin\theta}\right)}$$

$$=\frac{1+\sin\theta}{\sqrt{[(1+\sin\theta)(1-\sin\theta)]}}=\frac{1+\sin\theta}{\sqrt{(1-\sin^2\theta)}}=\frac{1+\sin\theta}{\cos\theta}$$

$$=\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}=\sec\theta+\tan\theta$$

25 On dividing by $\cos 9^\circ$

$\text{L H S}=\frac{1+\tan 9^\circ}{1-\tan 9^\circ}=\tan(45^\circ+9^\circ)=\tan 54^\circ$ as in Q 24 (*i*)

26 (*i*) $\text{L H S}=\frac{\sin A}{\cos A}+\frac{\cos A}{\sin A}=\frac{\sin^2 A+\cos^2 A}{\cos A\sin A}$

$$=\frac{2}{2\sin A\cos A}=\frac{2}{\sin 2A}=2\ \operatorname{cosec} 2A$$

(*ii*) $\text{L H S}=\frac{\cos A}{\sin A}-\frac{\sin A}{\cos A}=\frac{\cos^2 A-\sin^2 A}{\sin A\cos A}$

$$=\frac{2\cos 2A}{\sin 2A}=2\cot 2A$$

(*iii*) Transfer $\tan A$ to R H S

$2\tan 2A+4\tan 4A+8\cot 8A=\cot A-\tan A$

$=2\cot 2A$ by (*ii*)

$4\tan 4A+8\cot 8A=2(\cot 2A-\tan 2A)$

$=2\cdot 2\cot 4A$ as in (*ii*)

$8\cot 8A=4[\cot 4A-\tan 4A]=4\cdot 2\cot 8A$ as in (*ii*)

$=8\cot 8A$

(*iv*) Transposing, we have to prove that

$\operatorname{cosec} A-2\sin A=2\cot 2A\cos A$

$$\text{L H S}=\frac{1}{\sin A}-2\sin A=\frac{1-2\sin^2 A}{\sin A}\times\frac{2\cos A}{2\cos A}$$

$$=\frac{\cos 2A}{\sin 2A}\,2\cos A=2\cot 2A\cos A$$

27 $\text{L H S}=\frac{1}{\sin 2A}-\frac{\cos 2A}{\sin 2A}=\frac{1-\cos 2A}{\sin 2A}$

$$=\frac{2\sin^2 A}{2\sin A\cos A}=\tan A$$

(ii) and (iii) proceed as above

(iv) $2\sin A\cos A(\cos^2 A-\sin^2 A)=\sin 2A\cos 2A=\frac{1}{2}\sin 4A$

28 $\text{L H S}=\frac{(1-\cos 2\theta)+\sin 2\theta}{(1+\cos 2\theta)+\sin 2\theta}=\frac{2\sin^2\theta+2\sin\theta\cos\theta}{2\cos^2\theta+2\sin\theta\cos\theta}$

$$=\frac{2\sin\theta(\sin\theta+\cos\theta)}{2\cos\theta(\cos\theta+\sin\theta)}=\tan\theta$$

(ii) Proceed as above

(iii) $\text{L H S}=\sqrt{\{2+\sqrt{(2+2\cos 4\theta)}\}}$

$=\sqrt{[2+\sqrt{\{2(1+\cos 4\theta)\}}]}=\sqrt{\{2+\sqrt{(2\cdot 2\cos^2 2\theta)}\}}$

$=\sqrt{(2+2\cos 2\theta)}=\sqrt{\{2(1+\cos 2\theta)\}}=\sqrt{(2\cdot 2\cos^2\theta)}=2\cos\theta$

(iv) $\text{L H S}=2\cos^2 28°+(2\sin 62°\sin 4°)$

$=2\cos^2 28°+2\cos 28°\cos 86°$ $\qquad \sin\theta=\cos(90°-\theta)$

$=2\cos 28°[\cos 28°+\cos 86°]$

$=2\cos 28°(2\cos 57°\cos 29°)$

$=4\cos 28°\cos 29°\sin 33°$

$\cos 57°=\cos(99-33°)=\sin 33°$

29 (a) Changing in terms of cos, we get

$$\text{L H S}=\frac{1-\cos 8\theta}{\cos 8\theta}\cdot\frac{\cos 4\theta}{1-\cos 4\theta}=\frac{2\sin^2 4\theta\cos 4\theta}{\cos 8\theta\, 2\sin^2 2\theta}$$

$$=\frac{(2\sin 4\theta\cos 4\theta)(2\sin 2\theta\cos 2\theta)}{\cos 8\theta\, 2\sin^2 2\theta}$$

$$=\frac{\sin 8\theta}{\cos 8\theta}\cdot\frac{\cos 2\theta}{\sin 2\theta}=\tan 8\theta\cot 2\theta=\frac{\tan 8\theta}{\tan 2\theta}$$

(b) $\tan A(\sec 4A-1)=\tan 4A(\sec 2A-1)$

or $\quad \frac{\sec 4A-1}{\sec 2A-1}=\frac{\tan 4A}{\tan A}$ $\qquad$ As in (a)

30 We have to prove that

$$\cot\beta=2\frac{(\tan\alpha-\tan\beta)}{1+\tan\alpha\tan\beta}$$

or $\cot\beta+\tan\alpha=2\tan\alpha-2\tan\beta$

or $\cot\beta+2\tan\beta=\tan\alpha$ which is given

31 We know that

$$\cos 2A=\frac{1-\tan^2 A}{1+\tan^2 A}$$

Using this formula we get

$$\frac{1-\tan^2\alpha}{1+\tan^2\alpha}=\frac{3\,\dfrac{1-\tan^2\beta}{1+\tan^2\beta}-1}{3-\dfrac{1-\tan^2\beta}{1+\tan^2\beta}}=\frac{1-2\tan^2\beta}{1+2\tan^2\beta}$$

Apply componendo and dividendo, we get

$$\frac{2\tan^2\alpha}{2}=\frac{4\tan^2\beta}{2},\qquad \tan^2\alpha=2\tan^2\beta$$

or $\tan\alpha=\sqrt{2}\tan\beta$

32 $\text{L H S}=a\,\dfrac{1-\tan^2\theta}{1+\tan^2\theta}+b\,\dfrac{2\tan\theta}{1+\tan^2\theta}$, Put $\tan\theta=b/a$

$$=a\,\frac{a^2-b^2}{a^2+b^2}+b\,\frac{2ab}{a^2+b^2}=a\left(\frac{a^2-b^2+2b^2}{a^2+b^2}\right)=a$$

Alt $b\sin 2\theta=a(1-\cos 2\theta)$ or $\dfrac{b}{a}=\dfrac{2\sin^2\theta}{2\sin\theta\cos\theta}=\tan\theta$(given)

33 L H S $\sin A\,[\sin^2 60^\circ-\sin^2 A]=\sin A\,[3/4-\sin^2 A]$

$$=\tfrac{1}{4}(3\sin A\quad 4\sin^3 A)=\tfrac{1}{4}\sin 3A$$

(ii) proceed as above

(iii) $\cos 3\theta=4\cos^3\theta-3\cos\theta=\cos\theta\,[4\cos^2\theta-3]$

$$=\tfrac{1}{2}\left(a+\frac{1}{a}\right)\left[4\quad\frac{1}{4}\left(a+\frac{1}{a}\right)^2-3\right]$$

$$=\tfrac{1}{2}\left(a+\frac{1}{a}\right)\left[a^2+\frac{1}{a^2}-1\right]=\tfrac{1}{2}\left[a^3+\frac{1}{a^3}\right]$$

34 (i) $\cos 5\theta=\cos(3\theta+2\theta)=\cos 3\theta\cos 2\theta-\sin 3\theta\sin 2\theta$

$$=(4\cos^3\theta-3\cos\theta)(2\cos^2\theta-1)$$
$$-(3\sin\theta-4\sin^3\theta)\,2\sin\theta\cos\theta$$
$$=\cos\theta\,[(4\cos^2\theta-3)(2\cos^2\theta-1)-2(1-\cos^2\theta)\{3-4(1-\cos^2\theta)\}]$$
$$=\cos\theta\,[(8\cos^4\theta-10\cos^2\theta+3)+2(1-\cos^2\theta)(1-4\cos^2\theta)]$$
$$=\cos\theta\,[(8\cos^4\theta-10\cos^2\theta+3)+2(1-5\cos^2\theta+4\cos^4\theta)]$$
$$=16\cos^5\theta-20\cos^3\theta+5\cos\theta$$

(ii) $\sin 5\theta=\sin(3\theta+2\theta)=\sin 3\theta\cos 2\theta+\cos 3\theta\sin 2\theta$

$$=(3\sin\theta\quad 4\sin^3\theta)(1-2\sin^2\theta)$$
$$+(4\cos^3\theta-3\cos\theta)\,2\sin\theta\cos\theta$$
$$=\sin\theta[(3-4\sin^2\theta)(1-2\sin^2\theta)+2(1-\sin^2\theta)(4\cos^2\theta-3)]$$
$$=\sin\theta\,[3-10\sin^2\theta+8\sin^4\theta+2(1-\sin^2\theta)(1-4\sin^2\theta)]$$
$$=\sin\theta\,[3-10\sin^2\theta+8\sin^4\theta+2-10\sin^2\theta+8\sin^4\theta]$$

$= \sin\theta\,[5-20\sin^2\theta+16\sin^4\theta]$

$= 16\sin^5\theta-20\sin^3\theta+5\sin\theta$

Again R H S

$= \sin\theta\,[5\cos^4\theta-10\cos^2\theta\sin^2\theta+\sin^4\theta]$

$= \sin\theta\,[5(1-\sin^2\theta)^2-10(1-\sin^2\theta)\sin^2\theta+\sin^4\theta]$

$= \sin\theta\,[5-10\sin^2\theta+5\sin^4\theta-10\sin^2\theta+10\sin^4\theta+\sin^4\theta]$

$= \sin\theta\,[5-20\sin^2\theta+16\sin^4\theta]$

which is same as above

(iii) Already done in part (ii) and put $\theta=36°$

$\sin 5\theta=\sin 180°=0$ etc

Alternative solution

By De-Moivers Theorem we know that

$(\cos\theta+i\sin\theta)^5=\cos 5\theta+i\sin 5\theta$

L H S on expansion by binomial theorem is

$\cos^5\theta+5\cos^4\theta\,(i\sin\theta)+10\cos^3\theta\,(i\sin\theta)^2$
$+10\cos^2\theta\,(i\sin\theta)^3+5\cos\theta\,(i\sin\theta)^4+(i\sin\theta)^5$

Now equate real and imaginary parts and change $\sin^2\theta$ to $1-\cos^2\theta$ and $\cos^2\theta$ to $1-\sin^2\theta$ depending on the answer

35 (i) We know that $\cos 3A=4\cos^3A-3\cos A$

$\cos^3A=\frac{1}{4}(\cos 3A+3\cos A)$ Also $\cos(2n\pi+\theta)=\cos\theta$

Applying the above we have

L H S $=\frac{1}{4}[(3\cos\theta+\cos 3\theta)+3\cos(120°+\theta)$
$+\cos(360°+3\theta)+3\cos(240°+\theta)+\cos(720°+3\theta)]$

$=\frac{1}{4}[3\cos 3\theta]+\frac{3}{4}[\cos\theta+\cos(120°+\theta)+\cos(240°+\theta)]$

$=\frac{3}{4}\cos 3\theta+\frac{3}{4}[\cos\theta+2\cos(180°+\theta)\cos 60°]$

$=\frac{3}{4}\cos 3\theta+\frac{3}{4}[\cos\theta+2\cdot\frac{1}{2}(-\cos\theta)]=\frac{3}{4}\cos 3\theta$

(ii) L H S $=\frac{1}{2}[(1+\cos 2A)+1+\cos(2A+240°)+1$
$+\cos(2A-240°)]$

$=\frac{1}{2}[3+\cos 2A+2\cos 2A\cos 240°]$

$=\frac{1}{2}[3+\cos 2A-\cos 2A]=\frac{3}{2}$

[$\cos 240°=\cos(270°-30°)=-\sin 30°=-\frac{1}{2}$]

(iii) $\cos^2 A=\frac{1}{2}(1+\cos 2A)$

$\cos\frac{\pi}{4}=\frac{1}{\sqrt2}$ $\cos\frac{3\pi}{4}=\cos\left(\pi-\frac{\pi}{4}\right)=-\cos\frac{\pi}{4}=-\frac{1}{\sqrt2}$

$\cos\frac{5\pi}{4}=-\frac{1}{\sqrt2}$ $\cos\frac{7\pi}{4}=\cos\left(2\pi-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}=\frac{1}{\sqrt2}$

$$\cos^2\frac{\pi}{8}=\frac{1}{2}\left(1+\cos\frac{\pi}{4}\right)=\frac{\sqrt{2}+1}{2\sqrt{2}}$$

$$\cos^4\frac{\pi}{8}=\frac{1}{8}\left[(\sqrt{2}+1)^2\right]=\frac{1}{8}(3+2\sqrt{2})$$

Similarly $\cos^4\frac{3\pi}{8}=\frac{1}{8}\left[(\sqrt{2}-1)^2\right]=\frac{1}{8}(3-2\sqrt{2}),$

$$\cos^4\frac{5\pi}{8}=\frac{1}{8}\left[(\sqrt{2}-1)^2\right]=\frac{1}{8}(3-2\sqrt{2})$$

and $\cos^4\frac{7\pi}{8}=\frac{1}{8}\left[(\sqrt{2}+1)^2\right]=\frac{1}{8}(3+2\sqrt{2})$

$\Sigma\cos^4\pi/8=\frac{1}{8}[12]=3/2$ ***Proved***

(iv) Do Yourself

36 We know that $\tan A\tan B+1=\left(\frac{\sin A\sin B}{\cos A\cos B}+1\right)$

$$=\frac{\cos A\cos B+\sin A\sin B}{\cos A\cos B}=\frac{\cos(A-B)}{\cos A\cos B}$$

Now take -3 from R H S to L H S and write it as $1+1+1$

Hence we have to prove that

$$[\tan\theta\tan(\theta+60^\circ)+1]+[\tan\theta\tan(\theta-60^\circ)+1]$$
$$+[\tan(\theta+60^\circ)\tan(\theta-60^\circ)+1]=0$$

or
$$\frac{\cos(\theta-\theta-60^\circ)}{\cos\theta\cos(\theta+60^\circ)}+\frac{\cos(\theta-\theta+60^\circ)}{\cos\theta\cos(\theta-60^\circ)}$$
$$+\frac{\cos(\theta+60^\circ-\theta+60^\circ)}{\cos(\theta+60^\circ)\cos(\theta-60^\circ)}$$

$$=\frac{\cos 60^\circ\cos(\theta-60^\circ)+\cos 60^\circ\cos(\theta+60^\circ)+\cos 120^\circ\cos\theta}{\cos\theta\cos(\theta+60^\circ)\cos(\theta-60^\circ)}$$

$$=\frac{\frac{1}{2}[\cos(\theta-60^\circ)+\cos(\theta+60^\circ)]+\cos(90^\circ+30^\circ)\cos\theta}{D^r}$$

$$=\frac{\frac{1}{2}\,2\cos\theta\cos 60^\circ-\sin 30^\circ\cos\theta}{D^r}=\frac{1}{2}\,\frac{(\cos\theta-\cos\theta)}{D^r}=0$$

(b) $\cot A\cot B-1=\frac{\cos(A+B)}{\sin A\sin B}$, and $3=1+1+1$

37 (i) We know that $\tan 60^\circ=\sqrt{3}$,

$$\text{L H S}=\tan A+\frac{\sqrt{3}+\tan A}{1-\sqrt{3}\tan A}-\frac{\sqrt{3}-\tan A}{1+\sqrt{3}\tan A}$$

$$=\tan A+\frac{(\sqrt{3}+\tan A)(1+\sqrt{3}\tan A)-(\sqrt{3}-\tan A)(1-\sqrt{3}\tan A)}{1-3\tan^2 A}$$

$$=\tan A+\frac{8\tan A}{1-3\tan^2 A}=\frac{9\tan A-3\tan^3 A}{1-3\tan^2 A}$$

$$=3\,\frac{(3\tan A-\tan^3 A)}{1-3\tan^2 A}=3\tan 3A$$

Note You can also do it by the method given below in (ii)

(ii) $\text{L.H.S}=\frac{\cos\alpha}{\sin\alpha}+\frac{\cos(60°+\alpha)}{\sin(60°+\alpha)}-\frac{\cos(60°-\alpha)}{\sin(60°-\alpha)}$

$$=\frac{\cos\alpha}{\sin\alpha}+\frac{\sin(60°-\alpha)\cos(60°+\alpha)-\cos(60°-\alpha)\sin(60°+\alpha)}{\sin^2 60°-\sin^2\alpha}$$

$$=\frac{\cos\alpha\left(\frac{3}{4}-\sin^2\alpha\right)+\sin\alpha\sin(60°-\alpha-60°-\alpha)}{\sin\alpha\left\{\frac{3}{4}-\sin^2\alpha\right\}}$$

$$=\frac{\cos\alpha(3-4\sin^2\alpha)+4\sin\alpha\sin(-2\alpha)}{4\sin\alpha\left(\frac{3-4\sin^2\alpha}{4}\right)}$$

$$=\frac{\cos\alpha(3-4\sin^2\alpha)-4\sin\alpha\,2\sin\alpha\cos\alpha}{3\sin\alpha-4\sin^3\alpha}$$

$$=\frac{\cos\alpha\,[3-12(1-\cos^2\alpha)]}{\sin 3\alpha}$$

$$=3\,\frac{[4\cos^3\alpha-3\cos\alpha]}{\sin 3\alpha}=3\,\frac{\cos 3\alpha}{\sin 3\alpha}=3\cot 3\alpha$$

38 $\text{L H S}=\frac{\sin(A+B)\sin(A-B)}{\frac{1}{2}(\sin 2A-\sin 2B)}$

$$=\frac{2\sin(A+B)\sin(A-B)}{2\sin(A-B)\cos(A+B)}=\tan(A+B)$$

39 $7+1=3+5$

$\text{L.H S}=\frac{(\sin A+\sin 7A)+(\sin 3A+\sin 5A)}{(\cos A+\cos 7A)+(\cos 3A+\cos 5A)}$

$$=\frac{2\sin 4A\cos 3A+2\sin 4A\cos A}{2\cos 4A\cos 3A+2\cos 4A\cos A}$$

$$=\frac{\sin 4A(\cos 3A+\cos A)}{\cos 4A(\cos 3A+\cos A)}=\tan 4A$$

(ii) Proceed as in part (i)

(iii) $\text{L.H S}=\frac{(\cos 3A+\cos 7A)+2\cos 5A}{(\cos A+\cos 5A)+2\cos 3A}$

$$=\frac{2\cos 5A\cos 2A+2\cos 5A}{2\cos 3A\cos 2A+2\cos 3A}=\frac{\cos 5A}{\cos 3A}$$

$$=\frac{\cos(3A+2A)}{\cos 3A}=\frac{\cos 3A\cos 2A}{\cos 3A}-\frac{\sin 3A\sin 2A}{\cos 3A}$$

$$=\cos 2A-\tan 3A\sin 2A$$

(vi)

(*iv*) In the N^r write 6 as $1+5$ and 15 as $10+5$

$N^r = \cos 6\theta + \cos 4\theta + 5\cos 4\theta + 5\cos 2\theta + 10\cos 2\theta + 10$

$= 2\cos 5\theta\cos\theta + 5\cdot 2\cos 3\theta\cos\theta + 10\cdot 2\cos^2\theta$

$= 2\cos\theta\,[\cos 5\theta + 5\cos 3\theta + 10\cos\theta] = 2\cos\theta\,[D^r]$

$\dfrac{N^r}{D^r} = 2\cos\theta$ **Proved**

(*v*) R H S $= 2\cos x - 2\cos 4x\cos x = 2\cos x\,(1-\cos 4x)$

$= 2\cos x\cdot 2\sin^2 2x = 4\cos x\,(2\sin x\cos x)^2$

$= 16\cos^3 x\sin x$

(*vi*) Combine Ist two and last two

40 (*i*) L H S $= (\cos^2\alpha + \sin^2\alpha) + (\cos^2\beta + \sin^2\beta)$

$+ 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= 2[1+\cos(\alpha-\beta)] = 2\cdot 2\cos^2\dfrac{\alpha-\beta}{2} = 4\cos^2\dfrac{\alpha-\beta}{2}$

(*ii*) Proceed as above

(*iii*) L H S $= 2\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\alpha-\beta}{2} + 2\cos\dfrac{\alpha+\beta+2\gamma}{2}\cos\dfrac{\alpha+\beta}{2}$

$= 2\cos\dfrac{\alpha+\beta}{2}\left[\cos\dfrac{\alpha-\beta}{2} + \cos\dfrac{\alpha+\beta+2\gamma}{2}\right]$

$= 2\cos\dfrac{\alpha+\beta}{2}\, 2\cos\dfrac{\alpha+\gamma}{2}\cos\left(-\dfrac{\gamma+\beta}{2}\right)$

$= 4\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$, $\quad \cos(-\theta) = \cos\theta$

(*iv*) L H S

$$= \left[\frac{2\cos\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}\right]^n + \left\{\frac{2\sin\frac{A+B}{2}\cos\frac{A-B}{2}}{2\sin\frac{A+B}{2}\sin\frac{B-A}{2}}\right\}^n$$

$= \cot^n\dfrac{A-B}{2} + \left[-\cot\dfrac{A-B}{2}\right]^n = 2\cot^n\dfrac{A-B}{2}$ or 0

according as n is even or odd $\quad [\sin(-\theta) = -\sin\theta]$

(*v*) R H S $= \dfrac{2\sin\frac{A-C}{2}\cos\frac{A+C}{2}}{2\sin\frac{C+A}{2}\sin\frac{A-C}{2}} = \cot\dfrac{A+C}{2} = \cot B,$

$\left[\quad A, B, C \text{ being in A P}, \quad B = \dfrac{A+C}{2}\right]$

(*vi*) Since the quadrilateral *ABCD* is cyclic, we have

$A+C = 180°$ and $B+D = 180°$ **(Property)**

Hence $\cos A = \cos(180° - C) = -\cos C$ (1)
and $\cos B = \cos(180° - D) = -\cos D$ (2)

Adding (1) and (2), we get

$\cos A + \cos B = -\cos C - \cos D$

or $\cos A + \cos B + \cos C + \cos D = 0$

41 (i) $\cos 20° + \cos 100° + \cos 140°$

$= (\cos 20° + \cos 140°) + \cos(90° + 10°)$

$= 2\cos 80° \cos 60° - \sin 10° = \cos 80° - \sin 10°$

$= \sin 10° - \sin 10° = 0$

[$\cos 80° = \cos(90° - 10°) = \sin 10°$]

(ii) Proceed as above

(iii) L H S $= 2\cos\frac{\pi}{13}\cos\frac{9\pi}{13} + 2\cos\frac{4\pi}{13}\cos\frac{\pi}{13}$

$= 2\cos\frac{\pi}{13}\left[\cos\frac{9\pi}{13} + \cos\frac{4\pi}{13}\right]$

$= 2\cos\frac{\pi}{13}\, 2\cos\frac{\pi}{2}\cos\frac{5\pi}{26} = 0$ $\quad \cos(\pi/2) = 0$

(iv) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

L H S $= \frac{1}{2}[(\cos 120° + \cos 80°) + \cos(240° + \cos 40°)$

$= \frac{1}{2}[-\frac{1}{2} + \cos 80° - \frac{1}{2} + \cos 40° - \cos 20° - \frac{1}{2}]$ $-(\cos 340° + \cos 60°)]$

$= \frac{1}{2}[-\frac{3}{2} + 2\cos 60°\cos 20° - \cos 20°]$ See § 3 (31) Page 17

$= -\frac{3}{4} + \frac{1}{2}(\cos 20° - \cos 20°) = -\frac{3}{4}$

(v) L H S $= \frac{1}{2\sin\pi/7}\left[2\sin\frac{\pi}{7}\cos\frac{2\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{4\pi}{7} + 2\sin\frac{\pi}{7}\cos\frac{6\pi}{7}\right]$

$= \frac{1}{2\sin\pi/7}\left[\left(\sin\frac{3\pi}{7} - \sin\frac{\pi}{7}\right) + \left(\sin\frac{5\pi}{7} - \sin\frac{3\pi}{7}\right) + \left(\sin\frac{7\pi}{7} - \sin\frac{5\pi}{7}\right)\right]$

$= -\frac{1}{2}$ as $\sin\frac{7\pi}{7} = \sin\pi = 0$ and other terms cancel

(vi) We know that $\tan A + \tan B = \frac{\sin(A+B)}{\cos A\cos B}$

Hence we have to prove that [Rule 21 (a) P 15]

$\tan 10°+\tan 70°=\tan 60°+\tan 50°$ $\qquad \tan 60°=\sqrt{3}$

or $\dfrac{\sin 80°}{\cos 10° \cos 70°}=\dfrac{\sin 110°}{\cos 60° \cos 50°}$

But $\sin 80°=\cos 10°$, $\sin 110°=\cos 20°$

$\dfrac{1}{\cos 70°}=\dfrac{2 \cos 20°}{\cos 50°}$ $\qquad \cos 60°=1/2$

or $\cos 50°=2 \cos 70° \cos 20°=\cos 90°+\cos 50°$

$=\cos 50°$ **Proved**

(vii) L H S $=\frac{1}{2}[1+\cos 146°+1+\cos 94°+\cos 120°+\cos 26°]$

$=\frac{1}{2}[2-\frac{1}{2}+\cos 146°+\cos 94°+\cos 26°]$ $\qquad \cos 120°=-\frac{1}{2}$

$=\frac{1}{2}[\frac{3}{2}+\cos 146°+2 \cos 60° \cos 34°]$

$=\frac{1}{2}[\frac{3}{2}-\cos 34°+\cos 34°]=\frac{3}{4}$

(viii) Hint $\cos \dfrac{6\pi}{7}=-\cos \dfrac{\pi}{7}$, $\cos \dfrac{5\pi}{7}=-\cos \dfrac{2\pi}{7}$ etc

42 (i) Multiply above and below by 2 and apply

$2 \cos A \cos B=\cos (A+B)+\cos (A-B)$ and

$2 \sin A \sin B=\cos (A-B)-\cos (A+B)$

L H S

$=\dfrac{(\cos 5A+\cos A)-(\cos 9A+\cos 5A)+(\cos 11A+\cos 9A)}{(\cos A-\cos 7A)-(\cos 3A-\cos 7A)+(\cos 3A-\cos 11A)}$

$\dfrac{\cos A+\cos 11A}{\cos A-\cos 11A}=\dfrac{2 \cos 6A \cos 5A}{2 \sin 6A \sin 5A}=\cot 6A \cot 5A$

(ii) Proceed as above

(iii) We are given that $2 \cos A \sin C=\sin B$

or $\sin (A+C)-\sin (A-C)=\sin B$ (1)

But in a $\triangle$, $A+C=180°-B$ so that $\sin (A+C)=\sin B$

$\sin (A-C)=0$ by (1), $\qquad A-C=0$ or $A=C$

Hence the triangle is isosceles

43 Here there are four angles each being double of the preceeding one

We multiply above and below by $2^4 \sin A$ (Rule)

L H S $=\dfrac{1}{16 \sin A}[(2 \sin A \cos A)(2 \cos 2A)(2 \cos 4A)$

$(2 \cos 8A)]$

$=\dfrac{1}{16 \sin A}[(\sin 2A\ 2 \cos 2A)(2 \cos 4A)(2 \cos 8A)]$

$=\dfrac{1}{16 \sin A}[(\sin 4A\ 2 \cos 4A)(2 \cos 8A)]$

$$=\frac{1}{16\sin A}\,[(\sin 8A\;2\cos 8A)]$$

$$=\frac{1}{16\sin A}\sin 16A=\frac{1}{2^4\sin A}\sin(2^4A)$$

(i) $LHS=\frac{1}{2}\cos 20^\circ\cos 40^\circ\cos 80^\circ$ $\qquad\because\ \cos 60^\circ=\frac{1}{2}$

$$=\frac{1}{2}\,\frac{1}{2^3\sin 20^\circ}\sin(2^3\,20^\circ)$$

$$=\frac{1}{16\sin 20^\circ}\sin 160^\circ=\frac{1}{16\sin 20^\circ}\sin(180^\circ-20^\circ)$$ As above

$$=\frac{1}{16}\,\frac{\sin 20^\circ}{\sin 20^\circ}=\frac{1}{1}$$

Another form

By compliment rule $\sin\theta=\cos(90-\theta)$ and hence the question reduces to (i)

(ii) $LHS=\frac{1}{2^4\sin 7^\circ}\sin(2^4\;7^\circ)=\frac{1}{16\sin 7^\circ}\sin(112^\circ)$

$$=\frac{1}{16}\,\frac{\sin(180^\circ-68^\circ)}{\sin(90^\circ-83)}=\frac{1}{16}\,\frac{\sin 68^\circ}{\cos 83^\circ}$$

(iii) Here $A=\frac{2\pi}{15}=24^\circ$

$$LHS=\frac{1}{2^4\sin 24^\circ}\sin(2^4\;24^\circ)$$

$$=\frac{1}{16\sin 24^\circ}\sin(384^\circ)=\frac{1}{16}$$

$\sin 384^\circ=\sin(360^\circ+24^\circ)=\sin 24^\circ$

Again we know $\cos(2\pi-\theta)=\cos\theta$

$$\cos\frac{16\pi}{15}=\cos\left(2\pi-\frac{16\pi}{15}\right)=\cos\frac{14\pi}{15}$$

and hence the question reduces to (iii)

(iv) $LHS=\frac{1}{2^3\sin\pi/7}\sin(2^3\pi/7)$

$$=\frac{1}{8\sin\pi/7}\sin(\pi+\pi/7)=-\frac{1}{8}$$

$\sin(\pi+\theta)=-\sin\theta$

(v) $LHS=\frac{1}{2^2\sin\pi/5}\left[\sin 2^2\frac{\pi}{5}\right]$

$$=\frac{1}{4\sin\pi/5}\sin\left(\pi-\frac{\pi}{5}\right)=\frac{1}{4}$$

Also $\cos\frac{\pi}{5}+\cos\frac{3\pi}{5}=2\cos\frac{\pi}{5}\cos\frac{2\pi}{5}=2\,\frac{1}{4}=\frac{1}{2}$

(vi) $2 \sin A \sin B = \cos (A-B) - \cos (A+B)$

$$\text{L H S} = \frac{\sqrt{3}}{2} \cdot \frac{1}{2} (2 \sin 20° \sin 80°) \sin 40°$$

$$= \frac{\sqrt{3}}{4} (\cos 60° - \cos 100°) \sin 40°$$

$$= \frac{\sqrt{3}}{4} [\tfrac{1}{2} \sin 40° - \tfrac{1}{2} (2 \cos 100° \sin 40°)]$$

$$= \frac{\sqrt{3}}{4} [\tfrac{1}{2} \sin 40° - \tfrac{1}{2} (\sin 140° - \sin 60°)]$$

$$= \frac{\sqrt{3}}{8}\left[\sin 40° - \sin (\pi - 40°) + \frac{\sqrt{3}}{2}\right] = \frac{3}{16}$$

Note Using $2 \cos A \cos B = \cos (A+B) + \cos (A-B)$ we can prove the part (i) as above but the method given above is easier when each angle is double of the preceding one

(vii) It follows from (i) and (v)

Another method $\tan 60° = \sqrt{3}$

$$\text{L H S} = \sqrt{3} \tan 20° \tan 40° \tan 80°$$

$$= \sqrt{3} \tan 20° \tan (60° - 20°) \tan (60° + 20°)$$

$$= \sqrt{3} \tan 20° \; \frac{\sqrt{3} - \tan 20°}{1 + \sqrt{3} \tan 20°} \; \frac{\sqrt{3} + \tan 20°}{1 - \sqrt{3} \tan 20°}$$

$$= \sqrt{3} \tan 20° \; \frac{3 - \tan^2 20°}{1 - 3 \tan^2 20°}$$

$$= \sqrt{3} \; \frac{3 \tan 20° \;\; \tan^3 20°}{1 - 3 \tan^2 20°}$$

$$= \sqrt{3} \tan 3 (20°) = \sqrt{3} \tan 60° = \sqrt{3} \sqrt{3} = 3$$

We have used the formula for $\tan 3A$

(viii) $$\text{L H S} = \frac{1}{2 \cos \pi/14}\left(2 \cos \frac{\pi}{14} \sin \frac{\pi}{14} \sin \frac{3\pi}{14} \sin \frac{5\pi}{14}\right)$$

$$= \frac{1}{2 \cos \pi/14}\left[\sin 2\pi/14 \sin 3\pi/14 \sin 5\pi/14\right]$$

$$= \frac{1}{4 \cos \pi/14}\left[\cos \frac{3\pi}{14} - \cos \frac{7\pi}{14}\right] \sin \frac{3\pi}{14}$$

$$= \frac{1}{8 \cos \pi/14}\left[2 \sin \frac{3\pi}{14} \cos \frac{3\pi}{14} - 0\right] \qquad \cos \frac{\pi}{2} = 0$$

$$= \frac{1}{8 \cos \pi/14}\left[\sin \frac{6\pi}{14}\right] = \frac{1}{8},$$

$$\sin \frac{6\pi}{14} = \sin \left(\frac{\pi}{2} - \frac{\pi}{14}\right) = \cos \pi/14$$

44 Before doing this question remember the following

$\sin 18° = \cos 72° = \frac{\sqrt{5}-1}{4}$, $\cos 36° = \sin 54° = \frac{\sqrt{5}+1}{4}$

$\cos 18° = \sin 72° = \frac{1}{4}\sqrt{\left(10+2\sqrt{5}\right)}$

$\sin 36° = \cos 54° = \frac{1}{4}\sqrt{\left(10-2\sqrt{5}\right)}$

(i) (a) $\text{L H S} = \frac{1}{2}\,[2\,(\sin 12° \sin 48°) \sin 54°]$

$= \frac{1}{2}\,[(\cos 36° - \cos 60°) \cos 36°]$

$= \frac{1}{2}\left[\cos^2 36° - \frac{1}{2}\cos 36°\right]$

Put $\cos 36° = \frac{\sqrt{5}+1}{4}$

$= \frac{1}{2}\left[\frac{5+1+2\sqrt{5}}{16} - \frac{1}{2}\,\frac{\sqrt{5}+1}{4}\right]$

$= \frac{1}{2}\left[\frac{3+\sqrt{5}}{8} - \frac{\sqrt{5}+1}{8}\right] = \frac{1}{2}\;\frac{3-1}{8}$

$= \frac{1}{2}\;\frac{1}{4} = \frac{1}{8}$

(b) $\sin\frac{13\pi}{10} = \sin\left(\pi + \frac{3\pi}{10}\right) = -\sin\frac{3\pi}{10} = -\sin 54° = -\cos 36°$

(iii) is correct

(ii) $\text{L H S} = \frac{1}{8}\;\frac{\sqrt{5}+1}{4}\;\frac{1}{2}\,(2\cos 42° \cos 78°)$

$= \frac{1}{16}\,(\sqrt{5}+1)\,(\cos 120° + \cos 36°)$

$= \frac{1}{16}\,(\sqrt{5}+1)\left(-\frac{1}{2} + \frac{\sqrt{5}+1}{4}\right)$

$= \frac{1}{16}\,(\sqrt{5}+1)\;\frac{(\sqrt{5}-1)}{4} = \frac{1}{16}\;\frac{5-1}{4} = \frac{1}{16}$

(iii) $\text{L H S} = \frac{1}{4}\,[(2\sin 6° \sin 66°)\,(2\sin 42° \sin 78°)]$

$= \frac{1}{4}\,[(\cos 60° - \cos 72°)\,(\cos 36° - \cos 120°)]$

$= \frac{1}{4}\left[\left(\frac{1}{2} - \frac{\sqrt{5}-1}{4}\right)\left(\frac{\sqrt{5}+1}{4} + \frac{1}{2}\right)\right]$

$= \frac{1}{4}\;\frac{3-\sqrt{5}}{4}\;\frac{3+\sqrt{5}}{4} + \frac{1}{64}\,(9-5) = \frac{1}{16}$

(iv) Change in terms of sin and cos and then group 6° and 66° 42° and 78° as in part (iii)

(v) $\sin 36^\circ=\frac{\sqrt{(10-2\sqrt{5})}}{4}$, $\sin 72^\circ=\cos 18^\circ=\frac{\sqrt{(10+2\sqrt{5})}}{4}$

$\sin 108^\circ=\cos 18^\circ$, $\sin 144^\circ=\sin(180^\circ-36^\circ)=\sin 36^\circ$

$$\begin{aligned} \text{L H S} &=(\sin 36^\circ \sin 144^\circ)(\sin 72^\circ \sin 108^\circ)\\ &=(\sin 36^\circ \sin 36^\circ)(\cos 18^\circ \cos 18^\circ)\\ &=\sin^2 36^\circ \cos^2 18^\circ\\ &=\frac{10-2\sqrt{5}}{16}\cdot\frac{10+2\sqrt{5}}{16}=\frac{100-20}{16\times16}=\frac{80}{16\times16}=\frac{5}{16} \end{aligned}$$

(vi) $\cos\frac{3\pi}{8}=\cos\left\{\frac{\pi}{2}-\frac{\pi}{8}\right\}=\sin\frac{\pi}{8}$

$\cos\frac{5\pi}{8}=\cos\left\{\frac{\pi}{2}+\frac{\pi}{8}\right\}=-\sin\frac{\pi}{8}$

$\cos\frac{7\pi}{8}=\cos\left\{\pi-\frac{\pi}{8}\right\}=-\cos\frac{\pi}{8}$

$$\begin{aligned} \text{L H S} &=(1+\cos \pi/8)(1+\sin \pi/8)(1-\sin \pi/8)(1-\cos \pi/8)\\ &=(1-\cos^2 \pi/8)(1-\sin^2 \pi/8)=\sin^2 \pi/8 \cos^2 \pi/8\\ &=\tfrac{1}{4}\left[2\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right]^2=\tfrac{1}{4}\left[\sin\frac{\pi}{4}\right]^2\\ &=\tfrac{1}{4}\left\{\frac{1}{\sqrt{2}}\right\}^2=\frac{1}{8} \end{aligned}$$

(vii) We know that $\pi/15$ radian $=12^\circ$

L H S $=\cos 12^\circ \cos 24^\circ \cos 36^\circ \cos 48^\circ \cos 60^\circ \cos 72^\circ \cos 84^\circ$

$\cos 60^\circ=\frac{1}{2}$, $\cos 36^\circ=\frac{\sqrt{5}+1}{4}$, $\cos 72^\circ=\sin 18^\circ=\frac{\sqrt{5}-1}{4}$

$$\begin{aligned} \text{L H S} &=\frac{1}{2}\cdot\frac{\sqrt{5}+1}{4}\cdot\frac{\sqrt{5}-1}{4}[(\cos 12^\circ \cos 48^\circ)(\cos 24^\circ \cos 84^\circ)]\\ &=\frac{1}{32}(5-1)\left[\frac{\cos 60^\circ+\cos 36^\circ}{2}\cdot\frac{\cos 108^\circ+\cos 60^\circ}{2}\right]\\ &=\frac{1}{32}\left[\left(\frac{1}{2}+\frac{\sqrt{5}+1}{4}\right)\left(-\frac{\sqrt{5}-1}{4}+\frac{1}{2}\right)\right]\\ &=\frac{1}{32}\cdot\frac{3+\sqrt{5}}{4}\cdot\frac{3-\sqrt{5}}{4}=\frac{1}{32}\times\frac{9-5}{16}=\frac{1}{128} \end{aligned}$$

45 (a) (i) $\cos(\alpha+\beta)=\frac{4}{5}$ $\qquad \tan(\alpha+\beta)=\frac{\sqrt{(5^2-4^2)}}{4}=\frac{3}{4}$

$\sin(\alpha-\beta)=\frac{5}{13}$ $\qquad \tan(\alpha-\beta)=\frac{5}{\sqrt{(13^2-5^2)}}=\frac{5}{12}$

$$\tan 2\alpha=\tan(\alpha+\beta+\alpha-\beta)=\frac{\tan(\alpha+\beta)+\tan(\alpha-\beta)}{1-\tan(\alpha+\beta)\tan(\alpha-\beta)}$$

$$=\frac{\frac{3}{4}+\frac{5}{12}}{1-\frac{3}{4}\cdot\frac{5}{12}}=\frac{56}{33}$$

(ii) $\sin^2\frac{\alpha-\beta}{2}=\frac{1}{2}[1-\cos(\alpha-\beta)]$,

$\cos^2\frac{\alpha-\beta}{2}=\frac{1}{2}[1+\cos(\alpha-\beta)]$

46 (i) Dividing the given relations, we get

$$\frac{\sin x+\sin y}{\cos x+\cos y}=\frac{a}{c}\quad \text{or}\quad \frac{2\sin\frac{(x+y)}{2}\cos\frac{x-y}{2}}{2\cos\frac{x+y}{2}\cos\frac{x-y}{2}}=\frac{a}{b}\ .$$

$\tan\left(\frac{x+y}{2}\right)=\frac{a}{b}$ Now $\sin A=\frac{2\tan A/2}{1+\tan^2 A/2}$

$\sin(x+y)=\frac{2.a/b}{1+a^2/b^2}=\frac{2ab}{a^2+b^2}$

(ii) Squaring and adding the given relations
$(\sin^2 x+\sin^2 y+2\sin x\sin y)$
$+(\cos^2 x+\cos^2 y+2\cos x\cos y)=a^2+b^2$

or $2(\cos x\cos y+\sin x\sin y)=a^2+b^2-2$

or $\cos(x-y)=\frac{a^2+b^2-2}{2}$

(iii) $\tan^2\frac{A}{2}=\frac{1-\cos A}{1+\cos A}$, $\tan^2\frac{x-y}{2}=\frac{1-\frac{a^2+b^2-2}{2}}{1+\frac{a^2+b^2-2}{2}}$ (by ii)

or $\tan\left(\frac{x-y}{2}\right)=\pm\sqrt{\left(\frac{4-a^2-b^2}{a^2+b^2}\right)}$

(b) $\sin^6 x+\cos^6 x=(\sin^2 x+\cos^2 x)(\sin^4 x-\sin^2 x\cos^2 x+\cos^4 x)$

$=(\sin^2 x+\cos^2 x)^2-3\sin^2 x\cos^2 x$

$=1-3\sin^2 x\cos^2 x=\frac{1}{4}[4-3\sin^2 2x]$

$=\frac{1}{4}[4-3\{(\sin x+\cos x)^2-1\}^2]$

$=\frac{1}{4}[4-3(a^2-1)^2]$

47 (i) $\cot B-\cot A=y$, $\frac{1}{\tan B}-\frac{1}{\tan A}=y$

$\frac{\tan A-\tan B}{\tan A\tan B}=y$ $x/y=\tan A\tan B$

Now $\cot(A-B)=\frac{1}{\tan(A-B)}=\frac{1+\tan A\tan B}{\tan A-\tan B}$

$$=\frac{1+x/y}{x}=\frac{1}{x}+\frac{1}{y}$$

(*ii*) $2\cos A=x+\frac{1}{x}$, $\quad 4\sin^2 A=4-4\cos^2 A=4-\left(x+\frac{1}{x}\right)^2$

$$=-\left[\left(x+\frac{1}{x}\right)^2-4\right]$$

or $4\sin^2 A=i^2\left[x-\frac{1}{x}\right]^2$, $\qquad 2\sin A=i\left(x-\frac{1}{x}\right)$

Similarly $2\cos B=y+\frac{1}{y}$ $\qquad 2\sin B=i\left(y-\frac{1}{y}\right)$

Now $2\cos(A-B)=2(\cos A\cos B+\sin A\sin B)$

$$=2\cdot\tfrac{1}{4}\left[\left(x+\frac{1}{x}\right)\left(y+\frac{1}{y}\right)+i^2\left(x-\frac{1}{x}\right)\left(y-\frac{1}{y}\right)\right]$$

$$=\tfrac{1}{2}\left[\left(xy+\frac{1}{xy}+\frac{y}{x}+\frac{x}{y}\right)-1\left(xy+\frac{1}{xy}-\frac{y}{x}-\frac{x}{y}\right)\right]$$

$$=\tfrac{1}{2}\,2\left(\frac{y}{x}+\frac{x}{y}\right)=\frac{x}{y}+\frac{y}{x}$$

48 From the given relation we have

$a\cos\theta=c-b\sin\theta$ Square and change in terms of $\sin\theta$

$a^2(1-\sin^2\theta)=c^2-2bc\sin\theta+b^2\sin^2\theta$

$(a^2+b^2)\sin^2\theta-2bc\sin\theta+(c^2-a^2)=0$

Its roots are $\sin\alpha$ and $\sin\beta$ as α and β are the values of θ given

$$\sin\alpha+\sin\beta=\text{sum of roots}=\frac{2bc}{a^2+b^2}$$

$$\sin\alpha\sin\beta=\text{product of roots}=\frac{c^2-a^2}{a^2+b^2}$$

(*ii*) Here arrange as a quadratic in $\cos\theta$

49 Here $b\sec\theta=c-a\tan\theta$ Square

$b^2(1+\tan^2\theta)=c^2-2ca\tan\theta+a^2\tan^2\theta$

$(a^2-b^2)\tan^2\theta-2ca\tan\theta+(c^2-b^2)=0$

$$\tan\alpha+\tan\beta=\frac{2ca}{a^2-b^2},\ \tan\alpha\tan\beta=\frac{c^2-b^2}{a^2-b^2}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{2ca}{(a^2-b^2)-(c^2-b^2)}=\frac{2ac}{a^2-c^2}$$

(*ii*) Write $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$, $\sin 2\theta=\frac{2\tan\theta}{1+\tan^2\theta}$ etc

(iii) Arrange as a quadratic in $\tan \theta/2$ as above

$9 \tan^2 \theta/2 - 8 \tan \theta/2 + 3 = 0$ Its roots are complex as

$64 - 4 \cdot 3 \cdot 9 = -$ ive i.e. $b^2 - 4ac < 0$

Hence the given equation is wrong. If we choose 9/2 in R.H.S. then it becomes $15 \tan^2 \theta/2 - 16 \tan \theta/2 + 3 = 0$

$$\tan \frac{\alpha}{2} + \tan \frac{\beta}{2} = \frac{16}{15}, \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = \frac{1}{5}$$

$$\tan\left(\frac{\alpha+\beta}{2}\right) = \frac{\tan \alpha/2 + \tan \beta/2}{1 - \tan \alpha/2 \tan \beta/2} = \frac{16/15}{1-\frac{1}{5}} = \frac{4}{3}$$

$$\text{Now } \sin(\alpha+\beta) = \frac{2 \tan \frac{\alpha+\beta}{2}}{1 + \tan^2 \frac{\alpha+\beta}{2}} = \frac{2 \cdot \frac{4}{3}}{1 + \frac{16}{9}} = \frac{24}{25}$$

50 (i) In $7 \cos \theta + 24 \sin \theta$,
take $7 = r \cos \alpha$ and $24 = r \sin \alpha$

$$r = \sqrt{(7^2 + 24^2)} = 25$$

$$\text{L.H.S.} = r(\cos \theta \cos \alpha + \sin \theta \sin \alpha) = 25 \cos(\theta - \alpha)$$

Now max. and min. values of

$\cos(\theta - \alpha)$ are 1 and -1

Therefore the max. and min. values of given expression are 25 and -25 respectively.

(ii) Proceed as above.

(iii) The given expression is

$$5 \cos \theta + 3(\cos \theta \cos 60° - \sin \theta \sin 60°) + 3$$
$$= \tfrac{1}{2}[(13 \cos \theta - 3\sqrt{3} \sin \theta)] + 3$$

Put $13 = r \cos \alpha$, $3\sqrt{3} = r \sin \alpha$

$$r = \sqrt{(169 + 27)} = \sqrt{(196)} = 14$$

$$\text{Given expression} = \frac{r}{2}[\cos(\theta + \alpha)] + 3 = 7 \cos(\theta + \alpha) + 3$$

Hence max. and min. values of the expression are $7+3$ and $-7+3$ i.e. 10 and -4 respectively.

(iv) (iii) is correct.

(v) (iii) is correct.

(vi) $\text{L.H.S.} = \dfrac{1 \cos 10° - \sqrt{3} \sin 10°}{\sin 10° \cos 10°}$

Put $1 = r \cos \alpha$, $\sqrt{3} = r \sin \alpha$ $r^2 = 1 + 3 = 4$ $r = 2$

and $\tan \alpha = \sqrt{3}$, $\alpha = 60°$

$$\text{L.H.S.} = \frac{r(\cos 60° \cos 10° - \sin 60° \sin 10°)}{\sin 10° \cos 10°}$$

$$=\frac{2\cos(60°+10°)}{\frac{1}{2}\sin 20°}=4\ \frac{\cos 70°}{\sin 20°}=4\ \frac{\sin 20°}{\sin 20}=4$$

(*vii*) Proceed as in part (*vi*)

51 We have $\tan^{2/3}\alpha=\tan^2\frac{1}{2}\theta=\frac{1-\cos\theta}{1+\cos\theta}=\frac{\sqrt{3}-\sqrt{(a^2-1)}}{\sqrt{3}+\sqrt{(a^2-1)}}$

$$\frac{\sin^{2/3}\alpha}{\sqrt{3}-\sqrt{(a^2-1)}}=\frac{\cos^{2/3}\alpha}{\sqrt{3}+\sqrt{(a^2-1)}}=\frac{\sin^{2/3}\alpha+\cos^{2/3}\alpha}{2\sqrt{3}}$$

$=k$, say

$$1=\sin^2\alpha+\cos^2\alpha=k^3\,[\{\sqrt{3}-\sqrt{(a^2-1)}\}^3$$
$$+\{\sqrt{3}+\sqrt{(a^2+1)}\}^3]=k^3\,[6\sqrt{3}+6\sqrt{3}\,(a^2-1)]=k^3\,6\sqrt{3}\,a^2$$

or $\frac{[\sin^{2/3}\alpha+\cos^{2/3}\alpha]^3}{24\sqrt{3}}\ 6\sqrt{3}a^2=1$ or $(\sin^{2/3}\alpha+\cos^{2/3})^3=\frac{4}{a^2}$

or $\sin^{2/3}\alpha+\cos^{2/3}\alpha=\left(\frac{4}{a^2}\right)^{1/3}=\left(\frac{2}{a}\right)^{2/3}$

52 (*i*) We know that

$$\cos A=\frac{1-\tan^2 A/2}{1+\tan^2 A/2}$$

$$\cos\phi=\frac{1-\tan^2\phi/2}{1+\tan^2\phi/2}=\frac{1-\frac{1+e}{1-e}\tan^2\theta/2}{1+\frac{1+e}{1-e}\tan^2\theta/2},$$

from the given relation

$$=\frac{(1-\tan^2\theta/2)-e\,(1+\tan^2\theta/2)}{(1+\tan^2\theta/2)-e\,(1-\tan^2\theta/2)}$$

Divide above and below by $1+\tan^2\theta/2$

or $$\cos\phi=\frac{\cos\theta-e}{1-e\cos\theta}$$

(*ii*) $$\cos\theta=\frac{a\cos\phi+b}{a+b\cos\phi},$$

$$\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}=\frac{a\,(1-\tan^2\phi/2)+b\,(1+\tan^2\phi/2)}{a\,(1+\tan^2\phi/2)+b\,(1-\tan^2\phi/2)}$$

or $$\frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}=\frac{(a+b)-(a-b)\tan^2\phi/2}{(a+b)+(a-b)\tan^2\phi/2}$$

Apply componendo and dividendo

$$\frac{2\tan^2\theta/2}{2}=\frac{2\,(a-b)\tan^2\phi/2}{2\,(a+b)}$$

$$\tan\theta/2=\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\phi/2$$

(*iii*) Proceed as above

53 (*i*) $\tan^2 \theta/2=\frac{1-\cos\theta}{1+\cos\theta}=\left(1-\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}\right)$

$$\div\left(1+\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}\right)$$

$$=\frac{(1-\cos\alpha)+\cos\beta\,(1-\cos\alpha)}{(1+\cos\alpha)-\cos\beta\,(1+\cos\alpha)}$$

$$=\frac{(1-\cos\alpha)\,(1+\cos\beta)}{(1+\cos\alpha)\,(1-\cos\beta)}$$

or $\tan^2\theta/2=\frac{2\sin^2\alpha/2}{2\cos^2\alpha/2}\cdot\frac{2\cos^2\beta/2}{2\sin^2\beta/2}=\tan^2\alpha/2\cot^2\beta/2.$

$\tan\theta/2=\pm\tan\alpha/2\cot\beta/2$

(*ii*) The given relation is

$$\frac{\tan\alpha}{\tan\beta}=\frac{1}{\cos\theta}$$

Apply componendo and dividendo

$$\frac{\tan\alpha-\tan\beta}{\tan\alpha+\tan\beta}=\frac{1-\cos\theta}{1+\cos\theta}$$

or $\frac{\sin(\alpha-\beta)}{\sin(\alpha+\beta)}=\frac{2\sin^2\theta/2}{2\cos^2\theta/2}=\tan^2\theta/2$ etc

54 $\cos 2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}=\frac{1-(2\tan^2\phi+1)}{1+(2\tan^2\phi+1)}=\frac{-2\tan^2\phi}{2(1+\tan^2\phi)}$

or $\cos 2\theta=-\frac{\sin^2\phi}{\cos^2\phi}\cdot\frac{1}{\sec^2\phi}=-\sin^2\phi$

$$\cos 2\theta+\sin^2\phi=0$$

55 (*i*) Let the two parts be A and B so that $A+B=\theta$ and $A-B=\phi$ and $\tan A=m\tan B$

$\frac{\tan A}{\tan B}=\frac{m}{1}$ Apply componendo and dividendo

$\frac{\tan A-\tan B}{\tan A+\tan B}=\frac{m-1}{m+1}$ or $\frac{\sin(A-B)}{\sin(A+B)}=\frac{m-1}{m+1}$ [22 P 15]

or $\sin\phi=\frac{m-1}{m+1}\sin\theta$

(*ii*) From the given relation we have

$\frac{m}{n}=\frac{\cos(\theta-\alpha)}{\cos(\theta+\alpha)}$ Apply componendo and dividendo

$\frac{m-n}{m+n}=\frac{\cos(\theta-\alpha)-\cos(\theta+\alpha)}{\cos(\theta-\alpha)+\cos(\theta+\alpha)}=\frac{2\sin\theta\sin\alpha}{2\cos\theta\cos\alpha}=\frac{\tan\alpha}{\cot\theta}$

$(m-n)\cot\theta=(m+n)\tan\alpha$

(iii) and (iv) Proceed as above

(v) From the given relation on changing to sin and cos we have

$$\frac{\cos^2\theta}{\sin^2\theta}=\frac{\cos(\theta-\alpha)\cos(\theta-\beta)}{\sin(\theta-\alpha)\sin(\theta-\beta)}$$ Apply compo and divi

$$\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin^2\theta}=\frac{\cos(\theta-\alpha+\theta-\beta)}{\cos(\theta-\alpha-\theta+\beta)}$$ using formula of $\cos(A\pm B)$

or $$\cos 2\theta=\frac{\cos[2\theta-(\alpha+\beta)]}{\cos(\alpha-\beta)}\qquad [\ \cos(-\theta)=\cos\theta]$$

$$\cos 2\theta\cos(\alpha-\beta)=\cos 2\theta\cos(\alpha+\beta)+\sin 2\theta\sin(\alpha+\beta)$$

$$\cos 2\theta[\cos(\alpha-\beta)-\cos(\alpha+\beta)]=\sin 2\theta\sin(\alpha+\beta)$$

$$\frac{\cos 2\theta}{\sin 2\theta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{2\sin\alpha\sin\beta}$$

or $\cot 2\theta=\frac{1}{2}[\cot\beta+\cot\alpha]$

(vi) $\tan\beta=\dfrac{\sin(\alpha+\gamma)}{\cos(\alpha-\gamma)}$ on changing to sin and cos

$$\sin 2\beta=\frac{2\tan\beta}{1+\tan^2\beta}=\frac{2\sin(\alpha+\gamma)\cos(\alpha-\gamma)}{\cos^2(\alpha-\gamma)+\sin^2(\alpha+\gamma)}$$

$$=\frac{\sin 2\alpha+\sin 2\gamma}{\frac{1}{2}[1+\cos(2\alpha-2\gamma)+1-\cos(2\alpha+2\gamma)]}$$

$$=\frac{\sin 2\alpha+\sin 2\gamma}{1+\frac{1}{2}\,2\sin 2\alpha\sin 2\gamma}=\frac{\sin 2\alpha+\sin 2\gamma}{1+\sin 2\alpha\sin 2\gamma}$$

56 Multiplying both sides by $\sin\theta$ we have to prove that

$$2^{n-1}(2\sin\theta\cos\theta)\cos 2\theta\cos 2^2\theta\ \ \cos 2^{n-1}\theta=\sin\theta$$

or $2^{n-2}(2\sin 2\theta\cos 2\theta)\cos 2^2\theta\ \ \cos 2^{n-1}\theta=\sin\theta$

or $2^{n-3}(2\sin 2^2\theta\cos 2^2\theta)\ \ \cos 2^{n-1}\theta=\sin\theta$

or $2^{n-4}(2\sin 2^3\theta\cos 2^3\theta)\ \ \cos 2^{n-1}\theta=\sin\theta$

or $2\sin 2^{n-1}\theta\cos 2^{n-1}\theta=\sin\theta$

or $\sin 2^n\theta=\sin\theta$ put $\theta=\pi/(2^n+1)$

or $\sin\dfrac{2^n\pi}{2^n+1}=\sin\dfrac{\pi}{2^n+1}$

or $\sin\dfrac{(2^n+1-1)\pi}{2^n+1}=\sin\dfrac{\pi}{2^n+1}$ or $\sin\left(\pi-\dfrac{\pi}{2^n+1}\right)=\sin\dfrac{\pi}{2^n+1}$

Above is true as $\sin(\pi-A)=\sin A$

57 We know that if a, b, c are in A P, then

$$b-a=c-b$$

$$\sin(z+x-y)-\sin(y+z-x)=\sin(x+y-z)-\sin(z+x-y)$$

or $2\cos z\sin(x-y)=2\cos x\sin(y-z)$

Divide both sides by $2\cos x\cos y\cos z$

$$\frac{\sin x\cos y-\cos x\sin y}{\cos x\cos y}=\frac{\sin y\cos z-\cos y\sin z}{\cos y\cos z}$$

or $\tan x-\tan y=\tan y-\tan z$ or $2\tan y=\tan x+\tan z$

$\tan x$, $\tan y$, $\tan z$ are also in A P

58 Since $\sec(\phi-\alpha)$, $\sec\phi$ $\sec(\phi+\alpha)$ are in A P,

$$2\sec\phi=\sec(\phi-\alpha)+\sec(\phi+\alpha)$$

or $$\frac{2}{\cos\phi}=\frac{\cos(\phi+\alpha)+\cos(\phi+\alpha)}{\cos(\phi-\alpha)\cos(\phi+\alpha)}$$

or $2(\cos^2\phi-\sin^2\alpha)=\cos\phi\,[2\cos\phi\cos\alpha]$

or $\cos^2\phi\,(1-\cos\alpha)=\sin^2\alpha=(1-\cos^2\alpha)$

$\cos^2\phi=1+\cos\alpha=2\cos^2\alpha/2$

$\cos\phi=\sqrt{2}\cos\alpha/2$

59 R H S

$$=2\frac{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}{2\sin\frac{A-B}{2}\cos\frac{A-B}{2}-2\sin\frac{A+B}{2}\sin\frac{A-B}{2}}$$

$$=\frac{2\cos\frac{A+B}{2}}{\cos\frac{A-B}{2}-\sin\frac{A-B}{2}}\times\frac{\cos\frac{A-B}{2}+\sin\frac{A+B}{2}}{\cos\frac{A-B}{2}+\sin\frac{A+B}{2}}$$

$$=\frac{(\cos A+\cos B)+\sin(A+B)}{\cos^2\frac{A-B}{2}-\sin^2\frac{A+B}{2}}$$ ($2\sin\theta\cos\theta=\sin$

$$=\frac{(\cos A+\cos B)+\sin A\cos B+\cos A\sin B}{\cos A\cos B}$$

$\cos\ \alpha-\sin^2\beta=\cos(\alpha+\beta)\cos(\alpha-$

$$=\frac{\cos A(1+\sin B)+\cos B(1+\sin A)}{\cos A\cos B}$$

$$=\frac{1+\sin B}{\cos B}+\frac{1+\sin A}{\cos A}$$

$$=\frac{1+\sin A}{\cos A}+\frac{(1+\sin B)\cos B}{1-\sin B}=\frac{1+\sin A}{\cos A}+\frac{\cos B}{1-\sin B}$$

60 We know that $\tan\left(\frac{\pi}{4}+\alpha\right)=\frac{1+\tan\alpha}{1-\tan\alpha}=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$

$$\tan^2\left(\frac{\pi}{4}+\alpha\right)=\frac{1+\sin 2\alpha}{1-\sin 2\alpha}\qquad 2\sin\alpha\cos\alpha=\sin 2\alpha$$

Squaring the given relation we have to prove that

$$\frac{1+\sin y}{1-\sin y}=\frac{(1+\sin x)^3}{(1-\sin x)^3}$$ Apply comp and divi

$$\frac{2\sin y}{2}=\frac{2(3\sin x+\sin^3 x)}{2(1+3\sin^2 x)}$$

$$\sin y=\frac{\sin x(3+\sin^2 x)}{1+3\sin^2 x}$$

61 L H S $\sin^3 x(3\sin x-4\sin^3 x)+\cos^3 x(4\cos^3 x-3\cos x)$

$=4(\cos^6 x-\sin^6 x)-3(\cos^4 x-\sin^4 x)$

$=\cos 2x[4(\cos^4 x+\sin^4 x+\cos^2 x\sin^2 x)-3]$

$=\cos 2x[4\{(\cos^2 x-\sin^2 x)^2+3\cos^2 x\sin^2 x\}-3]$

$=\cos 2x[4\cos^2 2x-3(1-4\sin^2 x\cos^2 x)]$

$=\cos 2x[4\cos^2 2x-3(1-\sin^2 2x)]$

$=\cos 2x[4\cos^2 2x-3\cos^2 2x]$

$=\cos 2x\cos^2 2x\quad=\cos^3 2x$

62 We have to eliminate θ and ϕ

$\sin^2\alpha=4\sin^2\phi/2\ \sin^2\theta/2=(1-\cos\phi)(1-\cos\theta)$

or $1-\cos^2\alpha=\left(1-\frac{\cos\alpha}{\cos\beta}\right)\left(1-\frac{\cos\alpha}{\cos\gamma}\right)$

$$=1-\cos\alpha\left(\frac{1}{\cos\beta}+\frac{1}{\cos\gamma}\right)+\frac{\cos^2\alpha}{\cos\beta\cos\gamma}$$

or $\cos\alpha\left(\frac{\cos\beta+\cos\gamma}{\cos\beta\cos\gamma}\right)=\cos^2\alpha\,\frac{(1+\cos\beta\cos\gamma)}{\cos\beta\cos\gamma}$

$$\frac{\cos\beta+\cos\gamma}{1+\cos\beta\cos\gamma}=\frac{\cos\alpha}{1}$$ Apply comp and dividend

$$\frac{1-\cos\beta-\cos\gamma+\cos\beta\cos\gamma}{1+\cos\beta+\cos\gamma+\cos\beta\cos\gamma}=\frac{1-\cos\alpha}{1+\cos\alpha}$$

or $\frac{(1-\cos\beta)(1-\cos\gamma)}{(1+\cos\beta)(1+\cos\gamma)}=\frac{1-\cos\alpha}{1+\cos\alpha}$

or $\frac{(2\sin^2\beta/2)(2\sin^2\gamma/2)}{(2\cos^2\beta/2)(2\cos^2\gamma/2)}=\frac{2\sin^2\alpha/2}{2\cos^2\alpha/2}$

or $\tan^2\beta/2\tan^2\gamma/2=\tan^2\alpha/2$

63 The given expression

$=\frac12[a(1-\cos 2\theta)+b\sin 2\theta+c(1+\cos 2\theta)]$

$=\frac12(a+c)+\frac12 b\sin 2\theta+\frac12(c-a)\cos 2\theta$

Hence by § 2(37) the least and greatest values of the given expression are respectively

$\frac12(a+c)-\frac12\sqrt{\{b^2+(c-a)^2\}}$ and $\frac12(a+c)+\frac12\sqrt{\{b^2+(c-a)^2\}}$

so that the value of the expression lies between these values as required

64 (a) $\sin(\alpha+\beta+\gamma)=\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma$

$=\sin\alpha\cos\beta\cos\gamma+\cos\alpha\sin\beta\cos\gamma+\cos\alpha\cos\beta\sin\gamma$

$\sin\alpha\sin\beta\sin\gamma$

Now divide each term by $\cos\alpha\cos\beta\cos\gamma$ and you get L.H.S

(b) On putting the value of $\sin(\alpha+\beta+\gamma)$ we have

$\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$=\sin\alpha(1-\cos\beta\cos\gamma)+\sin\beta(1-\cos\gamma\cos\alpha)$

$+\sin\gamma(1-\cos\alpha\cos\beta)+\sin\alpha\sin\beta\sin\gamma>($

α, β, γ all lie in Ist quadrant . $\cos\alpha\cos\beta$ is +ive and less than 1 so that $1-\cos\alpha\cos\beta$ is +ive and also $\sin\gamma$ is +ive Thus every term in L H S is +ive

$\sin\alpha+\sin\beta+\sin\gamma>\sin(\alpha+\beta+\gamma)$ Proved

65 $\dfrac{\sin\theta\cos A+\cos\theta\sin A}{\sin\theta\cos B+\cos\theta\sin B}=\sqrt{\left\{\dfrac{2\sin A\cos A}{2\sin B\cos B}\right\}}$

or $\{\tan\theta\cos A+\sin A\}\sqrt{(\sin B\cos B)}$

$=(\tan\theta\cos B+\sin B)\sqrt{(\sin A\cos A)}$

Collecting the terms of $\tan\theta$ on one side and rest on the other we get

$\tan\theta\{(\cos A\sqrt{(\sin B\cos B)}-\cos B\sqrt{(\sin A\cos A)}\}$

$=\{\sin B\sqrt{(\sin A\cos A)}-\sin A\sqrt{(\sin B\cos B)}\}$

or $\tan\theta\sqrt{(\cos A\cos B)}\{\sqrt{(\cos A\sin B)}-\sqrt{(\cos B\sin A)}\}$

$=\sqrt{(\sin A\sin B)}\{\sqrt{(\cos A\sin B)}-\sqrt{(\cos B\sin A)}\}$

$\tan\theta=\sqrt{\{(\tan A\tan B)\}}$ or $\tan^2\theta=\tan A\tan B$

66 Making use of given relations we have

$\cos^2 x=\tan^2 y=\sec^2 y-1=\cot^2 z-1$

or $1+\cos^2 x=\dfrac{\cos^2 z}{1-\cos^2 z}=\dfrac{\tan^2 x}{1-\tan^2 z}=\dfrac{\sin^2 x}{\cos^2 x-\sin^2 x}$

or changing to $\sin x$ we get

$(2-\sin^2 x)(1-2\sin^2 x)=\sin^2 x$

or $2\sin^4 x-6\sin^2 x+2=0$ or $\sin^4 x-3\sin^2 x+1=0$

$\sin^2 x=\dfrac{3\pm\sqrt{(9-4)}}{2}=\dfrac{3\pm\sqrt{5}}{2}=\dfrac{3-\sqrt{5}}{2}$

We have rejected the value of $\dfrac{3+\sqrt{5}}{2}$ as it is greater than 1

or $\sin^2 x=\dfrac{6-2\sqrt{5}}{4}=\left(\dfrac{\sqrt{5}-1}{2}\right)^2$

$\sin x=\dfrac{\sqrt{5}-1}{2}=2\dfrac{(\sqrt{5}-1)}{4}=2\sin 18^\circ$

By symmetry we can say that $\sin x=\sin y=\sin z=2\sin 18^\circ$

67 From the given relation we have

$$3\cos x+\sin x=3\cos y-\sin y \qquad (1)$$

Put $3=r\cos\alpha \quad 1=r\sin\alpha \qquad r=\sqrt{10},\ \tan\alpha=\frac{1}{3}$

$r\cos(x-\alpha)=r\cos(y+\alpha)$

$x-\alpha=\pm(y+\alpha)$

$x=-y$ or $x=y+2\alpha$

Clearly $x=-y$ satisfies (1)

$3x=-3y$ or $\sin 3x=\sin(-3y)=-\sin 3y$

or $\sin 3x+\sin 3y=0$

68 Let us put $\tan\alpha=x,\ \tan\beta=y,\ \tan\gamma=z$

$$2x^2y^2z^2+x^2y^2+y^2z^2+z^2x^2=1 \text{ (given)} \qquad (1)$$

Now $\sin^2\alpha=\dfrac{\sin^2\alpha}{\cos^2\alpha+\sin^2\alpha}=\dfrac{\tan^2\alpha}{1+\tan^2\alpha}=\dfrac{x^2}{1+x^2}$

$\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\dfrac{x^2}{1+x^2}+\dfrac{y^2}{1+y^2}+\dfrac{z^2}{1+z^2}$

$\Sigma\sin^2\alpha=\dfrac{\Sigma x^2(1+y^2)(1+z^2)}{(1+x^2)(1+y^2)(1+z^2)}$

$N^r=x^2(1+y^2+z^2+y^2z^2)+y^2(1+z^2+x^2+z^2x^2)$
$\qquad +z^2(1+x^2+y^2+x^2y^2)$

$=\Sigma x^2+2\Sigma x^2y^2+3x^2y^2z^2$

$=\Sigma x^2+2(1-2x^2y^2z^2)+3x^2y^2z^2$ by (1)

$=2+\Sigma x^2-x^2y^2z^2 \qquad (2)$

$D^r=1+\Sigma x^2+\Sigma x^2y^2+x^2y^2z^2$

$=1+\Sigma x^2+(1-2x^2y^2z^2)+x^2y^2z^2$ by (1)

$=2+\Sigma x^2-x^2y^2z^2=N^r$ by (2) $\qquad (3)$

$\Sigma\sin^2\alpha=\dfrac{N^r}{D^r}=\dfrac{N^r}{N^r}=1$ by (3)

69 (*a*) Let us put $\tan\theta=t_1,\ \tan\phi=t_2 \qquad t_1^2t_2^2=\left(\dfrac{a-b}{a+b}\right) \quad (1)$

Also $\cos 2\theta=\dfrac{1-\tan^2\theta}{1+\tan^2\theta}=\dfrac{1-t_1^2}{1+t_1^2}$ etc

Now $a-b\cos 2\theta=a-b\,\dfrac{(1-t_1^2)}{(1+t_1^2)}$

$=\dfrac{(a-b)+(a+b)\,t_1^2}{(1+t_1^2)}=\dfrac{(a+b)}{(1+t_1^2)}\left[\dfrac{a-b}{a+b}+t_1^2\right]$

$=\dfrac{(a+b)}{(1+t_1^2)}[t_1^2t_2^2+t_1^2]=\dfrac{(a+b)\,t_1^2\,(1+t_2^2)}{(1+t_1^2)}$ by (1)

Similary $a-b\cos 2\phi=\dfrac{(a+b)}{1+t_2^2}t_2^2\,(1+t_1^2)$

$$(a-b\cos 2\theta)(a-b\cos 2\phi)=(a+b)^2\, t_1^2 t_2^2$$

$$=(a+b)^2\,\frac{(a-b)}{a+b}\,a^2-b^2$$

which is independent of θ

(b) A careful look at the question suggests that we ha[ve] to prove $x=y$. We know that $2\cos^2\theta=1+\cos 2\theta$, an[d] $2\sin^2\theta=1-\cos 2\theta$. Hence changing to double angle i[n] the given relation $\frac{\cos^4 x}{\cos^2 y}+\frac{\sin^4 x}{\sin^2 y}=1$, we get

$$\tfrac{1}{4}(1+\cos 2x)^2\,\tfrac{1}{2}(1-\cos 2y)+\tfrac{1}{4}(1-\cos 2x)\,\tfrac{1}{2}(1+\cos 2y)$$

$$=\tfrac{1}{2}(1+\cos 2y)\,\tfrac{1}{2}(1-\cos 2y)$$

or $\{(1+\cos 2x)^2+(1-\cos 2x)^2\}-\cos 2y\,\{(1+\cos 2x)^2-(1-\cos 2x)^2\}$

$=2(1-\cos^2 2y)$

or $2(1+\cos^2 2x)-\cos 2y\,(4\cos 2x)=2-2\cos^2 2y$

or $\cos^2 2x+\cos^2 2y-2\cos 2x\cos 2y=0$

or $(\cos 2x-\cos 2y)^2=0$ $\qquad \cos 2x=\cos 2y$ or $x=y$

$\frac{\cos^4 y}{\cos^2 x}+\frac{\sin^4 y}{\sin^2 x}=\cos^2 x+\sin^2 x=1$ (on putting $y=x$)

70 (a) Let us put $\cos(\alpha+\theta)\sin(\beta-\gamma)=a$, etc

$a+b+c=\Sigma(\cos\alpha\cos\theta-\sin\alpha\sin\theta)\sin(\beta-\gamma)$

$=\cos\theta\,\Sigma\cos\alpha\sin(\beta-\gamma)-\sin\theta\,\Sigma\sin\alpha\sin(\beta-\gamma)$

It is easy to observe on expansion that both the above sigmas are zero. Hence $a+b+c=0$ so that from Algebra we know that $a^3+b^3+c^3=3abc$

On putting the values of a, b, c we prove the given result

(b) Multiplying the given relations by $\sin B$ and $\cos B$ and subtracting we get

$-\sqrt{2}(\sin A\cos B-\cos A\sin B)=\sin B\cos B(\cos^2 B+\sin^2 B)$

$$\sin(A-B)=-\frac{1}{2\sqrt{2}}\sin 2B \qquad (1)$$

Again squaring and adding the given relations we get

$2\cdot 1=1+(\cos^6 B+\sin^6 B)+2(\cos^4 B-\sin^4 B)$

$1=1(\cos^4 B+\sin^4 B-\cos^2 B\sin^2 B)+2\cdot 1(\cos^2 B-\sin^2 B)$

or $1=(1-3\cos^2 B\sin^2 B)+2\cos 2B$

or $\tfrac{3}{4}(2\sin B\cos B)^2=2\cos 2B$

or $3\sin^2 2B=8\cos 2B$

or $3(1-\cos^2 2B)=8\cos 2B$

or $3\cos^2 2B+8\cos 2B-3=0$

or $3\cos^2 2B+9\cos 2B-\cos 2B-3=0$

or $(\cos 2B+3)(3\cos 2B-1)=0$ $\cos 2B=\frac{1}{3}$ as $\cos 2B\neq -3$

$\sin^2 2B=1-\cos^2 2B=1-\frac{1}{9}=\frac{8}{9}$

$\sin 2B=\pm\frac{2\sqrt{2}}{3}$

Hence from (1), $\sin(A-B)=\frac{1}{2\sqrt{2}}\sin 2B=\pm\frac{1}{3}$ **Proved**

§ 3 Angles of triangle

If A B C be the angles of a triangle then we know that

$A+B+C=180°$ or $B+C=180°-A$

(*a*) $\sin(B+C)=\sin(180°-A)=\sin A$

Similary $\sin(C+A)=\sin B$ $\sin(A+B)=\sin C$

Thus in a triangle sine of any one angle is equal to sin of sum of the remaining two angles

(*b*) $\cos(B+C)=\cos(180°-A)=-\cos A$

or $\cos A=-\cos(B+C)$

Similarly $\cos(C+A)=-\cos B$, $\cos B=-\cos(C+A)$

$\cos(A+B)=-\cos C$, $\cos C=-\cos(A+B)$

Thus the cosine of any one angle is equal to minus times the cosine of the remaining two angles

(*c*) $A+B+C=180°$, $\frac{1}{2}(A+B+C)=90°$

$$\sin\left(\frac{B+C}{2}\right)=\sin\left(90°-\frac{A}{2}\right)=\cos\frac{A}{2}$$

$$\cos\left(\frac{B+C}{2}\right)=\cos\left(90°-\frac{A}{2}\right)=\sin\frac{A}{2}$$

Thus the sine of half of any angle in a triangle is equal to the cosine of half the sum of the remaining two angles and similarly the cosine of half of any angle is equal to sine of half of the sum of the other angles

(*d*) $\tan(A+B+C)=\frac{S_1-S_3}{1-S_2}$

$$=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

If A,B,C are angles of a triangle then $\tan(A+B+C)=\tan\pi=0$ in this case, $\tan A+\tan B+\tan C=\tan A\tan B\tan C$ or $S_1=S_3$

(*e*) $\sin(A+B+C)=\sin A\cos B\cos C+\cos A\sin B\cos C$
$+\cos A\cos B\sin C-\sin A\sin B\sin C$

If A, B, C are angles of a triangle, then $\sin(A+B+C)=\sin\pi=0$. Hence in this case, $\sin A\sin B\sin C=\sin A\cos B\cos C$
$+\cos A\sin B\cos C+\cos A\cos B\sin C$

(f) $\cos(A+B+C)=\cos A\cos B\cos C-\cos A\sin B\sin C$
$-\sin A\cos B\sin C-\sin A\sin B\cos C$

If A, B, C are angles of a triangle, then $\cos(A+B+C)=\cos\pi$ $=-1$. Hence in this case $1+\cos A\cos B\cos C$
$-\cos A\sin B\sin C+\sin A\cos B\sin C+\sin A\sin B\cos C$

Problem Set (C)

1 If $A+B+C=180°$, then prove the following

(i) $\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$ **(Roorkee 79)**

(ii) $\sin 2A+\sin 2B-\sin 2C=4\cos A\cos B\sin C$

(iii) $\sin(B+C-A)+\sin(C+A-B)+\sin(A+B-C)$
$=4\sin A\sin B\sin C$

(iv) If $\sin^{-1}a+\sin^{-1}b+\sin^{-1}c=\pi$ then prove that
$a\sqrt{(1-a^2)}+b\sqrt{(1-b^2)}+c\sqrt{(1-c^2)}=2abc$

2 (i) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$

(ii) $\cos 2A+\cos 2B-\cos 2C=1-4\sin A\sin B\cos C$

3 If $A+B+C=180°$, prove that

(i) $\sin A+\sin B+\sin C=4\cos A/2\cos B/2\cos C/2$

(ii) $\sin A+\sin B-\sin C=4\sin A/2\sin B/2\cos C/2$

(iii) $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\sin A+\sin B+\sin C}=8\sin A/2\sin B/2\sin C/2$

(iv) $\dfrac{\cos A}{\sin B\sin C}+\dfrac{\cos B}{\sin C\sin A}+\dfrac{\cos C}{\sin A\sin B}=2$

(v) Express $\sin 3A+\sin 3B+\sin 3C$ as the product of three trigonometrical ratios where A, B, C are the angles of a triangle. If the given expression be zero then at least one angle of the triangle is $60°$

(vi) For any three angles A, B, C prove that
$\sin A+\sin B+\sin C-\sin(A+B+C)$
$=4\sin\dfrac{A+B}{2}\sin\dfrac{B+C}{2}\sin\dfrac{C+A}{2}$

(vii) If $A+B+C=180°$ show that
$\sin(B+2C)+\sin(C+2A)+\sin(A+2B)$
$=4\sin\dfrac{B-C}{2}\sin\dfrac{C-A}{2}\sin\dfrac{A-B}{2}$ **(Roorkee 68)**

(viii) In a triangle prove that

$$\sin^3 A+\sin^3 B+\sin^3 C=3\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$$

(ix) If $A+B+C=2\pi$, prove that

$$\sin^3 A+\sin^3 B+\sin^3 C=3\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}-\sin\frac{3A}{2}\sin\frac{3B}{2}\sin\frac{3C}{2}$$

(x) In a triangle ABC prove that

$$\sin^4 A+\sin^4 B+\sin^4 C=\frac{3}{2}+2\cos A\cos B\cos C+\frac{1}{2}\cos 2A\cos 2B\cos 2C$$

In any triangle A, B, C, prove the identities in Q 4, 5 and 6

4 (i) $\cos A+\cos B+\cos C=1+4\sin A/2\sin B/2\sin C/2$

(ii) $\cos A+\cos B-\cos C=-1+4\cos A/2\cos B/2\sin C/2$

(iii) $\dfrac{\sin 2A+\sin 2B+\sin 2C}{\cos A+\cos B+\cos C-1}=8\cos A/2\cos B/2\cos C/2$

(I I T 76)

5 (i) $\sin^2 A+\sin^2 B-\sin^2 C=2\sin A\sin B\cos C$

(ii) $\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$

(iii) $\sin^2 A+\sin^2 B+\sin^2 C=2+2\cos A\cos B\cos C$

6 (i) $\sin^2 A/2+\sin^2 B/2+\sin^2 C/2=1-2\sin A/2\sin B/2\sin C/2$

(ii) $\cos^2 A/2+\cos^2 B/2+\cos^2 C/2=2+2\sin A/2\sin B/2\sin C/2$

(iii) $\sin^2 A/2+\sin^2 B/2-\sin^2 C/2=1-2\cos A/2\cos B/2\sin C/2$

7 If $x+y+z=\pi/2$, show that

$\sin^2 x+\sin^2 y+\sin^2 z=1-2\sin x\sin y\sin z$

$\cos^2 x+\cos^2 y+\cos^2 z=2+2\sin x\sin y\sin z$

8 If $A+B+C=\pi$, prove that

(i) $\cos A/2+\cos B/2+\cos C/2=4\cos\dfrac{\pi-A}{4}\cos\dfrac{\pi-B}{4}\cos\dfrac{\pi-C}{4}$

$$=4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}$$

(ii) $\sin A/2+\sin B/2+\sin C/2$

$$=1+4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$$

(iii) $\cos A/2 + \cos B/2 - \cos C/2 = 4\cos\dfrac{\pi+A}{4}\cos\dfrac{\pi+B}{4}\cos\dfrac{\pi-C}{4}$

(i) If $\alpha+\beta=\gamma$, then
$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$

(ii) $\sin 2(\alpha-\beta)+\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)$
$= 4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)$

(iii) $\sin x\sin y\sin(x-y)+\sin y\sin z\sin(y-z)$
$+\sin z\sin x\sin(z-x)+\sin(x-y)\sin(y-z)\sin(z-x)$
$=0$

(iv) $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)-\sin(\alpha+\beta+\gamma)$
$=4\sin\alpha\sin\beta\sin\gamma$ (I I T 78)

10 If A, B, C are the angles of a triangle, show that

(i) $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) $\cot A\cot B+\cot B\cot C+\cot C\cot A=1$

(iii) $\dfrac{\cot A+\cot B}{\tan A+\tan B}+\dfrac{\cot B+\cot C}{\tan B+\tan C}+\dfrac{\cot C+\cot A}{\tan C+\tan A}=1$

11 (a) If $x+y+z=xyz$, prove that

(i) $\dfrac{2x}{1-x^2}+\dfrac{2y}{1-y^2}+\dfrac{2z}{1-z^2}=\dfrac{2x}{1-x^2}\,\dfrac{2y}{1-y^2}\,\dfrac{2z}{1-z^2}$

(ii) $\dfrac{3x-x^3}{1-3x^2}+\dfrac{3y-y^3}{1-3y^2}+\dfrac{3z-z^3}{1-3z^2}=\dfrac{3x-x^3}{1-3x^2}\,\dfrac{3y-y^3}{1-3y^2}\,\dfrac{3z-z^3}{1-3z^2}$ (Roorkee 83)

(iii) If $xy+yz+zx=1$, prove that
$\dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$

(iv) If $xy+yz+zx=1$ then prove that
$\dfrac{x}{1+x^2}+\dfrac{y}{1+y^2}+\dfrac{z}{1+z^2}=\dfrac{2}{[(1+x^2)(1+y^2)(1+z^2)]^{1/2}}$ (I I T 71)

(b) For any three angles A, B, C prove that
$\tan(B-C)+\tan(C-A)+\tan(A-B)$
$=\tan(B-C)\tan(C-A)\tan(A-B)$

12. If $A+B+C=\pi$ prove that

(i) $\tan A/2\tan B/2+\tan B/2\tan C/2+\tan C/2\tan A/2=1$

(ii) $\cot A/2+\cot B/2+\cot C/2=\cot A/2\cot B/2\cot C/2$,

Another form

If $x+y+z=\pi/2$ then

(i) $\tan y\tan z+\tan z\tan x+\tan x\tan y=1$

(ii) $\cot x+\cot y+\cot z=\cot x\cot y\cot z$

13 If $A+B+C=2S$, show that

(i) $\cos^2 S+\cos^2(S-A)+\cos^2(S-B)+\cos^2(S-C)$
$$=2+2\cos A\cos B\cos C$$
(I I T 62)

(ii) $\sin(S-A)+\sin(S-B)+\sin(S-C)-\sin S$
$$=4\sin A/2\sin B/2\sin C/2$$

14 If $A+B+C+D=2\pi$, show that
$\cos A-\cos B+\cos C-\cos D$
$$=4\sin\frac{A+B}{2}\sin\frac{A+D}{2}\cos\frac{A+C}{2}$$

15 $\tan^2 A/2+\tan^2 B/2+\tan^2 C/2\geqslant 1$ when $A+B+C=\pi$
Hence the minimum value of $\Sigma\tan^2 A/2$ is 1

16 In a triangle A, B, C, prove that
$\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)$
$$=3\sin A\sin B\sin C$$

17 $\sin 3A\cos^3(B-C)+\sin 3B\cos^3(C-A)+\sin 3C\cos^3(A-B)$
$$=\sin 3A\sin 3B\sin 3C$$

18 In a triangle if $\tan A/2$, $\tan B/2$, $\tan C/2$ are in A P then $\cos A$, $\cos B$, $\cos C$ are also in A P

19 If the tangents of the angles of a triangle are in A P, prove that the squares of the sides are in the ratio
$x^2(x^2+9)$: $(3+x^2)^2$: $9(1+x^2)$, where x is the least or greatest tangent

20 In a triangle ABC prove that
$a^3\cos B\cos C+b^3\cos C\cos A+c^3\cos A\cos B$
$$=abc(1-2\cos A\cos A\cos C)$$

Solutions to Problem set (C)

We shall make use of the formulae of § 3 Page 57

1 (i) L H S $=2\sin A\cos A+2\sin(B+C)\cos(B-C)$
$=2\sin A\cos A+2\sin A\cos(B-C)$
$=2\sin A[\cos A+\cos(B-C)]$
$=2\sin A[\cos(B-C)-\cos(B+C)]$
$=2\sin A\cdot 2\sin B\sin C$, $\quad\cos A=-\cos(B+C)$
$=4\sin A\sin B\sin C$

(ii) $\sin 2A+\sin 2B-\sin 2C$
L H S $=2\sin A\cos A+2\sin(B-C)\cos(B+C)$
$=2\sin A\cos A-2\sin(B-C)\cos A$
$=2\cos A[\sin A-\sin(B-C)]$
$=2\cos A[\sin(B+C)-\sin(B-C)]$

$=2\cos A\,[2\cos B\sin C]$

$=4\cos A\cos B\sin C$ $\qquad \sin A=\sin(B+C)$

(ii) $B+C-A=180^\circ-A-A=180^\circ-2A$

$\sin(B+C-A)=\sin(180^\circ-2A)=\sin 2A$

Hence the question reduces to

$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C$

To prove it proceed as in (ii)

(iv) Let $\sin^{-1} a=A$ $\qquad a=\sin A$

$a\sqrt{(1-a^2)}=\sin A\cos A=\frac{1}{2}\sin 2A$

2 (i) In the answer we want -1 and as such we write $\cos 2A$ as $2\cos^2 A-1$ and combine the other two terms. It becomes (i) part

L H S $=(2\cos^2 A-1)+2\cos(B+C)\cos(B-C)$

$=-1+2\cos^2 A-2\cos A\cos(B-C)$

$=-1+2\cos A\,[\cos A-\cos(B-C)]$

$=-1+2\cos A\,[-\cos(B+C)-\cos(B-C)]$

$=-1-2\cos A\,(2\cos B\cos C)$

$=-1-4\cos A\cos B\cos C$

(ii) Here we want $+1$ and as such we write $\cos 2A=1-2\sin^2 A$ and combine the other two terms

L H S $=1-2\sin^2 A+2\sin(B+C)\sin(C-B)$

$=1-2\sin^2 A-2\sin A\sin(B-C)$ $\qquad$ Note $[\sin(-\theta)=-\sin\theta]$

$=1-2\sin A\,[\sin A+\sin(B-C)]$

$=1-2\sin A\,[\sin(B+C)+\sin(B-C)]$

$=1-2\sin A\,(2\sin B\cos C)$

$=1-4\sin A\sin B\cos C$

3 (i) L H S

$=2\sin A/2\cos A/2+2\sin\frac{B+C}{2}\cos\frac{B-C}{2}$

$=2\sin A/2\cos A/2+2\cos A/2\cos\frac{B-C}{2}$

$=2\cos A/2\left[\sin A/2+\cos\frac{B-C}{2}\right]$

$=2\cos A/2\left[\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right]$

$=2\cos A/2\,(2\cos B/2\cos C/2)$

$=4\cos A/2\cos B/2\cos C/2$

(ii) Proceed as above

(*iii*) By the help of Q 1 (*i*) and Q 3 (*i*), we have

L H S

$$=\frac{4 \sin A \sin B \sin C}{4 \cos A/2 \cos B/2 \cos C/2}$$

$$=\frac{(2 \sin A/2 \cos A/2)(2 \sin B/2 \cos B/2)(2 \sin C/2 \cos C/2)}{\cos A/2 \cos B/2 \cos C/2}$$

$$= 8 \cos A/2 \cos B/2 \cos C/2$$

(*iv*) On simplification we have to prove that

$$\frac{\sin A \cos A+\sin B \cos B+\sin C \cos C}{\sin A \sin B \sin C}$$

Multiplying both sides by 2, we get

$$\sin 2A+\sin 2B+\sin 2C=4 \sin A \sin B \sin C$$

which is Q 1 (*i*)

(*v*) $A+B+C=\pi, \qquad 3A+3B+3C=3\pi$

$$\frac{3A+3B}{2}=\frac{3\pi}{2}-\frac{3C}{2}$$

$$\sin \frac{3A+3B}{2}=-\cos \frac{3C}{2} \text{ and } \cos \frac{3A+3B}{2}=-\sin \frac{3C}{2}$$

Note

$$\text{L H S}=2 \sin \frac{3A+3B}{2} \cos \frac{3A-3B}{2}+2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$=-2 \cos \frac{3C}{2} \cos \frac{3A-3B}{2}+2 \sin \frac{3C}{2} \cos \frac{3C}{2}$$

$$=-2 \cos \frac{3C}{2}\left[\cos \frac{3A-3B}{2}-\sin \frac{3C}{2}\right]$$

$$=-2 \cos \frac{3C}{2}\left[\cos \frac{3A-3B}{2}+\cos \frac{3A+3B}{2}\right]$$

$$=-4 \cos \frac{3A}{2} \cos \frac{3B}{2} \cos \frac{3C}{2}$$

If it be zero then at least one factor is zero, hence

$$\frac{3A}{2}=90^\circ \quad \text{or} \quad A=60^\circ \text{ etc}$$

(*vi*) Note Here A, B, C are not angles of a triangle i e $A+B+C\neq180^\circ$

Combining the first two and last two, we get

$$\text{L H S}=2 \sin \frac{A+B}{2} \cos \frac{A-B}{2}+2 \cos \frac{A+B+2C}{2} \sin\left(-\frac{A+B}{2}\right)$$

$$=2 \sin \frac{A+B}{2}\left[\cos \frac{A-B}{2}-\cos \frac{A+B+2C}{2}\right]$$

$\sin(-\theta)=-\sin\theta$

$$=2\sin\frac{A+B}{2}\left[2\sin\frac{A+C}{2}\sin\frac{B+C}{2}\right]$$

$$=4\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+A}{2}$$

(vii) Here $A+B+C=180°$

$B+2C=B+C+C=180°-A+C=180°+(C-A)$

$\sin(B+2C)=-\sin(C-A)$

L.H S $=-[\sin(C-A)+\sin(A-B)+\sin(B-C)$

$$=-\left[2\sin\frac{C-A}{2}\cos\frac{C-A}{2}+2\sin\frac{A-C}{2}\cos\frac{A+C-2B}{2}\right]$$

$$=-2\sin\frac{C-A}{2}\left[\cos\frac{C-A}{2}-\cos\frac{A+C-2B}{2}\right]$$

$$=-2\sin\frac{C-A}{2}\left[2\sin\frac{C-B}{2}\sin\frac{A-B}{2}\right]$$

$$=4\sin\frac{A-B}{2}\sin\frac{B-C}{2}\sin\frac{C-A}{2}$$

(viii) $\sin 3A=3\sin A-4\sin^3 A$

$$\sin^3 A=\frac{1}{4}(3\sin A+\sin 3A)$$

Hence we have to find the value of

$$\frac{3}{4}(\sin A+\sin B+\sin C)+\frac{1}{4}(\sin 3A+\sin 3B+\sin 3C)$$

$$=\frac{3}{4}\,4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+\frac{1}{4}\,4\cos\frac{3A}{2}\cos\frac{3B}{2}\cos\frac{3C}{2}$$

by Q 3 (i & v)

(ix) Proceed as above Note that $A+B+C=2\pi$

(x) $2\sin^2 A=1-\cos 2A$

$4\sin^4 A=1-2\cos 2A+\cos^2 2A$

or $\sin^4 A=\frac{1}{4}-\frac{1}{2}\cos 2A+\frac{1}{4}\cdot\frac{1+\cos 4A}{2}$

$$=\frac{3}{8}-\frac{1}{2}\cos 2A+\frac{1}{8}\cos 4A$$

$$\Sigma\sin^4 A=3\cdot\frac{3}{8}-\frac{1}{2}\Sigma\cos 2A+\frac{1}{8}\Sigma\cos 4A \quad (1)$$

Now by Q 2 (i)

$\Sigma\cos 2A=-1-4\cos A\cos B\cos C$ (2)

Similarly we may prove that $\Sigma\cos 4A$

$=-1+4\cos 2A\cos 2B\cos 2C$ (3)

Hence from (1) by the help of (2) and (3) we prove the result

4 (*i*) Here we want $+1$ and so we write $\cos A = 1 - 2\sin^2 A/2$

$\text{L H S} = 1 - 2\sin^2 A/2 + 2\cos\frac{B+C}{2}\cos\frac{B-C}{2}$

$= 1 - 2\sin^2 A/2 + 2\sin A/2\cos\frac{B-C}{2}$

$= 1 - 2\sin A/2\left[\sin A/2 - \cos\frac{B-C}{2}\right]$

$= 1 - 2\sin A/2\left[\cos\frac{B+C}{2} - \cos\frac{B-C}{2}\right]$

$= 1 - 2\sin A/2\,[2\sin B/2\sin(-C/2)]$

$= 1 + 4\sin A/2\sin B/2\sin C/2$ $\qquad\sin(-\theta) = -\sin\theta$

(*ii*) Here write $\cos A = 2\cos^2 A/2 - 1$ and proceed as above

(*iii*) With the help of 1 (*i*) and Q 4 (*i*),

$\text{L.H S} = \frac{4\sin A\sin B\sin C}{4\sin A/2\sin B/2\sin C/2} = 8\cos A/2\cos B/2\cos C/2$

$\sin A = 2\sin A/2\cos A/2$ etc

5 (*i*) $\text{L H S} = \sin^2 A + \sin(B+C)\sin(B-C)$

$= \sin^2 A + \sin A\sin(B-C)$

$= \sin A\,[\sin A + \sin(B-C)]$

$= \sin A\,[\sin(B+C) + \sin(B-C)]$

$= \sin A\,(2\sin B\cos C)$

$= 2\sin A\sin B\cos C$

(*ii*) We write $\cos^2 A = 1 - \sin^2 A$

$\text{L H S} = 1 - \sin^2 A + \cos^2 B + \cos^2 C$

$= 1 + (\cos^2 B - \sin^2 A) + \cos^2 C$

$= 1 + \cos(B+A)\cos(B-A) + \cos^2 C$

$= 1 - \cos C\cos(B-A) + \cos^2 C$

$= 1 - \cos C[\cos(B-A) - \cos C]$

$= 1 - \cos C[\cos(B-A) + \cos(B+A)]$

$= 1 - \cos C(2\cos A\cos B)$

$= 1 - 2\cos A\cos B\cos C$

(*iii*) On changing $\sin^2 A$ to $1 - \cos^2 A$ etc it reduces to part II

Proceeding directly as we need 2, we write

$\sin^2 A = 1 - \cos^2 A$ and $\sin^2 B = 1 - \cos^2 B$

$\text{L H S} = 2 - \cos^2 A - \cos^2 B + \sin^2 C$

$= 2 - (\cos^2 A - \sin^2 C) - \cos^2 B$

$= 2 - \cos(A+C)\cos(A-C) - \cos^2 B$

$= 2 + \cos B\cos(A-C) - \cos^2 B$

$= 2 + \cos B\,[\cos(A-C) - \cos B]$

$=2+\cos B\,[\cos(A-C)+\cos(A+C)]$

$=2+\cos B\,(2\cos A\cos C)$

$=2+2\cos A\cos B\cos C$

6 (i) We want $+1$ in R H S

$$L\,H\,S=1-\cos^2 A/2+\sin^2 B/2+\sin^2 C/2$$
$$=1-(\cos^2 A/2-\sin^2 B/2)+\sin^2 C/2$$
$$=1-\cos\frac{A+B}{2}\cos\frac{A-B}{2}+\sin^2\frac{C}{2}$$
$$=1-\sin C/2\cos\frac{A-B}{2}+\sin^2\frac{C}{2}$$
$$=1-\sin C/2\left[\cos\frac{A-B}{2}-\sin\frac{C}{2}\right]$$
$$=1-\sin\frac{C}{2}\left[\cos\frac{A-B}{2}-\cos\frac{A+B}{2}\right]$$
$$=1-\sin C/2\left[2\sin\frac{A}{2}\sin\frac{B}{2}\right]$$
$$=1-2\sin A/2\sin B/2\sin C/2$$

(ii) Either change $\cos^2$ to $\sin^2$ and it becomes part (i) or proceed directly as in Q 5 (iii)

(iii) We want $+1$ in R H S

$$L\,H\,S=1-\cos^2 A/2+\sin^2 B/2-\sin^2 C/2$$
$$=1-\cos^2\frac{A}{2}+\sin\frac{B+C}{2}\sin\frac{B-C}{2}$$
$$=1-\cos^2\frac{A}{2}+\cos\frac{A}{2}\sin\frac{B-C}{2}$$
$$=1-\cos\frac{A}{2}\left[\cos\frac{A}{2}-\sin\frac{B-C}{2}\right]$$
$$=1-\cos\frac{A}{2}\left[\sin\frac{B+C}{2}-\sin\frac{B-C}{2}\right]$$
$$=1-\cos\frac{A}{2}\left[2\cos\frac{B}{2}\sin\frac{C}{2}\right]$$
$$=1-2\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

(iii)

7 It is same as Q 6 If we choose $x=A/2$ $y=B/2$ $z=C/2$ so that $x+y+z=\pi/2$ leads to $A+B+C=\pi$ and then both parts reduce to those of Q 6 (i) and (ii) However you should proceed directly

8 (i) $A+B+C=\pi$ $\quad \frac{A+B}{4}=\frac{\pi-C}{4}$ etc for 2nd form (I)

$$\text{L.H.S.} = 2\cos\frac{A+B}{4}\cos\frac{A-B}{4} + \sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

$$= 2\cos\frac{\pi-C}{4}\cos\frac{A-B}{4} + 2\sin\frac{\pi-C}{4}\cos\frac{\pi-C}{4}$$

$\sin\theta = 2\sin\theta/2\cos\theta/2$

$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} + \sin\frac{A+B}{4}\right] \quad \text{(by 1)}$$

$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} + \cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

$$= 2\cos\frac{\pi-C}{4}\left[2\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\right]$$

$$= 4\cos\frac{\pi-A}{4}\cos\frac{\pi-B}{4}\cos\frac{\pi-C}{4}$$

(*ii*) $\text{L.H.S.} = 2\sin\frac{A+B}{4}\cos\frac{A-B}{4} + \cos\left(\frac{\pi}{2}-\frac{C}{2}\right)$ **Note**

$$= 2\sin\frac{\pi-C}{4}\cos\frac{A-B}{4} + 1 - 2\sin^2\frac{\pi-C}{4}$$

$$= 1 + 2\sin\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} - \sin\frac{A+B}{4}\right] \quad \text{by (1)}$$

$$= 1 + 2\sin\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} - \cos\left(\frac{\pi}{2}-\frac{A+B}{4}\right)\right]$$

$$= 1 + 2\sin\frac{\pi-C}{4}\left[2\sin\frac{\pi-B}{4}\sin\frac{\pi-A}{4}\right]$$

$$= 1 + 4\sin\frac{\pi-A}{4}\sin\frac{\pi-B}{4}\sin\frac{\pi-C}{4}$$

(*iii*) $\text{L.H.S.} = 2\cos\frac{A+B}{4}\cos\frac{A-B}{4} - \sin\left(\frac{\pi}{2}-\frac{C}{2}\right)$

$$= 2\cos\frac{\pi-C}{4}\cos\frac{A-B}{4} - 2\sin\frac{\pi-C}{4}\cos\frac{\pi-C}{4}$$

$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} - \sin\frac{\pi-C}{4}\right]$$

$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} - \sin\left(\frac{A+B}{4}\right)\right]$$

$$= 2\cos\frac{\pi-C}{4}\left[\cos\frac{A-B}{4} + \cos\left(\frac{\pi}{2}+\frac{A+B}{4}\right)\right]$$

$\cos(\pi/2+\theta) = -\sin\theta$

$$= 2\cos\frac{\pi-C}{4}\left[2\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\right]$$

$$= 2\cos\frac{\pi+A}{4}\cos\frac{\pi+B}{4}\cos\frac{\pi-C}{4}$$

9 (i) We want $+1$ in the R H S Alos $\alpha+\beta=\gamma$

L.H.S $=\cos^2\alpha+1-\sin^2\beta+\cos^2\gamma$

$=1+(\cos^2\alpha-\sin^2\beta)+\cos^2\gamma$

$=1+\cos(\alpha+\beta)\cos(\alpha-\beta)+\cos^2\gamma$

$=1+\cos\gamma\,[\cos(\alpha-\beta)+\cos^2\gamma$

$=1+\cos\gamma\,[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$ $\gamma=\alpha+$

$=1+\cos\gamma\,[2\cos\alpha\cos\beta]$

$=1+2\cos\alpha\cos\beta\cos\gamma$

(ii) Let $\alpha-\beta=x,\ \beta-\gamma=y,\ \gamma-\alpha=z$

$x+y+z=0$ or $x+y=-z$

$\sin(x+y)=-\sin z,\quad \cos(x+y)=\cos z$ (I)

Hence we have to prove that

$\sin 2x+\sin 2y+\sin 2z=-4\sin x\sin y\sin z$

L H S $=2\sin(x+y)\cos(x-y)+2\sin z\cos z$

$=-2\sin z\cos(x-y)+2\sin z\cos z$

$=-2\sin z\,[\cos(x-y)-\cos z]$ by (I)

$=-2\sin z\,[\cos(x-y)-\cos(x+y)]$

$=-2\sin z\,(2\sin x\sin y)$ by (I)

$=-4\sin x\sin y\sin z$

(iii) Combining the Ist two and last two terms, we get

L.H S $=\sin y\,[\sin x\sin(x-y)+\sin z\sin(y-z)]$

$+\sin(z-x)\,[\sin z\sin x+\sin(x-y)\sin(y-z)]$

$=\frac{1}{2}\sin y\,[\cos y-\cos(2x-y)+\cos(2z-y)-\cos y]$

$+\frac{1}{2}\sin(z-x)\,[\cos(z-x)-\cos(z+x)+\cos(x-2y+z)-\cos(x-z)]$

$=\frac{1}{2}\sin y\,[\cos(2z-y)-\cos(2x-y)]$

$+\frac{1}{2}\sin(z-x)\,[\cos(x-2y+z)-\cos(z+x)]$

$=\frac{1}{2}\sin y\,[2\sin(z+x-y)\sin(x-z)]$

$+\frac{1}{2}\sin(z-x)\,[2\sin(z+x-y)\sin y]$

$=0$ $[\because \sin(x-z)=-\sin(z-x)]$

(iv) Combine the Ist two and last two terms as in part (iii)

10 $\tan(A+B+C)=\tan\pi=0$

$\frac{S_1-S_3}{1-S_2}=0$ or $S_1=S_3$

(i) or $\tan A+\tan B+\tan C=\tan A\tan B\tan C$

(ii) Divide both sides by $\tan A\tan B\tan C$ we get

$\frac{1}{\tan B\tan C}+\frac{1}{\tan C\tan A}+\frac{1}{\tan A\tan B}=1$

$\therefore \cot B\cot C+\cot C\cot A+\cot A\cot B=1,$

(iii) On changing cot in terms of tan

$$\text{L H S} = \frac{\tan A+\tan B}{\tan A \tan B\,(\tan A+\tan B)} + \quad + \quad = 1$$

or $\dfrac{1}{\tan A \tan B}+\dfrac{1}{\tan B \tan C}+\dfrac{1}{\tan C \tan A}=1$

or $\tan C+\tan A+\tan B=\tan A \tan B \tan C$

which is true by (1)

11 (*i*) Put $x=\tan A$, $y=\tan B$ and $z=\tan C$

Since $x+y+z=xyz$,

$\tan A+\tan B+\tan C-\tan A \tan B \tan C=0$ or $S_1-S_3=0$

or $\tan(A+B+C)=\dfrac{S_1-S_3}{1-S_2}=\dfrac{0}{1-S_2}=0$

$A+B+C=0$ or $n\pi$

or $2A+2B+2C=0$ or $2n\pi$

$\tan(2A+2B+2C)=0$

$\dfrac{S_1-S_3}{1-S_2}=0$, $S_1=S_3$,

or $\tan 2A+\tan 2B+\tan 2C=\tan 2A \tan 2B \tan 2C$,

Now put $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}=\dfrac{2x}{1-x^2}$ etc,

(*ii*) Proceeding as above $\tan(3A+3B+3C)=0$

$\dfrac{S_1-S_3}{1-S_2}=0$ so that $S_1=S_3$

or $\tan 3A+\tan 3B+\tan 3C=\tan 3A \tan 3B \tan 3C$

Now put $\tan 3A=\dfrac{3\tan A-\tan^3 A}{1-3\tan^2 A}=\dfrac{3x-x^3}{1-3x^2}$ etc.

(*iii*) Put $x=\tan A$, $y=\tan B$, $z=\tan C$.

Then $xy+yz+zx=1$ gives

$\tan A \tan B+\tan B \tan C+\tan C \tan A-1=0$

or $S_2-1=0$ or $1-S_2=0$

Now $\tan(A+B+C)=\dfrac{S_1-S_3}{1-S_2}=\dfrac{S_1-S_3}{0}=\infty=\tan\dfrac{\pi}{2}$

$A+B+C=\pi/2$ or $2A+2B+2C=\pi$

$\tan(2A+2B+2C)=\tan\pi=0$

or $\dfrac{S_1-S_3}{1-S_2}=0$ $S_1=S_3$

or $\tan 2A+\tan 2B+\tan 2C=\tan 2A \tan 2B \tan 2C$

or $\dfrac{2x}{1-x^2}+\dfrac{2y}{1-y^2}+\dfrac{2z}{1-z^2}=\dfrac{8xyz}{(1-x^2)(1-y^2)(1-z^2)}$

or $\dfrac{x}{1-x^2}+\dfrac{y}{1-y^2}+\dfrac{z}{1-z^2}=\dfrac{4xyz}{(1-x^2)(1-y^2)(1-z^2)}$ Proved

(iv) Put $x=\tan\alpha$ etc so that $1-S_2=0$ given

$\tan(\alpha+\beta+\gamma)=\dfrac{S_1-S_3}{1-S_2}=\infty$ $\qquad \alpha+\beta+\gamma=\pi/2$

Also $\dfrac{x}{1+x^2}=\dfrac{\tan\alpha}{\sec^2\alpha}=\sin\alpha\cos\alpha=\frac{1}{2}\sin 2\alpha$

Hence we have to prove that

$\frac{1}{2}[\sin 2\alpha+\sin 2\beta+\sin 2\gamma]=\dfrac{2}{\sec\alpha\sec\beta\sec\gamma}$

or $\sin 2\alpha+\sin 2\beta+\sin 2\gamma=4\cos\alpha\cos\beta\cos\gamma$

when $\alpha+\beta+\gamma=\pi/2$ or $\sin(\beta+\gamma)=\sin(\pi/2-\alpha)=\cos\alpha$

Above relation can be easily proved

(b) Put $B-C=p$, $C-A=q$, $A-B=r$

$p+q+r=0$ or $\tan(p+q+r)=\tan 0°=0$

or $\dfrac{S_1-S_3}{1-S_2}=0$ $\qquad S_1=S_3$

or $\tan p+\tan q+\tan r=\tan p\tan q\tan r$

Now put the values of p, q and r

12 (i) $A+B+C=\pi$ $\qquad \dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}$

$\tan\left(\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}\right)=\tan\dfrac{\pi}{2}=\infty$

or $\dfrac{S_1-S_3}{1-S_2}=\infty$ $\qquad 1-S_2=0$ or $S_2=1$

or $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$

(ii) Dividing both sides by $\tan A/2\tan B/2\tan C/2$ we get

$\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}=\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

13 (i) Change in terms of double angles i e

$\cos^2 A=\frac{1}{2}(1+\cos 2A)$

Also $A+B+C=2S$

L H S $=\frac{1}{2}[1+\cos 2S+1+\cos(2S-2A)+1+\cos(2S-2B)+1+\cos(2S-2C)]$

$=2+\frac{1}{2}[2\cos(2S-A)\cos A+2\cos(2S-B-C)\cos(B-C)]$

$=2+\cos A(\cos(B+C)+\cos(B-C)]$, $\qquad 2S-A=B+C$

$=2+\cos A(2\cos B\cos C)$

$=2+2\cos A\cos B\cos C$

Alternate

$$\text{L H S} = 1 - \sin^2 S + 1 - \sin^2 (S-A) + \cos^2 (S-B) + \cos^2 (S-C)$$

$$= 2 + [\cos^2 (S-B) - \sin^2 S] + [\cos^2 (S-C) - \sin^2 (S-A)]$$

Now apply $\cos^2 A - \sin^2 B = \cos (A+B) \cos (A-B)$ etc

(ii) $\text{L H S} = 2 \sin \frac{2S-A-B}{2} \cos \frac{B-A}{2} + 2 \cos \frac{2S-C}{2} \sin\left(-\frac{C}{2}\right)$

$= 2 \sin \frac{C}{2}\left[\cos \frac{B-A}{2} - \cos \frac{A+B}{2}\right], \qquad 2S-C = A+B$

$= 2 \sin \frac{C}{2}\left[2 \sin \frac{A}{2} \sin \frac{B}{2}\right] = 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$

14 $A+B+C+D = 2\pi$

$\text{L H S} = (\cos A + \cos C) - (\cos B + \cos D)$

$= 2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} - 2 \cos \frac{B+D}{2} \cos \frac{B-D}{2}$

Now $\frac{B+D}{2} = \pi - \frac{A+C}{2}, \qquad \cos \frac{B+D}{2} = -\cos \frac{A+C}{2}$

$\text{L H S} = 2 \cos \frac{A+C}{2}\left[\cos \frac{A-C}{2} + \cos \frac{B-D}{2}\right]$

$= 2 \cos \frac{A+C}{2}\left[2 \cos \frac{A+B-C-D}{4} \cos \frac{A+D-C-B}{4}\right]$ (1)

Now $C+D = 2\pi - (A+B), \; C+B = 2\pi - (A+D)$

$\frac{(A+B)-(C+D)}{4} = \frac{(A+B)-2\pi+(A+B)}{4} = \frac{A+B}{2} - \frac{\pi}{2}$

$\cos \frac{(A+B)-(C+D)}{4} = \cos\left(\frac{A+B}{2} - \frac{\pi}{2}\right) = \sin \frac{A+B}{2}$ (2)

Again $\frac{(A+D)-(C+B)}{4} = \frac{(A+D)-2\pi+(A+D)}{4} = \left(\frac{A+D}{2} - \frac{\pi}{2}\right)$

$\cos \frac{(A+D-C-B)}{4} = \cos\left(\frac{A+D}{2} - \frac{\pi}{2}\right) = \sin \frac{A+D}{2}$ (3)

Hence from (1) by the help of (2) and (3), we get

$\text{L H S} = 4 \cos \frac{A+C}{2} \sin \frac{A+B}{2} \sin \frac{A+D}{2}$

15 We know from Q 12 (*i*), that when $A+B+C=\pi$, then $\tan A/2 \tan B/2 + \tan B/2 \tan C/2 + \tan C/2 \tan A/2 = 1$ (1) or $yz+zx+xy=1$, where $x = \tan A/2$ etc

Now we know that $(x-y)^2+(y-z)^2+(z-x)^2 \geqslant 0$

or $2\Sigma x^2-2\Sigma xy \geqslant 0$

or $\Sigma x^2 \geqslant \Sigma xy$ or $\Sigma x^2 \geqslant 1$, $\Sigma xy=1$

or $\tan^2 A/2+\tan^2 B/2+\tan^2 C/2 \geqslant 1$

16 $\sin^3 A \cos(B-C)=\sin^2 A \sin A \cos(B-C)$

$=\frac{1}{2}\sin^2 A\, 2\sin(B+C)\cos(B-C)$

$=\frac{1}{2}\sin^2 A(\sin 2B+\sin 2C)$

$=\sin^2 A(\sin B\cos B+\sin C\cos C)$

L H S $=\Sigma \sin^3 A\cos(B-C)$

$=\sin^2 A(\sin B\cos B+\sin C\cos C)$

$+\sin^2 B(\sin C\cos C+\sin A\cos A)$

$+\sin^2 C(\sin A\cos A+\sin B\cos B)$

$=\sin A\sin B(\sin A\cos B+\cos A\sin B)+ \quad +$

$=\sin A\sin B\sin(A+B)+ \quad +$

$=\sin A\sin B\sin C+ \quad +$

$=3\sin A\sin B\sin C$

17 $\cos 3\theta=4\cos^3\theta-3\cos\theta$, $\cos^3\theta=\frac{1}{4}(\cos 3\theta+3\cos\theta)$

$\sin 3A\cos^3(B-C)=\frac{1}{4}\sin 3A[\cos 3(B-C)$

$+3\cos(B-C)]$

$=\frac{1}{4}\sin 3(B+C)\cos 3(B-C)+\frac{3}{4}\sin(3B+3C)\cos(B-C)$

$=\frac{1}{8}(\sin 6B+\sin 6C)+\frac{3}{8}(\sin(4B+2C)+\sin(2B+4C))$

Now $4B+2C=2(B+C)+2B=2\pi-2A+2B$

$\sin(4B+2C)=\sin\{2\pi+2(B-A)\}=\sin 2(B-A)$

L H S $=\frac{1}{8}\Sigma(\sin 6B+\sin 6C)+\frac{3}{8}\Sigma[\sin 2(B-A)$

$+\sin 2(C-B)]$

$\Sigma \sin 3A\cos^3(B-C)=\frac{1}{8}\, 2\Sigma\sin 6A+\frac{3}{8}\,0$

$=\frac{1}{4}(\sin 6A+\sin 6B+\sin 6C)$

$=\frac{1}{4}\,4\sin 3A\sin 3B\sin 3C$ as in Q 1

$=\sin 3A\sin 3B\sin 3C$

18 $\tan A/2-\tan B/2=\tan B/2-\tan C/2$

or $\dfrac{\sin\frac{A-B}{2}}{\cos A/2\cos B/2}=\dfrac{\sin\frac{B-C}{2}}{\cos B/2\cos C/2}$

or $\sin\frac{A-B}{2}\cos\frac{C}{2}=\sin\frac{B-C}{2}\cos A/2$

or $2\sin\frac{A-B}{2}\sin\frac{A+B}{2}=2\sin\frac{B-C}{2}\sin\frac{B+C}{2}$

or $\cos B-\cos A=\cos C-\cos B$

$\cos A, \cos B, \cos C$ are in A.P.

19. Since $\tan A, \tan B, \tan C$ are in A.P., we have

$$2\tan B = \tan A + \tan C \qquad (1)$$

where $\tan A = x$ so that $2\tan B = x + \tan C \qquad (2)$

Now in a triangle, we always have

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$ [Q. 10 (i)]

By (1), $3\tan B = x \tan B \tan C$

or $3\tan B = x\tan B\,(2\tan B - x)$ by (2)

∴ This gives $\tan B = \dfrac{3+x^2}{2x}$, and $\tan C = 2\tan B - x = \dfrac{3}{x}$

∴ $a^2 : b^2 : c^2 = \sin^2 A : \sin^2 B : \sin^2 C$

$$= \frac{x^2}{1+x^2} : \frac{(3+x^2)^2}{4x^2+(3+x^2)^2} : \frac{9}{x^2+9}$$

$$= \frac{x^2}{1+x^2} : \frac{(3+x^2)^2}{(1+x^2)(9+x^2)} : \frac{9}{x^2+9}$$

$$= x^2(x^2+9) : (x^2+3)^2 : 9(1+x^2)$$

$\left[\because \sin^2\theta = \tan^2\theta\cos^2\theta = \dfrac{\tan^2\theta}{\sec^2\theta} = \dfrac{\tan^2\theta}{1+\tan^2\theta}\right]$

20. L.H.S. $= k^3[\sin^3 A\cos B\cos C + \sin^3 B\cos C\cos A + \sin^3 C\cos A\cos B]$

$= k^3[\sin A(1-\cos^2 A)\cos B\cos C + \sin B(1-\cos^2 B)\cos C\cos A + \sin C(1-\cos^2 C)\cos A\cos B]$

$= k^3[\sin A\cos B\cos C + \sin B\cos C\cos A + \sin C\cos A\cos B]$

$\quad - k^3\dfrac{\cos A\cos B\cos C}{2}(\sin 2A + \sin 2B + \sin 2C]$

$= k^3[\sin A\cos B\cos C + \cos A\sin(B+C)]$

$\quad - k^3\dfrac{\cos A\cos B\cos C}{2}\,4\sin A\sin B\sin C$ Q. 1 (i)

$= k^3\sin A[\cos B\cos C + \cos A]$

$\quad - k^3\cos A\cos B\cos C\sin A\sin B\sin C$ [$\because \sin(B+C) = \sin A$]

$= k^3\sin A\sin B\sin C[1 - 2\cos A\cos B\cos C]$ $\because \cos A = -\cos(B+C)$

$= abc[1 - 2\cos A\cos B\cos C]$

2 Trigonometrical Equations

§ 1 General Solutions of Trigonometrical Equations

(i) The equation $\sin\theta = k$ $(-1 \leqslant k \leqslant 1)$

(a) Let $-1 < k < 1$. In this case, there are two values of θ of the form α and $\pi-\alpha$ in the interval $0 \leqslant \theta < 2\pi$, which satisfy the equation. Adding $2r\pi$ $(r \in I)$, we get $2r\pi+\alpha$ and $(2r+1)\pi-\alpha$. Both these solutions are included in $n\pi+(-1)^n\alpha$, $n \in I$, where I is the set of integers. Hence the general solution in this case is

$$\theta = n\pi + (-1)^n \alpha,$$

where $n \in I$ that is, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$

In particular if $k=0$ then $\sin\theta = 0 = \sin 0°$ $\qquad \theta = n\pi$

(b) Let $k=1$. In this case, there is only one value $\frac{\pi}{2}$ in the interval $0 \leqslant \theta < 2\pi$ satisfying the given equation. Hence general solution in this case is given by

$$\theta = 2n\pi + \frac{\pi}{2}, \quad (n \in I)$$

(c) Let $k=-1$. In this case there is only one value $\frac{3\pi}{2}$ in the interval $0 \leqslant \theta < 2\pi$ which satisfies the given equation. Hence general solution in this case is

$$\theta = 2n\pi + \frac{3\pi}{2}, \quad n \in I$$

(ii) The equation $\cos\theta = k$ $(-1 \leqslant k \leqslant 1)$

(a) Let $-1 < k < 1$, $k \neq 0$. In this case, there are two values of θ of the form $\pm\alpha$, in the interval $-\pi < \theta \leqslant \pi$ which satisfy the given equation. Hence the general solution in this case is given by

$$\theta = 2n\pi \pm \alpha$$

(b) Let $k=0$. The solution in this case is

$$\theta = 2r\pi \pm \frac{\pi}{2} \quad \text{or} \quad \theta = (4r \pm 1)\frac{\pi}{2}, \quad (r \in I)$$

Thus in this case, the solution set consists of all odd multiples of $\frac{\pi}{2}$. Hence we may write the solution as

$$\theta=(2n+1)\frac{\pi}{2} \quad \text{or} \quad \theta=n\pi+\frac{\pi}{2}, \quad n \in I$$

(c) Let $k=1$. In this case there is only one value 0 of θ in the interval $-\pi < \theta \leqslant \pi$ which satisfies the equation. Hence the solution in this case is

$$\theta=2n\pi, \quad n \in I$$

(*d*) $k=-1$. Here $\theta=\pi$ is the only solution in the interval $-\pi < \theta \leqslant \pi$. Hence the general solution in this case is

$$\theta=(2n+1)\pi, \quad n \in I$$

(iii) The equation $\tan\theta=k$, $(-\infty < k < \infty)$

For any real value of k there are two values of the form α and $\pi+\alpha$, in the interval $0 \leqslant \theta < 2\pi$ which satisfy the given equation. Adding $2r\pi$, we get $2r\pi+\alpha$ and $(2r+1)\pi+\alpha$, $r \in I$. Both these solutions are included in the solution

$$\theta=n\pi+\alpha, \quad n \in I$$

In particular, if $k=0$, then the solution is

$$\theta=n\pi$$

(iv) The equation $\sin^2\theta=k=\sin^2\alpha$

Since values of $\sin\theta$ are $\pm\sin\alpha$, and $-\sin\alpha=\sin(-\alpha)$, therefore $\theta=\pm\alpha$, $\pi\mp\alpha$

Adding $2r\pi$, we get $2r\pi\pm\alpha$ and $(2r+1)\pi\mp\alpha$. Both these solutions are included in $n\pi\pm\alpha$ $(n \in I)$. Hence the general solution in this case is given by

$$\theta=n\pi\pm\alpha$$

(v) The equation $\cos^2\theta=k=\cos^2\alpha$

In this case also the solution is given by

$$\theta=n\pi\pm\alpha$$

(vi) The equation $\tan^2\theta=\tan^2\alpha$

The solution set is given by

$$\theta=n\pi\pm\alpha$$

Note Students are advised to derive the solution sets in (v) and (vi)

§ 2 Expression $a\cos\theta\pm b\sin\theta=c$

In such cases, we put

$$a=r\cos\alpha, \quad b=r\sin\alpha$$

so that $$r=\sqrt{(a^2+b^2)}, \quad \tan\alpha=b/a \quad \text{or} \quad \alpha=\tan^{-1} b/a$$

The given expression becomes

$$r[\cos\theta\cos\alpha \pm \sin\theta\sin\alpha] = c$$

or

$$\cos(\theta \pm \alpha) = c/r$$

Problem Set

1 Solve the following equations

(a) $2\sin^2 x + \sqrt{3}\cos x + 1 = 0$

(b) $\sin^2\theta - \cos\theta = \frac{1}{4}$ in the interval $0 \leq \theta \leq 2\pi$ (IIT 74)

(c) $2\sin^2\theta - 3\cos\theta$ in the interval $0 \leq \theta \leq 2\pi$ (IIT 63)

(d) $4\cos^2\theta + \sqrt{3} = 2[\sqrt{3}+1]\cos\theta$

(e) $\tan^2 x + [1-\sqrt{3}]\tan x - \sqrt{3} = 0$

(f) $4\cos\theta - 3\sec\theta = 2\tan\theta$

(g) $4\cos^2 x\sin x - 2\sin^2 x = 3\sin x$

(h) $3\cos^2\theta - 2\sqrt{3}\sin\theta\cos\theta - 3\sin^2\theta = 0$

(i) $(1-\tan\theta)(1+\sin 2\theta) = 1+\tan\theta$ (IIT 83)

(j) $5\cos 2\theta + 2\cos^2(\frac{1}{2}\theta) + 1 = 0, -\pi < \theta < \pi$ (Roorkee 84)

2 (a) What is the most general value of θ which satisfies both the equations

(a) $\sin\theta = -\frac{1}{2}$ and $\tan\theta = 1/\sqrt{3}$

(b) $\cos\theta = -1/\sqrt{2}$, and $\tan\theta = 1$

(c) $\cos\theta = 1/\sqrt{2}$ $\tan\theta = -1$ (MNR 80)

(d) $\tan\theta = \sqrt{3}$ and $\operatorname{cosec}\theta = -2/\sqrt{3}$ (MNR 82)

3 (a) $\tan^2\theta + \cot^2\theta = 2$

(b) $\cot\theta + \tan\theta = 2\operatorname{cosec}\theta$

(b_1) $\cot\theta - \tan\theta = \sec\theta$

(c) $2\sin 2x - \sin x = 0$

(d) $2\sin^2 x + \sin^2 2x = 2$

(e) $\cot\theta - \tan\theta = 2$ (Roorkee 79)

(f) $\tan\theta + \tan 2\theta + \sqrt{3}\tan 2\theta\tan\theta = \sqrt{3}$

(g) $\tan\theta + \tan 4\theta + \tan 7\theta = \tan\theta\tan 4\theta\tan 7\theta$

4 (a) $\sin 3x = 4\sin\alpha\sin(x+\alpha)\sin(x-\alpha)$

(b) $\cot x/2 - \operatorname{cosec} x/2 = \cot x$

(c) $\sec\theta - 1 = (\sqrt{2}-1)\tan\theta$

(d) $\sqrt{2}\sec\theta + \tan\theta = 1$

(d_1) $\operatorname{cosec} x = 1 + \cot x$

(e) $\cos 2\theta = (\sqrt{2}+1)(\cos\theta - 1/\sqrt{2})$

(f) $r\sin\theta = \sqrt{3}, r + 4\sin\theta = 2(\sqrt{3}+1), 0 \leq \theta \leq 2\pi$ (Roorkee 81)

(f_1) $r\sin\theta =$, $r = 4(1+\sin\theta)$ $0 \leq \theta \leq 2\pi$ (Roorkee 77)

(g) $3\tan^2\theta - 2\sin\theta = 0$ (IIT 64)

(Roorkee 74)

5 (a) $\sin 9\theta=\sin \theta$

(b) $\sin 5x=\cos 2x \quad 0 \leqslant x \leqslant 180°$ (I I T 62)

(b_1) $\sec x \cos 5x+1=0,\ 0 < x < 2\pi$ (Roorkee 78)

(c) $\sin 2\theta=\cos 3\theta \quad 0 \leqslant x \leqslant 360°$ (I I T 68)
Use this to find the value of $\sin 18°$

(d) $\tan 3\theta=\cot \theta$

(d_1) $\tan 2\theta \tan \theta=1$ (Roorkee 80)

(e) $\tan p\theta=\cot q\theta$

(f) $\tan mx+\cot nx=0$

(g) $\sin m\theta+\sin n\theta=0$

(h) Find the coordinates of the point of intersection of the curves
$y=\cos x,\ y=\sin 3x$ if $-\pi/2 \leqslant x \leqslant \pi/2$ (I I T 82)

6 (a) $\cos \theta+\cos 2\theta+\cos 3\theta=0$

(b) $\cos \theta \cos 2\theta \cos 3\theta=\frac{1}{4}, \quad 0 \leqslant \theta \leqslant \pi$ (I I T 61)

(c) $\cos 6\theta+\cos 4\theta+\cos 2\theta+1=0 \quad 0 \leqslant \theta \leqslant \pi$ (I I T 69)

(d) $\sin \theta+\sin 3\theta+\sin 5\theta=0 \quad 0 \leqslant \theta \leqslant \frac{1}{2}\pi$ (M N R 85)

(e) $\sin \theta+\sin 5\theta=\sin 3\theta \quad 0 \leqslant \theta \leqslant \pi$ (I I T 67)

(f) $\cos \theta+\cos 7\theta+\cos 3\theta+\cos 5\theta=0$

(g) $\cos \theta+\sin \theta=\cos 2\theta+\sin 2\theta$

(h) $\cos 3x+\cos 2x=\sin (3x/2)+\sin (x/2) \quad 0 \leqslant x < 2\pi$ (I I T 71, 73)

(i) $4 \sin x \sin 2x \sin 4x=\sin 3x$

7 (a) $\sin 7\theta+\sin 4\theta+\sin \theta=0 \quad 0 \leqslant \theta \leqslant \pi/2$ (I I T 72)

(b) $\sin 6\theta=\sin 4\theta-\sin 2\theta$ (I I T 77)

(b_1) $\cos \theta-\cos 2\theta=\sin 3\theta$ (Roorkee 79)

(c) $\sin \theta-\sin 2\theta+\sin 3\theta+\sin 4\theta=0$

(d) $\sin \frac{n+1}{2} \theta=\sin \frac{n-1}{2} \theta+\sin \theta$

(e) $\sin (3\theta+\alpha)+\sin (3\theta-\alpha)+\sin (\alpha-\theta)-\sin (\alpha+\theta)=\cos \alpha$

(f) $\sin 5\theta-\sin \theta=4 \cos^2 \theta-2$

(g) $\sec 4\theta-\sec 2\theta=2$

(h) $\sin^2 n\theta-\sin^2 (n-1) \theta=\sin^2 \theta$

8 (a) $\sin x+\sqrt{3} \cos x=\sqrt{2}$

(b) $\sqrt{3} \cos \theta+\sin \theta=1$ for $-2\pi < \theta < 2\pi$

(c) $\sqrt{3} \sin \theta-\cos \theta=\sqrt{2}$

(d) $3-2 \cos \theta-4 \sin \theta-\cos 2\theta+\sin 2\theta=0$ (I I T 59)

(d_1) $\sin x+\cos x=1$ (M N R 87)

(e) $\tan \theta+\sec \theta=\sqrt{3}$ (I I T 61)

(e_1) $\cot \theta+\operatorname{cosec} \theta=\sqrt{3}$ (Roorkee 73)

(f) $(2+\sqrt{3})\cos\theta=1-\sin\theta$

(g) $\tan\left(\frac{\pi}{2}\sin\theta\right)-\cot\left(\frac{\pi}{2}\cos\theta\right)$

(h) If $\tan(\pi\cos\theta)=\cot(\pi\sin\theta)$, prove that

$$\cos\left(\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$$

9 (a) $\tan\theta+\tan 2\theta+\tan 3\theta=0$

(b) $\tan\theta+\tan 2\theta=\tan 3\theta$

(c) $\tan 3\theta+\tan\theta=2\tan 2\theta$

(d) $4\cot 2\theta=\cot^2\theta-\tan^2\theta$

(e) $3\tan(\theta-15^\circ)=\tan(\theta+15^\circ)$

(f) $\tan\theta+\tan\left(\theta+\frac{\pi}{3}\right)+\tan\left(\theta+\frac{2\pi}{3}\right)=3$

10 (a) If $\sin A=\sin B$ and $\cos A=\cos B$, find the values of A in terms of B

(b) If $\tan(A-B)=1$, $\sec(A+B)=2/\sqrt{3}$, find the smallest +ive values of A and B and also their most general values (IIT 71)

(c) If $\cos(A-B)=\frac{1}{2}$ and $\sin(A+B)=\frac{1}{2}$ find the smallest +ive values of A and B and also their most general values (IIT 70)

(d) $\cos(2\theta+3\phi)=\frac{1}{2}$ and $\cos(3\theta-2\phi)=\sqrt{3}/2$

(e) $\sin(\theta+\phi)=1/\sqrt{2}$ and $\cos(\theta-\phi)=1/\sqrt{2}$

11 (a) $\sin^2 x+\sin x\cos x+\cos^2 x=1$

(b) $\sqrt{3}(\cos\theta-\sqrt{3}\sin\theta)=4\sin 2\theta\cos 3\theta$

(c) $\frac{\sqrt{3}}{2}\sin x-\cos x=\cos^2 x$

(d) $2(\cos x+\cos 2x)+\sin 2x(1+2\cos x)=2\sin x$

(d_1) $2(\sin x-\cos 2x)-\sin 2x(1+2\sin x)+2\cos x=0$ (IIT 78)

(e) Find the range of y such that the following equation in x $y+\cos x=\sin x$ has a real solution For $y=1$, find x such that $0<x<2\pi$ (Roorkee 87)

(f) Find real θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is

(a) real (b) purely imaginary ((IIT 75)

(g) Does the equation $2\cos^2 x/2\sin^2 x=x^2+x^{-2}$ $0\leqslant x\leqslant\pi/2$ has a real solution (IIT 76)

(IIT 80)

12 Solve $\tan 3x=\tan 5x$

13 Find all the solutions of the equation
$\sin x+(\sin \pi/8)\sqrt{\{(1-\cos x)^2+\sin^2 x\}}=0$

14 Solve $\cos x \cos 6x=-1$

15 Solve the system of equations
(i) $x+y=\pi/4$, $\tan x+\tan y=1$,
(ii) $2^{\sin x+\cos y}=1$, $16^{\sin^2 x+\cos^2 y}=4$

16 Consider the system of linear equations in x, y and z
$(\sin 3\theta)\, x-y+z=0$
$(\cos 2\theta)\, x+4y+3z=0$
$2x+7y+7z=0$
Find the values of θ for which the system has non-trivial solution **(I I T 86)**

17 (*a*) The solution set of the system of equations
$$x+y=\frac{2\pi}{3},\ \cos x+\cos y=3/2$$
where x and y are real is **(I I T 86)**
(*b*) The set of all x in the interval $[0,\pi]$ for which
$2\sin^2 x-3\sin x+1\geqslant 0$ **(I I T 87)**

18 If the expression $\dfrac{\left[\sin\frac{x}{2}+\cos\frac{x}{2}-i\tan x\right]}{\left(1+2i\sin\frac{x}{2}\right)}$ is real, then the set of all values of x is **(I I T 87)**

19 (*a*) The number of all possible triplets $(a_1\ a_2\ a_3)$ such that $a_1+a_2\cos 2x+a_3\sin^2 x=0$ for all x is
(*a*) Zero (*a*) One (*c*) Three (*d*) Infinite (*d*) None of these **(I I T 87)**
(*b*) Show that $x=0$ is the only solution satisfying the equation $1+\sin^2 ax=\cos x$, where a is irrational **(Roorkee 87)**

20 Solve for x and y
$x\cos^3 y+3x\cos y\sin^2 y=14$
$x\sin^3 y+3x\cos^2 y\sin y=13$ **(Roorkee 88)**

Solutions

1 (*a*) Changing $\sin^2 x$ into $1-\cos^2 x$, we get
$$2(1-\cos^2 x)+\sqrt{3}\cos x+1=0$$
or $$2\cos^2 x-\sqrt{3}\cos x-3=0$$
$$\cos x=\frac{\sqrt{3}\pm\sqrt{(3+24)}}{4}=\frac{\sqrt{3}\pm 3\sqrt{3}}{4}=\sqrt{3}\ \text{ or }\ \frac{-\sqrt{3}}{2}$$

Since $\sqrt{3}$ is greater than 1 it is not admissible as $\cos x$ can not be greater than 1

$$\cos x=\frac{-\sqrt{3}}{2}=-\cos\frac{\pi}{6}=\cos\left(\pi-\frac{\pi}{6}\right)=\cos\frac{5\pi}{6}$$

$$x=2n\pi\pm\frac{5\pi}{6}$$

(b) As above the equation reduces to

$4\cos^2\theta+4\cos\theta-3=0$ or $(2\cos\theta+3)(2\cos\theta-1)=0$

But $\cos\theta=-3/2$ (rejected) $\cos\theta=\frac{1}{2}=\cos\pi/3$

$\theta=2n\pi\pm\pi/3$

We have to choose values of θ s t $0\leqslant\theta\leqslant 2\pi$

$\theta=\pi/3,\ 2\pi-\pi/3=5\pi/3$

(c) As above $(\cos\theta+2)(2\cos\theta-1)=0$

$\cos\theta=-2$ (rejected), $\cos\theta=\frac{1}{2}=\cos\pi/3$

$\theta=2n\pi\pm\pi/3$

We have to choose values of θ s t $0\leqslant\theta\leqslant 2\pi$

$\theta=\pi/3\quad 2\pi-\pi/3=5\pi/3$

(d) $(2\cos\theta-1)(2\cos\theta-\sqrt{3})$

$\cos\theta=1/2=\cos\pi/3\quad \cos\theta=\sqrt{3}/2=\cos\pi/6$

$\theta=2n\pi\pm\pi/3\quad \theta=2n\pi\pm\pi/6$

(e) $(\tan x+1)(\tan x-\sqrt{3})=0$

$\tan x=-1=-\tan\pi/4=\tan(-\pi/4)$

$x=n\pi-\pi/4$

or $\tan x=\sqrt{3}=\tan\pi/3$ $x=n\pi+\pi/3$

(f) $4\cos\theta-\dfrac{3}{\cos\theta}=\dfrac{2\sin\theta}{\cos\theta}$

or $4(1-\sin^2\theta)=2\sin\theta+3$

or $4\sin^2\theta+2\sin\theta-1=0$

$$\sin\theta=\frac{-2\pm\sqrt{(4+16)}}{8}=\frac{-2\pm2\sqrt{5}}{8}=\frac{\sqrt{5}-1}{4},-\left(\frac{\sqrt{5}+1}{4}\right)$$

or $\sin\theta=\sin 18°$ i e $\sin\pi/10$ $\theta=n\pi+(-1)^n\pi/10$

or $\sin\theta=-\cos 36°=-\sin 54°=\sin\left(-\dfrac{3\pi}{10}\right)$

$\theta=n\pi-(-1)^n 3\pi/10$

(g) $\sin x(4\cos^2x-2\sin x-3)=0$

or $-\sin x(4\sin^2x+2\sin x-1)=0$

Now see part (f)

(h) Dividing by $\cos^2\theta$ we get

$3-2\sqrt{3}\tan\theta-3\tan^2\theta=0$

$\sqrt{3}\tan^2\theta+2\tan\theta-\sqrt{3}=0$

or $\sqrt{3}\tan^2\theta+3\tan\theta-\tan\theta-\sqrt{3}=0$

$$\sqrt{3}\tan\theta(\tan\theta+\sqrt{3})-1(\tan\theta+\sqrt{3})=0$$

or $(\tan\theta+\sqrt{3})(\sqrt{3}\tan\theta-1)=0$

$\tan\theta=-\sqrt{3}=-\tan\pi/3=\tan(-\pi/3)$

$\theta=n\pi-\pi/3$

$\tan\theta=1/\sqrt{3}=\tan\pi/6,$ $\qquad\theta=n\pi+\pi/6$

Alt The given equation is

$3(\cos^2\theta-\sin^2\theta)=\sqrt{3}\ (2\sin\theta\cos\theta)$

or $3\cos 2\theta=\sqrt{3}\sin 2\theta$

$$\tan 2\theta=\sqrt{3}=\tan\frac{\pi}{3}$$

$$2\theta=n\pi+\frac{\pi}{3}\quad\text{or}\quad\theta=\frac{n\pi}{2}+\frac{\pi}{6}$$

(*i*) Replace $\sin 2\theta$ by $\dfrac{2\tan\theta}{1+\tan^2\theta}$ and question reduces to

$$\frac{1+\tan\theta}{1-\tan\theta}=1+\frac{2\tan\theta}{1+\tan^2\theta}=\frac{(1+\tan\theta)^2}{1+\tan^2\theta}$$

or $(1+\tan\theta)(2\tan^2\theta)=0$

$\tan\theta=0\quad\tan\theta=-1=\tan(-\pi/4)$

$\theta=n\pi,\ n\pi-\pi/4$

(*j*) $5(2\cos^2\theta-1)+(1+\cos\theta)+1=0$

or $10\cos^2\theta+\cos\theta-3=0$

$(5\cos\theta+3)(2\cos\theta-1)=0$ $\qquad\theta=\pi/3,\ \pi-\cos^{-1}3/5$

2 (*a*) We shall first consider values of θ between 0 and 2π

$\sin\theta=-\frac{1}{2}=-\sin\pi/6=\sin(\pi+\pi/6)$ or $\sin(2\pi-\pi/6)$

$$\theta=\frac{7\pi}{6},\ \frac{11\pi}{6}$$

$$\tan\theta=\frac{1}{\sqrt{3}}=\tan\frac{\pi}{6}=\tan\left(\pi+\frac{\pi}{6}\right)$$

$\theta=\pi/6,\ 7\pi/6$

Hence the value of θ between 0 and 2π which satisfies both the equations is $7\pi/6$

Hence the general value of θ is $2n\pi+7\pi/6$, where $n\in I$

(*b*) $\cos\theta=-\dfrac{1}{\sqrt{2}}=-\cos\dfrac{\pi}{4}=\cos\left(\pi-\dfrac{\pi}{4}\right)$ or $\cos\left(\pi+\dfrac{\pi}{4}\right)$

$$\theta=\frac{3\pi}{4}\quad\frac{5\pi}{4}$$

$\tan\theta=1=\tan\pi/4,\ \tan(\pi+\pi/4)$

$\theta=\pi/4,\ 5\pi/4$

Hence the value of θ between 0 and 2π which satisfies both the equations is $5\pi/4$

General value is $2n\pi+5\pi/4=(2n+1)\pi+\pi/4$

(c) $2n\pi+\frac{7\pi}{4}$

(d) $\tan\theta=\sqrt{3}=\tan\pi/3=\tan(\pi+\pi/3)$

$\theta=\pi/3 \quad 4\pi/3$

$\operatorname{cosec}\theta=\frac{-2}{\sqrt{3}}$ or $\sin\theta=\frac{-\sqrt{3}}{2}=-\sin\frac{\pi}{3}$

or $\sin\theta=\sin(\pi+\pi/3)$ or $\sin(2\pi-\pi/3)$ as in part (a)

$\theta=4\pi/3,\ 5\pi/3$

Hence the value of θ between 0 to 2π which satisfies both the equations is $4\pi/3$

General value is $\theta=2n\pi+4\pi/3=(2n+1)\pi+\pi/3$

3 (a) $\tan^2\theta+\frac{1}{\tan^2\theta}-2=0$ or $\tan^4\theta-2\tan^2\theta+1=0$

or $(\tan^2\theta-1)^2=0$ $\tan^2\theta=1$

$\tan\theta=\pm1=\pm\tan\pi/4$

$\theta=n\pi\pm\pi/4$

(b) $\frac{\cos\theta}{\sin\theta}+\frac{\sin\theta}{\cos\theta}=\frac{2}{\sin\theta}$ or $\frac{\cos^2\theta+\sin^2\theta}{\sin\theta\cos\theta}=\frac{2}{\sin\theta}$

or $\frac{1}{\sin\theta\cos\theta}=\frac{2}{\sin\theta}$ or $\sin\theta(2\cos\theta-1)=0$

$\sin\theta=0$ $\theta=n\pi$ $\cos\theta=\frac{1}{2}=\cos\pi/3$ $\theta=2n\pi\pm\pi/3$

But $\theta=n\pi$ does not satisfy the given equation Hence the only solution is $\theta=2n\pi\pm\pi/3$

(b_1) $\cos\theta(\cos 2\theta-\sin\theta)=0$ or $\cos\theta(1-2\sin^2\theta-\sin\theta)=0$

$\cos\theta(2\sin\theta-1)(\sin\theta+1)=0$

Either $\cos\theta=0$ or $\sin\theta=\frac{1}{2}$ and $\sin\theta=-1$

$\theta=n\pi+\pi/2$ or $\theta=n\pi+(-1)^n\pi/6$ or $\theta=2n\pi+3\pi/2$

But $\theta=n\pi+\pi/2$ and $\theta=2n\pi+3\pi/2$ do not satisfy the given equation Hence the only solution is $\theta=n\pi+(-1)^n\frac{\pi}{6}$

(c) $2\sin 2x-\sin x=0$

$4\sin x\cos x-\sin x=0$ $\sin x(4\cos x-1)=0$

$\sin x=0$ $x=n\pi$ $\cos x=\frac{1}{4}=\cos\alpha$ say

$x=2n\pi\pm\alpha$ where $\alpha=\cos^{-1}\frac{1}{4}$

(d) $\sin^2 2x=2(1-\sin^2 x)$ or $(2\sin x\cos x)^2=2\cos^2 x$

or $2\sin^2 x\cos^2 x-\cos^2 x=0$

$\cos^2 x(2\sin^2 x-1)=0$

$\cos^2 x = 0 = \cos^2 \pi/2 \qquad x = n\pi \pm \pi/2$

$\sin^2 x = \frac{1}{2} \qquad \sin x = \pm \frac{1}{\sqrt{2}} = \pm \sin \pi/4 \qquad x = n\pi \pm \pi/4$

(e) $\frac{\cos \theta}{\sin \theta} - \frac{\sin \theta}{\cos \theta} = 2$

or $\cos^2 \theta - \sin^2 \theta = 2 \sin \theta \cos \theta$

or $\cos 2\theta = \sin 2\theta \qquad \tan 2\theta = 1 = \tan \pi/4$

$2\theta = n\pi + \pi/4$

or $\theta = \frac{n\pi}{2} + \frac{\pi}{8}$

(f) $\tan \theta + \tan 2\theta = \sqrt{3}\,(1 - \tan \theta \tan 2\theta)$

or $\frac{\tan \theta + \tan 2\theta}{1 - \tan \theta \tan 2\theta} = \sqrt{3}$

or $\tan(\theta + 2\theta) = \tan \pi/3$

or $3\theta = n\pi + \pi/3$ or $\theta = n\pi/3 + \pi/9$

(g) $\tan(A+B+C) = \frac{S_1 - S_3}{1 - S_2} = 0 \qquad S_1 = S_3$ given

$\tan(\theta + 4\theta + 7\theta) = 0$ or $\tan 12\theta = 0$

$12\theta = n\pi$ or $\theta = \frac{n\pi}{12}$

4 (a) $3 \sin \alpha - 4 \sin^3 \alpha = 4 \sin \alpha (\sin^2 x - \sin^2 \alpha)$ **L S R 77**

$\sin^2 x = 3/4$

or $\sin x = \pm \frac{\sqrt{3}}{2} = \pm \sin \frac{\pi}{3}$

$x = n\pi \pm \pi/3$

(b) $\frac{\cos(x/2)}{\sin(x/2)} - \frac{\cos x}{2 \sin(x/2) \cos(x/2)} = \frac{1}{\sin(x/2)}$

or $2 \cos^2 x/2 - \cos x = 2 \cos x/2$

or $1 + \cos x - \cos x = 2 \cos x/2$

$\cos x/2 = 1/2 = \cos \pi/3 \qquad x/2 = 2n\pi \pm \pi/3$

or $x = 4n\pi \pm 2\pi/3$

(c) $\sec \theta = (\sqrt{2} - 1) \tan \theta + 1$ square

$1 + \tan^2 \theta = (3 - 2\sqrt{2}) \tan^2 \theta + 2(\sqrt{2} - 1) \tan \theta + 1$

$(2 - 2\sqrt{2}) \tan^2 \theta - (2 - 2\sqrt{2}) \tan \theta = 0$

or $\tan^2 \theta - \tan \theta = 0$ or $\tan \theta (\tan \theta - 1) = 0$

$\therefore \tan \theta = 0 = \tan 0 \qquad \theta = n\pi$

$\tan \theta = 1 = \tan \pi/4 \qquad \theta = n\pi + \pi/4$

But $\theta = n\pi$ does not satisfy the equation when n is odd. Hence the solution is given by $\theta = 2m\pi$ and $\theta = n\pi + \pi/4$, $m, n \in I$

(d) $\sqrt{2}\sec\theta = 1-\tan\theta$ Square

$2(1+\tan^2\theta) = 1+\tan^2\theta - 2\tan\theta$

$\tan^2\theta + 2\tan\theta + 1 = 0$ or $(\tan\theta+1)^2 = 0$

$\tan\theta = -1 = -\tan\pi/4 = \tan(-\pi/4)$

$\theta = n\pi - \pi/4$

But for odd values of n the given equation is not ...

Hence the required solution is $\theta = 2m\pi - \pi/4,\ m \in I$

(d_1) Do yourself

Ans $\theta = 2n\pi + \pi/2$

[Note that $\theta = 2n\pi$ does not satisfy the equation]

(e) $2\cos^2\theta - 1 = (\sqrt{2}+1)\cos\theta - 1 - 1/\sqrt{2}$

or $2\cos^2\theta - \sqrt{2}\cos\theta - \cos\theta + 1/\sqrt{2} = 0$

$\sqrt{2}\cos\theta(\sqrt{2}\cos\theta - 1) - 1/\sqrt{2}(\sqrt{2}\cos\theta - 1) = 0$

$(\sqrt{2}\cos\theta - 1/\sqrt{2})(\sqrt{2}\cos\theta - 1) = 0$

$\cos\theta = 1/2 = \cos\pi/3$ $\theta = 2n\pi \pm \pi/3$

$\cos\theta = 1/\sqrt{2} = \cos\pi/4$ $\theta = 2n\pi \pm \pi/4$

(f) Eliminating r between the given equations, we get

$\frac{\sqrt{3}}{\sin\theta} + 4\sin\theta = 2(\sqrt{3}+1)$

or $4\sin^2\theta - 2\sqrt{3}\sin\theta - 2\sin\theta + \sqrt{3} = 0$

$2\sin\theta(2\sin\theta - \sqrt{3}) - 1(2\sin\theta - \sqrt{3}) = 0$

$(2\sin\theta - 1)(2\sin\theta - \sqrt{3}) = 0$

$\sin\theta = 1/2 = \sin\pi/6$ $\theta = n\pi + (-1)^n\pi/6$

$\sin\theta = \sqrt{3}/2 = \sin\pi/3$ $\theta = n\pi + (-1)^n\pi/3$

We have to find out values of θ lying between 0 and 2π. The values corresponding to $n=0$, $n=1$ are

$\pi/6$ $\pi/3$ $\pi-\pi/6$ $\pi-\pi/3$ or $\pi/6$ $\pi/3$ $5\pi/6$ $2\pi/3$

The values corresponding to $n=2$ and greater than 2 wil be between 0 and 2π

(f_1) Eliminating r we get $4\sin^2\theta + 4\sin\theta - 3 = 0$

or $(2\sin\theta+3)(2\sin\theta-1) = 0$ $\sin\theta = \frac{1}{2} = \sin\pi/6$

$\theta = \pi/6,\ \pi - \pi/6$ for $0 \leqslant \theta \leqslant 2\pi$

The other value is rejected as $\sin\theta$ cannot be greater than 1

(g) $3\frac{\sin^2\theta}{\cos^2\theta} - 2\sin\theta = 0$ $\cos\theta \neq 0$

Multiplying throughout by $\cos^2\theta$ and then changing it into $\sin^2\theta$ we get

$3\sin^2\theta - 2\sin\theta(1-\sin^2\theta) = 0$

$\sin\theta(3\sin\theta - 2 + 2\sin^2\theta) = 0$

$\sin\theta\,(2\sin^2\theta+4\sin\theta-\sin\theta-2)=0$

$\sin\theta\,(\sin\theta+2)\,(2\sin\theta-1)=0$

$\sin\theta=0, \qquad \theta=n\pi$

$\sin\theta=1/2=\sin\pi/6, \qquad \theta=n\pi+(-1)^n\,\pi/6$

$\sin\theta=-2$ (rejected as $\sin\theta$ cannot be greater than one numerically)

5 (*a*) $\sin 9\theta=\sin\theta$

$$9\theta=n\pi+(-1)^n\,\theta \tag{1}$$

If n be even say $2r$ then $9\theta=2r\pi+\theta \qquad \theta=r\pi/4$

If n be odd say $2r+1$ then $9\theta=(2r+1)\,\pi-\theta$

or $\theta=(2r+1)\,\pi/10$

(*b*) $\cos 2x=\sin 5x$ or $\cos 2x=\cos(\pi/2-5x)$

$2x=2n\pi\pm(\pi/2-5x)$

$7x=(2n\pi+\pi/2)$ or $-3x=n\pi-\pi/2$

or $x=(4n+1)\,\pi/14$ or $x=-(4n-1)\,\pi/6 \qquad n\in I$

We have to find values of x lying between 0 and 180°

Putting $n=0, 1, 2, 3,$ in the first, we get

$x=\frac{\pi}{14}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{13\pi}{14}$ and other values will be greater than 180°

Putting $n=0, -1$ in the second, we get

$x=\frac{\pi}{6}, \frac{5\pi}{6}$ and values corresponding to other values of n will not lie between 0 and π

$$x=\frac{\pi}{14}, \frac{\pi}{6}, \frac{5\pi}{14}, \frac{9\pi}{14}, \frac{5\pi}{6}, \frac{13\pi}{14}$$

(b_1) $\cos 5x=\cos x=\cos(\pi-x)$ etc

(*c*) $\sin 2\theta=\cos 3\theta$ or $\cos 3\theta=\cos(\pi/2-2\theta)$

$3\theta=2n\pi\pm(\pi/2-2\theta)$

or $3\theta=(4n-1)\,\pi/2-2\theta$, and $3\theta=(4n-1)\,\pi/2+2\theta$

$\theta=(4n+1)\,\pi/10$ or $\theta=(4n-1)\,\pi/2$

From 1st putting $n=0, 1, 2, 3, 4,$ we get

$\theta=\frac{\pi}{10}, \frac{5\pi}{10}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

or $\theta=18°, 90°, 162°, 234°, 306°$ The values of θ corresponding to other values of n will not lie between 0° and 360°

From the second putting $n=1$ we get $\theta=3\pi/2$ i e 270°

2nd Part Let $\theta=18°$, then $2\theta=90°-3\theta=36°$

$\sin 2\theta=\sin(90°-3\theta)=\cos 3\theta$

or $2\sin\theta\cos\theta=4\cos^3\theta-3\cos\theta$

$\cos\theta\,[2\sin\theta-4(1-\sin^2\theta)-3]=0$

$\therefore$ $\cos\theta=0$ i.e., $\theta=90°$ But $\theta=18°$ and hence rejected

or $4\sin^2\theta+2\sin\theta-1=0$

$$\sin\theta=\frac{-2\pm\sqrt{(4+16)}}{8}=\frac{-1\pm\sqrt{5}}{4}=\frac{\sqrt{5}-1}{4}$$

The other value of $\sin\theta$ being $-$ive is rejected as sin 1[illegible] +ive

$$\sin 18°=\frac{\sqrt{5}-1}{4}$$

(*d*) $\tan 3\theta=\cot\theta=\tan(\pi/2-\theta)$

$3\theta=n\pi+(\pi/2-\theta)$ or $4\theta=(2n+1)\pi/2$ $\qquad\theta=(2n$

(d_1) $\dfrac{2\tan\theta}{1-\tan^2\theta}\tan\theta=1$ $\qquad 3\tan^2\theta=1$

or $\tan\theta=\pm 1/\sqrt{3}=\pm\tan\pi/6$ $\qquad\theta=n\pi\pm\pi/6$

(*e*) $\dfrac{\sin p\theta}{\cos p\theta}=\dfrac{\cos q\theta}{\sin q\theta}$

or $\cos p\theta\cos q\theta-\sin p\theta\sin q\theta=0$

or $\cos(p+q)\theta=0=\cos\pi/2$

$(p+q)\theta=2n\pi\pm\pi/2=(4n\pm1)\pi/2=(\text{odd integer})\,\pi/2=(2n+1)\pi/2$

$$\theta=\frac{(2n+1)}{(p+q)}\pi/2,\qquad n\in I$$

(*f*) Just as in part (*e*) $\qquad\cos(m-n)x=0=\cos\pi/2$

$$x=\frac{(2r+1)}{m-n}\,\frac{\pi}{2}\qquad r\in I$$

(*g*) $\sin m\theta=-\sin n\theta=\sin(-n\theta)$

$m\theta=r\pi+(-1)^r(-n\theta)=r\pi-(-1)^r n\theta$

If r=even=$2k$ say then $\qquad m\theta=2k\pi-n\theta$

or $(m+n)\theta=2k\pi$ or $\theta=\dfrac{2k\pi}{m+n}$

r=odd=$(2k+1)$, say, then $m\theta=(2k+1)\pi+n\theta$

$(m-n)\theta=(2k+1)\pi$ or $\theta=\dfrac{(2k+1)}{m-n}\pi$

(*h*) $\sin 3x=\cos x=\sin(\pi/2-x)$

$3x=n\pi+(-1)^n(\pi/2-x)$

n even$=2r$ $\qquad 3x=2r\pi+\pi/2-x$ $\qquad x=(4r+1)\pi/8$

n odd$=2r+1$ $\qquad 3x=(2r+1)\pi-\pi/2+x$

or $x=(4r+1)\pi/4$

Also $-\pi/2\leqslant x\leqslant\pi/2$

The points of intersection are $\qquad x=\pi/8,\ \pi/4,\ -3\pi/8$

$$\left(\frac{\pi}{8}, \cos\frac{\pi}{8}\right), \left(\frac{\pi}{4}, \cos\frac{\pi}{4}\right) \text{ and } \left(-\frac{3\pi}{8}, \cos\left(-\frac{3\pi}{8}\right)\right)$$

i e $\left(\frac{\pi}{8}, \frac{\sqrt{(2+\sqrt{2})}}{2}\right), \left(\frac{\pi}{4}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{3\pi}{8}, \frac{\sqrt{(2-\sqrt{2})}}{2}\right)$

since $\cos\frac{\pi}{8}=\sqrt{(2+\sqrt{2})}/2$ and

$$\cos\left(-\frac{3\pi}{8}\right)=\cos\frac{3\pi}{8}=\cos\left(\frac{\pi}{2}-\frac{\pi}{8}\right)=\sin\frac{\pi}{8}=\sqrt{(2-\sqrt{2})}/2$$

[Note that $2\cos^2\frac{\pi}{8}=1+\cos\frac{\pi}{4}=1+\frac{1}{\sqrt{2}}=\frac{\sqrt{2}+1}{\sqrt{2}}=\frac{2+\sqrt{2}}{2}$

$$\cos\frac{\pi}{8}\ \frac{\sqrt{(2+\sqrt{2})}}{2}$$

$$\sin^2\frac{\pi}{8}=1-\cos^2\frac{\pi}{8}=1-\frac{2+\sqrt{2}}{4}=\frac{(2-\sqrt{2})}{4}$$

$$\sin\frac{\pi}{8}=\frac{\sqrt{(2-\sqrt{2})}}{2}\text{]}$$

6 (*a*) Combining $\cos\theta$ and $\cos 3\theta$,

$(\cos\theta+\cos 3\theta)+\cos 2\theta=0$

$2\cos 2\theta\cos\theta+\cos 2\theta=0$

or $\cos 2\theta\,(2\cos\theta+1)=0$

$\cos 2\theta=0=\cos\pi/2$ $\quad 2\theta=2n\pi+\pi/2$ or $\theta=n\pi+\pi/4$

$2\cos\theta+1=0$ $\quad \cos\theta=-\frac{1}{2}=-\cos\pi/3=\cos(\pi-\pi/3)$

$=\cos 2\pi/3$,

$\theta=2n\pi\pm 2\pi/3$

(*b*) Combining $\cos\theta$ and $\cos 3\theta$, we get

$\frac{1}{2}(2\cos\theta\cos 3\theta)\cos 2\theta=\frac{1}{4}$

or $(\cos 2\theta+\cos 4\theta)\cos 2\theta=\frac{1}{2}$

$\frac{1}{2}[2\cos^2 2\theta+2\cos 4\theta\cos 2\theta]=\frac{1}{2}$

or $1+\cos 4\theta+2\cos 4\theta\cos 2\theta=1$

$\cos 4\theta\,(1+2\cos 2\theta)=0$

$\cos 4\theta=0=\cos\pi/2$ $\quad 4\theta=2n\pi\pm\pi/2$ $\quad \theta=n\pi/2\pm\pi/8$

$n=0,\ \theta=\frac{\pi}{8},\ n=1,\ \theta=\frac{\pi}{2}\pm\frac{\pi}{8}=\frac{3\pi}{8}, \frac{5\pi}{8},\ n=2,\ \theta=\pi-\frac{\pi}{8}=\frac{7\pi}{8}$

$0\leqslant\theta\leqslant\pi$

From the second equation

$\cos 2\theta=-\frac{1}{2}=-\cos\pi/3=\cos(\pi-\pi/3)=\cos 2\pi/3$

$2\theta=2n\pi\pm 2\pi/3$ $\quad \theta=n\pi\pm\pi/3$

$n=0,\ \theta=\pi/3,\ n=1,\ \theta=\pi-\pi/3=2\pi/3,$

$\theta=\frac{\pi}{8}, \frac{\pi}{3}, \frac{3\pi}{8}, \frac{5\pi}{8}, \frac{2\pi}{3}, \frac{7\pi}{8}$ $\quad 0\leqslant\theta\leqslant\pi$

(c) $(\cos 6\theta+\cos 4\theta)+(1+\cos 2\theta)=0$

$2\cos 5\theta\cos\theta+2\cos^2\theta=0$

or $2\cos\theta(\cos 5\theta+\cos\theta)=0$ or $4\cos\theta\cos 2\theta\cos 3\theta=0$ $\quad 0\leqslant\theta\leqslant\pi$

$\cos 2\theta=0$ $\quad \theta=n\pi+\pi/2$

$n=0$ $\quad \theta=\pi/2$

$\cos 2\theta=0$ $\quad 2\theta=n\pi+\pi/2$

$n=0$ $\theta=\pi/4$ $n=1$ $\theta=\pi/2+\pi/4=3\pi/4$ $\quad \theta=(n/2)\pi+\pi/4$

$\cos 3\theta=0$ $\quad 3\theta=n\pi+\dfrac{\pi}{2}$

$n=0$, $\theta=\pi/6$, $n=1$, $\theta=\pi/3+\pi/6$ or $\theta=\pi/2$, $\quad \theta=\dfrac{n\pi}{3}+\dfrac{\pi}{6}$

$n=2$, $\theta=2\pi/3+\pi/6=5\pi/6$

$\theta=30°$, $45°$, $90°$, $135°$, $150°$

(d) On simplification $\sin 3\theta(2\cos 2\theta+1)=0$

$\theta=n\pi/3$ $\quad \theta=n\pi\pm\pi/3$ for given range $\theta=0$ $\quad 2\theta=120°$

$\theta=\pi/3$

(e) On simplifications we have

$\sin 3\theta(2\cos 2\theta-1)=0$

$\theta=0$, $\pi/6$, $\pi/3$ $\quad 2\pi/3$, $5\pi/6$ and π

(f) Combining θ and 7θ 3θ and 5θ, we get

$2\cos 4\theta(\cos 3\theta+\cos\theta)=0$

$4\cos 4\theta\cos 2\theta\cos\theta=0$

$\theta=n\pi/4+\pi/8$ $\quad (n/2)\pi+\pi/4$, $n\pi+\pi/2$

(g) $(\cos\theta-\cos 2\theta)$ $\quad (\sin 2\theta-\sin\theta)=0$

$2\sin 3\theta/2\sin\theta/2-2\cos 3\theta/2\sin\theta/2=0$

$\sin\theta/2(\sin 3\theta/2-\cos 3\theta/2)=0$

$\sin\theta/2=0$ $\quad \theta/2=n\pi$ or $\theta=2n\pi$

$\sin(3\theta/2)-\cos(3\theta/2)=0$ $\quad \tan(3\theta/2)=1=\tan(\pi/4)$

$\dfrac{3\theta}{2}=n\pi+\dfrac{\pi}{4}$ or $\theta=\dfrac{2n\pi}{3}+\dfrac{\pi}{6}$

(h) $\cos 3x+\cos 2x=\sin(3x/2)+\sin(x/2)$

$2\cos\dfrac{5x}{2}\cos\dfrac{x}{2}-2\sin x\cos\dfrac{x}{2}=0$

$2\cos x/2(\cos 5x/2-\sin x)=0$

$\cos x/2=0$ $\quad x/2=n\pi+\pi/2$ $\quad x=2n\pi+\pi$

$n=0$, $x=\pi$ as $0\leqslant x\leqslant 2\pi$

$\cos\dfrac{5x}{2}=\sin x=\cos\left(\dfrac{\pi}{2}-x\right)$

$\dfrac{5x}{2}=2n\pi\pm\left(\dfrac{\pi}{2}-x\right)$

Taking + sign

$$\frac{5x}{2}+x=2n\pi+\frac{\pi}{2}, \quad \text{or} \quad \frac{7x}{2}=(4n+1)\frac{\pi}{2}$$

$x=(4n+1)\pi/7$

$n=0,\ x=\frac{\pi}{7},\ n=1,\ x=\frac{5\pi}{7},\ n=2,\ x=\frac{9\pi}{7},$

$n=3,\ x=13\pi/7$

Now taking − ve sign,

$$\frac{5x}{2}-x=2n\pi-\pi/2 \quad \text{or} \quad \frac{3x}{2}=(4n-1)\frac{\pi}{2}$$

$x=(4n-1)\frac{\pi}{3} \qquad n=1,\ x=\pi$

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(i) The given equation can be written as

$4\sin x\sin(3x-x)\sin(3x+x)=3\sin x-4\sin^3 x$

$4\sin x\,[\sin^2 3x-\sin^2 x]=3\sin x-4\sin^3 x$

or $4\sin x\sin^2 3x-4\sin^3 x=3\sin x-4\sin^3 x$

or $\sin x\,[4\sin^2 3x-3]=0$

$\sin x=0 \qquad x=n\pi$

$4\sin^2 3x=3 \qquad \sin 3x=\pm\sqrt{3}/2=\pm\sin\pi/3$

$3x=n\pi\pm\pi/3$ or $x=n\pi/3\pm\pi/9$ See rule IV Page 67

7 (a) Combining 7θ and θ, we get $\quad 0\leqslant\theta\leqslant\pi/2$

$2\sin 4\theta\cos 3\theta+\sin 4\theta=0$

$\sin 4\theta\,(2\cos 3\theta+1)=0$

$\sin 4\theta=0=\sin 0 \qquad 4\theta=n\pi$ or $\theta=n\pi/4$

$\theta=0,\ \frac{\pi}{4},\ \frac{\pi}{2}$, corresponding to $n=0, 1, 2$

$$\cos 3\theta=-\tfrac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$$

$3\theta=2n\pi\pm\frac{2\pi}{3} \qquad \theta=(6n\pm2)\frac{\pi}{9}$

$n=0 \quad \theta=\frac{2\pi}{9}$ and all other values will be greater than $\frac{\pi}{2}$

$\theta=0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{2\pi}{9}$

(b) $\sin 6x-(\sin 4x-\sin 2x)=0$

$2\sin 3x\cos 3x-2\cos 3x\sin x=0$

$2\cos 3x\,(\sin 3x-\sin x)=0$

(c) $(\cos 6\theta+\cos 4\theta)+(1+\cos 2\theta)=0$ $\qquad 0 \leqslant \theta \leqslant \pi$

$2 \cos 5\theta \cos \theta+2 \cos^2 \theta=0$

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$\cos 2\theta=0 \qquad \theta=n\pi+\pi/2$

$n=0 \qquad \theta=\pi/2$

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$n=0, \theta=\pi/4, n=1 \quad \theta=\pi/2+\pi/4=3\pi/4 \qquad \theta=(n/2)\pi+\pi/4$

$\cos 3\theta=0 \qquad 3\theta=n\pi+\dfrac{\pi}{2}$

$n=0, \theta=\pi/6, n=1, \theta=\pi/3+\pi/6 \qquad \theta=\dfrac{n\pi}{3}+\dfrac{\pi}{6}$

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$x=(4n-1)\frac{\pi}{3} \qquad n=1,\ x=\pi,$

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(*i*) The given equation can be written as

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$4\sin x[\sin^2 3x-\sin^2 x]=3\sin x-4\sin^3 x$

or $4\sin x\sin^2 3x-4\sin^3 x=3\sin x-4\sin^3 x$

or $\sin x[4\sin^2 3x-3]=0$

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$\theta=0,\ \frac{\pi}{4},\ \frac{\pi}{2}$, *corresponding to* $n=0,\ 1,\ 2$

$$\cos 3\theta=-\tfrac{1}{2}=-\cos\frac{\pi}{3}=\cos\left(\pi-\frac{\pi}{3}\right)=\cos\frac{2\pi}{3}$$

$3\theta=2n\pi\pm\frac{2\pi}{3} \qquad \theta=(6n\pm 2)\frac{\pi}{9}$

$n=0,\ \theta=\frac{2\pi}{9}$ and all other values will be greater than $\frac{\pi}{2}$

$\theta=0,\ \frac{\pi}{4},\ \frac{\pi}{2},\ \frac{2\pi}{9}$

(*b*) $\sin 6x-(\sin 4x-\sin 2x)=0$

$2\sin 3x\cos 3x-2\cos 3x\sin x=0$

$2\cos 3x(\sin 3x-\sin x)=0$

$\cos 3x = 0 = \cos\frac{\pi}{2}$ $\quad 3x = n\pi + \frac{\pi}{2}$ $\quad x = \frac{n\pi}{3} + \frac{\pi}{6}$

$\sin 3x = \sin x$ $\quad 3x = n\pi + (-1)^n x$

n even $= 2r$, say, $3x = 2r\pi + x$ $\quad x = r\pi,$

n odd $= 2r+1$ $\quad 3x = (2r+1)\pi - x,$ $\quad x = (2r+1)\pi/4$

(b_1) Do yourself

(c) $(\sin\theta + \sin 3\theta) + (\sin 2\theta + \sin 4\theta) = 0$

$2\sin 2\theta\cos\theta + 2\sin 3\theta\cos\theta = 0$

$2\cos\theta(\sin 2\theta + \sin 3\theta) = 0$

$4\cos\theta\sin\frac{5\theta}{2}\cos\frac{\theta}{2} = 0$

$\cos\theta = 0 = \cos\frac{\pi}{2}$ $\quad \theta = n\pi + \frac{\pi}{2}$

$\cos\frac{\theta}{2} = 0 = \cos\frac{\pi}{2}$ $\quad \frac{\theta}{2} = n\pi + \frac{\pi}{2}$ $\quad \theta = (2n+1)\pi$

$\sin\frac{5\theta}{2} = 0 = \sin 0$ $\quad \frac{5\theta}{2} = n\pi$ or $\theta = \frac{2n\pi}{5}$

(d) $\left[\sin\left(\frac{n+1}{2}\right)\theta - \sin\left(\frac{n-1}{2}\right)\theta\right] - \sin\theta = 0$

$2\cos\frac{n\theta}{2}\sin\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} = 0$

$2\sin\frac{\theta}{2}\left[\cos\frac{n\theta}{2} - \cos\frac{\theta}{2}\right] = 0$

$\sin\frac{\theta}{2} = 0$ $\quad \frac{\theta}{2} = r\pi$ or $\theta = 2r\pi$

$\cos\frac{n\theta}{2} = \cos\frac{\theta}{2}$ $\quad \frac{n\theta}{2} = 2r\pi \pm \frac{\theta}{2}$

or $(n\pm1)\frac{\theta}{2} = 2r\pi$ $\quad \theta = \frac{4r\pi}{n\pm1}$

(e) Combining the first two and last two terms we get

$2\sin 3\theta\cos\alpha + 2\cos\alpha\sin(-\theta) = \cos\alpha$

$2\sin 3\theta - 2\sin\theta = 1$

or $2(3\sin\theta - 4\sin^3\theta) - 2\sin\theta = 1$

or $8\sin^3\theta - 4\sin\theta + 1 = 0$ $\quad$ Clearly $\sin\theta = \frac{1}{2}$ satisfies it

$(2\sin\theta - 1)(4\sin^2\theta + 2\sin\theta - 1) = 0$

$\sin\theta = \frac{1}{2} = \sin\pi/6$ $\quad \theta = n\pi + (-1)^n\pi/6$

Also $\sin\theta = \frac{-2\pm\sqrt{(4+16)}}{2} = \frac{\sqrt{5}-1}{4}$ $\quad -\frac{(\sqrt{5}+1)}{4}$

44 (i) If $a_1, a_2, a_3, \ldots, a_n$ be in A P of non zero t
prove that

(i) $$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1}=\frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

(ii) $$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$$

45 The number of terms of an A P is even, the sur
the odd terms is 24, of the even terms 30, and the last
exceeds the first by $10\frac{1}{2}$, find the number of terms and th ser

46 Let the sequence $a_1, a_2 \ldots a_n$ form an A P and let a
prove that

$$\frac{a_1}{a_2}+\frac{a_2}{a_3}+\frac{a_3}{a_4}+\ldots+\frac{a_n}{a_{n-1}}-a_1\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right)$$
$$=\frac{a_{n-1}}{a_1}+\frac{}{a}$$

47 Find the sum of all two digit numbers which v
divided by 4, yield unity as remainder

48 Find an A P in which sum of any number of term
always three times the squared number of these terms

49 150 workers were engaged to finish a piece of work
a certain number of days Four workers dropped the sec
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50 Along a road lie an odd number of stones placed
intervals of 10 meters These stones have to be assembled aro
the middle stone A person can carry only one stone at a ti
A man carried the job with one of the end stones by carry
them in succession In carrying all the stones he covered a
tance of 3 km Find the number of stones

51 $x^{18}=y^{21}=z^{28}$, prove that 3 3 $\log_y x$, 3 $\log_z y$, 7 lo
form an A P

52 (a) Prove that if p, q, r $(p\neq q)$ are terms (
necessarily consecutive) of an A P then there exists a ratio
number k such that $(r-q)/(q-p)=k$

$\therefore\ S=\frac{q}{2}[2(a+pd)+(q-1)d]=\frac{q}{2}[2a+(p-1)d+(p+q)d]$

$=\frac{q}{2}\left[0-(p+q)\frac{2a}{p-1}\right]=-a\ \frac{(p+q)q}{p-1}$ by (1)

20 (i) $S_p=q$ and $S_q=p$

$\frac{p}{2}[2a+(p-1)d]=q,\ \frac{q}{2}[2a+(q-1)d]=p$

Subtracting, $\frac{p-q}{2}\left[(2a-d)+\frac{p^2-q^2}{2}d\right]=q-p,$

or $2a-d+(p+q)d=2(-1)$, cancelling $p-q$

or $2a+(p+q-1)d=-2$ (1)

$\therefore\ S_{p+q}=\frac{p+q}{2}[2a+(p+q-1)d]$

$=\frac{p+q}{2}(-2)=-(p+q)$ by (1)

(ii) Do yourself

21 Here $a=1$ for all and $d=1, 2, 3$ respectively for S_1, S_2, S_3, $n=n$ for all

$S_1+S_3=\frac{n}{2}[2\cdot1+(n-1)\cdot1]+\frac{n}{2}[2\cdot1+(n-1)\cdot3]$

$=\frac{n}{2}[n+1]+\frac{n}{2}[3n-1]=\frac{n}{2}\cdot4n=2n^2$

$2S_2=2\cdot\frac{n}{2}[2\cdot1+(n-1)\cdot2]=n\cdot(2n)=2n^2$

$S_1+S_3=2S_2$

22 (a) Here a and d are same as there is same A P and $n=n, 2n, 3n$ for S_1, S_2 and S_3 respectively —

$3(S_2-S_1)=3\left[\frac{2n}{2}[2a+(2n-1)d]-\frac{n}{2}[2a+(n-1)d]\right]$

$=\frac{3n}{2}[(4a-2a)+(4n-2-n+1)d]$

$=\frac{3n}{2}[(2a+(3n-1)d]=S_3$ i e sum of $3n$ terms

(b) We have $\frac{n}{2}(a+T_n)=pn^2$ and $\frac{m}{2}(a+T_m)=pm^2$

Eliminating a, we get $T_n-T_m=2p(n-m)$

or $[a+(n-1)d]-[a+(m-1)d]=2p(n-m)$

or $d(n-m)=2p(n-m)$ or $d=2p$

Thus $a+T_n=2np$ gives

52 (b) Prove that the numbers $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$ cannot be the terms of a single A P with non zero common difference

53 (i) Find the number of terms common to the two A P s 3, 7, 11, 407 and 2, 9, 16 709

(ii) Certain numbers appear in both arithmatic progressions 17, 21, 25, and 16, 21, 26, , Find the sum of first hundred numbers appearing in both progressions

54 Each of the two triplets of numbers log a log b, log c and log a—log $2b$, log $2b$—log $3c$, log $3c$—log a in an A P Can the numbers a, b, c, be the lengths of the sides of a triangle ? If they can, what kind of triangle is it ? Find the angles of the triangle provided that it exists

Solutions To Problem Set (A)

1 $T_5=a+4d=1$, $T_{31}=a+30d=-77$

Solving the above two, we get $a=13$ and $d=-3$ (1)

$S_{15}=\frac{n}{2}[2a+(n-1)d]=\frac{15}{2}[26+14(-3)]=-120$

Let $T_n=-17$ Then $a+(n-1)d=-17$

$13+(n-1)(-3)=-17$ $3n=33$ or $n=11$

Let $S_n=20$ Then $\frac{n}{2}[2a+(n-1)d]=20$

$n[2\times 13+(n-1)(-3)]=40$ by (1)

$3n^2-29n+40=0$ $(n-8)(3n-5)=0$ $n=8$

The value of n cannot be fractional

2 $a=20$, $d=-\frac{2}{3}$ and let $S_n=300$

$\frac{n}{2}[2\cdot 20+(n-1)(-2/3)]=300$

Simplifying $n^2-61n+900=0$ or $(n-25)(n-36)=0$

$n=25$ or 36,

Since common ratio is negative and $S_{25}=S_{36}=300$, it shows that the sum of last eleven i e T_{26}, T_{27} T_{36} is zero

3 (a) Putting $n=1, 2, 3$ in $T_n=\frac{3+n}{4}$ we get the series as

1, 5/4, 3/2, $a=1, d=\frac{1}{4}$

$S_{105}=\frac{105}{2}[2\cdot 1+(104)\frac{1}{4}]=\frac{105}{2}\times 28=1470$

$2a+(n-1)\ d=2np$

or $2a+(n-1)\ 2p=2np$, given $a=p$

Hence $S_p=\frac{p}{2}\ [2a+(p-1)\ d]=\frac{p}{2}[2p+(p-1)\ 2p]=p^3$

23 (a) Here $a=1$ for all, and $d=1, 2, 3,\ \ n$ respectively for the n A P's

We have to find sum of their nth terms

$\Sigma T_n=[1+(n-1)\ 1]+[1+(n-1)\ 2]+\ \ +[1+(n-1)\ n]$

$=[1+1+1\ \ n \text{ terms}]+(n-1)\ [1+2+3+\ \ n \text{ terms}]$

$=n+(n-1)\frac{n}{2}[1+n]=\frac{n}{2}[2+n^2-1)$

$=\frac{n}{2}[n^2+1]$

(b) Do yourself

(c) $S_1=\frac{n}{2}[2\ 1+(n-1)\ 1]$, $S_2=\frac{n}{2}[2\ 2+(n-1)\ 2]$

$S_m=\frac{n}{2}[2m+(n-1)\ m]$

$S_1+S_2+\ \ +S_m$

$=n\ (1+2+3+\ \ m)+\frac{n\ (n-1)}{2}\ (1+2+3+\ \ m)$

$=\frac{m\ (m+1)}{2}\left[n+\frac{n^2-n}{2}\right]=\frac{m\ (m+1)}{2}\ \ \frac{n\ (n+1)}{2}$

$=\frac{1}{4}mn\ (m+1)\ (n+1)$

Here we have used

$1+2+3+\ \ m=\frac{m}{2}\ [1+m]$ i e $S=\frac{n}{2}[a+l]$

24 Here $a=1, 2, 3\ \ m$ respectively
and $d=1, 3, 5\ \ 2m-1$, and $n=n$

$S_1+S_2+S_3+\ \ S_m$,

$S_1=\frac{n}{2}[2\ 1+(n-1)\ 1]$, $S_2=\frac{n}{2}\ (2\ 2+(n-1)\ \ 3]$

$S_m=\frac{n}{2}\ [2\ m+(n-1)\ (2m-1)]$

$S_1+S_2+S_3+\ \ +S_m$

$=n\ (1+2+3+\ \ m)+\frac{n\ (n-1)}{2}\ [1+3+5+\ \ +(2m-1)]$

$=n\ \frac{m\ (m+1)}{2}+\frac{n\ (n-1)}{2}\ \frac{m}{2}\ [1+2m-1]$, using $S=\frac{n}{2}[a+l]$

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(i) $$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1} = \frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

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$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\ldots+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}$$

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51 $x^{18}=y^{21}=z^{28}$ prove that 3 3 $\log_y x$, 3 $\log_z y$, 7 $\log_x z$ form an A P

52 (a) Prove that if p q r $(p\neq q)$ are terms (not necessarily consecutive) of an A P then there exists a rational number k such that $(r-q)/(q-p)=k$

$$=\frac{mn}{2}[m+1+m(n-1)]=\tfrac{1}{2}mn(mn+1)$$

25 (a) The various groups of natural numbers are

$G_1=(1), G_2=(2, 3, 4), G_3=(5, 6, 7, 8, 9)$

The number of terms in successive groups are

1, 3, 5 which are in A.P whose nth terms will be

$a+(n-1)d=1+(n-1)2=2n-1$

Hence the number of terms in nth group are $(2n-1)$ $=N$ (1)

Ist term of $G_1=1=1+(1-1)^2$

Ist term of $G_2=2=1+1+(2-1)^2$

Ist term of $G_3=5=1+4=1+(3-1)^2$

Ist term of $G_4=10=1+9=1+(4-1)^2$

Hence Ist term of $G_n=1+(n-1)^2=A$ (2)

Also the terms in each group form an A P of common difference 1 $D=1$ (3)

Hence sum of terms which will be in A P in the nth group

$$=\frac{N}{2}[2A+(N-1)D]$$

$$=\frac{2n-1}{2}[2\{1+(n-1)^2\}+(2n-1-1)1] \text{ by (1), (2) and (3),}$$

$$=(2n-1)[n^2-2n+2+n-1]$$

$$=(2n-1)(n^2-n+1)$$

$$=2n^3-3n^2+3n-1=n^3+(n^3-3n^2+3n-1)$$

$$=n^3+(n-1)^3$$

Alternative method for first term of G_n

The first terms are 1 2 5, 10

whose successive differences are 1, 3, 5, 7 etc which are in A P

Let $S=1+2+5+10+ \quad +T_n$

$S= \quad 1+2+5+ \quad +T_{n-1}+T_n$

Subtracting, $0=\ 1+(1+3+5+ \quad (n-1) \text{ terms})-T_n$

$$T_n=1+\frac{n-1}{2}[2\cdot1+(n-1-1)2]$$

$$=1+(n-1)(n-1)=1+(n-1)^2$$

Hence the first term of nth group is $1+(n-1)$ as found above in (2) by another method of inspection

(b) As in part (a) the number of terms in nth group is $2n-1$ which are in A P of common difference 1 and first term of successive groups are 1 2 3 , and hence of nth group is n.

Sum of terms of nth group

$$=\frac{2n-1}{2}[2n+(2n-1-1)1]=\frac{2n-1}{2}(4n-2)=(2n-1)^2$$

52 (b) Prove that the numbers $\sqrt{2}$, $\sqrt{3}$ $\sqrt{5}$ cannot be the terms of a single A P with non zero common difference

53 (i) Find the number of terms common to the two A P s 3, 7, 11, 407 and 2, 9, 16 709

(ii) Certain numbers appear in both arithmatic progressions 17, 21, 25, and 16, 21, 26, Find the sum of first hundred numbers appearing in both progressions

54 Each of the two triplets of numbers log a log b, log c and log a—log $2b$, log $2b$—log $3c$ log $3c$—log a in an A P Can the numbers a, b, c, be the lengths of the sides of a triangle? If they can, what kind of triangle is it? Find the angles of the triangle provided that it exists

Solutions To Problem Set (A)

1 $T_5=a+4d=1$, $T_{31}=a+30d=-77$

Solving the above two, we get $a=13$ and $d=-3$ (1)

$S_{15}=\frac{n}{2}[2a+(n-1)d]=\frac{15}{2}[26+14(-3)]=-120$

Let $T_n=-17$ Then $a+(n-1)d=-17$

$13+(n-1)(-3)=-17$ $3n=33$ or $n=11$

Let $S_n=20$ Then $\frac{n}{2}[2a+(n-1)d]=20$

$n[2\times13+(n-1)(-3)]=40$ by (1)

$3n^2-29n+40=0$ $(n-8)(3n-5)=0$ $n=8$

The value of n cannot be fractional

2 $a=20$, $d=-\frac{2}{3}$ and let $S_n=300$

$\frac{n}{2}[2\ 20+(n-1)(-2/3)]=300$

Simplifying $n^2-61n+900=0$ or $(n-25)(n-36)=0$

$n=25$ or 36,

Since common ratio is negative and $S_{25}=S_{36}=300$, it shows that the sum of last eleven i e T_{26} T_{27}, T_{36} is zero

3 (a) Putting $n=1, 2, 3$ in $T_n=\frac{3+n}{4}$ we get the series as

1, 5/4, 3/2, $a=1$, $d=\frac{1}{4}$

$S_{105}=\frac{105}{2}[2\ 1+(104)\frac{1}{4}]=\frac{105}{2}\times28=1470$

6 Do yourself Ist terms of nth group $=1+\frac{n(n-1)}{2}$

∴ Ist term of 50th group $=1226$ Ans 62525

27 The number of terms in successive groups are 1, 2 3 , and hence in nth there will be n terms in A P of common difference 2

The only thing we have to find is the first term

The successive first terms are 1, 3, 7, 13

whose successive differences 2, 4, 6 are in A P

$$S=1+3+7+13+\quad +T_n$$

$$S=\quad 1+3+7\quad +T_{n-1}+T_n \qquad \text{Subtract}$$

$$0=1+(2+4+6\quad (n-1)\text{ terms})-T_n$$

$$T_n=1+\frac{n-1}{2}[2\ 2+(n-2)\ 2]$$

$$=1+(n-1)n=n^2-n+1$$

The terms in nth group form an A P

or which $a=n^2-n+1,\ d=2,\ n=n$

$$S_n=\frac{n}{2}[2(n^2-n+1)+(n-1)\ 2]$$

$$=n[n^2-n+1+n-1]=n\ n^2=n^3$$

Let the three numbers in A P be $a-d\ a,\ a+d$

Sum $=3a=15$ ∴ $a=5$

Sum of their square $=(a-d)^2+a^2+(a+d)^2=83$

or $3a^2+2d^2=83$ or $2d^2=83-3\ 25$ $d^2=4$ or $d=\pm 2$

Hence the numbers are 3 5, 7 or 7, 5 3

Here $3a=12$ $a=4$

Also $(a-d)^3+a^3+(a+d)^3=288$

$3a^3+6ad^2=288$ or $24d^2=288-3\times 64=96$

$d^2=4$ or $d=\pm 2$

Hence the numbers are

2, 4, 6 or 6, 4, 2

(a) Let the four numbers in A P be

$a-3d,\ a-d,\ a+d,\ a+3d$

Sum $=4a=20$ $a=5$

Sum of their squares $=4a^2+20d^2=120$

$20d^2=120-4\times 25=20$ $d^2=1$ or $d=\pm 1$

Hence the numbers are 2, 4, 6, 8 or 8, 6, 4, 2

(b) $a=8,\ d=\pm 1$ Numbers are 5, 7, 9, 11 or 11, 9, 7, 5

Here $4a=28$ $a=7$

Also $\frac{(a-3d)(a+d)}{(a-d)(a+3d)}=\frac{8}{15}$

44 (i) If $a_1, a_2, a_3, \ldots, a_n$ be in A P of non zero terms prove that

(i) $$\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+\frac{1}{a_3a_{n-2}}+\ldots+\frac{1}{a_na_1} = \frac{2}{a_1+a_n}\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)$$

(ii) $$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\ldots+\frac{1}{a_{n-1}a_n}=\frac{n-1}{a_1a_n}$$

45 The number of terms of an A P is even, the sum of the odd terms is 24, of the even terms 30, and the last term exceeds the first by $10\frac{1}{2}$, find the number of terms and th series

46 Let the sequence $a_1, a_2, \ldots a_n$ form an A P and let $a_1=0$ prove that

$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\ldots+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots+\frac{1}{a_{n-2}}\right) = \frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}$$

47 Find the sum of all two digit numbers which when divided by 4, yield unity as remainder

48 Find an A P in which sum of any number of terms is always three times the squared number of these terms

49 150 workers were engaged to finish a piece of work in a certain number of days Four workers dropped the second day, four more workers dropped the third day and so on It takes 8 more days to finish the work now Find the number of days in which the work was completed

50 Along a road lie an odd number of stones placed at intervals of 10 meters These stones have to be assembled around the middle stone A person can carry only one stone at a time A man carried the job with one of the end stones by carrying them in succession In carrying all the stones he covered a distance of 3 km Find the number of stones

51 $x^{18}=y^{21}=z^{28}$ prove that 3, 3 $\log_y x$ 3 $\log_z y$, 7 $\log_x$ form an A P

52 (a) Prove that if p, q, r $(p\neq q)$ are terms (not necessarily consecutive) of an A P, then there exists a rational number k such that $(r-q)/(q-p)=k$

or $15\,[a^2-3d^2-2ad]=8\,[a^2-3d^2+2ad]$

or $7\,(a^2-3d^2)=46ad$ or $7\,(49-3d^2)=46\times 7\,d$

or $49-3d^2=46d$ or $3d^2+46d-49=0$

$(d-1)\,(3d+49)=0$, $d=1$

Required numbers are 4, 6, 8, 10

32. Let the means be x_1 x_2 ... x_m so that
$1, x_1, x_2, \ldots x_m, 31$ is an A.P. of $(m+2)$ terms.
$31=T_{m+2}=a+(m+1)\,d=1+(m+1)\,d$

$$d=\frac{30}{m+1} \qquad (1)$$

We are given that

$$\frac{x_7}{x_{m-1}}=\frac{5}{9} \qquad \frac{T_8}{T_m}=\frac{a+7d}{a+(m-1)\,d}=\frac{5}{9}$$

or $9a+63d=5a+(5m-5)\,d$

or $4\cdot 1=(5m-68)\,\dfrac{30}{m+1}$ by (1)

or $2m+2=75m-1020$

$73\,m=1022$ $\quad m=\dfrac{1022}{73}=14$

33. $a, x_1, x_2, \ldots x_n, b$ is an A.P. of $n+2$ terms
where $x_1, x_2, \ldots x_n$ are n A.M.'s between a and b

$$b=T_{n+2}=a+(n+1)\,d \qquad \frac{b-a}{n+1}=d \qquad (1)$$

We have to find $x_1+x_2+\ldots+x_n$

$=T_2+T_3+\ldots+T_{n+1}$

$=(a+d)+(a+2d)+\ldots+(a+nd)$

$=na+d\,(1+2+3+\ldots n)$

$=na+d\dfrac{n}{2}\,(1+n)$

$=na+\dfrac{b-a}{n+1}\dfrac{n}{2}\,(1+n)=\dfrac{n}{2}\,(2a+b-a)$

$=n\dfrac{(a+b)}{2}=n$ (A.M. of a and b)

A.M. of a and b is $\dfrac{a+b}{2}$

34. By the given condition

$$\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=\frac{a+b}{2}$$

$T_1=a,\ T_2=a+d=b \qquad d=b-a$

$l=T_n=a+(n-1)\ d=c$

$n-1=\frac{c-a}{d}=\frac{c-a}{b-a}$

$n=\frac{c-a}{b-a}+1=\frac{c+b-2a}{b-a}$ (1)

$S_n=\frac{n}{2}\ [a+l]=\frac{(c+b-2a)}{2\ (b-a)}\ (a+c)$ by (1), (2)

$T_n=S_n-S_{n-1}$

Now $S_n=3n^2+4n$

∴ $S_{n-1}=3\ (n-1)^2+4\ (n-1)=3n^2-2n-1$

$T_n=(3n^2+4n)-(3n^2-2n-1)$ by (1) $=6n+1$

∴ $T_1=7,\ T_2=13,\ T_3=19$

∴ $a=7,\ d=6 \qquad T_n=6n+1$

2 $\frac{S_m}{S_n}=\frac{m^2}{n^2} \qquad \frac{\frac{m}{2}\ [2a+(m-1)\ d]}{\frac{n}{2}\ [2a+(n-1)\ d]}=\frac{m^2}{n^2}$

$\frac{2a+(m-1)\ d}{2a+(n-1)\ d}=\frac{m}{n}$ or $2a\ (n-m)=a\ \{m\ (n-1)-n\ (m-1)\}$

or $2a\ (n-m)=d\ (n-m) \qquad d=2a$

$\frac{T_m}{T_n}=\frac{a+(m-1)\ d}{a+(n-1)\ d}=\frac{a+(m-1)\ 2a}{a+(n-1)\ 2a}=\frac{2m-1}{2n-1}$

1 (a) $\frac{S_n}{S'_n}=\frac{\frac{n}{2}[2a+(n-1)\ d]}{\frac{n}{2}[2a'+(n-1)\ d']}=\frac{3n+8}{7n+15}$

or $\frac{a+\frac{n-1}{2}d}{a'+\frac{n-1}{2}\ d'}=\frac{3n+8}{7n+15}$ (1)

We have to find $\frac{T_{12}}{T'_{12}}=\frac{a+11d}{a'+11d'}$

Ch[anging] $\frac{n-1}{2}=11$ or $n=23$ in (1) we get

$\frac{T_{12}}{T'_{12}}=\frac{a+11d}{a'+11d'}=\frac{3\ (23)+8}{7\ (23)+15}=\frac{77}{176}=\frac{7}{16}$

(b) $\frac{S_n}{S_n'}=\frac{a+\frac{n-1}{2}\ d}{a'+\frac{n-1}{2}d'}=\frac{7n+1}{4n+27}$ (1)

or $2a^{n+1}+2b^{n+1}=a^{n+1}+b^{n+1}+ab^n+ba^n$

or $a^{n+1}-a^nb=b^na-b^{n+1}$ or $a^n(a-b)=b^n(a-b)$

∴ $a^n=b^n$ Above is possible only when $n=0$

$a^0=b^0=1$

35 Let a and b be the two numbers so that $a+b=\frac{13}{6}$ given (1)
Let $2n$ (even) means be inserted between them so that $a, x_1, x_2, \ldots x_{2n}, b$ is an A P of $(2n+2)$ terms whose first term is a and last term is b and whose sum is

$$\frac{2n+2}{2}[a+b]=(n+1)(a+b) \quad (2)$$

∴ Sum of the means = Sum of the series $-(a+b)$

$(2n+1)$ (given) $=(n+1)(a+b)-(a+b)$

$=n(a+b)$

or $2n+1=n\,\frac{13}{6}$ by (1) or $12n+6=13n$ $n=6$

Hence the number of means inserted $=2n=12$

36 Do yourself

37 Do yourself

38 Do yourself

39 Do yourself

40 $a_n=T_n=a_1+(n-1)d$

or $\frac{(a_n-a_1)}{n-1}=d$ or $\frac{[\sqrt{(a_n)}+\sqrt{(a_1)}][\sqrt{(a_n)}-\sqrt{(a_1)}]}{n-1}=d$ (1)

$a_2-a_1=d$ $[\sqrt{(a_2)}+\sqrt{(a_1)}][\sqrt{(a_2)}-\sqrt{(a_1)}]=d$

$$d\left[\frac{1}{\sqrt{(a_1)}+\sqrt{(a_2)}}\right]=\sqrt{(a_2)}-\sqrt{(a_1)}$$

$$d\left[\frac{1}{\sqrt{(a_2)}+\sqrt{(a_3)}}\right]=\sqrt{(a_3)}-\sqrt{(a_2)}$$

$$d\left[\frac{1}{\sqrt{(a_{n-1})}+\sqrt{(a_n)}}\right]=\sqrt{(a_n)}-\sqrt{(a_{n-1})}$$

Adding we get

$$d\,S=[\sqrt{(a_n)}-\sqrt{(a_1)}]=\frac{(n-1)\,d}{\sqrt{(a_n)}+\sqrt{(a_1)}} \text{ by (1)}$$

where S denotes the sum of the given series

$$\therefore \quad S=\frac{(n-1)}{\sqrt{(a_n)}+\sqrt{(a_1)}}$$

41 (a) Since the three terms are in A P

$2\log_{10}(2^x-1)=\log_{10}2+\log_{10}(2^x+3)$

or $(2^x-1)^2=2(2^x+3)$

(b) Using the property e (1) of P 105, we get

$a_5+a_{20}=a_1+a_{24},\ a_{10}+a_{15}=a_1+a_{24}$

Hence the given relations reduces to

$$3\,(a_1+a_{24})=225,\quad \text{giving } a_1+a_{24}=75,$$

Hence $S_{24}=\frac{24}{2}(a_1+a_{24})=12\times75=900$

(c) Ans 98

4 $S_8=64$ $\quad 2a+7d=16$

$S_{19}=361$ $\quad \therefore\ 2a+18d=38$ Solving $a=1,\ d=2$

$$S_n=\frac{n}{2}[2\cdot1+(n-1)\,2]=n^2$$

5 (a) $S_{40}=3600,\ S_{30}=3600-\frac{1}{3}(3600)=2400,$ because after paying 30 instalments (1/3)rd of debt is still left The above two equations give

$$2a+39d=180 \text{ and } 2a+29d=160$$

Solving them we get $d=2$ and $a=51$

i e the value of first instalment is Rs 51

(b) Do yourself Ans 16

6 Sum of the interior angles of a polygon of n sides

$$=(2n-4)\,\pi/2=(n-2)\,\pi$$

Also $a=120°,\ d=5°$

$$\frac{n}{2}[2\cdot120°+(n-1)\,5°]=(n-2)\,180°$$ Cancel 5 and simplify

$n^2-25n+144=0 \qquad (n-9)(n-16)=0$

$\therefore\ n=9,\ 16$ But when

$n=16,\ T_n=a+(16-1)\,d=120°+15\times5°=195°$

This is not possible as interior angle cannot be greater than 180°

$n=9$ is the correct answer

7 Ans $\frac{1}{2}n\,[n\log(a/b)+\log ab]$

8 $T_m=a+(m-1)\,d=n$ $\qquad$ Solving we get

$T_n=a+(n-1)\,d=m$ $\qquad d=-1$ and $a=m+n-1$

$Tp=a+(p-1)\,d=m+n-1+(p-1)(-1)=m+n-p$

$T_{m+n}=a+(m+n-1)\,d=(m+n-1)+(m+n-1)(-1)=0$

9 $S=\frac{n}{2}(a+l)$ or $\frac{2S}{a+l}=n$ (1)

$l=a+(n-1)\,d$ so that $d=\frac{l-a}{n-1}$ (2)

Now put for n from (1) in (2)

Alt $\frac{l^2-a^2}{2S-(l+a)}=\frac{l^2-a^2}{n\,(l+a)-(l+a)}$ by (1) $=\frac{l-a}{n-1}=d$ by (2) or $(y-1)^2=2(y+3)$ where $y=2^x$

or $y^2-4y-5=0$

$\therefore$ $(y-5)(y+1)=0$ or $y=5$, $2^x=y\neq-1$

or $2^x=5$ $x=\log_2 5$

(b) Since $5^{1+x}+5^{1-x}$, $a/2$, 25^x+25^{-x} are in A P, we have

$2\,a/2=5^{1+x}+5^{1-x}+25^x+25^{-x}$,

Now put $5^x=t$ so that $t>0$, we then have

$a=5t+5/t+t^2+1/t^2=(t^2+1/t^2)+5(t+1/t)$

or $a=(t-1/t)^2+2+5[(\sqrt{t}-1/\sqrt{t})^2+2]$

$=(t-1/t)^2+5(\sqrt{t}-\sqrt{t})^2+12\geqslant 12$

Thus values of a are given by the inequality $a\geqslant 12$

42 $T_1+T_2+T_3+\ldots T_m=T_{m+1}+T_{m+2}+\ldots n$ terms

i e next n terms

Add $T_1+T_2+\ldots T_m$ to both sides, we get

$2S_m=T_1+T_2+\ldots+T_{m+1}+\ldots(m+n)$ terms

$2\,S_m=S_{m+n}$ and similarly

$2\,S_m=S_{m+p}$

$$2\,\frac{m}{2}\,[2a+(m-1)\,d]=\frac{m+n}{2}\,[2a+(m+n-1)\,d] \quad (1)$$

$$2\,\frac{m}{2}\,[2a+(m-1)\,d]=\frac{m+p}{2}\,[2a+(m+p-1)\,d] \quad (2)$$

We have to eliminate the unknown quantities a and d

From (1) $\dfrac{2a+(m+n-1)\,d}{2a+(m-1)\,d}=\dfrac{2m}{m+n}$

Now subtract (1) from each side

$$\frac{nd}{2a+(m-1)\,d}=\frac{m-n}{m+n} \quad (3)$$

Similarly from (2) on replacing n by p in (3), we get

$$\frac{pd}{2a+(m-1)\,d}=\frac{m-p}{m+p} \quad (4)$$

Dividing (3) and (4), we get

$$\frac{n}{p}=\left(\frac{m-n}{m+n}\right)\left(\frac{m+p}{m-p}\right)$$

$$\frac{(m+n)\,(m-p)}{p}=\frac{(m-n)\,(m+p)}{n}$$

$$(m+n)\frac{m-p}{mp}=(m+n)\,\frac{(m-n)}{mn}$$

or $(m+n)\left(\frac{1}{p}-\frac{1}{m}\right)=(m+p)\,\left(\frac{1}{n}-\frac{1}{m}\right)$

or $(m+n)\,\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$

10 $T_1=a$, $T_2=a+d=b$ $\quad d=b-a$

$$l=T_n=a+(n-1)\ d=c$$

$$n-1=\frac{c-a}{d}=\frac{c-a}{b-a}$$

$$n=\frac{c-a}{b-a}+1=\frac{c+b-2a}{b-a} \qquad (1)$$

$$S_n=\frac{n}{2}\ [a+l]=\frac{(c+b-2a)}{2\ (b-a)}\ (a+c) \text{ by (1)}, \qquad (2)$$

11 $T_n=S_n-S_{n-1}$

Now $S_n=3n^2+4n$

$S_{n-1}=3\ (n-1)^2+4\ (n-1)=3n^2-2n-1$

$T_n=(3n^2+4n)-(3n^2-2n-1)$ by (1) $=6n+1$

$T_1=7$, $T_2=13$, $T_3=19$

$a=7$, $d=6$ $\quad T_n=6n+1$

12 $\dfrac{S_m}{S_n}=\dfrac{m^2}{n^2}$ $\quad \therefore \dfrac{\frac{m}{2}\ [2a+(m-1)\ d]}{\frac{n}{2}\ [2a+(n-1)\ d]}=\dfrac{m^2}{n^2}$

$\therefore \dfrac{2a+(m-1)\ d}{2a+(n-1)\ d}=\dfrac{m}{n}$ or $2a\ (n-m)=a\ \{m\ (n-1)-n\ (m-1)\}$

or $2a\ (n-m)=d\ (n-m)$ $\quad d=2a$

$$\frac{T_m}{T_n}=\frac{a+(m-1)\ d}{a+(n-1)\ d}=\frac{a+(m-1)\ 2a}{a+(n-1)\ 2a}=\frac{2m-1}{2n-1}$$

13 (a) $\dfrac{S_n}{S'_n}=\dfrac{\frac{n}{2}[2a+(n-1)\ d]}{\frac{n}{2}[2a'+(n-1)\ d']}=\dfrac{3n+8}{7n+15}$

or $\dfrac{a+\frac{n-1}{2}d}{a'+\frac{n-1}{2}\ d'}=\dfrac{3n+8}{7n+15}$ $\qquad (1)$

We have to find $\dfrac{T_{12}}{T'_{12}}=\dfrac{a+11d}{a'+11d'}$

Changing $\dfrac{n-1}{2}=11$ or $n=23$ in (1) we get

$$\frac{T_{12}}{T'_{12}}=\frac{a+11d}{a'+11d'}=\frac{3\ (23)+8}{7\ (23)+15}=\frac{77}{176}=\frac{7}{16}$$

(b) $\dfrac{S_n}{S_n'}=\dfrac{a+\frac{n-1}{2}\ d}{a'+\frac{n-1}{2}d'}=\dfrac{7n+1}{4n+27}$ $\qquad (1)$

43 Now $a_1^2-a_2^2=(a_1-a_2)(a_1+a_2)=-d(a_1+a_2)$

$$a_3^2-a_4^2=(a_3-a_4)(a_3+a_4)=-d(a_3+a_4)$$

$$a_{2k-1}^2-a_{2k}^2= \quad =-d(a_{2k-1}+a_{2k})$$

Adding we get

$$S=-d[a_1+a_2+ \quad +a_{2k}]$$

$$S=-d\,S_{2k}=-d\frac{2k}{2}[a_1+a_{2k}] \qquad (1)$$

or $\quad S=-dk(a_1+a_{2k})$

We have to eliminate d

Now $\quad a_{2k}=a_1+(2k-1)d \quad$ or, $\frac{a_1-a_{2k}}{2k-1}=-d$

Putting the value of $-d$ in (1), we get

$$S=\frac{k(a_1-a_{2k})}{2k-1}(a_1+a_{2k})=\frac{k}{2k-1}(a_1^2-a_{2k}^2)$$

44 (i) $\frac{1}{a_1}+\frac{1}{a_n}=\frac{a_n+a_1}{a_1a_n}$

$$\frac{1}{a_2}+\frac{1}{a_{n-1}}=\frac{a_{n-1}+a_2}{a_2a_{n-1}}=\frac{a_n-d+a_1+d}{a_2a_{n-1}}=\frac{a_n+a_1}{a_2a_{n-1}}$$

$$\frac{1}{a_n}+\frac{1}{a_1}=\frac{a_1+a_n}{a_na_1}=\frac{a_1+a_n}{a_na_1}$$

Adding $2\left(\frac{1}{a_1}+\frac{1}{a_2}+ \quad \frac{1}{a_n}\right)$

$$=(a_n+a_1)\left[\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+ \quad \frac{1}{a_na_1}\right]$$

or $\frac{2}{a_n+a_1}\left(\frac{1}{a_1}+\frac{1}{a_2}+ \quad \frac{1}{a_n}\right)$

$$=\left[\frac{1}{a_1a_n}+\frac{1}{a_2a_{n-1}}+ \quad \frac{1}{a_na_1}\right]$$

(ii) $\frac{1}{a_1a_2}=\frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_2}\right]$

$$\frac{1}{a_2a_3}=\frac{1}{d}\left[\frac{1}{a_2}-\frac{1}{a_3}\right]$$

$$\frac{1}{a_{n-1}a_n}=\frac{1}{d}\left[\frac{1}{a_{n-1}}-\frac{1}{a_n}\right]$$

Whence by addition, we get

$$\frac{T_p}{T_p'}=\frac{c+(p-1)\,d}{a+(p-1)\,d} \quad \text{Put } \frac{n-1}{2}=p-1$$

or $\quad n=2p-1$ in (1)

$$\frac{T_p}{T_p'}=\frac{7(2p-1)+1}{4(2p-1)+27}=\frac{14p-6}{8p+23}$$

Now replace p by n in the above

(c) Proceed as above,

14 Clearly the numbers between 250 and 1000 which are divisible by 3 are 252, 255, 258 999

$\therefore\ T_n=999=a+(n-1)\,d=252+(n-1)\,3$

$333=84+n-1 \quad \therefore\ n=333-83=250$

$$S=\frac{n}{2}\,[a+l]=\frac{250}{2}\,[252+999]=125\times1251=156375$$

15 (i) Odd numbers between 2 and 1000 are

3, 5, 7, 9, 11, 13, 15 993, 995, 997, 999,

Those odd numbers which are divisible by 3 are

3, 9, 15, 21 ,993, 999

They form an A P of of which $a=3$, $d=6$, $l=999$

$l=T_n=a+(n-1)\,d \qquad 999=3+(n-1)\,6$

$333=1+2n-2$ or $334=2n \qquad n=167$

$$S=\frac{n}{2}\,[a+l]=\frac{167}{2}\,[3+999]\frac{167}{2}\times1002=167\times501$$

$$=167\times500+167=83500+167=83667$$

(ii) Ans n^2 (iii) Ans 867

(iv) Let the odd integers be $2m+1$, $2m+3$, $2m+5$, and let their number be n Then

$$57^2-13^2=\frac{n}{2}\,[2\,(2m+1)+(n-1)\,2]$$

$$=n\,(2m+n)=2mn+n^2$$

or $\quad 57^2-13^2=(n+m)^2-m^2$

Hence $m=13$ and $n+m=57$ or $n=57-13=44$

Hence the required odd integers are

27, 29, 31, 113

16 Let the first term be A and common difference be D

$$S_{p+q}=\frac{p+q}{2}[2A+(p+q-1)\,D] \qquad \text{(1)}$$

It is given that $\quad T_p=a$ and $T_q=b$

$\left.\begin{array}{l}A+(p-1)\,D=a\\ A+(q-1)\,D=b\end{array}\right\}$ Subtracting $(p-q)\,D=a-b$

$$\frac{1}{a_1a_2}+\frac{1}{a_2a_3}+\quad +\frac{1}{a_{n-1}a_n}=\frac{1}{d}\left[\frac{1}{a_1}-\frac{1}{a_n}\right]=\frac{1}{d}\left[\frac{a_n}{a_1}-\frac{a_1}{a_n}\right]$$
$$=\frac{1}{d}\ \frac{(n-1)\,d}{a_1\,a_n}=\frac{n-1}{a_1a_n}$$

45 Ans 8 terms, series is $1\frac{1}{2}$, 3, $4\frac{1}{2}$,

46 Let d be the common difference of the given A P

Then since $a_1=0$, we have $a_2=d$, $a_3=2d$, $a_n=(n-1)\,d$

Hence L H S $=\frac{2d}{d}+\frac{3d}{2d}+\frac{4d}{3d}+\quad +\frac{(n-1)\,d}{(n-2)\,d}$

$$-d\left(\frac{1}{d}+\frac{1}{2d}+\frac{1}{3d}+\quad +\frac{1}{(n-3)\,d}\right)$$
$$=(1+1)+\left(1+\frac{1}{2}\right)+\left(1+\frac{1}{3}\right)+\quad +\left(1+\frac{1}{n-3}\right)$$
$$+\left(1+\frac{1}{n-2}\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\quad +\frac{1}{n-3}\right)$$
$$=1+1+1+\text{ to } (n-2) \text{ terms}+\frac{1}{n-2}$$
$$=(n-2)+\frac{1}{n-2}=\frac{a_{n-1}}{a_3}+\frac{a_3}{a_{n-1}}$$

47 The first two digit number which when divided by 4 leaves remainder 1 is 4 3+1=13 and last is 4 24+1=97, form (4k+1)

Thus we have to find the sum of $13+17+21+\quad 97$

which is an A P $\qquad 97=13+(n-1)\,4 \qquad n=22$

$$S=\frac{n}{2}\,[a+l]=11\,[13+97]=11\times 110=1210$$

48 $S_n=3n^2$ by given condition

$$\therefore\ \frac{n}{2}\,[2a+(n-1)\,d]=3n^2 \text{ or } 2a-d=n\,(6-d) \qquad (1)$$

The relation (1) is to hold good for all values of n L H S is constant whereas R H S varies with n if $6-d\neq 0$ which is not possible Hence we must have $6-d=0$ i e $d=6$ and then $2a-d=0$

$2a=d=6$ or $a=3$ $\qquad$ A P is $3+9+15+21+$

49 Let the work finish in n days when the workers started dropping so that the total number of workers who worked all these days is the sum of A P

$150+146+142+\quad n$ terms

$$=\frac{n}{2}\,[2(150)+(n-1)\,(-4)]=n\,(152-2n) \qquad (1)$$

$$D=\frac{a-b}{p-q} \quad (2)$$

Adding $\quad 2A+(p+q-2)\ D=a+b,$

or $\quad 2A+(p+q-1)\ D=a+b+D=a+b+\frac{a-b}{p-q}$ by (2) $\quad (3)$

Hence from (1), and (3) $S_{p+q}=\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$

17 Let A be the first term and D the common difference of A P

$$T_p=a=A+(p-1)\ D=(A-D)+pD \quad (1)$$
$$T_q=b=A+(q-1)\ D=(A-D)+qD \quad (2)$$
$$T_r=c=A+(r-1)\ D=(A-D)+rD \quad (3)$$

Here we have got two unknowns A and D which are to be eliminated

We multiply (1), (2) and (3) by $q-r$, $r-p$ and $p-q$ respectively and add $a(q-r)+b(r-p)+c(p-q)=(A-D)\{q-r+r-p+p-q)\}+D\{p(q-r)+q(r-p)+r(p-q)\}=0$

18 $S_p=a=\frac{p}{2}[2A+(p-1)\ D]$

$$\frac{a}{p}=\left(A-\frac{D}{2}\right)+\frac{p}{2}\ D \quad (1)$$

Similarly writing for S_q and S_r, we have

$$\frac{b}{q}=\left(A-\frac{D}{2}\right)+\frac{q}{2}\ D \quad (2)$$

$$\frac{c}{r}=\left(A-\frac{D}{2}\right)+\frac{r}{2}\ D \quad (3)$$

Multiply (1), (2) and (3) by $q-r$, $r-p$ and $p-q$ respectively and add $\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$

19 (i) We are given that $\quad S_p=S_q$

$$\frac{p}{2}[2a+(p-1)\ d]=\frac{q}{2}[2a+(q-1)\ d]$$

or $\quad (2a-d)(p-q)+(p^2-q^2)\ d=0$, cancel $p-q$ as $p\neq q$

or $\quad 2a-d+(p+q)\ d=0$

or $\quad 2a+(p+q-1)\ d=0 \quad (1)$

$$S_{p+q}=\frac{p+q}{2}[2a+(p+q-1)\ d]=\frac{p+q}{2}\ 0=0 \text{ by (1)}$$

(ii) $S_p=0 \quad \therefore \frac{p}{2}[2a+(p-1)\ d]=0 \quad d=\frac{-2a}{p-1} \quad (1)$

Sum of next q terms $=$ sum of an A P whose first term will be $T_{p+1}=a+pd$

Had the workers not dropped then the work would have finished in $(n-8)$ days with 150 workers working on each day

$$i\ e\ 150\,(n-8) \qquad (2)$$

$\therefore$ $n(152-2n)=150(n-8)$ or $n^2-n-600=0$

or $(n-25)(n+24)=0$ $\quad n=25$

50 Let the number of stones be $2n+1$ so that there is one mid stone and n stones each on either side

There will be $(n+1)$ stones on the left and $(n+1)$ stones on right side or n intervals each of 10 meters both on the right and left side of mid stone Now he starts from one of the end stones, picks it up goes to mid stone drops it and goes to last stone on the other side picks it and come back to mid stone In all he travels n interval of 10 meters each 3 times Now from centre he will go to 2nd stone on L H S then come back and then go to 2nd last on R H S and again come back Thus he will travel $(n-1)$ intervals of 10 meters each 4 times Similarly $(n-2)$ intervals of 10 meters each 4 times for the 3rd and so on for the last

Hence the total distance covered$=3$k m $=3000=3000$

or $3\cdot 10\,n+4\,[10\,(n-1)+10\,(n-2)+\ldots+10]$

$=30n+40\,[1+2+3+\ldots(n-1)]=3000$

or $30n+40\,\frac{n-1}{2}\,[1+n-1]=3000$ or $2n^2+n-300=0$

or $(n-12)(2n+25)=0$ $\quad n=12$

Hence the number of stones$=2n+1=25$

51 Let $x^{18}=y^{21}=z^{28}=k$, say then on taking log we get

$18\log x=21\log y=28\log z=\log k \qquad (1)$

If the given terms are a, b, c, d then

$$a=3,\ b=3\log_y x=3\left(\frac{\log x}{\log y}\right)=3\,\frac{21}{18}=\frac{21}{6}=\frac{7}{2}=3\frac{1}{2}$$

Similarly $c=3\,\frac{28}{21}=4$ and $d=7\,\frac{18}{28}=\frac{9}{2}=4\frac{1}{2}$

Hence the four numbers are 3, $3\frac{1}{2}$, 4, $4\frac{1}{2}$ which are clearly in A P of common difference $\frac{1}{2}$

52 (a) Let p, q, r be the lth, mth and nth terms of an A P, then $p=a+(l-1)d$, $q=a+(m-1)d$ and $r=a+(n-1)d$,

when $r-q=(n-m)d$ and $q-p=(m-l)d$

so that $\frac{r-q}{q-p}=\frac{(n-m)d}{(m-l)d}=\frac{n-m}{m-l}$ $\quad(\because d\neq 0)$

$\sin\theta = \sin 18° = \sin \pi/10 \qquad \theta = n\pi + (-1)^n \pi/10$

$\sin\theta = -\sin 54° = \sin\left(\frac{-3\pi}{10}\right) \qquad \theta = n\pi - (-1)^n \frac{3\pi}{10}$

(*f*) We have $\quad 2\sin\theta\cos 2\theta = 2(2\cos^2\theta - 1) = 2\cos 2\theta$

$\cos 2\theta(\sin\theta - 1) = 0 \qquad \sin\theta = 1, \cos 2\theta = 0$

$\theta = n\pi + (-1)^n \pi/2,\ 2\theta = 2n\pi \pm \frac{\pi}{2} \quad \text{or} \quad \theta = n\pi \pm \frac{\pi}{4}$

(*g*) $\qquad \frac{1}{\cos 4\theta} - \frac{1}{\cos 2\theta} = 2$

$\cos 2\theta - \cos 4\theta = 2\cos 2\theta\cos 4\theta = \cos 2\theta + \cos 6\theta$

$\cos 6\theta + \cos 4\theta = 0 \quad \text{or} \quad 2\cos 5\theta\cos\theta = 0$

$\cos\theta = 0 = \cos\pi/2 \qquad \theta = n\pi + \pi/2$

$\cos 5\theta = 0 = \cos\frac{\pi}{2} \qquad 5\theta = n\pi + \frac{\pi}{2}$

or $\qquad \theta = \frac{n\pi}{5} + \frac{\pi}{10}$

(*h*) $\qquad \sin^2 A - \sin^2 B = \sin(A+B)\sin(A-B)$

$\sin(2n-1)\theta\sin\theta - \sin^2\theta = 0$

$\sin\theta[\sin(2n-1)\theta - \sin\theta] = 0$

$\sin\theta = 0 \qquad \theta = r\pi$

$\sin(2n-1)\theta = \sin\theta \qquad (2n-1)\theta + r\pi + (=1)^r\theta$

r even $= 2k$, $(2n-2)\theta = 2k\pi \qquad \theta = \frac{\lambda\pi}{n-1}$

r odd $= 2k+1$, $2n\theta = (2k+1)\pi \qquad \theta = (k+\frac{1}{2})\frac{\pi}{n}$

8 (*a*) Dividing both sides by $\sqrt{(1+3)} = 2$, we get

$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\cos x\cos\frac{\pi}{6} + \sin x\sin\frac{\pi}{6} = \cos\frac{\pi}{4}$

$\cos\left(x - \frac{\pi}{6}\right) = \cos\frac{\pi}{4}$

$x - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{4} \qquad x = 2n\pi + \frac{\pi}{6} \pm \frac{\pi}{4}$

(*b*) As above $\cos\left(\theta - \frac{\pi}{6}\right) = \frac{1}{2} = \cos\frac{\pi}{3}$

$\theta - \frac{\pi}{6} = 2n\pi \pm \frac{\pi}{3}$

$\theta = 2n\pi + \frac{\pi}{6} + \frac{\pi}{3} \quad \text{or} \quad 2n\pi + \frac{\pi}{6} - \frac{\pi}{3}$

$= 2n\pi - \frac{\pi}{2}$ or $2n\pi - \frac{\pi}{6}$

$\theta = (4n+1)\frac{\pi}{2}$ or $(12n-1)\frac{\pi}{6}$

We have to find values of θ such that $-2\pi < \theta < 2\pi$

$n = 0 \quad \theta = \frac{\pi}{2} \quad \frac{\pi}{6}$

$n = 1 \quad \theta = \frac{5\pi}{2}$ (rejected) $\frac{11\pi}{6}$

for $n > 1$ the values of θ will be outside the given range

$n = -1 \quad \theta = \frac{-3\pi}{2} \quad \theta = \frac{-13\pi}{6}$ (rejected)

$\theta = \frac{-3\pi}{2}, \frac{-\pi}{6}, \frac{\pi}{2}$ and $\frac{11\pi}{6}$

(c) As in part (a) we have on dividing by 2

$$\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta = -\frac{1}{\sqrt{2}}$$

$$\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3} = -\cos\frac{\pi}{4} = \cos\left(\pi - \frac{\pi}{4}\right)$$

$$\cos\left(\theta + \frac{\pi}{3}\right) = \cos\frac{3\pi}{4}$$

$$\theta + \frac{\pi}{3} = 2n\pi \pm \frac{3\pi}{4}$$

or $\theta = 2n\pi - \frac{\pi}{3} \pm \frac{3\pi}{4}$

Another Method

$$\frac{\sqrt{3}}{2}\sin\theta - \frac{1}{2}\cos\theta = \frac{1}{\sqrt{2}}$$

$$\sin\theta\cos\frac{\pi}{6} - \cos\theta\sin\frac{\pi}{6} = \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}$$

$$\sin\left(\theta - \frac{\pi}{6}\right) = \sin\frac{\pi}{4}$$

$$\theta - \frac{\pi}{6} = n\pi + (-1)^n\frac{\pi}{4}$$

$$\theta = n\pi + (-1)^n\frac{\pi}{4} + \frac{\pi}{6}$$

(d) $3 - 2\cos\theta - 4\sin\theta - (1 - 2\sin^2\theta) + 2\sin\theta\cos\theta = 0$

$2\sin^2\theta - 4\sin\theta + 2 - 2\cos\theta + 2\sin\theta\cos\theta = 0$

$(\sin\theta - 2\sin\theta + 1) + \cos\theta(\sin\theta - 1) = 0$

Since l, m, n are +ve integers and $m \neq l$, $(n-m)/(m-l)$ is a rational number

(b) $T_p = \sqrt{2}, T_q = \sqrt{3}, T_r = \sqrt{5}$

$\sqrt{3} - \sqrt{2} = T_q - T_p = (q-p)d, \ \sqrt{5} - \sqrt{3} = (r-p)d$

$$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{5}-\sqrt{3}} = \frac{q-p}{r-p} = k \text{ say} \qquad \text{(I)}$$

As p, q, r are +ve integers so k is a rational number in (I)

Squaring (I) we get $5 - 2\sqrt{6} = k^2 (8 - 2\sqrt{15})$ (2)

or $\sqrt{(15)}\, k^2 - \sqrt{6} = (8k^2 - 5)/2 = s$ say

Here s is again a rational number Square again

∴ $15 k^4 + 6 - 2\sqrt{[90)]}\, k^2 = s^2$

o $15k^4 + 6 - s^2 = 6\sqrt{(10)}\, k^2$ (3)

or $(15k^4 - s^2 + 6)/6k^2 = \sqrt{(10)}$

L H S of (3) is rational whereas R H S $\sqrt{(10)}$ is irrational which is not possible Hence $\sqrt{2}, \sqrt{3}, \sqrt{5}$ cannot be three terms of an A P

53 (i) It is easy to observe that both the series consist of 102 terms

Let $T_p = 3 + 4(p-1) = 4p - 1$ and

$T_q = 2 + 7(q-1) = 7q - 5$ be the general terms of the two series where both p and q lie between 1 and 102 We have to find the values of p and q for which $T_p = T_q$

i e $4p - 1 = 7q - 5$ or $4(p+1) = 7q$ (1)

Now p and q are +ive integers and hence from (1) we conclude that q is multiple of 4 and so let $q = 4s$ and as q lies between 1 and 102 therefore s lies between 1 and 25

Now putting $q = 4s$ in (1) we get $4(p+1) = 7(4s)$

or $p = 7s - 1$ (2)

Again p lies between 1 and 102 and hence from (2), s should lie between 1 and 14 (instead of 1 and 25) Thus there are only 14 terms common to both the A P 's

$T_p = 4p - 1 = 4(7s - 1) - 1$ by (2) $= 28s - 5$

Now put $s = 1, 2, 3$ 14 and the common terms are 23, 51, 79 etc

(ii) Denoting the nth and mth terms of the two progressions by T_n and T'_m, we have $T_n = 17 + (n+1)\, 4 = 4n + 13$ and $T'_m = 16 + (m-1)\, 5 = 5m + 11$

For common terms, we must have

$T_n = T'_m \Rightarrow 4n + 13 = 5m + 11 \Rightarrow 5m = 2(2n+1)$

This shows that $2n + 1 = 5k, k = 1, 3, 5,$

$= a^n r^{1+2+3+\cdots+n-1} = a^n r^{(n-1)n/2}$

$\therefore P^2 = a^{2n} r^{n(n-1)}$ (2)

$R = \frac{1}{a} + \frac{1}{ar} + \frac{1}{ar^2} + \cdots + \frac{1}{ar^{n-1}}$ (n terms)

$$R = \frac{1}{a}\frac{\left(1-\frac{1}{r^n}\right)}{\left(1-\frac{1}{r}\right)} = \frac{(r^n-1)}{r-1}\frac{1}{ar^{n-1}} \quad (3)$$

$\therefore \frac{S}{R} = a\frac{(1-r^n)}{1-r}\frac{r-1}{r^n-1} a\, r^{n-1} = a^2 r^{(n-1)}$ by (1) and (3)

$\left(\frac{S}{R}\right)^n = a^{2n} r^{n(n-1)} = P^2$ by (2)

21 (a) $T_p = AR^{p-1} = x,\ T_q = AR^{q-1} = y,\ T_r = AR^{r-1} = z$

$\log x = \log A + (p-1)\log R$ (1)

$\log y = \log A + (q-1)\log R$ (2)

$\log z = \log A + (r-1)\log R$ (3)

Multiply (1), (2) and (3) by $q-r$, $r-p$ and $p-q$ respectively and add

$(q-r)\log x + (r-p)\log y + (p-q)\log z$
$= 0\log A + 0\log R = 0$

(b) We are given that T_p, T_q and T_r of an A P are in G P

$\frac{T_q}{T_p} = \frac{T_r}{T_q} = R$ (common ratio of G P)

or $\frac{a+(q-1)d}{a+(p-1)d} = \frac{a+(r-1)d}{a+(q-1)d} = R$

Also we know that if $\frac{x}{y} = \frac{z}{u}$ then each is equal to

$\frac{x-z}{y-u}$ or $\frac{x+z}{y+u}$

$$R = \frac{[a+(q-1)d]-[a+(r-1)d]}{[a+(p-1)d]-[a+(q-1)d]} = \frac{(q-r)d}{(p-q)d} = \frac{q-r}{p-q}$$

22 (i) $S_1 = \frac{a(r^n-1)}{r-1},\ S_2 = \frac{a(r^{2n}-1)}{r-1},\ S_3 = \frac{a(r^{3n}-1)}{r-1}$

$S_3 - S_2 = \frac{a}{r-1}(r^{3n}-r^{2n}) = \frac{a(r^n-1)}{r-1}r^{2n}$

$S_1(S_3-S_2) = \frac{a(r^n-1)}{r-1}\frac{a(r^n-1)}{r-1}r^{2n} = \left[\frac{a(r^n-1)}{r-1}r^n\right]^2$ (1)

$S_2 - S_1 = \frac{a}{r-1}(r^{2n}-r^n) = \frac{a(r^n-1)}{r-1}r^n$ (2)

Hence $5m = 10k$ or $m = 2k$, $k = 1, 3, 5,$
Hence the common terms are given by
$T'_{2k} = 5 \cdot 2k + 11 = 10k + 11$, $k = 1, 3, 5,$
Sum of first 100 common terms
$= 21 + 41 + 61 +$ to 100 terms
$= \frac{100}{2} [2 \times 21 + (100-1) 20 = 101100$

54 We have

$2 \log b = \log a + \log c$ or $b^2 = ac$ (1)
and $2 [\log 2b - \log 3c] = \log 3c - \log 2b$ or $2b = 3c$ (2)
Solving (1) and (2) for a and b, we get $a = 9c/4$, $b = 3c/2$
Thus the triple of numbers that satisfies the given conditions is $9c/4$, $3c/2$, c $(c \neq 0)$ Now $a = 9c/4$, $b = 3c/2$ and c will form a triangle if (i) $a + b > c$ (ii) $b + c > a$ and (iii) $c + a > b$ But since $a + b = 15c/4 > c$, $b + c = 5c/2 > 9c/4 = a$, and $a + c = 13c/4 > 3c/2 = b$ $(c > 0)$, a triangle with sides a, b and c exists and since $a^2 > b^2 + c^2$, it is obtuse To find the angles, we use the cosine formula
Thus $\cos A = \frac{b^2 + c^2 - a^2}{2bc} = -\frac{29}{48}$, $\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{61}{72}$

and $\cos C = \frac{a^2 + b^2 - c^2}{2ab} = \frac{101}{108}$

Thus, bearing in mind that A, B, C are the angles of a triangle, we get

$$A = \cos^{-1}\left(-\frac{29}{48}\right) = \pi - \cos^{-1}\frac{29}{48}$$

$$B = \cos^{-1}\left(\frac{61}{72}\right) \text{ and } C = \cos^{-1}\left(\frac{101}{108}\right)$$

§ 2 Geometrical Progression

Definition A series in which each term is same multiple of the preceeding term is called a geometrical progression In other words, a series in which the ratio of successive terms is constant is called a G P This constant ratio is called **common ratio** and is denoted by r e g

(i) 1 4, 16 , (ii) 9, 6, 4 , (iii) a, ar, ar^2

All the above series are geometrical progression in which common ratios are 4, $\frac{2}{3}$ and r respectively

nth term of a G P

$S_1(S_3-S_2)=(S_2-S_1)^2$ by (1) and (2)

(ii) We have to prove

$$S_1^2+S_2^2=S_1(S_2+S_3) \text{ i.e. } \frac{S_3-S_2}{S_2-S_1}=\frac{S_2}{S_1}$$

Now $S_3-S_1=\frac{a}{r-1}[(r^{3n}-1)-(r^n-1)]$

$$=\frac{a}{r-1}(r^n-1)[r^{2n}+r^n+1-1)]$$

Thus $S_3-S_1=\frac{a(r^n-1)r^n(r^n+1)}{r-1}=\frac{ar^n(r^n-1)(r^n+1)}{r-1}$

Also by part (i), $S_2-S_1=\frac{ar^n(r^n-1)}{r-1}$

Hence by division $\frac{S_3-S_1}{S_2-S_1}=r^n+1$ (1)

Again $\frac{S_2}{S_1}=\frac{a(r^{2n}-1)}{r-1}\times\frac{r-1}{a(r^n-1)}=r^n+1$ (2)

Hence by (1) and (2), $\frac{S_3-S_1}{S_2-S_1}=\frac{S_2}{S_1}$ *as required*

23 (i) S_n = sum of an infinite G P whose first term is n and common ratio $r=\frac{1}{n+1}$

$$S_n=\frac{a}{1-r}=\frac{n}{1-\frac{1}{n+1}}=n+1$$

Putting $n=1, 2, 3, \dots n$

$$S_1+S_2+S_3+\dots+S_n=2+3+4+\dots n+1$$

$$=\frac{n}{2}[2\cdot 2+(n-1)\cdot 1]=\tfrac{1}{2}n(n+3)$$

(ii) Do yourself

24 $S_n=\frac{a(1-r^n)}{1-r}=\frac{a}{1-r}-a\frac{r^n}{1-r}$ (1)

Putting $n=1, 2, 3 \dots n$ and adding we get

(i) $S_1+S_2+S_3+\dots+S_n$

$$=n\left(\frac{a}{1-r}\right)-\frac{a}{1-r}(r+r^2+r^3+\dots+r^n)$$

$$=\frac{na}{1-r}-\frac{a}{1-r}r\cdot\frac{(1-r^n)}{1-r}$$

$$=\frac{na}{1-r}-\frac{ar}{(1-r)^2}(1-r^n)$$

8 Sum of a certain number of terms of the series

$$\frac{2}{9}, -\frac{1}{3}, \frac{1}{2} \quad \text{is } \frac{55}{72}$$

Find the number

9 How many terms of the series 1, 4, 16, must be taken to have their sum equal to 341

10 In a G P sum of n terms is 255, the last term is 128 and the common ratio is 2 Find n

11 In a G P sum of n terms is 364, first term is 1 and the common ratio is 3 Find n

12 Sum upto n terms the series

(i) $7+77+777+$

(ii) $6+66+666+$ (I I T 74)

(iii) $8+88+888+$ (Roorkee 78)

13 Express the recurrent decimal 0 125 125 125 as a rational number (I I T 77)

14 Find the value of 123 regarding it as a geometric series

15 Find the value of 423 (I I T 73 Roorkee 77)

16 After striking a floor a certain ball rebounds 4/5th of the height from which it has fallen Find the total distance that it travels before coming to rest, if it is gently dropped from a height of 120 meters

17 A ball is dropped from a height of 48 ft and rebounds two third of the distance it falls If it continues to fall and rebound in this way, how far will it travel before coming to rest ?

18 (i) In a G P if the $(m+n)$th term be p and $(m-n)$th term be q, then prove that its mth term is $\sqrt{(pq)}$

(ii) Find the sum of $2n$ terms of a series of which every even term is a times the term before it, and every odd term c times the term before it the first term being unity

19 The rth sth and tth terms of a certain G P are R, S and T respectively Prove that $R^{s-t}\ S^{t-r}\ T^{r-s}=1$ (Roorkee 59)

20 (i) If S be the sum, P the product and R the sum of the reciprocals of n terms of a G P prove that

$$\left(\frac{S}{R}\right)^n = P^2 \qquad \text{(Roorkee 81 57, I I T 67)}$$

(ii) If $A=1+r^a+r^{2a}+$ to ∞ and

(ii) $S_1+S_3+S_5+\dots+S_{2n-1}$ i e n terms

$$=a\,\frac{a}{1-r}-\frac{a}{1-r}\left[r+r^3+r^5+\dots+r^{2n-1}\right]$$

$$=n\,\frac{a}{1-r}-\frac{a}{1-r}\cdot\frac{r\,(1-r^{2n})}{1-r^2}$$

$$=\frac{na}{1-r}-\frac{ar}{(1-r)^2\,(1+r)}\,(1-r^{2n})$$

25 Since a, b, c, d are in G P

$\therefore$ $b=ar,\ c=ar^2,\ d=ar^3$

L H S $=a^2\,(1+r^2+r^4)\,a^2\,(r^2+r^4+r^6)$

$=a^4\,r^2\,(1+r^2+r^4)^2$

R H S $=[a^2\,(r+r^3+r^5)]^2=a^4\,r^2\,(1+r^2+r^4)^2=$ L H S

26 Just as in Q 25

L H S $=\dfrac{a^2\,[1+r+r^2]}{a^2\,[r^3+r^2+r]}=\dfrac{1}{r}$

R H S $=\dfrac{a\,(r+1)}{a\,(r^2+r)}=\dfrac{1}{r}=$ L H S

27 Let the three numbers in G P be ar, a, $\frac{a}{r}$

$$\text{Sum}=a\left(r+1+\frac{1}{r}\right)=65$$

$$a\,(r^2+r+1)=65r \qquad (1)$$

Product $ar\,a\,\dfrac{a}{r}=3375$ or $a^3=(15)^3$ $\qquad a=15$

Putting for a in (1) we get $15\,(r^2+r+1)=65r$

or $3r^2+3r+3=13r$ or $3r^2-10r+3=0$

$(r-3)\,(3r-1)=0$ $\qquad r=3,$ or $1/3$

Hence the numbers are 45, 15, 5 or 5 15 45

28 (a) Here as above $a=5$

Also $a^2\left(r\,1+1\,\dfrac{1}{r}+r\cdot\dfrac{1}{r}\right)=\dfrac{175}{2}$

or $25\,(r^2+r+1)=\dfrac{175}{2}\,r$ or $2\,(r^2+r+1)=7r$

or $2r^2-5r+2=0$ or $(r-2)\,(2r-1)=0$

$\therefore$ $r=2,\ \frac{1}{2}$ Hence the numbers are

10, 5, 5/2 or 5/2, 5, 10

(b) 18, 6, 2 or 2, 6, 18

$B = 1 + r^b + r^{2b} + \ldots$ to ∞, prove that

$$r = \left(\frac{A-1}{A}\right)^{1/a} = \left(\frac{B-1}{B}\right)^{1/b}$$

21 (a) If x, y, z be respectively the pth, qth and rth terms of a G P, then prove that

$$(q-r)\log x + (r-p)\log y + (p-q)\log z = 0$$

(b) If the pth, qth, rth terms of an A. P are in G P, show that common ratio of the G P is $\frac{q-r}{p-q}$

22 If S_1, S_2, S_3 be respectively the sums of n, $2n$, $3n$ terms of a G P then prove that

(i) $S_1(S_3 - S_2) = (S_2 - S_1)^2$, (ii) $S_1^2 + S_2^2 = S_1(S_2 + S_3)$

23 (i) If $S_1, S_2 \ldots S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3 … n and common ratios are

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4} \ldots \frac{1}{n+1}$ respectively then prove that

$$S_1 + S_2 + S_3 + \ldots + S_n = \tfrac{1}{2}n(n+3)$$

(ii) If S_p denotes the sum of the series $1 + r^p + r^{2p} + \ldots$, to ∞ and s_p the sum of the series $1 - r^p + r^{2p} - \ldots$ to ∞ prove the $S_p + s_p = 2S_{2p}$

24 If S_n represents the sum to n terms of a G P whose first term and common ratio are a and r respectively, then prove that

(i) $S_1 + S_2 + S_3 + \ldots + S_n = \frac{na}{1-r} - \frac{ar(1-r^n)}{(1-r)^2}$

(ii) $S_1 + S_3 + S_5 + \ldots + S_{2n-1} = \frac{an}{1-r} - \frac{ar(1-r^{2n})}{(1-r)^2(1+r)}$

25 If a, b, c, d, be in G P, prove that

$$(a^2 + ac + c^2)(b^2 + bd + d^2) = (ab + bc + cd)^2$$

26 If a, b, c be in G P then prove that

$$\frac{a^2 + ab + b^2}{bc + ca + ab} = \frac{b+a}{c+b}$$

27 Find three numbers in G P whose sum is 65 and their product is 3375

28 (a) The product of three numbers in G P is 125 and sum of their products taken in pairs is $87\frac{1}{2}$ Find them

(b) If the continued product of three numbers in G P is 216 and the sum of the products taken in pairs is 156 find the numbers (M N R, 78)

29 Let the numbers be a, ar, ar^2

$$\text{Sum} = a(1+r+r^2) = 70 \quad (1)$$

Also $4a, 5ar, 4ar^2$ are in A P by given condition

$$2(5ar) = 4a + 4ar^2 \quad \text{or} \quad \tfrac{5}{2}r = 1 + r^2 \quad (2)$$

We have to solve (1) and (2) for a and r

From (2), $\frac{5}{2}r + r = 1 + r + r^2$ or $\frac{7}{2}r = 1 + r + r^2$

Putting in (1) we get $a\,\frac{7}{2}r = 70$

$$ar = 20 \quad \text{or} \quad a = \frac{20}{r}$$

Hence from (1) on putting for a we get

$$\frac{20}{r}(1+r+r^2) = 70 \quad \text{or} \quad 2+2r+2r^2 = 7r$$

or $2r^2 - 5r + 2 = 0$ or $(r-2)(2r-1) = 0$

$r = 2, \frac{1}{2}$

$a = 10, 40$

Hence the numbers are 10, 20, 40 or 40, 20, 10

30 Proceeding as above $a(1+r+r^2) = 14$ (1)

and $2(ar+1) = (a+1) + (ar^2 - 1)$

or $a(r^2 - 2r + 1) = 2$ (2)

Divide (1) and (2) and simplifying

$2r^2 - 5r + 2 = 0$ $r = 2, \frac{1}{2}$ and hence $a = 2, 8$

Numbers are 2, 4, 8, or 8, 4, 2

31 The three numbers in A P are $a-d, a, a+d$

Sum $= 3a = 15$ $a = 5$

The numbers are $5-d, 5, 5+d$

Adding 1, 4, 19 respectively to these they become

$6-d, 9, 24+d$ and these are given to be in G P

$9^2 = (6-d)(24+d)$ or $d^2 + 18d - 63 = 0$

$(d+21)(d-3) = 0$ $d = -21$ or 3

Hence the numbers are 26, 5, −16 or 2, 5, 8

32 The last three of the four numbers are in A P and hence they may be chosen as $a-d, a, a+d$

Also the first number is same as the last one i e $a+d$

Therefore the four numbers are $a+d, a-d, a, a+d$

The first three of the above four are in G P

$(a-d) = a(a+d)$ But $d = 6$ given

$(a-6)^2 = a(a+6)$

or $a^2 - 12a + 36 = a^2 + 6a$ or $18a = 36$ $a = 2$

29 Three numbers are in G P whose sum is 70. If the extremes be each multiplied by 4 and the means by 5, they will be in A P Find the numbers

30 The sum of three numbers in G P is 14 If the first two terms are each increased by 1 and the third term decreased by 1, the resulting numbers are in A P Find the numbers (Roorkee 73)

31 Three numbers whose sum is 15 are in A. P If 1, 4, 19 be added to them respectively, then they are in G P Find the numbers

32 In a set of four numbers the first three are in G P and the last three in A P with common difference 6 If the first number is the same as the fourth, find the four numbers (I I T 74)

33 Find four numbers in G P whose sum is 85 and product is 4096

34 Insert five geometric means between 486 and 2/3

35 If A and G be the A M and G M between two numbers prove that the numbers are

$$A \pm \sqrt{[(A+G)(A-G)]}$$

36 Construct a quadratic in x such that A M of its roots is A and G M is G (I I T 68)

37 If one G M G and two arithmetic means p and q be inserted between any two given numbers then show that

$$G^2=(2p-q)(2q-p)$$

38 If one A M A and two geometric means p and q be inserted between any two given numbers then show that

$$p^3+q^3=Apq$$

39 If n geometric means be inserted between a and b then prove that their product is $(ab)^{n/2}$

40 For what value of n, $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the geometric mean of a and b

41 (a) Does there exist a geometric progression containing 27, 8 and 12 as three of its terms ? If it exists, how many progressions are possible ? (I.I.T

(b) Show that the numbers 10, 11, 12 cannot be the terms of a single G P with common ratio not equal to 1

Putting for a and d the four numbers are
8, −4, 2, 8, which satisfy the given conditions

33 Let the four numbers in G P be

$$\frac{a}{r^3}, \frac{a}{r}, ar, ar^3,$$

Product$=a^4=4096=8\times512=8\times8\times64=8^4$ $\qquad a=8$

$$\text{Sum}=8\left(\frac{1}{r^3}+\frac{1}{r}+r+r^3\right)=85$$

$$8\left(r^3+\frac{1}{r^3}\right)+8\left(r+\frac{1}{r}\right)-85=0 \qquad (1)$$

Let $r+\frac{1}{r}=z \qquad \left(r+\frac{1}{r}\right)^3=z^3$

or $\quad r^3+\frac{1}{r^3}+3r\,\frac{1}{r}\left(r+\frac{1}{r}\right)=z^3$

$$r^3+\frac{1}{r^3}=z^3-3z$$

Hence (1) becomes $\quad 8(z^3-3z)+8z-85=0$

or $\qquad 8z^3-16z-85=0 \qquad (2)$

By inspection we find that $z=\frac{5}{2}$ satisfies it, or $2z-5$ is a factor of (2) Hence (2) can be written as

$$(2z-5)(4z^2+10z+17)=0$$

The second factor has its discriminant
$B^2-4AC=100-4\ 4\ 17=-$ive so that its roots are imaginary

Now $z=\frac{5}{2}$ gives $r+\frac{1}{r}=\frac{1}{2}$ or $2r^2-5r+2=0$

$(r-2)(2r-1)=0 \qquad r=2, \frac{1}{2}$, and $a=8$

Hence the four numbers are
1, 4, 16 64 or 64, 16 4 1

34 486, x_1, x , x_3, x_4, x_5, $\frac{2}{3}$ will be a G P of 7 terms

$\therefore \frac{2}{3}=T_7=ar^6=(486)\,r^6$

$$r^6=\frac{1}{729}=\frac{1}{3^6} \qquad r=\pm\frac{1}{3}$$

$x_1=T_2=ar=486\ (\pm\frac{1}{3})=\pm162$

$x_2=T_3=ar^2=486\ (\frac{1}{9})=54$, $x_3=\pm18$, $x_4=6$, $x_5=\pm2$.

Hence the five geometric means are
162 54, 18, 6 2 or −162, 54, −18, 6, −2

The third term of a G P is 4 The product of first five terms is

(i) 4^3, (ii) 4^5 (iii) 4^4, (iv) none of these

(I I T 82)

A, G P consists of an even number of terms The sum of all the terms is three times that of the odd terms Determine the common ratio of the G P

Find the sum of the terms of an infinitely decreasing G P in which all the terms are positive, the first terms is 4, and the difference between the third and fifth term is equal to 32/81

(a) In a G P the first, third and fifth terms may be considered as the first, fourth and sixteenth terms of an A P Determine the fourth term of the A. P, knowing that its first term is 5

(b) The sum of first ten terms of an A P is equal to 155, and the sum of the first two terms of a G P is 9, find these progressions if the first term of A P is equal to common ratio of G P and the first term of G P is equal to common difference of A P (Roorkee 87)

Find the three numbers constituting a G P if it is known that the sum of the numbers is equal to 26 and that when 1, 6 and 3 are added to them respectively, the new numbers are obtained which form an A P

Three numbers form a G P If the 3rd term is decreased by 64 then the three numbers thus obtained will constitute an A P If the second term of this A P is decreased by 8, a G P will be formed again Determine the numbers

The length of a side of a square is a meters A second square is formed by joining the middle points of this square Then a third square is formed by joining the middle points of the second square and so on The process is carried on adinfinitum Find the sum of the areas of the squares

If a b, c are all real numbers, and

$$(a^2+b^2)x^2-2b\,(a+c)\,x+(b^2+c^2)=0,$$

then a b, c are in G P and x is their common ratio

35 Let the numbers be a and b

then $A = \text{A M} = \dfrac{a+b}{2}$ or $a+b=2A$ (1)

$G = \text{G M} = \sqrt{(ab)}$ ∴ $ab = G^2$ (2)

From (1) and (2) we find that a and b are the roots of

$$t^2-2At+G^2=0$$

$$t=\frac{2A\pm\sqrt{(4A^2-4G^2)}}{2}=A\pm\sqrt{\{(A-G)(A+G)\}}$$

36 It is same as Q 35 $t^2-2At+G^2=0$

37 Let G be the geometric mean of a and b then $G^2=ab$ (1)

Let p and q be two arithmetic means between a and b then

a, p, q, b are in A P

$2p=a+q$ and $2q=p+b$

or $a=2p-q$ and $2q-p=b$

Hence from (1) $G^2=ab=(2p-q)(2q-p)$

38 Let the numbers be x and y,

A is their A M $2A=x+y$ (1)

Also p, q are two geometric means between x and y

x, p, q, y are in G P

Since x, p, q are in G P $p^2=xq$ or $p^3=xpq$

Since p, q, y are in G P $q^2=py$ or $q^3=ypq$

Adding $p^3+q^3=pq(x+y)=2Apq$ by (1)

$p^3+q^3=2Apq$

39 $a, x_1, x_2, \ldots x_n, b$ is a G P of $n+2$ terms

$b=T_{n+2}=ar^{n+1}$ or $r=\left(\dfrac{b}{a}\right)^{1/n+1}$

Now product of n means $=x_1x_2\ldots x_n$ (1)

$=(ar)(ar^2)(ar^3)\ldots(ar^n)$

$=a^n r^{1+2+3+\ldots+n}=a^n r^{n(n+1)/2}$

$=a^n\left(\dfrac{b}{a}\right)^{n/2}$ by (1) $=a^{n/2}b^{n/2}=(ab)^{n/2}$

40 $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}=\sqrt{(ab)}=a^{1/2}b^{1/2}$ by given condition

$a^{n+1}+b^{n+1}=a^n a^{1/2}b^{1/2}+b^n b^{1/2}a^{1/2}=a^{n+1/2}\sqrt{b}+b^{n+1/2}\sqrt{a}$

$a^{n+1/2}(\sqrt{a}-\sqrt{b})=b^{n+1/2}(\sqrt{a}-\sqrt{b})$

∴ $a^{n+1/2}=b^{n+1/2}$

Above is possible only if $n+\frac{1}{2}=0$ ∴ $n=-\frac{1}{2}$

41 (a) Suppose, if possible, 27, 8, 12 are respectively the p^{th}, q^{th} and r^{th} terms of the G P

$a+aR+aR^2+aR^3+\ldots$

∴ $T_p=aR^{p-1}=27$ (1)

$T_q=aR^{q-1}=8$ (2)

50 Prove the equality

$$\underbrace{(666\ \ldots\ 6)^2}_{n \text{ digits}} + \underbrace{888\ \ldots\ 8}_{n \text{ digits}} = \underbrace{444\ \ldots\ 4}_{2n \text{ digits}}$$

Solutions to Problem Set (B)

1 $T_5 = ar^4 = 81$, $T_2 = ar = 24$

Dividing, $r^3 = \frac{81}{24} = \frac{27}{8}$ $\quad r = \frac{3}{2} > 1$ and hence

$$S_8 = \frac{a(r^n - 1)}{r - 1} = \frac{16}{3/2 - 1}\,[(3/2)^8 - 1] = 32\,\frac{(3^8 - 2^8)}{2^8}$$

$$= \tfrac{1}{8}(3^4 - 2^4)(3^4 + 2^4) = \tfrac{1}{8}(65 \times 97)$$

2 $\frac{S_3}{S_6} = \frac{125}{152}$ $\quad \frac{a(r^3-1)/(r-1)}{a(r^6-1)/(r-1)} = \frac{125}{152}$

$$\frac{1}{r^3+1} = \frac{125}{152} \text{ or } 152 = 125r^3 + 125$$

or $125r^3 = 27$ $\quad r = \frac{3}{5}$

3 (i) Split into two series one of which is G P and the A P

Ans $a\frac{(a^n - 1)}{a - 1} + b\,\frac{n(n+1)}{2}$

(ii) Open the brackets and split into three series

$$(x^2 + x^4 + x^6 \ldots) + \left(\frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^6} + \ldots\right) + (2 + 2 + 2 \ldots)$$

$$= x^2\,\frac{(x^{2n} - 1)}{x^2 - 1} + \frac{1}{x^2}\,\frac{\left(\frac{1}{x^{2n}} - 1\right)}{\frac{1}{x^2} - 1} + 2n$$

$$= \frac{x^2\,(x^{2n} - 1)}{x^2 - 1} + \frac{x^{2n} - 1}{x^{2n}\,(x^2 - 1)} + 2n$$

$$= \frac{x^{2n} - 1}{x^2 - 1}\left[x^2 + \frac{1}{x^{2n}}\right] + 2n = \frac{x^{2n} - 1}{x^2 - 1}\left[\frac{x^{2n+2} + 1}{x^{2n}}\right]$$

4 (i) nth term of the series is $T_n = 1 + x + x^2 + x^3 + \ldots$ n terms

$$T_n = \frac{1\,(1 - x^n)}{1 - x}.$$

Putting $n = 1, 2, 3 \ldots n$ and adding we get

$$S_n = \frac{1}{1 - x}\,[(1 + 1 + 1 + \ldots) - (x + x^2 + x^3 + \ldots)]$$

$$= \frac{1}{1 - x}\left[n - \frac{x\,(1 - x^n)}{1 - x}\right]$$

and $T_r = aR^{r-1} = 12$ (3)

Dividing (1) by (2) and by (3), we get

$$R^{p-q} = \frac{27}{8} = (\tfrac{3}{2})^3 \quad (4)$$

and $R^{q-r} = \frac{3}{2}$ (5)

Cubing (5), $R^{3(q-r)} = (\frac{3}{2})^3$ (6)

Now multiplying (4) and (6), we get

$$R^{p-q} R^{q-3r} = (\tfrac{3}{2})^3 (\tfrac{2}{3})^3$$

$$R^{(p+2q-3r)} = 1$$

Hence $p+2q-3r=0$ (7)

Since p, q, r are positive integers, we have to find the positive integral solutions of (7) consistent with the given conditions. Since (7) involves three unknowns, we can give arbitrary positive integral values to any two of them, say q and r. For example if $q=1$, $r=2$, then $p=4$

This means that $a=8$, $aR=12$, $aR^3=27$ $R=3/2$

Hence one G P satisfying the conditions is

$$8+8\times(\tfrac{3}{2})+8\times(\tfrac{3}{2})^2+8(\tfrac{3}{2})^3+$$

It is evident that infinite number of soultions of (7) are possible satisfying the given condition. Hence there will be infinite number of G P 's satisfying the given conditions

(b) Let $T_p=10$, $T_q=11$, $T_r=12$

$AR^{p-1}=10$, $AR^{q-1}=11$, $AR^{r-1}=12$

$$R^{p-q}=\frac{10}{11},\ R^{q-r}=\frac{11}{12}$$

or $$\left(\frac{10}{11}\right)^{1/(p-q)}=\left(\frac{11}{12}\right)^{1/(q-r)}=R$$

or $$\left(\frac{10}{11}\right)^{q-r}=\left(\frac{11}{12}\right)^{p-q}$$

or $(10)^{q-r}(12)^{p-q}=(11)^{p-q+q-r}=11^{p-r}$

Now L H S of above is cleary even whereas R H S is an odd number which is not possible. Hence 10, 11, 12 cannot be the terms of a single G P

42 Ans (ii)

Let the G P be $a+ar+ar^2+ar^3+$ (1)

Then as given $T_3=ar^2=4$

$$T_1T_2T_3T_4T_5=a\ ar\ ar^2\ ar^3\ ar^4=a^5r^{10}$$

$$=(ar^2)^5=4^5 \quad \text{by (1)}$$

43 By given condition we have

$S_{2n}=3\,[T_1+T_3+T_5+ \;\; +T_{2n-1}]=3\,[a+ar^2+ar^4+ \;\; n \text{ terms}]$

$$=\frac{1}{(1-x)^2}[n(1-x)-x(1-x^n)]$$

(ii) Ans $x^2\frac{(1-x^{2n})}{1-x^2}+xy\frac{(1-x^ny^n)}{1-xy}$

5 (a) $x=\frac{1}{1-a}$, $y=\frac{1}{1-b}$ Sum of an infinite G P

$$\therefore \quad a=\frac{x-1}{x},\quad b=\frac{y-1}{y}$$

$$\therefore \quad 1+ab+a^2b^2+\dots\infty=\frac{1}{1-ab}=\frac{1}{1-\frac{(x-1)(y-1)}{xy}}$$

$$=\frac{xy}{x+y-1}$$

(b) Since here common ratio is 2 which is greater than 1 Hence S is infinite so the first step i e $2S=S-1$ is correct but conclusion from this that $S=-1$ is false since we cannot cancel S i e ∞ from both sides

6 (i) Given that $T_p=(T_{p+1}+T_{p+2}+\dots\infty)$
or $ar^{p-1}=ar^p+ar^{p+1}\dots\infty$ But $a=1$

$$r^{p-1}=\frac{r^p}{1-r}\quad \text{Sum of an infinite G P}$$

$$1-r=r \quad \text{or} \quad r=\tfrac{1}{2}$$

Hence the series is $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16} \dots \infty$

(ii) Do yourself

7 Here $a+ar=5$, $T_p=3\,(T_{p+1}+T_{p+2}+\dots\infty)$

$$ar^{p-1}=3\,\frac{ar^p}{1-r}\qquad 1-r=3r$$

or $r=\frac{1}{4}$ hence $a=4$ etc

8 Here $a=\frac{2}{9}$ and $r=-\frac{1}{3}\cdot\frac{9}{2}=-\frac{3}{2}$ Let the number of terms be n

Also $S_n=\frac{a(1-r^n)}{1-r}=\frac{55}{72}$

$$\frac{2}{9}\,\frac{[1-(-3/2)^n]}{1-(-3/2)}=\frac{55}{72}$$

or $\quad 1-(-3/2)^n=\frac{55}{72}\times\frac{9}{2}\times\frac{5}{2}=\frac{275}{32}$

$$1-\frac{275}{32}=\left(-\frac{3}{2}\right)^n \text{ or } -\frac{243}{32}=\left(-\frac{3}{2}\right)^n$$

or $\left(-\frac{3}{2}\right)^5=\left(-\frac{3}{2}\right)^n \qquad n=5$

$\frac{a}{1-r}(1-r^{2n})=3\ \frac{a}{1-r^2}[1-(r^2)^n]$ or $1=\frac{3}{1+r}$ ∴ $r=2$

44 $a=4,\ T_2-T_4=\frac{32}{81}$ or $a(r^2-r^4)=\frac{32}{81}$

or $r^4-r^2+8/81=0$ or $81r^4-81r^2+8=0$

or $(9r^2-8)(9r^2-1)=0$

$r^2=8/9,\ 1/9$. The value of r is to be +ive since all the terms are +ive Out of the two we shall choose $r=\frac{1}{3}$ as the G P is infinitely decreasing

∴ $S_\infty=\frac{a}{1-r}=\frac{4}{1-\frac{1}{3}}=\frac{4\ 3}{2}=6$

45 (a) $a=5,\ ar^2=a+3d,\ ar^4=a+15d$

∴ $5r^2=5+3d,\ 5r^4=5+15d$ or $r^4=1+3d$

$25r^4=25+75d$ or $(5+3d)^2=25+75d$

or $25+30d+9d^2=25+75d$ ∴ $9d^2-45d=0$ $d=5,\ 0$

∴ $T_4=a+3d=5+15=20=ar^2=5r^2$ $r=2,\ -2$

5, 20, 80 are the terms

(b) Let the A P be $a+(a+d)+(a+2d)+$

By given condition G P is $d+da+da^2+$

S_{10} of A.P $=155$ S_2 of G P $=9$

∴ $2a+9d=31$ and $d+da=9$

Solving we get $a=2,\ d=3$ or $a=\frac{25}{2}$, and $d=\frac{2}{3}$

∴ A.P $2+5+8+11$ G P is $3+6+12+24$

or A.P $\frac{25}{2}+\frac{79}{6}+\frac{83}{6}$ G P is $\frac{2}{3}+\frac{25}{3}+\frac{625}{6}$

46 Let $a,\ ar,\ ar^2$ be in G P where $a(1+r+r^2)=26$ (1)

Also $a+1,\ ar+6,\ ar^2+3$ are in A P

$2(ar+6)=(a+1)+(ar^2+3)$ or $a(r^2-2r+1)=8$ (2)

Dividing (1) and (2) and simplifying we get

$18(r^2+1)=60r$ or $3r^2-10r+3=0$

$(r-3)(3r-1)=0$ $r=3,\ \frac{1}{3}$ Putting in (1) $a=2,\ 18$

The numbers are 2, 6, 18 or 18 6, 2

47 $a,\ ar,\ ar^2$ are in G P

$a,\ ar,\ ar^2-64$ are in A P $2ar=a+ar^2-64$

or $a(r^2-2r+1)=64$ (1)

Again $a,\ ar-8,\ ar^2-64$ are in G P $(ar-8)^2=a(ar^2-64)$

or $-16ar+64=-64a$ or $a(16r-64)=64$ (2)

∴ $r^2-2r+1=16r-64$ by (1) and (2)

or $r^2-18r+65=0$ $(r-5)(r-13)=0$

∴ $r=5$ hence $a=4$

9 $a=1$, $r=4$, and $S_n=341$

$$\frac{a(r^n-1)}{r-1}=341 \quad \text{or} \quad \frac{1(4^n-1)}{4-1}=341$$

$$4^n=341\times3+1=1024=16\times64=4^5 \qquad n=5$$

10 $l=T_n=ar^{n-1}=128$, $r=2$

$$S_n=\frac{a(r^n-1)}{r-1}=255 \quad \text{or} \quad \frac{ar^{n-1}r-a}{r-1}=255$$

or $\frac{128\,2-a}{2-1}=255$, $\quad 256-255=a$ or $a=1$

Now $ar^{n-1}=128 \qquad 1(2)^{n-1}=2^7 \qquad n-1=7$ or $n=8$

11 $n=6$

12 $S=7+77+777+\ \ n$ terms

$$=\frac{7}{9}[9+99+999+\ \ ]$$

$$=\frac{7}{9}\left[\left(1-\frac{1}{10}\right)+\left(1-\frac{1}{100}\right)+\left(1-\frac{1}{1000}\right)+\ \ \right]$$

$$=\frac{7}{9}\left[(1+1+1+\ \ )-\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}+\ \ \right)\right]$$

$$=\frac{7}{9}\left[n-\frac{1}{10}\frac{\left(1-\frac{1}{10^n}\right)}{1-\frac{1}{10}}\right]=\frac{7n}{9}-\frac{7}{81}\left(1-\frac{1}{10^n}\right)$$

(ii) $S=6+66+666+$

$$=\tfrac{6}{9}[9+99+999+\ \ ]$$

$$=\tfrac{6}{9}[(10-1)+(10^2-1)+(10^3-1)+\ \ ]$$

$$=\tfrac{6}{9}[(10+10^2+10^3+\ \ )-(1+1+1+\ \ )]$$

$$=\tfrac{6}{9}\left[10\frac{10^n-1}{10-1}-n\right]=\frac{6}{81}[10^{n+1}-9n-10]$$

(iii) $\frac{8}{81}\left[[10^{n+1}-9n-10\right]$

13 Let $\qquad x=\ 0\,125125125$

$\qquad 1000x=\ 125\,125125125$

Subtract $999x=\ 125\,0000000=\ 125$

$\therefore\ x=\frac{125}{999}$

14 $123=123232323$

$$=\frac{1}{10}+\frac{23}{1000}+\frac{23}{100000}+$$

$$=\frac{1}{10}+23\left[\frac{1}{10^3}+\frac{1}{10^5}+\frac{1}{10^7}+\ \ \infty\right]$$

• Numbers are 4, 20, 100 G P and 4, 20, 36 A.P and 4, 12, 36 again G P

48 Sides are $a, \frac{a}{\sqrt{2}}, \frac{a}{2}, \frac{a}{2\sqrt{2}}$,

Areas are $a^2, \frac{a^2}{2}, \frac{a^2}{4}, \frac{a^2}{8}$, etc Ans $2a^2$

49 Discriminant of the equation in x will be found to be $-4(b^2-ac)^2$ which is $\leqslant 0$ But for real x, it can not be negative and so we must have $b^2-ac=0$, showing that a, b, c are in G P Under this condition the equation has only one root given by

$$x=\frac{2b(a+c)}{2(a^2+b^2)}=\frac{b(a+c)}{a^2+ac}=\frac{b}{a}, \qquad b^2=ac$$

which is the common ratio

50 Hint Represent each number as a G P e g we write 666 6 as $6+6\ 10+6\ 10^2+ \quad +6\ 10^{n-1}$

$$=\frac{6(10^n-1)}{10-1}=\frac{2}{3}(10^n-1) \text{ etc}$$

§ 3 **Arithmetico geometric series**

Definition A type of series in which each term is the product of the corresponding terms of an A P and a G P is called arithmetico geometric series For example

$$1+3x+5x^2+7x^3+$$
$$a+(a+d)r+(a+2d)r^2+(a+3d)r^3+$$

In the above series 1, 3, 5, 7 , $a, a+d, a+2d$ are in A P and $1, x, x^2, x^3$ and 1 r, r^2, r^3 are in G P

Sum of n terms of an arithmetico geometric series

Let $S=a+(a+d)r+(a+2d)r^2+ \quad +[a+(n-1)d]r^{n-1}$ (1)

Multiply both sides of (1) by common ratio r and writing as below *i e* starting the vulue of rS by writing its first term below 2nd term of S

$$rS=ar \qquad +(a+d)r^2 \qquad + \quad +[a+(n-2)d]r^{n-1} +[a+(n-1)d]r^n$$

Subtracting, we get

$$S(1-r)=a+[dr+dr^2+ \quad +dr^{n-1} \text{ i e } (n-1) \text{ terms}] -[a+(n-1)d]r^n$$

The middle bracket is a G P of $(n-1)$ terms

$$S(1-r)=a+\frac{dr(1-r^{n-1})}{1-r}-[a+(n-1)d]r^n$$

$$=\frac{1}{10}+23\ \frac{\frac{1}{10^3}}{1-\frac{1}{10^2}}=\frac{1}{10}+\frac{23}{990}\qquad S=\frac{a}{1-r}$$

$$=\frac{99+23}{990}=\frac{122}{990}=\frac{61}{495}$$

15 Proceed as above Ans $\frac{419}{990}$

16 See Q 17 below Ans $120+960=1080$

17 After striking the floor falling from a distance of 48 ft it rebounds to a height of $\frac{2}{3}$ (48) Again it falls the same distance i e $\frac{2}{3}$ (48) and after rebounding it goes to a height of $\frac{2}{3}(\frac{2}{3})(48)=(\frac{2}{3})^2\ 48$ It falls the same distance and after rebounding goes to a height of $\frac{2}{3}[(\frac{2}{3})^2\ 48]=(\frac{2}{3})^3\ 48$ and so on Hence the total distance travelled is

$$48+2[\tfrac{2}{3}(48)+(\tfrac{2}{3})^2\ 48+(\tfrac{2}{3})^3\ 48+\ \ \infty]$$

We have taken twice as it goes to particular height and then falls the same distace

$$S=48+2\ \frac{\frac{2}{3}\ 48}{1-\frac{2}{3}}\quad i\ e\quad S=\frac{a}{1-r}$$

$$=48+2\,(96)=48+192=240 \text{ ft}$$

18 (i) $T_{m+n}=ar^{m+n-1}=p$

$T_{m-n}=ar^{m-n-1}=q$ and $T_m=ar^{m-1}$

Multiplying $a^2r^{2m-2}=pq$ $\quad T_m=ar^{m-1}=\sqrt{(pq)}$

(ii) $T_{2p}=a\quad T_{2p-1},\ T_{2p-1}=c\quad T_{2p-2}$ Multiplying

$$T_{2p}T_{2p-1}=acT_{2p-1}\ T_{2p-2}\qquad \frac{T_{2p}}{T_{2p-2}}=ac\quad \text{or}\quad r^2=ac$$

$$S_{2n}=\frac{A\,(r^{2n}-1)}{r-1}=1\ \frac{(ac)^n-1}{(\sqrt{(ac)}-1)}=\frac{\sqrt{((ac)+1)}\,(a^nc^n-1)}{(ac-1)}$$

19 Let the common ratio be taken as k and a be the first term

$R=T_r=ak^{r-1}$

$R^{s-t}=a^{s-t}\ k^{(r-1)\ (s-t)}$

$S^{t-r}=a^{t-r}\ k^{(s-1)\ (t-r)}$ Similarly

$T^{r-s}=a^{r-s}\ k^{(t-1)\ (r-s)}$

Multiplying the above three and knowing that

$A^m\ A^n\ A^p=A^{m+n+p}$

$R^{s-t}\ S^{t-r}\ T^{r-s}=a^0\ k^0=1$ **Proved**

20 (i) $S=a+ar+ar^2+ar^3+\ \ +ar^{n-1}$ i e n terms

$$S=\frac{a\,(1-r^n)}{1-r}\qquad (1)$$

$P=$Product$=a\ ar\ ar^2\ \ ar^{n-1}$

$$S=\frac{a}{1-r}+\frac{dr(1-r^{n-1})}{(1-r)^2}-\frac{[a+(n-1)d]r^n}{1-r}$$

Sum upto infinity

If $|r|<1$ and $n\to\infty$, then $\lim_{n\to\infty} r^n=0$

$$S=\frac{a}{1-r}+\frac{dr}{1-r}$$

§ 4 Natural Numbers

$$\Sigma n=1+2+3+\ldots n=\frac{n(n+1)}{2}$$

$$\Sigma n^2=1^2+2^2+3^2+\ldots n^2=\frac{n(n+1)(2n+1)}{6}$$

$$\Sigma n^3=1^3+2^3+3^3+\ldots n^3=\left[\frac{n(n+1)}{2}\right]^2$$

$$\Sigma a=a+a+a+a+\ldots+n \text{ terms}=na$$

Application

If the nth term of a series be given as

$$T_n=an^2+bn+c$$

In order to find the sum of n terms we put $n=1, 2, 3, \ldots n$ and then add them

$T_1=a\cdot 1^2+b\cdot 1+c$

$T_2=a\cdot 2^2+b\cdot 2+c$

$T_3=a\cdot 3^2+b\cdot 3+c$ and so on

$S_n=a(1^2+2^2+3^2+\ldots n^2)+b(1+2+3+\ldots n)$
$\quad+(c+c+c+\ldots n \text{ terms})$

$=a\,\Sigma n^2+b\,\Sigma n+c\cdot n$

Now putting the values of Σn^2 and Σn in the above we shall have the value of S_n

§ 5 Method of difference

Consider the series

$3+7+14+24+37+\ldots n$ terms

Here the difference between the successive terms are

$7-3,\ 14-7,\ 24-14,\ 37-24$

i e $\quad 4,\quad 7,\quad 10,\quad 13$ which are in A P

The above differences could be in G P also

In order to find its sum we follow the method given below

Let $S=3+7+14+24+37+\ldots+T_n$

$S=\quad 3+7+14+24+\ldots+T_{n-1}+T_n$

Subtracting, we get

$0=3+[4+7+10+13+\ldots(n-1) \text{ terms}]-T_n$

$$(\sin\theta-1)^2+\cos\theta(\sin\theta-1)=0$$
$$(\sin\theta-1)(\sin\theta-1+\cos\theta)=0$$
$$\sin\theta=1=\sin\frac{\pi}{2}\qquad \theta=n\pi+(-1)^n\frac{\pi}{2}$$

$\cos\theta+\sin\theta=1$ Divide by $\sqrt{(1+1)}=\sqrt{2}$

$$\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\frac{1}{\sqrt{2}} \text{ or } \cos\theta\cos\frac{\pi}{4}+\sin\theta\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$$

or $\cos\left(\theta-\frac{\pi}{4}\right)=\cos\frac{\pi}{4}\qquad \theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$

$$\theta=2n\pi \text{ or } 2n\pi+\frac{\pi}{4}+\frac{\pi}{4}=2n\pi+\frac{\pi}{2}$$

We could also write it as $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$

$$\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4}\qquad \theta=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$$

(d_1) We have already done it in part (d)

(e) $\tan\theta+\sec\theta=\sqrt{3}$ or $\frac{\sin\theta}{\cos\theta}+\frac{1}{\cos\theta}=\sqrt{3}$

or $\sqrt{3}\cos\theta-\sin\theta=1$ [$\cos\theta\neq0$) Divide by $\sqrt{(3+1)}=2$

$$\frac{\sqrt{3}}{2}\cos\theta-\tfrac{1}{2}\sin\theta=\tfrac{1}{2} \quad\text{or}\quad \cos\theta\cos\frac{\pi}{6}-\sin\theta\sin\frac{\pi}{6}=\tfrac{1}{2}$$

$$\cos\left(\theta+\frac{\pi}{6}\right)=\cos\frac{\pi}{3}$$

$$\theta+\frac{\pi}{6}=2n\pi\pm\frac{\pi}{3},$$

$$\theta=2n\pi+\frac{\pi}{6} \text{ or } 2n\pi-\frac{\pi}{2}$$

If $0\leqslant\theta\leqslant2\pi$ then

$n=0$, $\theta=\frac{\pi}{6}$, $\frac{-\pi}{2}$ The second value is rejected

$n=1$ $\theta=2\pi-\frac{\pi}{2}=\frac{3\pi}{2}$ but $\cos\frac{3\pi}{2}=0$ and as $\cos\theta\neq0$

hence rejected Therefore the only solution within given range is $\theta=30°=\pi/6$

Note You could also do this question as in Q 4 (*c d*) Page 83

(e_1) Proceed as above Ans $\theta=2n\pi+\pi/3$

(f) $(2-\sqrt{3})\cos\theta=1-\sin\theta$

Here $(2+\sqrt{3})\cos\theta-1=-\sin\theta$ Square

$(4+3+4\sqrt{3})\cos^2\theta+1-2(2+\sqrt{3})\cos\theta=1-\cos^2\theta$

$$(8+4\sqrt{3})\cos^2\theta-2(2+\sqrt{3})\cos\theta=0$$

$$(4+2\sqrt{3})\cos\theta\,(2\cos\theta-1)=0$$

$$\cos\theta=0 \qquad \theta=n\pi+\pi/2$$

$$\cos\theta=\tfrac{1}{2}=\cos\frac{\pi}{3} \qquad \theta=2m\pi\pm\frac{\pi}{3} \qquad \text{(I)}$$

But $\theta=n\pi+\pi/2$ does not satisfy the given equation when n is odd. Also $\theta=2m\pi+\pi/3$ does not satisfy it. Hence the required solution is $\theta=2m\pi-\pi/3$ and $\theta=2m\pi+\pi/2$, $m,n\in I$

(g) $\tan\left(\frac{\pi}{2}\sin\theta\right)=\cot\left(\frac{\pi}{2}\cos\theta\right)$

$$\tan\left(\frac{\pi}{2}\sin\theta\right)=\tan\left(\frac{\pi}{2}-\frac{\pi}{2}\cos\theta\right)$$

$$\frac{\pi}{2}\sin\theta=n\pi+\frac{\pi}{2}-\frac{\pi}{2}\cos\theta$$

$$\sin\theta+\cos\theta=2n+1$$

$$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{2n+1}{\sqrt{2}} \qquad \text{Divide by } \sqrt{(1+1)}$$

$$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{2n+1}{\sqrt{2}}=\cos\alpha \text{ say}$$

$$\theta-\frac{\pi}{4}=2r\pi\pm\alpha \qquad \theta=2r\pi+\frac{\pi}{4}\pm\cos^{-1}\frac{2n+1}{\sqrt{2}},$$

where $n=0$ or -1 and $r\in I$

$$\theta=2r\pi+\pi/4\pm\pi/4 \quad r\in I$$

We reject $+$sign since $\theta=2r\pi+\pi/2$ does not satisfy the equation. Hence the only solutions is $\theta=2r\pi$

(h) Proceeding as in part (g) we shall get

$$\cos\left(\theta-\frac{\pi}{4}\right)=\frac{2n+1}{2\sqrt{2}} \qquad n=0 \text{ or } -1$$

Hence $\left(\cos\theta-\frac{\pi}{4}\right)=\pm\frac{1}{2\sqrt{2}}$

9 (a) $\tan\theta+\tan 2\theta+\tan(\theta+2\theta)=0$

$$(\tan\theta+\tan 2\theta)+\frac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=0$$

$$(\tan\theta+\tan 2\theta)(1-\tan\theta\tan 2\theta+1)=0$$

$$\tan 2\theta+\tan\theta=0 \qquad \tan 2\theta=-\tan\theta=\tan(-\theta)$$

$$2\theta=n\pi-\theta \text{ or } 3\theta=n\pi \qquad \theta=n\pi/3$$

$$\tan\theta\tan 2\theta=2 \text{ or } \tan\theta\,\frac{2\tan\theta}{1-\tan^2\theta}=2$$

or $\tan^2\theta=1-\tan^2\theta$ or $2\tan^2\theta=1$

$T_n = 3 + S_{n-1}$ of an A.P whose $a=4$ and $d=3$

$$T_n = 3 + \frac{n-1}{2}\,[2\ 4 + (n-2)\ 3]$$

$$= \frac{6+(n-1)\,(3n+2)}{4}$$

or $T_n = \frac{1}{2}\,(3n^2 - n + 4)$

Now putting $n = 1, 2, 3 \quad n$ and adding

$$S_n = \frac{1}{2}\,[3\Sigma\, n^2 - \Sigma\, n + 4n]$$

$$= \frac{1}{2}\left[\ 3\ \frac{n\,(n+1)\,(2n+1)}{6} - \frac{n\,(n+1)}{2} + 4n\ \right]$$

$$= \frac{n}{2}\ (n^2 + n + 4)$$

Problem Set (C)

Sum the series

1 $1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} +$ to n terms and to ∞

2 $1 + \frac{1}{5} + \frac{3}{5^2} + \frac{5}{5^3} +$ to ∞

3 (a) $1 + \frac{3}{2} + \frac{5}{4} + \frac{7}{8} +$ n terms

(b) $1 + 2\ 2 + 3\ 2^2 + 4\ 2^3 + \quad + 100\ 2^{99}$ (Roorkee 82)

4 $1 + (1+2) + (1+2+3) + (1+2+3+4) +$ to n terms

5 $1^2 + (1^2+2^2) + (1^2+2^2+3^2) +$ to n terms

6 (a) $1\ 2^2 + 2\ 3^2 + 3\ 4^2 +$ to n terms (Roorkee 79)

(b) $1\ 2\ 3 + 2\ 3\ 4 + 3\ 4\ 5 +$ to n terms (I I T 73)

7 $1\ 3^2 + 2\ 5^2 + 3\ 7^2 +$ to 20 terms

8 $\frac{1^3}{1} + \frac{1^3+2^3}{1+3} + \frac{1^3+2^3+3^3}{1+3+5} +$ to 16 terms (I I T 76)

9 $1+3+7+15+31+$ to n terms

10 $1+3+6+10+15+$ to n terms

11 $4+6+9+13+18+$ to n terms

12 Find the sum of all possible products of the first n natural numbers taken two by two

Another Form

Find the sum of the products of the integers 1, 2, 3, …, n taken two at a time, and hence show that it is equal to half the excess of the cubes of the given integers over the sum of their squares be equal, then prove that a, b, c are in H P

26 If the mth term of an H P is n and nth term be m then prove that $(m+n)$th term is $\frac{mn}{m+n}$.

27 The 7th term of H P is 1/10 and 12th term is 1/25, find the 20th term

28 If $\frac{1}{a(b+c)}, \frac{1}{b(c+a)}, \frac{1}{c(a+b)}$ be in H P then a, b, c are also in H P

29 If $b+c$, $c+a$, $a+b$ are in H P then prove that

$$\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b} \text{ are in A P}$$

30 If a, b, c be in H P prove that

(i) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$ a e in H P (Roorkee 80)

(ii) $\frac{a}{b+c-a}, \frac{b}{c+a-b}, \frac{c}{a+b-c}$ are in H P

(iii) $\frac{1}{a}+\frac{1}{b+c}, \frac{1}{b}+\frac{1}{c+a}, \frac{1}{c}+\frac{1}{a+b}$ are in H P

31 If a, b, c are in A P prove that

$$\frac{bc}{ca+ab}, \frac{ca}{bc+ab}, \frac{ab}{bc+ac} \text{ are in H P}$$

32 If $b+c$, $c+a$, $a+b$ are in H P then prove that

$$a^2, b^2, c^2 \text{ are in A P}$$

33 If a be A.M of b and c, b, the G M of c and a then prove that c is the H M of a and b

34 First three of the four numbers are in A P the last three in H P prove that the four numbers are proportional

35 If a, b, c be in A P b, c, a be in H P then prove that c, a, b are in G P

36 If x, y, z are in A P ax, by, cz in G P and a, b, c in H P prove that $\frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$

37 (a) (i) If $a^x=b^y=c^z$ and a, b, c be in G P then prove that x, y, z are in H P

(ii) If $a^{1/x}=b^{1/y}=c^{1/z}$ and a, b, c be in G P then prove that x, y z are in A P (IIT 70)

(iii) If a b, c be in G P then prove that $\log_a n$, $\log_b n$, $\log_c n$ are in H P nother form

Find the coefficient of x^{n-2} in the polynomial

$(x-1)(x-2)(x-3) \quad (x-n)$

2 (b) Coefficient of x^{99} in the polynomial

$(x-1)(x-2)(x-3) \quad (x-100)$ is (I I.T 82)

3 Sum to n terms the series

$1^2-2^2+3^2-4^2+5^2-6^2+$

4 Find the sum of the series

$31^2+32^2+ \quad +50^2$

5 If s and t are respectively the sum and the sum of the squares of n successive positive integers beginning with a then show that $nt-s^2$ is independent of a

6 Balls are arranged in rows to form an equilateral triangle The first row consists of one ball, the second row of two balls and so on If 669 more balls are added then all the balls can be arranged in the shape of a square and each of the sides then contains 8 balls less than each side of the triangle did Determine the initial number of balls (Roorkee 85)

7 Sum the series

$n \cdot 1+(n-1) \cdot 2+(n-2) \cdot 3+ \quad 1 \cdot n$

18 On the ground are placed n stones the distance between the first and second is one yard, between the 2nd and 3rd is 3 yds, between the 3rd and 4th, 5 yds and so on How far will a person have to travel who shall bring them one by one to a basket placed at the first stone

19 Find the sum of the infinite series

$1+(1+a)r+(1+a+a^2)r^2+(1+a+a^2+a^3)r^3+ \quad$, r and a being proper fractions

20 Find the sum of n terms of the series the rth term of which is $(2r+1)2^r$

Solutions to Problem Set (C)

1 $1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+ \quad +T_n$

Above series is known as arithmetico geometric series as

1, 4, 7, 10 are in A.P and $1, \frac{1}{5}, \frac{1}{5^2}, \frac{1}{5^3}$ are in G P

nth term of A. P $=a+(n-1)d=1+(n-1)3=3n-2$

nth term of G P $=ar^{n-1}=1 \cdot \frac{1}{5^{n-1}}$

(iv) If a, b, c be in G P then prove that
$\log a^n, \log b^n, \log c^n$ are in A P

(b) Given $a^x = b^y = c^z = d^u$ and a, b, c, d are in G P, show that x, y, z, u are in H P (Roorkee 84)

38 If the mth, nth and pth terms of an A P and G P be equal and be respectively x, y and z, then prove that
$$x^{y-z}, y^{z-x} \; z^{x-y} = 1$$ (I I T 79)

39 (i) If a, b, c, d, e be five numbers such that $a, b\ c$ are in A P b, c, d are in G P and c, d, e are in H P Prove that

(i) a, c, e are in G P and

(ii) $e = \dfrac{(2b-a)^2}{a}$

(iii) If $a=2$ and $e=18$, find all possible values of b, c and d (I I T 76)

40 If a, b, c are in H P, b, c, d are in G P and $c, d\ e$ are in A P show that $e = \dfrac{ab^2}{(2a-b)^2}$

41 If a, b, c are in G P and x, y respectively be arithmetic means between a, b and b, c then prove that
$$\frac{a}{x} + \frac{c}{y} = 2 \quad \text{and} \quad \frac{1}{x} + \frac{1}{y} = \frac{2}{b}$$

42 If $\dfrac{a-x}{px} = \dfrac{a-y}{qy} = \dfrac{a-z}{rz}$ and p, q, r be in A P then prove that x, y, z are in H P

43 If a, b, c be in A P and a^2, b^2, c^2 in H P then prove that either $-a/2\ b, c$ are in G P or $a=b=c$ (I I T 77)

44 p, q, r are three numbers in G P Prove that the first term of an A P whose pth, qth and rth terms are in H P is to the common difference as $q+1\ \ 1$

45 A G P and H P have the same pth, qth and rth terms as a, b, c respectively Show that
$$a(b-c)\log a + b(c\ \ a)\log b + c(a-b)\log c = 0$$

46 An A P, a G P and H P have a and b for their first two terms Show that their $(n+2)$th terms will be in G P if
$$\frac{b^{2n+2} - a^{2n+2}}{ab\,(b^{2n} - a^{2n})} = \frac{n+1}{n}$$

47 An A P and a H P, have the same first term, the same last term, and the same number of terms prove that the product of the rth term from the beginning in one series and the rth term from the end in the other is independent of r

Let S be the sum of the series

$$S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\quad+\frac{(3n-2)}{5^{n-1}}$$

Multiply both sides by $r=\frac{1}{5}$

$$\frac{1}{5}S=\quad\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\quad+\frac{3n-5}{5^{n-1}}+\frac{3n-2}{5^n}$$

Subtract

$$\left(1-\frac{1}{5}\right)S=1+3\left[\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\quad(n-1)\text{ terms}\right]-\frac{3n-2}{5^n}$$

or $\frac{4}{5}S=1+3\quad\frac{1}{5}\frac{\left[1-\left(\frac{1}{5}\right)^{n-1}\right]}{1-\frac{1}{5}}-\frac{3n-2}{5^n}$

$$=1+\frac{3}{4}-\frac{3}{4}\frac{1}{5^{n-1}}-\frac{3n-2}{5\,5^{n-1}}$$

or $\quad\frac{4}{5}S=\frac{7}{4}-\frac{1}{5^{n-1}}\left[\frac{3}{4}+\frac{3n-2}{5}\right]$

or $\quad\frac{4}{5}S=\frac{7}{4}-\frac{1}{5^{n-1}}\quad\frac{7+12n}{20}$

$$S=\frac{35}{16}-\frac{12n+7}{16\,5^{n-1}}$$

To find the sum to infinity, we proceed as below

Let $\quad S=1+\frac{4}{5}+\frac{7}{5^2}+\frac{10}{5^3}+\quad\infty$

Then $\frac{1}{5}S=\frac{1}{5}+\frac{4}{5^2}+\frac{7}{5^3}+\quad\infty$

$$S\left(1-\frac{1}{5}\right)=1+3\left[\frac{1}{5}+\frac{1}{5^2}+\frac{1}{5^3}+\quad\infty\right]$$

or $\quad\frac{4}{5}S=1+3\quad\frac{\frac{1}{5}}{1-\frac{1}{5}}=1+\frac{3}{4}=\frac{7}{4}$

$\therefore S=\frac{35}{16}$

2 The given series is $1+S$ where S is an infinite arithmetic geometric series whose sum as in Ex 1 is $\frac{3}{8}$

Required sum$=1+\frac{3}{8}=\frac{11}{8}$

3 Proceed as in Ex 1

(ii) Do yourself

6 $\frac{a-b}{b-c}=\frac{a}{a}$ gives $b=\frac{a+c}{2}$ $\therefore$ in A.P

$\frac{a-b}{b-c}=\frac{a}{b}$ gives $b^2=ac$ $\therefore$ in G.P.

$\frac{a-b}{b-c}=\frac{a}{c}$ gives $b=\frac{2ac}{a+c}$ $\therefore$ in H.P.

7 $A=\frac{a+b}{2}$, $G=\sqrt{(ab)}$, $H=\frac{2ab}{a+b}$

$\therefore$ $AH=ab=G^2$ $\therefore$ A, G, H are in G.P

$A-G=\frac{a+b}{2}-\sqrt{(ab)}=\frac{a+b-2\sqrt{(ab)}}{2}=\frac{1}{2}(\sqrt{}$

$\therefore$ $A > G$ or $\frac{A}{G} > 1$

Also from above $AH=G^2$ $\therefore$ $\frac{A}{G}=\frac{G}{H}>1$

$\therefore$ $\frac{G}{H} > 1$ or $G > H$

Hence from (1) and (2), $A > G > H$

8. $A=27$ and $H=12$

But we know that A, G, H are in G.P $\therefore$ $G=3\times$

$\therefore$ $G^2=AH=27\times12=9\times36$ $\therefore$

9 $A=\frac{a+b}{2}$ or $a+b=2A$ $G=\sqrt{(ab)}$

or $H=\frac{2ab}{a+b}=4$, $\therefore$ $G^2=AH$ give

Also $2A+G^2=27$ or $2A+4A=27$

$\therefore$ $\frac{a+b}{2}=\frac{9}{2}$ or $a+b=9$

Also $G^2=4A=4.\frac{9}{2}=18$ or $ab=18$

(2) we conclude that a b

or $(x-6)(x-3)=0$

are 6 and 3.

H.P.

A.M.

$$S=6-\frac{2n+3}{2^{n-1}}$$

(b) $2^n(n-1)+1=99\ 2^{100}+1$ as $n=100$

4 $T_n=(1+2+3+\ \ n)=\frac{n(n+1)}{2}=\frac{1}{2}(n^2+n)$

Putting $n=1, 2, 3\ \ n$ and adding we get

$$S_n=\tfrac{1}{2}(\Sigma n^2+\Sigma n)$$

$$=\tfrac{1}{2}\left[\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$=\tfrac{1}{4}n(n+1)\left[\frac{2n+1}{3}+1\right]=\frac{n(n+1)(n+2)}{6}$$

5 $T_n=1^2+2^2+3^2+\ \ +n^2=\frac{n(n+1)(2n+1)}{6}$

$=\frac{1}{6}[2n^3+3n^2+n]$

Putting $n=1, 2, 3\ \ n$ and adding

$S_n=\frac{1}{6}[2\Sigma n^3+3\Sigma n^2+\Sigma n]$

$$=\frac{1}{6}\left[2\left(\frac{n(n+1)}{2}\right)^2+3\,\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}\right]$$

$$\frac{1}{12}n(n+1)[n(n+1)+(2n+1)+1]$$

$$=\frac{1}{12}n(n+1)[n^2+3n+2]=\frac{1}{12}n(n+1)(n+1)(n+2)$$

$$=\frac{1}{12}n(n+1)^2(n+2)$$

6 (a) Here $T_n=n(n+1)^2=n^3+2n^2+n$

Proceeding as in Q 5 $S_n=\frac{1}{12}n(n+1)(n+2)(3n+5)$

(b) Ans $\frac{1}{4}n(n+1)(n+2)(n+3)$,

7 T_n of A P 1 2, 3 is n

T_n of A P 3, 5, 7 is $3+(n-1)\,2=2n+1$

T_n of given series is $n(2n+1)^2$

or $T_n=4n^3+4n^2+n$

Putting $n=1, 2, 3\ \ n$, and add

$$S_n=4\Sigma n^3+4\Sigma n^2+\Sigma n$$

$$S_n=4\left[\frac{n(n+1)}{2}\right]^2+4\,\frac{n(n+1)(2n+1)}{6}+\frac{n(n+1)}{2}$$

$$=\tfrac{1}{6}n(n+1)(6n^2+14n+7)$$

Putting $n=20$, we get

2 Put $b=\frac{2ac}{a+c}$ as a, b, c are in H P and simplify

3 By the given condition if the numbers be a and b then

$$\frac{2ab}{a+b} \quad \sqrt{(ab)}=12 \quad 13 \quad \text{or} \quad 2\frac{\sqrt{(ab)}}{a+b}=\frac{12}{13}$$

or $26\sqrt{(ab)}=12\,(a+b)$ Divide by $2b$

$$13\sqrt{\frac{a}{b}}=6\left(\frac{a}{b}+1\right) \quad \text{or } 6x^2-13x+6=0 \quad \text{where } x=\sqrt{\frac{a}{b}}$$

$$(2x-3)\,(3x-2)=0 \qquad x=\frac{3}{2} \text{ or } \frac{2}{3}=\sqrt{\frac{a}{b}}$$

$$\frac{a}{b}=\frac{9}{4} \text{ or } \frac{4}{9}$$

Hence the numbers are in the ratio 4 , 9

4 Since x, y, z are in H P $\qquad y=\frac{2xz}{x+z}$

$$x-2y+z=x+z-\frac{4xz}{x+z}=\frac{(x+z)^2-4xz}{x+z}=\frac{(x-z)^2}{x+z}$$

$$\log(x-2y+z)=2\log(x-z)-\log(x+z)$$

or $\log(x+z)+\log(x-2y+z)=2\log(x-z)$ Proved

(*b*) Replacing x by $1/x$ we get

$$6/x^3-11/x^2+6/x-1=0 \quad \text{or} \quad x^3-6x^2+11x-6=0$$

Its roots are in A P say α, β, γ

$$2\beta=\alpha+\gamma \quad \text{or} \quad 3\beta=\alpha+\beta+\gamma=6 \quad \text{or} \quad \beta=2$$

Hence $(x-2)$ is a factor of above

$$x^3-6x^2+11x-6=(x-2)\,(x^2-4x+3)$$
$$=(x-2)\,(x-3)\,(x-1)$$

Roots 1, 2, 3 are in A P

or $1, \frac{1}{2}, \frac{1}{3}$ are in H P

5 (i) We are given that $\frac{a+b}{2}=2\sqrt{(ab)}$ Divide by b

or $\frac{a}{b}+1=4\sqrt{\left(\frac{a}{b}\right)}$ or $x^2-4x=-1$

$x^2-4x+4=4-1=3$ or $(x-2)^2=3$ $\qquad x-2=\pm\sqrt{3}$

$$x=2+\sqrt{3},\ 2-\sqrt{3}$$

$$\sqrt{\frac{a}{b}}=2+\sqrt{3} \quad \therefore \quad \frac{a}{b}=(2+\sqrt{3})^2$$

$$=\frac{(2+\sqrt{3})\,(2+\sqrt{3})\,(2-\sqrt{3})}{(2-\sqrt{3})}$$

$$=\frac{(2+\sqrt{3})\,(4-3)}{2-\sqrt{3}}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$S_{20}=\frac{1}{6}(20)(21)(6\ 20^2+14\ 20+7)=188090$

8 $T_n=\frac{1^3+2^3+3^3+\quad+n^3}{1+3+5+\quad n \text{ terms}}=\frac{\Sigma n^3}{\frac{n}{2}\ 1+(n-1)\ 2]\ 2]}$

or $T_n=\frac{1}{4}\frac{n^2(n+1)^2}{n^2}=\frac{1}{4}(n^2+2n+1)$

Putting $n=1, 2, 3$ and adding we get

$S_n=\frac{1}{4}(\Sigma n^2+2\Sigma n+n)$

$=\frac{1}{4}\left[\frac{n(n+1)(2n+1)}{6}+2\ \frac{n(n+1)}{2}+n\right]$

$=\frac{n}{24}[2n^2+3n+1+6n+6+6]$

$=\frac{n}{24}[2n^2+9n+13]$

Putting $n=16$ we get

$S_{16}=\frac{16}{24}[2(256)+144+13]=\frac{2}{3}(669)=446$

9 Here the successive differences are 2, 4, 8, 16 which ar G P

$S=1+3+7+15+31+\quad+T_n$

$S=\quad 1+3+7+15+\quad+T_{n-1}+T_n$

Subtracting we get

$0=1+[2+4+8+16 \text{ to } (n-1) \text{ terms}]-T_n$

$T_n=1+2+2^2+2^3+2^4+\quad \text{to } n \text{ terms}$

or $T_n=1\ \frac{2^n-1}{2-1}=2^n-1$

Now putting $n=1, 2, 3\quad n$ and adding

$S_n=(2+2^2+2^3+\quad+2^n)-n$

$=2\ \frac{(2^n-1)}{2-1}-n=2^{n+1}-2-n$

10 Do yourself Ans $\frac{n(n+1)(n+2)}{6}$

11 Here as above $T_n=\frac{1}{2}(n^2+n+6)$

$S_n=\frac{1}{6}n(n^2+3n+20)$

12 We know that

$(a_1+a_2+\quad+a_n)^2=\Sigma a_i^2+2\Sigma a_i a_j$

Now put $a_1=1,\ a_2=2,\quad a_n=n$

$\therefore\ (1+2+3+\quad+n)^2=(1^2+2^2+3^2+\quad+n^2)+2\Sigma a_i a_j$

48 α, β, γ are the geometric means between ca, ab, ab, bc, bc, ca respectively Prove that if a, b, c are in A P, then $\alpha^2, \beta^2, \gamma^2$ are also in A P, and $\beta+\gamma, \gamma+\alpha, \alpha+\beta$ are in H P

49 If the $(m+1)^{th}$, $(n+1)^{th}$ and $(r+1)^{th}$ terms of an A.P are in G P, m, n, r are in H P show that the ratio of the common difference to the first term in the A P is

$$-\frac{2}{n}$$

50 If $2(y-a)$ is the H M between $y-x$ and $y-z$, then show that $x-a, y-a, z-a$ are in G P

51 If S_1, S_2, S_3 denote the sums of n terms of three A P,s whose first terms are unity and common differences in H P, prove that

$$n=\frac{2\,S_3S_1-S_1S_2-S_2S_3}{S_1-2\,S_2+S_3}$$

52 Prove that $a_2a_3-a_1a_4$ is positive, zero as negative according as a_1, a_2, a_3, a_4 are in A P, G P or H P

53 If a, b, c are in A P, α, β, γ in H P, $a\alpha, b\beta, c\gamma$ in G P, (with common ratio not equal to 1), then prove that

$$a\ \ b\ \ c=\frac{1}{\gamma}\ \ \frac{1}{\beta}\ \ \frac{1}{\alpha}$$

54 If the A M, the G,M and the H M of first and last terms of the series 25, 26, 27, , $N-1$, N are the terms of the series, find the value of N

55 Prove that the three successive terms of a G P will form the sides of a triangle if the common ratio r satisfies the inequality

$$\tfrac{1}{2}(\sqrt{5}-1) < r < \tfrac{1}{2}(\sqrt{5}+1)$$

Solutions to Problem Set (D)

1 $H=\dfrac{2ab}{a+b}$ $\quad H-a=\dfrac{2ab}{a+b}-a=\dfrac{ab-a^2}{a+b}$

$$H-b=\frac{2ab}{a+b}-b=\frac{ab-b^2}{a+b}$$

$$\frac{1}{H-a}+\frac{1}{H-b}=(a+b)\left[\frac{1}{a(b-a)}+\frac{1}{b(a-b)}\right]$$

$$=(a+b)\frac{(b-a)}{ab\,(b-a)}=\frac{a+b}{ab}$$

$$=\frac{1}{a}+\frac{1}{b}$$

Proved

$$(\Sigma n)^2-\Sigma n^2=2S \qquad (1)$$

$$S=\tfrac{1}{2}\left[\frac{n^2(n+1)^2}{4}-\frac{n(n+1)(2n+1)}{6}\right]$$

$$=\tfrac{1}{24}n(n+1)[3n^2+3n-4n-2]$$

$$=\tfrac{1}{24}n(n+1)(3n^2-n-2)$$

$$=\tfrac{1}{24}n(n+1)(n-1)(3n+2) \qquad \text{Proved}$$

Another form

$$S=\tfrac{1}{2}[(\Sigma n)^2-\Sigma n^2]=\tfrac{1}{2}[\Sigma n^3-\Sigma n^2]$$

$$(\Sigma n^3)=(\Sigma n)^2$$

Another form

$$(x-1)(x-2)(x-3)\quad(x-n)$$

$$=x^n-x^{n-1}(1+2+3+\quad n)+x^{n-2}\,\Sigma ab$$

where a and b are natural numbers such that $a\neq b$

(b) From above coefficient of x^{n-1} is

$$-\Sigma n=-\frac{n(n+1)}{2} \quad \text{Put } x=100$$

$$=-\frac{100\times 101}{2}=-50{'}0$$

13 We consider two cases

(i) Let n be even Then

$$(1^2-2^2)+(3^2-4^2)+(5^2-6^2)+\quad+\{(n-1)^2-n^2\}$$

$$=(1-2)(1+2)+(3-4)(3+4)+(5-6)(5+6)+\quad+\{(n-1)-n\}$$

$$\{(n-1)+n\}=-1\,(1+2+3+\quad+n)=-n\frac{(n+1)}{2}$$

(ii) Let n be odd In this case,

The series $=(1^2-2^2)+(3^2-4^2)+\quad+\{(n-2)^2-(n-1)^2\}+n^2$

$$-1\,\{1+2+3+\quad(n-1)\}+n^2$$

$$=-\frac{(n-1)n}{2}+n^2$$

$$=\frac{n(n+1)}{2}$$

14 $S=(1^3+2^3+\quad+50^3)-(1^3+2^3+\quad+30^3)$

$$\left(\frac{50\times 51}{2}\right)^2-\left(\frac{30\times 31}{2}\right)^2\left[\text{using } \Sigma n^3=\left(\frac{n(n+1)}{2}\right)^2\right]$$

$$\tfrac{1}{4}(50\times 51-30\times 31)(50\times 51+30\times 31)=1409400$$

15 $a,\ a+1,\ a+2,\quad,(a+n-1)$

$$s=na+\frac{n(n-1)}{2}$$

2 Put $b=\frac{2ac}{a+c}$ as a, b, c are in H P and simplify

3 By the given condition if the numbers be a and b then

$$\frac{2ab}{a+b} \quad \sqrt{(ab)}=12 \quad 13 \text{ or } 2\frac{\sqrt{(ab)}}{a+b}=\frac{12}{13}$$

or $26\sqrt{(ab)}=12(a+b)$ Divide by $2b$

$$13\sqrt{\frac{a}{b}}=6\left(\frac{a}{b}+1\right) \text{ or } 6x^2-13x+6=0 \text{ where } x=\sqrt{\frac{a}{b}}$$

$$(2x-3)(3x-2)=0 \qquad x=\frac{3}{2} \text{ or } \frac{2}{3}=\sqrt{\frac{a}{b}}$$

$$\therefore \frac{a}{b}=\frac{9}{4} \text{ or } \frac{4}{9}$$

Hence the numbers are in the ratio 4 , 9

4 Since x, y, z are in H P $\quad y=\frac{2xz}{x+z}$

$$x-2y+z=x+z-\frac{4xz}{x+z}=\frac{(x+z)^2-4xz}{x+z}=\frac{(x-z)^2}{x+z}$$

$\therefore$ $\log(x-2y+z)=2\log(x-z)-\log(x+z)$

or $\log(x+z)+\log(x-2y+z)=2\log(x-z)$ Proved

(*b*) Replacing x by $1/x$ we get

$6/x^3-11/x^2+6/x-1=0$ or $x^3-6x^2+11x-6=0$

Its roots are in A P say $\alpha, \beta\ \gamma$

$2\beta=\alpha+\gamma$ or $3\beta=\alpha+\beta+\gamma=6$ or $\beta=2$

Hence $(x-2)$ is a factor of above

$$x^3-6x^2+11x-6=(x-2)(x^2-4x+3)$$
$$=(x-2)(x-3)(x-1)$$

Roots 1 2, 3 are in A P

or $\quad 1, \frac{1}{2}, \frac{1}{3}$ are in H P

5 (i) We are given that $\frac{a+b}{2}=2\sqrt{(ab)}$ Divide by b

or $\frac{a}{b}+1=4\sqrt{\left(\frac{a}{b}\right)}$ or $x^2-4x=-1$

$x^2-4x+4=4-1=3$ or $(x-2)^2=3 \qquad x-2=\pm\sqrt{3}$

$x=2+\sqrt{3}, 2-\sqrt{3}$

$$\sqrt{\frac{a}{b}}=2+\sqrt{3} \qquad \frac{a}{b}=(2+\sqrt{3})^2$$

$$=\frac{(2+\sqrt{3})(2+\sqrt{3})(2-\sqrt{3})}{(2-\sqrt{3})}$$

$$=\frac{(2+\sqrt{3})(4-3)}{2-\sqrt{3}}=\frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$t=a^2+(a+1)^2+ \quad +(a+n-1)^2$ (1)

$=na^2+2a\ \frac{n(n-1)}{2}+\sum_{n-1}^{n-1} N^2$

or $nt=n^2a^2+an^2(n-1)+n\ \Sigma\ N^2$

and $s^2=n^2a^2+an^2(n-1)+\frac{n^2(n-1)^2}{4}$ by (1)

$nt-s^2$ is clearly independent of a

16 $S=1+2+3+4 \quad +n=\Sigma n=\frac{n(n+1)}{2}$

$S+669=(n-8)^2$ or $\frac{n(n+1)}{2}+669=n^2-16n+64$

or $n^2-33n-1210=0$ or $n^2-55n+22n-1210=0$

$(n-55)(n+22)=0$ $n=55$

∴ Number of balls is $\frac{55\ 56}{2}=1540$

Also $1540+669=2209=(55-8)^2=47^2$

17 Ans $\frac{n(n+1)(n+2)}{6}$

18 Ans $\frac{1}{3}(n-1)n(2n-1)$ yds

19 Ans $\frac{1}{(1-r)(1-ar)}$

20 Ans $n\ 2^{n+2}-2^{n+1}+2$

§ 6 Harmonical Progression (H P)

Definition A series of quantities is said to be in harmonical progression when their reciprocals are in arithmetical progression e g $\frac{1}{3}, \frac{1}{5}, \frac{1}{7}$, and $\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}$, are H Ps' as their reciprocals 3, 5, 7 , and $a, a+d, a+2d$ are in A P

nth terms of H P

Find the nth terms of the corresponding A. P and then take its reciprocal

If the H P be as $\frac{1}{a}, \frac{1}{a+d}\ \frac{1}{a+2d}$

then corresponding A P is $a, a+d, a+2d,$

T_n of A. P is $a+(n-1)d$

T_n of H P is $\frac{1}{a+(n-1)d}$

(ii) Do yourself

6 $\frac{a-b}{b-c}=\frac{a}{a}$ gives $b=\frac{a+c}{2}$ in A P

$\frac{a-b}{b-c}=\frac{a}{b}$ gives $b^2=ac$ ∴ in G P

$\frac{a-b}{b-c}=\frac{a}{c}$ gives $b=\frac{2ac}{a+c}$ in H P

7 $A=\frac{a+b}{2}$, $G=\sqrt{(ab)}$, $H=\frac{2ab}{a+b}$

∴ $A\,H=ab=G^2$ A, G, H are in $G\,P$

$A-G=\frac{a+b}{2}-\sqrt{(ab)}=\frac{a+b-2\sqrt{(ab)}}{2}=\frac{1}{2}(\sqrt{a}-\sqrt{b})^2=+\text{ive}$

∴ $A > G$ or $\frac{A}{G}>1$ (1)

Also from above $A\,H=G^2$ $\frac{A}{G}=\frac{H}{H}>1$

$\frac{G}{H}>1$ or $G>H$ (2)

Hence from (1) and (2), $A>G>H$

8 $A=27$ and $H=12$

But we know that A, G, H are in $G\,P$

∴ $G^2=A\,H=27\times 12=9\times 36$ $G=3\times 6=18$

9 $A=\frac{a+b}{2}$ or $a+b=2A$, $G=\sqrt{(ab)}$ $G^2=ab$

or $H=\frac{2ab}{a+b}=4$, $G^2=AH$ gives $G^2=4A$

Also $2A+G^2=27$ or $2A+4A=27$ $A=\frac{27}{6}=\frac{9}{2}$

∴ $\frac{a+b}{2}=\frac{9}{2}$ or $a+b=9$ (1)

Also $G^2=4A=4\cdot\frac{9}{2}=18$ or $ab=18$ (2)

From (1) and (2) we conclude that a and b are the roots of

$t^2-9t+18=0$ or $(t-6)(t-3)=0$ $t=6, 3$

Hence the numbers are 6 and 3

10 a, b, c, d are in $H\,P$

b is $H\,M$ of a and c and their $A\,M$ is $\frac{a+c}{2}$ and $G\,M$ is $\sqrt{(a$

In order to solve the questions on H P, we should form the corresponding A P

Harmonic mean (Single)

The harmonic mean between two quantities a and b will be x if a, x, b be in harmonical progression

$\therefore \quad \frac{1}{a}, \frac{1}{x}, \frac{1}{b}$ will be in A P

$$2\ \frac{1}{x} = \frac{1}{a} + \frac{1}{b} = \frac{a+b}{ab} \qquad x = \frac{2ab}{a+b} = \text{H M}$$

n harmonic means between a and b

Let $x_1, x_2 \ x_n$, be n harmonic means between a and b

$a, x_1, x_2 \ x_n \ b$ are in H P

or $\frac{1}{a}, \frac{1}{x_1} \frac{1}{x_2} \quad \frac{1}{x_n}, \frac{1}{b}$ are in A P

$$\frac{1}{b} = T_{n+2} \text{ of A P} = \frac{1}{a} + (n+1)\ d$$

$$(n+1)\ d = \frac{1}{b} - \frac{1}{a} = \frac{a-b}{ab} \text{ or } d = \frac{a-b}{ab}\ \frac{1}{n+1}$$

$$\frac{1}{x_1} = T_2 = \frac{1}{a} + d, \frac{1}{x_2} = T_3 = \frac{1}{a} + 2d, \ \frac{1}{x_n} = T_{n+1} = \frac{1}{a} + nd$$

On putting the value of d we shall find the values of

$$\frac{1}{x_1}, \frac{1}{x_2}, \quad \frac{1}{x_n}$$

and on taking their reciprocals we shall find the values of

$$x_1, x_2 \quad x_n$$

Problem Set (D)

If H be the harmonic mean between a and b then prove that

$$\frac{1}{H-a} + \frac{1}{H-b} = \frac{1}{a} + \frac{1}{b}$$

If a, b, c be in H P prove that

(i) $\frac{1}{b-a} + \frac{1}{b-c} = \frac{2}{b}$ (ii) $\frac{b+a}{b-a} + \frac{b+c}{b-c} = 2$

If the harmonic mean of two numbers is to their geometric mean as 12 13, prove that the numbers are in the ratio of 4 9

If x, y, z are in H P, prove that
$\log(x+z) + \log(x+z-2y) = 2 \log(x-z)$

c is H M of b and d and their A M is $\frac{b+d}{2}$ and G M is $\sqrt{(bd)}$

Also $A > C > H$ where A G, H are respective means

$\therefore \quad \frac{a+c}{2} > b$, and $\frac{b+d}{2} > c \qquad A > H$

Add $\quad \frac{a+c}{2}+\frac{b+d}{2} > b+c$ or $a+b+c+d > 2b+2c$

or $\qquad a+d > b+c$

$\sqrt{(ac)} > b$ and $\sqrt{(bd)} > c \qquad G > H$

Multiply $\quad \sqrt{(ac)}\sqrt{(bd)} > bc$ or $acbd > b^2c^2$

or $\qquad ad > bc$

(iii) We know that $\frac{a^n+c^n}{2} > \left(\frac{a+c}{2}\right)^n \qquad (1)$

Now since a b, c are in H P, b is the H M between a and c Also $\frac{a+c}{2}$ is the A M between a and c

Hence $\frac{a+c}{2} > b$ and so $\left(\frac{a+c}{2}\right)^n > b^n \qquad (2)$

$\therefore$ From (1) and (2), we get

$\frac{a^n+c^n}{2} > b^n$ i e $a^n+c^n > 2b^n$

11 (a) a x y, z b are in A P and let its common difference be d

$b=T_5=a+4d \qquad d=\frac{b-a}{4} \qquad (1)$

$x+y+z=T_2+T_3+T_4=a+d+a+2d+a+3d$

or $15=3a+6d=3a+\frac{6}{4}(b-a)=\frac{3}{2}(a+b)$

$a+b=10 \qquad (2)$

a, x, y, z, b are in H P

$\frac{1}{a}, \frac{1}{x}, \frac{1}{y}, \frac{1}{z}, \frac{1}{b}$ are in A P of common difference say D

$\frac{1}{b}= T_5 = \frac{1}{a}+4D \qquad 4D = \frac{1}{b}-\frac{1}{a}=\frac{a-b}{ab} \qquad (3)$

$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=T_2+T_3+T_4+\frac{1}{a}+D=\frac{1}{a}+2D+\frac{1}{a}+3D$

or $\frac{5}{3}=\frac{3}{a}+6D = \frac{3}{a}+\frac{6}{4}\left(\frac{1}{b}-\frac{1}{a}\right)$ by (3)

or $\frac{5}{3}=\frac{3}{2}\left(\frac{1}{a}+\frac{1}{b}\right)$ or $10=9\ \frac{(a+b)}{ab}=\frac{9\ 10}{ab}$ by (2)

$ab=9 \qquad (4)$

(*b*) Solve the equation $6x^3-11x^2+6x-1=0$ if its roots are in harmonical progression (Roorkee 76)

5 If the A M between a and b is twice as great as their G M show that

$$a\ \ b=(2+\sqrt{3})\ \ (2-\sqrt{\ }\ )$$ (Roorkee 53)

(ii) The A. M of a and b is to their G M as m to n, show

$$a\ \ b=m+\sqrt{(m^2-n^2)}\ \ m-\sqrt{(m^2-n^2)}$$

6 Prove that a, b, c are in A. P , G P or H P according as the value of $\frac{a-b}{b-c}$ is equal to $\frac{a}{a}$, $\frac{a}{b}$ or $\frac{a}{c}$ respectively

7 If A, G, H be respectively the A M , G M and H M. between two given quantities a and b, then prove that

(i) A, G, H, are in G P

(ii) $A > G > H$

8 A M and H M between two quantities are 27 and 12 respectively, find their G M

9 The harmonic mean of two numbers is 4, their A. M A and G M G satisfy the relation $2A+G^2=27$ Find the two numbers (I.I.T 79)

10 If a, b, c, d are in H P show that

(i) $ad > bc$

(ii) $a+d > b+c$ where a, b, c, d are given real numbers (I.I.T 70)

(iii) $a^n+c^n > 2b^n$, n being a +ive integer

11 (*a*) The value of $x+y+z$ is 15 if a, x, y, z, b are in AP while the value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $\frac{5}{3}$ if a, x, y, z, b are in H P find a and b (I.I.T 78)

(*b*) Find three numbers a, b, c between 2 and 18 such that (i) their sum is 25 (ii) the numbers 2, a, b are consecutive terms of an A P and (iii) the numbers b, c, 18 are consecutive terms of a G P (I.I.T 83)

12 The A M of two numbers exceeds their G M by 15 and H.M by 27, find the numbers

13 If the A. M between two numbers exceeds their G M by 2 and the G M exceeds their H. M by 8/5, find the numbers

14 (*a*) If A be the A M and H the H M between two

Hence from (1) and (4) we have $a+b=10$ and $ab=9$

a and b are the roots of $t^2-10t+9=0$

$(t-9)(t-1)=0$

$t=9, 1$ are the required numbers a, and b (i)

(b) We have $a+b+c=25$ (1) $2a=b+2$

and $c^2=18b$ (3) Eliminating a from (1) and (2),

$b = 16-\frac{2c}{3}$ Then from (3), $c^2=18\left(16-\frac{2c}{3}\right)$

or $c^2+12c-18\times16=0$

or $(c-12)(c+24)=0$

$c=-24$ is rejected since it does not lie between 2 and 18

Hence $c=12$ Then (3) gives $b=8$ and finally (2) gives $a=5$ (1)

Thus $a=5$, $b=8$ and $c=12$

12 Let the numbers be a, b and their A M, G M, and H M be denoted by A, G and H respectively Also we know that (1)

A, G H are in G P or $G^2=AH$

Since $A-G=15$ and $A-H=27$

$(A-15)^2=G^2=AH$ by (1) $=A(A-27)$

or $A=75=\frac{a+b}{2}$ $a+b=150$ (2)

Since $A-G=15$ $75-G=15$ or $G=60=\sqrt{(ab)}$ (3)

$ab=3600$

Hence from (2) and (3) we conclude that a and b are the roots of $t^2-150t+3600=0$

or $(t-120)(t-30)=0$ $t=120, 30$

Hence the two numbers are 120 and 30

13 If the numbers be a and b then

$A-G=2$ $\frac{a+b}{2}-\sqrt{ab}=2$ or $(\sqrt{a}-\sqrt{b})^2=4$ (1)

$G-H=\frac{8}{5}$ $\sqrt{ab}-\frac{2ab}{a+b}=8/5$

or $\sqrt{ab}\,\frac{[a+b-2\sqrt{ab}]}{a+b}=8/5$

or $5\sqrt{ab}\,(\sqrt{a}-\sqrt{b})^2=8(a+b)$

or $5\sqrt{ab}\;4=8(a+b)$ by (1)

$2a-5\sqrt{ab}+2b=0$

or $(2\sqrt{a}-\sqrt{b})(\sqrt{a}-2\sqrt{b})=0$ $\sqrt{a}=2\sqrt{b}$ or $a=4b$

From (1) $(2\sqrt{b}-\sqrt{b})^2=4$ or $b=4$

numbers a and b then $\frac{a-A}{a-H}\times\frac{b-A}{b-H}=\frac{A}{H}$

(*b*) If q arithmetic and harmonic means be inserted between 2 and 3, prove that $A+6/H=5$
Where A is any of the A M 's and H the corresponding H M (Dhanbad 1987)

15 If A_1, A_2, G_1, G_2, and H_1, H_2 be two A M's and G M's and H M's between two quantities then prove that

$$\frac{G_1G_2}{H_1H_2}=\frac{A_1+A_2}{H_1+H_2}$$ (Roorkee 83)

16 If a be the arithmetic mean of b and c and G_1, G_2 be the two geometric means between them then prove that

$$G_1^3+G_2^3=2abc$$

17 If $a_1, a_2, a_3 \quad a_n$ are in harmonic progression, prove that
$a_1a_2+a_2a_3+ \quad +a_{n-1}a_n=(n-1)a_1a_n$ (I I T 75)

18 If p be the first of n arithmetic means between two numbers and q be the first of n harmonic means between the same two numbers, prove that the value of q cannot be between p and

$$\left(\frac{n+1}{n-1}\right)^2 p$$ (I I T 75)

19 If n harmonic means are inserted between 1 and r then show that $\frac{\text{Ist mean}}{n\text{th mean}}=\frac{n+r}{nr+1}$

20 (a) If $H_1, H_2 \quad H_n$ be n harmonic means between a and b show that $\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=2n$

(b) If n be a root of the equation

$$x^2(1-ab)-x(a^2+b^2)-(1+ab)=0,$$

prove that $H_1-H_n=ab(a-b)$

21 Insert six harmonic means between 3 and 6/23

22 For what value of n is $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the harmonic mean of a and b

23 If a, b, c be respectively the pth, qth, and rth terms of an H P then prove that

$$bc(q-r)+ca(r-p)+ab(p-q)=0$$

24 If pth term of an H P is qr and qth term is rp, prove that rth term is pq

25 If the roots of the equation

$$a(b-c)x^2+b(c-a)x+c(a-b)=0$$

$\sqrt{a}=2\sqrt{b}=4$ or $a=16$

Hence the numbers are 16 and 4 The other factor shall give numbers as 4 and 16

14 (a) Substitute $A=\frac{a+b}{2}$ and $H=\frac{2ab}{a+b}$ and simplify

Let A_i, H_i ($i=1, 2, \ldots 9$) denote the 9 A M s and 9 H M 's between 2 and 3 If d denote the common difference of A P, Then

$$3=2+10\,d \quad \text{or} \quad d=\frac{1}{10}$$

Let A denote the ith mean, then $A=2+di=2+\frac{i}{10}$

Again Let 2, H_1 H_2 , H_n 3 are in H P

or $\frac{1}{2}, \frac{1}{H_1}, \frac{1}{H_2}, \quad , \frac{1}{H_9}, \frac{1}{3}$ are in A P If d_1 is the common difference of this A P Then

$\frac{1}{3}=\frac{1}{2}+10\,d_1$ or $d_1=-\frac{1}{60}$

If H is the ith H M, then

$\frac{1}{H}=\frac{1}{2}+d_1 i=\frac{1}{2}-\frac{i}{60}$

Now $A+6/H=\left(2+\frac{i}{10}\right)+6\left(\frac{1}{2}-\frac{i}{60}\right)=5+\frac{i}{10}-\frac{i}{10}=5$

15 Let the two quantities be a and b then

$a, A_1\ A_2, b$ are in A P

$A_1-a=b-A_2$ common difference

$A_1+A_2=a+b$ (2)

Again a, G_1, G_2, b are in G P

$\frac{G_1}{a}=\frac{b}{G_2}=$common ratio

$G_1G_2=ab$

Also a, H_1, H_2, b are in H P

$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{b}$ are in A P

$\frac{1}{H_1}-\frac{1}{a}=\frac{1}{b}-\frac{1}{H_2}=$ common difference of A P

$\frac{1}{H_1}+\frac{1}{H_2}=\frac{1}{a}+\frac{1}{b}$ or $\frac{H_1+H_2}{H_1H_2}=\frac{a+b}{ab}$ (3)

$$\tan\theta=\pm\frac{1}{\sqrt{2}}=\pm\tan\alpha \text{ where } \tan\alpha=\frac{1}{\sqrt{2}}$$

$$\theta=n\pi\pm\alpha, \text{ where } \alpha=\tan^{-1}\frac{1}{\sqrt{2}}$$

(*b*) $(\tan\theta+\tan 2\theta)-\dfrac{\tan\theta+\tan 2\theta}{1-\tan\theta\tan 2\theta}=0$

$(\tan\theta+\tan 2\theta)(1-\tan\theta\tan 2\theta-1)=0$

or $\tan\theta\tan 2\theta(\tan\theta+\tan 2\theta)=0$

$\tan\theta=0,\quad \theta=n\pi,\quad \tan 2\theta=0,\quad 2\theta=n\pi$ or $\theta=n\pi/2$

$\tan\theta+\tan 2\theta=0$ gives as above in part (*a*), $\theta=n\pi/3$

But for odd values of n, both values of θ given by $\theta=n\pi/2$ do not satisfy the given equation

Also for even values of n, $\theta=\frac{n\pi}{3}$ does not satisfy the given equation Hence the required solution is

$$\theta=\frac{2m\pi}{2}=m\pi\quad \theta=(2\lambda+1)\pi/3,\quad m, \lambda\in I$$

Also for odd m, the solution $\theta=m\pi$ is included in the second solution Hence the solution is $\theta=2r\pi\quad \theta=(2k+1)\pi/3\quad r, k\in I$

(*c*) $\tan 3\theta+\tan\theta=2\tan 2\theta$

$\tan 3\theta-\tan 2\theta=\tan 2\theta-\tan\theta$

$$\frac{\sin(3\theta-2\theta)}{\cos 3\theta\cos 2\theta}=\frac{\sin(2\theta-\theta)}{\cos 2\theta\cos\theta}$$

or $\sin\theta(\cos\theta-\cos 3\theta)=0$

or $\sin\theta\, 2\sin\theta\sin 2\theta=0$

$\sin\theta=0,\quad \theta=n\pi$

$\sin 2\theta=0\quad 2\theta=n\pi$ or $\theta=\frac{n\pi}{2}$

But $\theta=n\pi$ is included in $\theta=\frac{1}{2}n\pi$ Hence sol is $\theta=\frac{1}{2}n\pi$

(*d*) $\dfrac{4}{\tan 2\theta}=\dfrac{1}{\tan^2\theta}-\tan^2\theta$ Put $\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}$

$$\frac{4(1-\tan^2\theta)}{2\tan\theta}=\frac{1-\tan^4\theta}{\tan^2\theta}$$

$(1-\tan^2\theta)[2\tan\theta-(1+\tan^2\theta)]=0$

$(1-\tan^2\theta)(\tan^2\theta-2\tan\theta+1)=0$

$(1-\tan^2\theta)(\tan\theta-1)^2=0$

$\tan^2\theta=1=\tan\frac{\pi}{4}$ or $\tan\theta=\pm\tan\frac{\pi}{4},\quad \theta=n\pi\pm\frac{\pi}{4}$

$\tan\theta-1=0,\quad \tan\theta=1=\tan\frac{\pi}{4}\quad \theta=n\pi+\frac{\pi}{4}$

But this value of θ is included in

$$\theta = n\pi \pm \frac{\pi}{4}$$

(e) $$\frac{3 \sin(\theta - 15^\circ)}{\cos(\theta - 15^\circ)} = \frac{\sin(\theta + 15^\circ)}{\cos(\theta + 15^\circ)}$$

or $$\frac{2 \sin(\theta - 15^\circ)\cos(\theta + 15^\circ)}{2 \sin(\theta + 15^\circ)\cos(\theta - 15^\circ)} = \frac{1}{3}$$

or $$\frac{\sin 2\theta - \sin 30^\circ}{\sin 2\theta + \sin 30^\circ} = \frac{1}{3}$$

or $$3 \sin 2\theta - 3 \cdot \tfrac{1}{2} = \sin 2\theta + \tfrac{1}{2}$$

$$2 \sin 2\theta = 2 \qquad \sin 2\theta = 1 = \sin \frac{\pi}{2}$$

$$2\theta = 2n\pi + \frac{\pi}{2} \quad \text{or} \quad \theta = n\pi + \frac{\pi}{4}$$

(f) $\tan \frac{\pi}{3} = \sqrt{3}$ $\quad \tan \frac{2\pi}{3} = \tan\left(\pi - \frac{\pi}{3}\right) = -\tan \frac{\pi}{3} = -\sqrt{3}$ [See § (i) b P 74]

Now $$\tan\theta + \tan\left(\theta + \frac{\pi}{3}\right) + \tan\left(\theta + \frac{2\pi}{3}\right)$$

$$= \tan\theta + \frac{\tan\theta + \sqrt{3}}{1 - \sqrt{3}\tan\theta} + \frac{\tan\theta - \sqrt{3}}{1 + \sqrt{3}\tan\theta}$$

$$= \frac{\tan\theta(1 - 3\tan^2\theta) + (\tan\theta + \sqrt{3})(1 + \sqrt{3}\tan\theta) + (\tan\theta - \sqrt{3})(1 - \sqrt{3}\tan\theta)}{1 - 3\tan^2\theta}$$

$$= \frac{\tan\theta - 3\tan^3\theta + 2\tan\theta + \sqrt{3}(2\sqrt{3}\tan\theta)}{1 - 3\tan^2\theta}$$

$$\frac{9\tan\theta - 3\tan^3\theta}{1 - 3\tan^2\theta} = 3\,\frac{(3\tan\theta - \tan^3\theta)}{1 - 3\tan^2\theta} = 3\tan 3\theta$$

Hence the given equation reduces to

$$3 \tan 3\theta = 3$$

$$\tan 3\theta = 1 = \tan\frac{\pi}{4} \qquad 3\theta = n\pi + \frac{\pi}{4} \quad \text{or} \quad \theta = \frac{n\pi}{3} + \frac{\pi}{12}$$

10 (a) $\sin A - \sin B = 0 \qquad \cos A - \cos B = 0$

$$2 \sin\frac{A - B}{2}\cos\frac{A + B}{2} = 0 \qquad \sin\frac{A + B}{2}\sin\frac{B - A}{2} = 0$$

or $$-2 \sin\frac{A + B}{2}\sin\frac{A - B}{2} = 0$$

From the two we observe that the common factor gives

$$\sin\frac{A - B}{2} = 0 \qquad \frac{A - B}{2} = n\pi$$

$$A - B = 2n\pi \qquad A = 2n\pi + B$$

Now $\dfrac{A_1+A_2}{G_1G_2}=\dfrac{a+b}{ab}$ [by (1) and (2)] $=\dfrac{H_1+H_2}{H_1H_2}$ by (3)

$$\frac{A_1+A_2}{H_1+H_2}=\frac{G_1G_2}{H_1H_2}$$ Proved. (1)

16 $2a=b+c$ as a is A M of b and c

Also b, G_1, G_2, c are in G P

Hence as in Q 38 (G P) page 132, $G_1=b^{2/3}c^{1/3}, G_2=b^{1/3}c^{2/3}$

$G_1^3+G_2^3=b^2c+bc^2=bc\,(b+c)=bc\,2a=2abc$ by (1)

17 Since $a_1, a_2 \ldots a_n$ are in H P

$\therefore \dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3} \ldots \dfrac{1}{a_n}$ are in A P of common difference say d

$\dfrac{1}{a_2}-\dfrac{1}{a_1}=d, \dfrac{1}{a_3}-\dfrac{1}{a_2}=d \ldots \dfrac{1}{a_n}-\dfrac{1}{a_{n-1}}=d$

or $a_1-a_2=d\,(a_1a_2), a_2-a_3=d\,(a_2a_3), \ldots, (a_{n-1}-a_n)=d(a_{n-1}a_n)$

Adding the above relations we get (1)

$a_1-a_n=d\,(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)$

Now we have to find the value of d

$$\frac{1}{a_n}=T_n \text{ of A P }=\frac{1}{a_1}+(n-1)\,d$$ (2)

$\dfrac{1}{a_n}-\dfrac{1}{a_1}=(n-1)d$ or $(a_1-a_n)=(n-1)\,d\,a_n\,a_1$

Putting the value of a_1-a_n from (2) in (1) we get

$(n-1)\,a_na_1d=d\,(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)$ Pro

$(n-1)\,a_na_1=a_1a_2+a_2a_3+\ldots+a_{n-1}a_n$

18 Let the two numbers be a and b

$a, x_1, x_2 \ldots x_n, b$ are in A P of $(n+2)$ terms

$b=T_{n+2}=a+(n+1)\,d \qquad d=\dfrac{b-a}{n+1}$

$p=$ Ist A M $x_1=T_2=a+d=a+\dfrac{b-a}{n+1}=\dfrac{na+b}{n+1}$ (1)

$a, y_1, y_2, \ldots y_n, b$ are in H P

i e $\dfrac{1}{a}, \dfrac{1}{y_1}, \dfrac{1}{y_2} \ldots \dfrac{1}{y_n}, \dfrac{1}{b}$ are in A P of common diff say D

$\dfrac{1}{b}=T_{n+2}=\dfrac{1}{a}+(n+1)\,D \qquad D=\dfrac{a-b}{(n+1)\,ab}$

$\dfrac{1}{y_1}=T_2=\dfrac{1}{a}+D=\dfrac{1}{a}+\dfrac{a-b}{(n+1)\,ab}=\dfrac{nb+a}{(n+1)\,ab}$ (

$q=$ Ist H M $=y_1=\dfrac{(n+1)\,ab}{nb+a}$

$$=2\left[\frac{ab+ac+ac+bc}{ab+ac+b^2+bc}\right]=2 \quad \because b^2=ac \text{ by (1)} \quad \text{Proved}$$

Again $\frac{1}{x}+\frac{1}{y}=\frac{2}{a+b}+\frac{2}{b+c}=\frac{2(a+c+2b)}{ab+ac+b^2+ac}$ Put $b^2=ac$

$$=\frac{2(a+c+2b)}{ab+2b^2+bc}=\frac{2(a+c+2b)}{b(a+c+2b)}=\frac{2}{b}$$

42 $2q=p+r \quad \because p, q\ r$ are in A P (1)

Let $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}=k$ say (2)

We have to prove that x, y, z are in H P

Putting for p, q, r from (2) in (1) we get

$2\left(\frac{a-y}{ky}\right)=\frac{a-x}{kx}+\frac{a-z}{kz}$ or $2\left(\frac{a}{y}\right)-2=\frac{a}{x}-1+\frac{a}{z}-1$

or $\frac{2}{y}=\frac{1}{x}+\frac{1}{z} \quad \frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A P or x, y, z are in H P

43 $2b=a+c \quad a, b, c$ are in A P (1)

$b^2=\frac{2a^2c^2}{a^2+c^2} \quad a^2, b^2, c^2$ are in A P (2)

$b^2[(a+c)^2-2ac]=2a^2c^2$ by (2),

or $b^2[(4b^2-2ac]=2a^2c^2$ by (1)

or $2b^4-b^2ac-a^2c^2=0$ factorize

$(b^2-ac)(2b^2+ac)=0$

Either $b^2-ac=0$ or $2b^2+ac=0$

If $b^2=ac$, then $\left(\frac{a+c}{2}\right)^2=ac$ by (1)

or $(a+c)^2-4ac=0$ or $(a-c)^2=0 \quad a=c$

and $2b=a+c=a+a=2a \quad b=a$

Hence $a=b=c$

If $2b^2+ac=0$ then $b^2=-\frac{a}{2}c$

$-\frac{a}{2}, b, c$ are in G P

44 $q^2=pr$ (1) as p, q, r are in G P

T_p, T_q, T_r of an A P are in H P

$\frac{1}{T_p}, \frac{1}{T_q}, \frac{1}{T_r}$ are in A P

$\frac{1}{T_q}-\frac{1}{T_p}=\frac{1}{T_r}-\frac{1}{T_q}$

We have to prove that q cannot lie between p and $\frac{(n+1)^2}{(n-1)^2}p$

Now $n+1 > n-1$ or $\left(\frac{n+1}{n-1}\right)^2 > 1$

$$\left(\frac{n+1}{n-1}\right)^2 p > p$$

or $$p < \left(\frac{n+1}{n-1}\right)^2 p \tag{3}$$

Now we will show that

$$\frac{p}{q} > 1 \text{ or } p > q \text{ or } q < p$$

We have $\frac{p}{q} = \frac{na+b}{n+1} \cdot \frac{nb+a}{(n+1)ab}$

$$= \frac{n(a^2+b^2)+ab(n^2+1)}{(n+1)^2 ab}$$

$$= \frac{n(a^2+b^2-2ab)+ab(n^2+1+2n)}{(n+1)^2 ab}$$

$$= \frac{n}{(n+1)^2}\left(\frac{a-b}{\sqrt{ab}}\right)^2 + \frac{(n+1)^2}{(n+1)^2}$$

or $$\frac{p}{q} = \frac{n}{(n+1)^2}\left(\sqrt{\frac{a}{b}} - \sqrt{\frac{b}{a}}\right)^2 + 1$$

The first factor is + ive and hence from above

$$\frac{p}{q} > 1 \quad \text{or} \quad p > q \quad \text{or} \quad q < p \tag{4}$$

Hence from (3) and (4) we get

$$q < p < \left(\frac{n+1}{n-1}\right)^2 p$$

$\therefore$ q cannot lie between p and $\left(\frac{n+1}{n-1}\right)^2 p$

19 We have calculated Ist mean in (2) of last question

$$y_1 = \frac{(n+1)}{nb+a} \tag{1}$$

Also from Q 18, $\frac{1}{y_n} = T_{n+1} = \frac{1}{a} + nD$

or $\frac{1}{y_n} = \frac{1}{a} + n\frac{(a-b)}{(n+1)ab} = \frac{(n+1)b+n(a-b)}{(n+1)ab} = \frac{b+na}{(n+1)ab}$

$y_n = n\text{th H M} = \frac{(n+1)ab}{b+na}$

$\frac{\text{Ist Mean}}{n\text{th Mean}} = \frac{y_1}{y_n} = \frac{b+na}{nb+a} = \frac{r+n \cdot 1}{nr+1} = \frac{n+r}{nr+1}$ $[a=1, b=r]$

or $\frac{T_p-T_q}{T_p}=\frac{T_q-T_r}{T_r}$

$$\frac{(p-q)\,d}{a+(p-1)\,d}=\frac{(q-r)\,d}{a+(r-1)\,d}$$

or $a\,(p+r-2q)=d\,\{(p-1)\,(q-r)-(r-1)\,p-q)\}$

$$\frac{a}{d}=\frac{pq-pr-q+r-(rp-rq-p+q)}{p+r-2q}$$

$$\frac{p\,(q+1)+r\,(q+1)-2pr-2q}{p+r-2q}$$

But $pr=q^2$

$$\frac{a}{d}=(q+1)\,\frac{(p+r-2q)}{p+r-2q}=q+1$$

45 $a=AR^{p-1},\ b=AR^{q-1},\ c=AR^{r-1}$ for G P

$$a=\frac{1}{A_1+(p-1)\,D},\ b=\frac{1}{A_1+(q-1)\,D},\ c=\frac{1}{A_1+(r-1)\,D}\ \text{H P}$$

$$\frac{1}{b}-\frac{1}{c}=(q-r)\,D \quad \text{or} \quad \frac{c-b}{bc}=(q-r)\,D$$

or $$\frac{a\,(b-c)}{q-r}=-Dabc=\frac{b\,(c-a)}{r-p}=\frac{c\,(a-b)}{p-q} \qquad (1)$$

Also $a^{q-r}\,b^{r-p}\,c^{p-q}=A^oR^o=1$ as in Q 19 G P page 130

$$(q-r)\log a+(r-p)\log b+(p-q)\log c=\log 1=0$$

or $a\,(b-c)\log a+b\,(c-a)\log b+c\,(a-b)\log c=0$

by (1)

46 $d=b-a$ for A P, $r=b/a$ for G P

$D=\frac{1}{b}-\frac{1}{a}$ for H P

$T_{n+2}=a+(n+1)\,d=a+(n+1)\,(b-a)=-na+(n+1)\,b$

for A P

$$T_{n+2}=ar^{n+1}=a\,\frac{b^{n+1}}{a^{n+1}}=\frac{b^{n+1}}{a^n} \text{ for G P}$$

$$T_{n+2}=\frac{1}{\frac{1}{a}+(n+1)\,D}=\frac{1}{\frac{1}{a}+(n+1)\left(\frac{1}{b}-\frac{1}{a}\right)}$$

$$=\frac{1}{-\frac{n}{a}+\frac{(n+1)}{b}}=\frac{ab}{(n+1)\,a-nb} \text{ for H P}$$

The above three terms are themselves in G P

$$\left(\frac{b^{n+1}}{a^n}\right)^2=[-na+(n+1)\,b]\,\frac{ab}{(n+1)\,a-nb}$$

$$\frac{b^{2n+1}}{a^{2n+1}}=\frac{-na+(n+1)\,b}{(n+1)\,a-nb}$$

20 (a) We have calculated H_1 and H_n in Q 1 and 19

$$H_1=\frac{(n+1)\,ab}{nb+a} \qquad H_n=\frac{(n+1)\,ab}{b+na}$$

$$\frac{H_1}{a}=\frac{(n+1)\,b}{nb+a} \qquad \frac{H_n}{b}=\frac{(n+1)\,a}{b+na}$$

Apply Componendo and Dividendo

$$\frac{H_1+a}{H_1-a}=\frac{(2n+1)\,b+a}{b-a},\ \frac{H_n+b}{H_n-b}=\frac{(2n+1)\,a+b}{a-b}$$

$$\frac{H_1+a}{H_1-a}+\frac{H_n+b}{H_n-b}=\frac{1}{b-a}\,[(2n+1)\,b+a-(2n+1)\,a-b]$$

$$=\frac{1}{b-a}\,[2nb-2na]=2n\frac{(b-a)}{(b-a)}=2n$$

(b) $$H_n-H_1=\frac{(n+1)\,ab}{na+b}-\frac{(n+1)\,ab}{(nb+a)}$$

$$=\frac{(n+1)\,ab\,[nb+a-na-b]}{n^2ab+n\,(a^2+b^2)+ab}$$

$$=\frac{(n+1)\,ab\,(1-n)\,(a-b)}{n^2-1}=-ab(a-b) \text{ by (1) below}$$

$$H_1-H_n=ab\,(a-b)$$

Since n is a root of the given equation

$$n^2\,(1-ab)-n\,(a^2+b^2)-(1+ab)=0 \qquad (1)$$

or $n^2-1=n\ ab+n\,(a^2+b^2)+ab$

21 $\frac{6}{5}, \frac{3}{4}, \frac{6}{11}, \frac{3}{7}, \frac{6}{17}, \frac{3}{10}$

22 $\frac{a^{n+1}+b^{n+1}}{a^n+b^n}=$ H M of a and $b=\frac{2ab}{a+b}$

$$a\,a^{n+1}+ab^{n+1}+ba^{n+1}+b\,b^{n+1}=2a^{n+1}\,b+2b^{n+1}\,a$$

$$a\,a^{n+1}+b\,b^{n+1}=a^{n+1}\,b+b^{n+1}\,a$$

or $a^{n+1}\,(a-b)=b^{n+1}\,(a-b)$

or $a^{n+1}=b^{n+1}$

or $\left(\frac{a}{b}\right)^{n+1}=1 \qquad n+1=0$ or $n=-1$

23 Convert into A P and proceed

24 T_p of A P $=\frac{1}{qr}$ T_q of A P is $\frac{1}{rp}$,

$A+(p-1)\,D=\frac{1}{qr}$, $A+(q-1)\,D=\frac{1}{rp}$ Subtract

$(p-q)\,D=\frac{1}{qr}-\frac{1}{rp}=\frac{p-q}{pqr} \qquad D=\frac{1}{qpr}$

Putting for D, we get

$$=2\left[\frac{ab+ac+ac+bc}{ab+ac+b^2+bc}\right]=2 \qquad b^2=ac \text{ by (1)} \qquad \text{Proved}$$

Again $\frac{1}{x}+\frac{1}{y}=\frac{2}{a+b}+\frac{2}{b+c}=\frac{2(a+c+2b)}{ab+ac+b^2+ac}$

Put $b^2=ac$

$$=\frac{2(a+c+2b)}{ab+2b^2+bc}=\frac{2(a+c+2b)}{b(a+c+2b)}=\frac{2}{b}$$

42 $2q=p+r$. p q, r are in A P (1)

Let $\frac{a-x}{px}=\frac{a-y}{qy}=\frac{a-z}{rz}=k$ say (2)

We have to prove that x, y, z are in H P

Putting for p, q, r from (2) in (1) we get

$$2\left(\frac{a-y}{ky}\right)=\frac{a-x}{kx}+\frac{a-z}{kz} \text{ or } 2\left(\frac{a}{y}\right)-2=\frac{a}{x}-1+\frac{a}{z}-1$$

or $\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$ $\quad \frac{1}{x},\frac{1}{y},\frac{1}{z}$ are in A P or x, y, z are in H P

43 $2b=a+c$ $\quad a, b, c$ are in A P (1)

$b^2=\frac{2a^2c^2}{a^2+c^2}$ $\quad a^2, b^2, c^2$ are in A P (2)

$b^2\,[(a+c)^2-2ac]=2a^2c^2$ by (2),

or $b^2\,[(4b^2-2ac]=2a^2c^2$ by (1)

or $2b^4-b^2\,ac-a^2c^2=0$ factorize

$(b^2-ac)\,(2b^2+ac)=0$

Either $b^2-ac=0$ or $2b^2+ac=0$

If $b^2=ac$, then $\left(\frac{a+c}{2}\right)^2=ac$ by (1)

or $(a+c)^2-4ac=0$ or $(a-c)^2=0$ $\quad a=c$

and $2b=a+c=a+a=2a$ $\quad b=a$

Hence $a=b=c$

If $2b^2+ac=0$ then $b^2=-\frac{a}{2}c$

$-\frac{a}{2}, b$ c are in G P

44 $q^2=pr$ (1) as p, q r are in G P

T_p, T_q, T_r of an A P are in H P

$\therefore$ $\frac{1}{T_p}, \frac{1}{T_q}, \frac{1}{T_r}$ are in A P

$$\frac{1}{T_q}-\frac{1}{T_p}=\frac{1}{T_r}-\frac{1}{T_q}$$

$$A=\frac{1}{qr}-\frac{p-1}{pqr}=\frac{1}{pqr}$$

T_r of H P $=A+(r-1)\ D=\dfrac{1+(r-1)}{pqr}=\dfrac{1}{pq}$

∴ T_r of H P $=pq$

25 If the roots of the given equation be equal, then

$$B^2-4AC=0$$

$$b^2(c-a)^2-4ac\ (b-c)\ (a-b)=0$$

$$b^2\ [c^2+a^2-2ca+4ca]-4ac\ (-ac+ab+bc)=0$$

$$b^2\ (c+a)^2+4a^2c^2-4abc\ (a+c)=0$$

$$[b\ (c+a)]^2+[2ac]^2-2\ (2ac)\ b\ (a+c)=0$$

Above is of the form $X^2+Y^2-2XY=0$

or $(X-Y)^2=0$ ∴ $X=Y$

∴ $b\ (c+a)=2ac$ ∴ $b=\dfrac{2ac}{a+c}$

Thus b is the harmonic mean of a and c In other words it means that a, b, c are in H P

26 T_m of A P $=\dfrac{1}{n}$ and T_n of A P $=\dfrac{1}{m}$

$A+(m-1)\ D=\dfrac{1}{n}$ and $A+(n-1)\ D=\dfrac{1}{m}$

Solving as is Q 24 $A=D=\dfrac{1}{mn}$

T_{m+n} of A P $=A+(m+n-1)\ D=\dfrac{1+m+n-1}{mn}=\dfrac{m+n}{mn}$

∴ T_{m+n} of H P is $\dfrac{mn}{m+n}$

27 Proceed as above in Q 26 or Q 24

$$d=3,\ a=-8,\ T_9=\frac{1}{49}$$

28 Taking reciprocals we are given that

$a\ (b+c)$, $b\ (c+a)$, $c\ (a+b)$ are in A P

$$(bc+ba)-(ab+ca)=(ca+cb)-(bc+ba)$$

or $c\ (b-a)=a\ (c-b)$

Divide by abc

$$\frac{b-a}{ab}=\frac{c-b}{bc}\quad\text{or}\quad\frac{1}{a}-\frac{1}{b}=\frac{1}{c}-\frac{1}{b}$$

∴ $\dfrac{1}{a}$ $\dfrac{1}{b}$ $\dfrac{1}{c}$ are in A P or a, b, c are in H P

29 $\dfrac{a}{b+c}$ $\dfrac{b}{c+a}$, $\dfrac{c}{a+b}$ will be in A P, if on adding 1 to each

or $\frac{T_p - T_q}{T_p} = \frac{T_q - T_r}{T_r}$

$$\frac{(p-q)\,d}{a+(p-1)\,d} = \frac{(q-r)\,d}{a+(r-1)\,d}$$

or $a\,(p+r-2q) = d\,\{(p-1)\,(q-r)-(r-1)\,p-q)\}$

$$\frac{a}{d} = \frac{pq-pr-q+r-(rp-rq-p+q)}{p+r-2q}$$

$$\frac{p\,(q+1)+r\,(q+1)-2pr-2q}{p+r-2q}$$

But $pr = q^2$

$$\frac{a}{d} = (q+1)\,\frac{(p+r-2q)}{p+r-2q} = q+1$$

45 $a = AR^{p-1}$, $b = AR^{q-1}$, $c = AR^{r-1}$ for G P

$$a = \frac{1}{A_1+(p-1)\,D},\ b = \frac{1}{A_1+(q-1)\,D}\quad c = \frac{1}{A_1+(r-1)\,D}\text{ H P}$$

$$\frac{1}{b} - \frac{1}{c} = (q-r)\,D \quad\text{or}\quad \frac{c-b}{bc} = (q-r)\,D$$

or $$\frac{a\,(b-c)}{q-r} = -Dabc = \frac{b\,(c-a)}{r-p} = \frac{c\,(a-b)}{p-q} \qquad (1)$$

Also $a^{q-r}\,b^{r-p}\,c^{p-q} = A\,R^{\circ} = 1$ as in Q 19 G P page 130

$$(q-r)\log a + (r-p)\log b + (p-q)\log c = \log 1 = 0$$

or $a\,(b-c)\log a + b\,(c-a)\log b + c\,(a-b)\log c = 0$

by (1)

46 $d = b-a$ for A P, $r = b/a$ for G P

$$D = \frac{1}{b} - \frac{1}{a} \text{ for H P}$$

$$T_{n+2} = a+(n+1)\,d = a+(n+1)\,(b-a) = -na+(n+1)\,b$$

for A P

$$T_{n+2} = ar^{n+1} = a\,\frac{b^{n+1}}{a^{n+1}} = \frac{b^{n+1}}{a^n} \text{ for G P}$$

$$T_{n+2} = \frac{1}{\frac{1}{a}+(n+1)\,D} = \frac{1}{\frac{1}{a}+(n+1)\left(\frac{1}{b}-\frac{1}{a}\right)}$$

$$= \frac{1}{-\frac{n}{a}+\frac{(n+1)}{b}} = \frac{ab}{(n+1)\,a-nb} \text{ for H P}$$

The above three terms are themselves in G P

$$\left(\frac{b^{n+1}}{a^n}\right)^2 = [-na+(n+1)\,b]\,\frac{ab}{(n+1)\,a-nb}$$

$$\frac{b^{2n+1}}{a^{2n+1}} = \frac{-na+(n+1)\,b}{(n+1)\,a-nb}$$

i e $\frac{a+b+c}{b+c}, \frac{b+c+a}{c+a}, \frac{c+a+b}{a+b}$ are in A P

or $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A P on dividing by $a+b+c$

or $b+c, c+a, a+b$ are in H P on taking reciprocal

But this is given to us

30 (i) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$, will be in H P

if $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A P

Add 1 and cancal $a+b+c$

that is, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A P

or if a b, c are in H P which is given

(ii) Take reciprocal and add 2 and cancel $a+b+c$ etc as in (I)

(iii) We have to prove that

$$\frac{a+b+c}{a(b+c)}, \frac{a+b+c}{b(c+a)}, \frac{a+b+c}{c(a+b)}$$ to be in H P

Taking reciprocal and caucel $a+b+c$

We have to prove that

$a(b+c), b(c+a), c(a+b)$ to be in A P

or $(ab+bc+ca)-bc, \Sigma ab-ca, \Sigma ab-ab$ to be in A P

or $-bc, -ca, -ab$ to be in A P

or $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ to be in A P Dividing by $-abc$

or a, b, c to be in H P which is true

31 Take reciprocal add 1, cancel $a+b+c$, multiply by abc etc.

32 Taking reciprocal we are given that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A P}$$

$$\therefore \frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

or $$\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$$

Cancel $c+a$ and cross multiply

$$b^2-a^2=c^2-b^2$$

a^2 b^2 c^2 are in A P

33 We are given that $a=\frac{b+c}{2}$ $b^2=ca$

Cross multiply

$$(n+1)\,[ab^{2n+1}-ba^{2n+1}]=n\,(b^{2n+2}-a^{2n+2})$$

or $\dfrac{n+1}{n}=\dfrac{b^{2n+2}-a^{2n+2}}{ab(b^{2n}-a^{2n})}$ **Proved**

47 T_r of A P $=a+(r-1)\,d$ where $b=a+(n-1)\,d$

$\therefore$ T_r of A P $=a+(r-1)\dfrac{b-a}{n-1}=\dfrac{a\,(n-r)+b\,(r-1)}{n-1}$ (1)

T_r from end of H P, $a,\ \ ,b$ (n terms)

$=$Reciprocal of T_r from end of A P $\dfrac{1}{a}\quad\dfrac{1}{b}$

$=$Reciprocal of T_r from beginning of A P $\dfrac{1}{b}\quad\dfrac{1}{a}$($n$ terms)

$=$Reciprocal of $\dfrac{1}{b}+(r-1)\,D$ where $\dfrac{1}{a}=\dfrac{1}{b}+(n-1)\,D$

$=$Reciprocal of $\dfrac{1}{b}+(r-1)\left(\dfrac{1}{a}-\dfrac{1}{b}\right)\dfrac{1}{n-1}$

$$=\frac{1}{n-1}\left[\frac{1}{b}(n-r)+(r-1)\frac{1}{a}\right]$$

$=$Reciprocal of $=\dfrac{1}{n-1}\,\dfrac{a\,(n-r)+b\,(r-1)}{ab}=\dfrac{ab\,(n-1)}{a\,(n-r)+b\,(r-1)}$ (2)

Multiplying (1) and (2) we get the product$=ab$, which is independent of r

48 By given condition

$\alpha^2=a^2bc$, $\beta^2=b^2ca$ $\gamma^2=c^2ab$ and $2b=a+c$

We have to prove that $2\beta^2=\alpha^2+\gamma$

Putting the values

$2abc\ b=abc\,(a+c)$ or $2b=a+c$ which is true

Again $\beta+\gamma$, $\gamma+\alpha$ $\alpha+\beta$ will be in H P if

$\dfrac{1}{\beta+\gamma}, \dfrac{1}{\gamma+\alpha}, \dfrac{1}{\alpha+\beta}$ are in A P or $\dfrac{1}{\gamma+\alpha}-\dfrac{1}{\beta+\gamma}=\dfrac{1}{\alpha+\beta}-\dfrac{1}{\gamma+\alpha}$

or $\dfrac{\beta-\alpha}{\beta+\gamma}=\dfrac{\gamma-\beta}{\alpha+\beta}$ or $\beta^2-\alpha^2=\gamma^2-\beta^2$

or $\alpha^2\ \beta^2\ \gamma^2$ are in A P which we have already proved

49 $a+md$ $a+nd$, $a+rd$ are in G P

$$(a+nd)^2=(a+md)\,(a+rd)$$

or $a\,(2n-m-r)=d\,(mr-n^2)$

$\dfrac{d}{a}=\dfrac{2n-(m+r)}{mr-n^2}$ (1)

and we have to prove that

$$c=\frac{2ab}{a+b} \quad \text{or} \quad ca+bc=2ab \qquad (1)$$

L H S of (1) $\quad b^2+bc=b(b+c) \qquad ca=b^2$

R H S of (1) $\quad (b+c)\,b=$ L H S $\qquad 2a=b+c$

34, Let the four numbers be a, b, c, d

$$2b=a+c \qquad (1)$$

as a, b, c are in A.P

$$c=\frac{2bd}{b+d} \qquad (2)$$

. b, c, d are in H P

$c(b+d=d(a+c)$ by (1) and (2)

$bc+cd=da+cd \qquad bc=ad$

or $\frac{a}{b}=\frac{c}{d}$ *i e* four numbers are proportional

35 $\quad a, b, c$ are in A.P, we have

$2b=a+c \qquad$ -(1)

b, c, a are in H P, we get

$$c=\frac{2ab}{a+b} \qquad (2)$$

$c(a+b)=a(a+c)$, by (1) and (2)

$ac+bc=a^2+ac$ or $a^2=bc$

c, a, b are in G P

36 We have $\quad 2y=x+z \qquad$ [x, y, z are in A P] (1)

$b^2y^2=ax\,cz \qquad$ [ax, by, cz are in G P] (2)

and $\quad b=\frac{2ac}{a+c} \qquad$ [. $a\ b, c$ are in H P] (3)

Substituting for y and b form (1) and (3) in (2), we get

$$\frac{4a^2c^2}{(a+c)^2}\,\frac{(x+z)^2}{4}=axcz$$

or $\quad \frac{(x+z)^2}{xz}=\frac{(a+c)^2}{ac}$

or $\quad \frac{x^2+z^2+2xz}{xz}=\frac{a^2+c^2+2ac}{ac}$

or $\quad \frac{x}{z}+\frac{z}{x}+2=\frac{a}{c}+\frac{c}{a}+2$

or $\quad \frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$

But m, n, r are in H P

$$n=\frac{2mr}{m+r} \qquad (2)$$

$$\therefore \frac{d}{a}=\frac{2n-(m+r)}{(m+r)\frac{n}{2}-n^2}=\frac{2}{n}\left[\frac{2n-(m+r)}{(m+r)-2n}\right]=-\frac{2}{n}$$

50 $(y-x)$, $2(y-a)$, $(y-z)$ are in H P

$\therefore \frac{1}{y-x}, \frac{1}{2(y-a)}, \frac{1}{y-z}$ are in A P

or $\frac{1}{2(y-a)}-\frac{1}{y-x}=\frac{1}{y-z}-\frac{1}{2(y-a)}$

or $\frac{2a-y-x}{y-x}=\frac{y+z-2a}{y-z}$

or $\frac{(x-a)+(y-a)}{(x-a)-(y-a)}=\frac{(y-a)+(z-a)}{(y-a)-(z-a)}$ (Note)

Apply componendo and dividendo

$\frac{2(x-a)}{2(y-a)}=\frac{2(y-a)}{2(z-a)}$

or $(y-a)^2=(x-a)(z-a)$

$(x-a), (y-a), (z-a)$ are in G P

51 $S_1=\frac{n}{2}[2\cdot 1+(n-1)d_1]$ $\qquad \frac{2(S_1-n)}{n(n-1)}=d_1$

or $\frac{1}{d_1}=\frac{n(n-1)}{2(S_1-n)}$

Similarly for $\frac{1}{d_2}$ and $\frac{1}{d_3}$

Since d_1, d_2, d_3 are given to be in H P therefore $\frac{1}{d_1}, \frac{1}{d_2}, \frac{1}{d_3}$ are in A P or on cancelling $\frac{n(n-1)}{2}$ we can say that

$\frac{1}{S_1-n}, \frac{1}{S_2-n}, \frac{1}{S_3-n}$ are in A P

$$\frac{1}{S_2-n}-\frac{1}{S_1-n}=\frac{1}{S_3-n}-\frac{1}{S_2-n}$$

or $\frac{S_1-S_2}{S_1-n}=\frac{S_2-S_3}{(S_3-n)}$ cross multiply

or $(S_1-S_2)S_3-(S_2-S_3)S_1=n\{S_1-S_2-S_2+S_3\}$

$n=$ as given

53 (i) a_1, a_2, a_3, a_4 are in A P

i e $\frac{a+b+c}{b+c}, \frac{b+c+a}{c+a}, \frac{c+a+b}{a+b}$ are in A P

or $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$ are in A P on dividing by $a+b+c$

or $b+c, c+a, a+b$ are in H P on taking reciprocal

But this is given to us

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if $\frac{b+c}{a}, \frac{c+a}{b}, \frac{a+b}{c}$ are in A P

Add 1 and cancal $a+b+c$

that is, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are in A P

or if a b, c are in H P which is given

(ii) Take reciprocal and add 2 and cancel $a+b+c$ etc as in (I)

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or $(ab+bc+ca)-bc, \Sigma ab-ca, \Sigma ab-ab$ to be in A P

or $-bc, -ca, -ab$ to be in A P

or $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ to be in A P Dividing by $-abc$

or a, b c to be in H P which is true

31 Take reciprocal add 1, cancel $a+b+c$, multiply by abc etc.

32 Taking reciprocal we are given that

$$\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b} \text{ are in A P}$$

$$\frac{1}{c+a}-\frac{1}{b+c}=\frac{1}{a+b}-\frac{1}{c+a}$$

or $$\frac{b-a}{(c+a)(b+c)}=\frac{c-b}{(a+b)(c+a)}$$

Cancel $c+a$ and cross multiply

$$b^2-a^2=c^2-b^2$$

a^2 b^2 c^2 are in A P

33 We are given that $a=\frac{b+c}{2}$, $b^2=ca$

$a_2=a_1+d,\ a_3=a_1+2d,\ a_4=a_1+3d$

$\therefore\ a_2a_3-a_1a_4=(a_1+d)\,(a_1+2d)-a_1\,(a_1+3d)$

$=(a_1^2+3a_1d+2d^2)-(a_1^2+3a_1d)=2d^2=$ +ive

(ii) a_1, a_2, a_3, a_4 are in G P

$\therefore\ \dfrac{a_2}{a_1}=\dfrac{a_4}{a_3} \qquad a_2\,a_3-a_1\,a_4=0$

(iii) a_1, a_2, a_3, a_4 are in H P

$\dfrac{1}{a_1}, \dfrac{1}{a_2}, \dfrac{1}{a_3}, \dfrac{1}{a_4}$ are in A.P of common difference D say

$\dfrac{1}{a_4}-\dfrac{1}{a_1}=3D$ or $a_1-a_4=3D\,a_1a_4$ (1)

$\dfrac{1}{a_2}=\dfrac{1}{a_1}+D=\dfrac{1+a_1\,D}{a_1} \qquad a_2=\dfrac{a_1}{1+a_1\,D}$

$\dfrac{1}{a_3}=\dfrac{1}{a_4}-D=\dfrac{1-a_4\,D}{a_4} \qquad a_3=\dfrac{a_4}{1-a_4\,D}$ (Note)

$$a_2a_3-a_1a_4=a_1a_4\left[\frac{1}{(1+a_1D)\,(1-a_4\,D)}-1\right]$$

$$=\frac{a_1a_4\,[1-\{1+(a_1-a_4)\,D-a_1a_4\,D^2\}]}{1+(a_1-a_4)\,D-a_1a_4\,D^2}$$ Now use (1)

$$=\frac{a_1a_4\,[-3D^2\,a_1a_4+a_1a_4\,D^2]}{1+3D^2a_1a_4-a_1a_4\,D^2}=-\frac{2(a_1a_4\,D)^2}{1+2D^2a_1a_4}$$

$=-$ive

53 $2b=a+c,\ \beta=\dfrac{2\alpha\gamma}{\alpha+\gamma}$ and $b^2\beta^2=ac\alpha\gamma$

Eliminating b and β from these, we get

$\dfrac{(a+c)^2}{4}\ \dfrac{4\alpha^2\gamma^2}{(\alpha+\gamma)^2}=ac\alpha\gamma$ or $\dfrac{(a+c)^2}{ac}=\dfrac{(\alpha+\gamma)^2}{\alpha\gamma}$

This gives $\dfrac{a}{c}+\dfrac{c}{a}=\dfrac{\alpha}{\gamma}+\dfrac{\gamma}{\alpha}$ Multiply by $\dfrac{a}{c}$

or $\left(\dfrac{a}{c}\right)^2-\dfrac{a}{c}\left(\dfrac{\alpha}{\gamma}+\dfrac{\gamma}{\alpha}\right)+1=0$

or $\left(\dfrac{a}{c}-\dfrac{\alpha}{\gamma}\right)\left(\dfrac{a}{c}-\dfrac{\gamma}{\alpha}\right)=0$

$\dfrac{a}{c}=\dfrac{\alpha}{\gamma}$ i e $\dfrac{a}{1/\gamma}=\dfrac{c}{1/\alpha}$ (1)

or $\dfrac{a}{c}=\dfrac{\gamma}{\alpha}$ i e $a\alpha=c\gamma$

The condition (2) is ruled out since this with the help of $b^2\beta^2=ac\alpha\gamma$ gives $b\beta=c\gamma$ so that $a\alpha=b\beta=c\gamma$ and so $a\alpha, b\beta, c\gamma$ are in G P with common ratio 1 which contradicts the and we have to prove that

$$c=\frac{2ab}{a+b} \quad \text{or} \quad ca+bc=2ab \qquad (1)$$

L H S of (1) $\quad b^2+bc=b\,(b+c) \qquad ca=b^2$

R H S of (1) $\quad (b+c)\,b=$ L H S $\qquad \because 2a=b+c$

34, Let the four numbers be a, b, c, d

$$2b=a+c \qquad (1)$$

as a, b, c are in A.P

$$c=\frac{2bd}{b+d} \qquad (2)$$

$\because$ b, c, d are in H P

$c\,(b+d)=d\,(a+c)$ by (1) and (2)

$bc+cd=da+cd \qquad bc=ad$

or $\frac{a}{b}=\frac{c}{d}$ i e four numbers are proportional

35 $\quad a, b, c$ are in A P, we have

$$2b=a+c \qquad (1)$$

b, c, a are in H P, we get

$$c=\frac{2ab}{a+b} \qquad (2)$$

$c\,(a+b)=a\,(a+c)$, by (1) and (2)

$ac+bc=a^2+ac$ or $a^2=bc$

$\because$ c, a, b are in G P

36 We have $\quad 2y=x+z \qquad [\because x, y, z$ are in A P] (1)

$b^2y^2=ax\,cz \qquad [\because ax, by, cz$ are in G P] (2)

and $\quad b=\frac{2ac}{a+c} \qquad [\because a\ b, c$ are in H P] (3)

Substituting for y and b form (1) and (3) in (2), we get

$$\frac{4a^2c^2}{(a+c)^2}\,\frac{(x+z)^2}{4}=axcz$$

or $$\frac{(x+z)^2}{xz}=\frac{(a+c)^2}{ac}$$

or $$\frac{x^2+z^2+2xz}{xz}=\frac{a^2+c^2+2ac}{ac}$$

or $$\frac{x}{z}+\frac{z}{x}+2=\frac{a}{c}+\frac{c}{a}+2$$

or $$\frac{x}{z}+\frac{z}{x}=\frac{a}{c}+\frac{c}{a}$$

hypothesis

Now using (1) and $b^2\beta^2=ac\alpha\gamma$, we get

$b^2\beta^2=c^2\alpha^2$ or $b\beta=c\alpha$

Thus $\frac{b}{1/\beta}=\frac{a}{1/\alpha}$ (3)

Finally from (1) and (3), we get

$$\frac{a}{1/\gamma}=\frac{b}{1/\beta}=\frac{c}{1/\alpha}$$

54 Let A=Arithmetic Mean$=\frac{25+N}{2}$,

G=Geometric Mean$=\sqrt{(25N)}=5\sqrt{N}$

and H=Harmonic Mean$=\frac{2\times 25N}{25+N}=\frac{50N}{25+N}$

Let A, G, H be the p^{th}, q^{th} and r^{th} terms of the series

Then $25+(p-1)\,1=\frac{25+N}{2}$ or $N=2p+23$

Since p is a positive integer, putting $p=1, 2, 3,$, we get $N=25, 27, 29, 31$, (1)

Again $25+(q-1)\,1=5\sqrt{N}$ or $\sqrt{N}=\frac{q+24}{5}$

Since N is a +ive integer we give to q those values which make N a positive integer These values are clearly

$q=1, 6, 11, 16, 21,$

Then $N=25, 36, 49, 64, 81, 100, 121, 144,$ (2)

Finally $25+(r-1)\,1=\frac{50N}{25+N}$

or $625+25N+25r+Nr-25-N=50N$

or $N=\frac{25\,(24+r)}{26-r}$

We give those positive integral values to r which make N a positive integer

These values of r are 1,21 and 25 only (Note that higher values of r make N negative)

Hence $N=25, 225$ (3)

The value of N which satisfies (1), (2) and (3) is 225

Therefore $N=225$

53 First we note that for common ratio equal to 1, any three terms of G P (with positive first term) can be the sides of a triangle Now suppose a, ar, ar^2 $(r\neq 1)$ are three successive terms of a G P As we know, three line segments can

37 (i) ∵ a, b, c are in G P, $b^2=ac$

Also $a^x=b^y=c^z=k$, say

$a=k^{1/x}$, $b=k^{1/y}$, $c=k^{1/z}$

Put the values of a, b, c in (1)

∴ $k^{2/y}=k^{1/x} \cdot k^{1/z}=k^{1/x+1/z}$

$$\frac{2}{y}=\frac{1}{x}+\frac{1}{z}$$

or $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ are in A P

x, y, z are in H P

(ii) Proceed exactly as above and you will have $2y=x+z$

i e x, y, z are in A P

(iii) a, b, c are in G P $b^2=ac$

Let $\log_a n=x$ $n=a^x$ and similarly $n=b^y$, und $n=c^z$ (1)

Determine a b c as in part (i) and out in (1)

(iv) Given a, b c are in C P $b^2=ac$ (1)

we have to prove that $\log a^n$, $\log b^n$, $\log c^n$ to be in A.P

or $n \log a$, $n \log b$, $n \log c$ to be in A P

or $2 \log b=\log a+\log c$

or $\log b^2=\log ac$ or $b^2=ac$ which is true by (1) We have cancelled n.

(b) Proceeding as in part a (i) we can say that x, y, z are in H P y z, u are in H P x, y z, u are in H P

38 $x=a+(m-1)\,d=AR^{m-1}$

T_m of both A P and G P are equal

$y=a+(n-1)\,d=AR^{n-1}$

T_n of both A P and G P are equal

$z=a+(p-1)\,d=AR^{p-1}$

∴ T_p of both A P, and G P are equal

$y-z=(n-p)\,d$ $z-x=(p-m)\,d$, $x-y=(m-n)\,d$ (1)

$x^{y-z}\,y^{z-x}\,z^{x-y}$

$=(AR^{m-1})^{(n-p)d}\,(AR^{n-1})^{(p-m)d}\,(AR^{p-1})^{(m-n)d}$

$=A^0\,R^0=1$

$(n-p+p-m+m-n)\,d=0$

$d\{(m-1)(n-p)+(n-1)(p-m)+(p-1)(m-n)\}=0$

39 $2b=a+c$ a, b c are in A P (1)

$c^2=bd$ ∴ b, c, d are in G P (2)

$d=\frac{2ce}{c+e}$ c d, e are in $H.P$ (3)

Fror

(1) and (2)

form a triangle if and only if each of them is less than the sum of the other two Hence it suffices to verify that largest one is less than the sum of the other two
Clearly for $r > 1$, the number ar^2 is the largest and for $r < 1$ a is the largest so we consider two cases
(a) Let $r > 1$ Then the inequality $ar^2 < ar+a$ must hold true since $a > 0$, we have

$$r^2-r-1 < 0$$

which gives $\frac{-\sqrt{5}+1}{2} < r < \frac{\sqrt{5}+1}{2}$
since $r > 1$, we see that r can vary in the interval

$$1 < r < \tfrac{1}{2}(\sqrt{5}+1) \quad (1)$$

(b) Let $0 < r < 1$ Proceeding as in case (a), we shall get

$$\tfrac{1}{2}(\sqrt{5}-1) < r < 1 \quad (2)$$

Hence from our remark in the beginning and from the inequalities (1) and (2), we find that the common ratio r of G P can vary in the interval

$$\tfrac{1}{2}(\sqrt{5}-1) < r < \tfrac{1}{2}(\sqrt{5}+1)$$

Problem Set (E)

Objective Questions

1 $\frac{1}{q+r}, \frac{1}{r+p}, \frac{1}{p+q}$ are in A P, then
(a) p, q, r are in A P, (b) p^2, q^2, r^2 are in A P,
(c) $\frac{1}{p}, \frac{1}{q}, \frac{1}{r}$ are in A P (d) None of these

2 99^{th} term of the series
$2+7+14+23+34+$ is
(i) 9998, (ii) 9999, (iii) 10000,
(iv) None of these

3 The sum of 40 terms of an A P whose first term is 2 and common difference 4, will be
(ii) 3200, (ii) 1600, (iii) 200, (iv) 2800 (M N R 78)

4 α, β be the roots of

$$x^2-3x+a=0$$

and γ, δ the roots of

$$x^2-12x+b=0$$

and numbers $\alpha, \beta, \gamma, \delta$ (in order) form an increasing G P, then
(ii) $a=3, b=12$, (ii) $a=12, b=3$,

$c(c+e)=e(a+c)$ or $c^2+ce=ea+ec$

or $c^2=ae$ $\quad a, c, e$ are in G P Proved (4)

(ii) We have proved that $c^2=ae$

$$e=\frac{c^2}{a}=\frac{(2b-a)^2}{a} \text{ by (1)}$$

(iii) $a=2$ and $e=18$

We have $c^2=a\,e$ by (4)

or $c^2=36$ $\quad c=6, -6$

$2b=a+c$ by (1)

or $2b=2+6$ or $2-6$

$b=4, -2$

$c^2=bd$ by (2)

$36=4d$ or $(-2)\,d$

$d=9, -18$

Hence b, c, d are 4 6, 9 or $-2, -6, -18$

∴ The five numbers a, b, c, d, e are

2, 4, 6 9, 18 or 2, -2, -6, -18 18

40 $b=\dfrac{2ac}{a+c}$, $\quad a, b, c$ are in H P (1)

$c^2=bd$, $\quad b, c, d$ are in G P (2)

$2d=c+e$ $\quad c, d, e$ are in A P (3)

We have to eliminate c and d and prove that

$$e=\frac{ab^2}{(2c-b)^2}$$

Now $e=2d-c$ by (3) $=\dfrac{2c^2}{b}-c$ by (2) (4)

and $b\,(a+c)=2ac$ by (1), $\quad ab=c\,(2a-b)$

or $c=\dfrac{ab}{2a-b}$ (5)

Eliminate c between (4) and (5)

From (4), $e=c\left(\dfrac{2c}{b}-1\right)=\dfrac{ab}{2a-b}\left(\dfrac{2}{b}\,\dfrac{ab}{2a-b}-1\right)$

$$=\frac{ab}{2a\quad b}\,\frac{b}{2a-b}=\frac{ab^2}{(2a-b)^2}$$ Proved

41 $b^2=ac$ $\quad a, b, c$ are in G P (1)

$2x=a+b$ $\quad x$ is A M of a and b (2)

$2y=b+c$ $\quad y$ is A M of b and c (3)

$$\frac{a}{x}+\frac{c}{y}=a\,\frac{2}{a+b}+c\,\frac{2}{b+c} \text{ by (2) and (3)}$$

(iii) $a=2, b=32$, (iv) $a=4, b=16$

5 The numbers 49, 4489, 444889, obtained by inserting 48 into the middle of the preceding numbers are squares of integers

(a) True, (b) False

6 If a, b, c are in A.P as well as in G P then

(i) $a=b\neq c$ (ii) $a\neq b=c$, (iii) $a\neq b\neq c$, (iv) $a=b=c$

(M N R 81)

7 If the harmonic mean between two positive numbers is to their Geometric mean as 12 13, the numbers are in the ratio

(a) 12 13, (b) 1/12 1/13,
(c) 4 9, (d) 1/4 1/9

8 The harmonic, geometric and arithmetic means between two positive quantities are in ascending order of magnitude

(a) True, (b) False

9 In an A P the sum of terms equidistant from the beginning and end is equal to

(i) first term, (ii) second term,
(iii) sum of first and last terms, (iv) last term

10 If $x, 2x+2, 3x+3$, are in G P, then the fourth term is

(a) 27, (b) -27, (c) 13 5, (d) -13 5 (M N.R 80)

11 The nth term of the series $3, \sqrt{3}, 1$, is $\frac{1}{243}$, then n is

(a) 12, (b) 13, (c) 14, (d) 15

12 If a, b, c are in G P, then

(i) $a(b^2+a^2)=c(b^2+c^2)$, (ii) $a(b^2+c^2)=c(a^2+b^2)$,
(iii) $a^2(b+c)=c^2(a+b)$, (iv) none of these

13 The value of $9^{1/3}\times 9^{1/9}\times 9^{1/27}\times$ ∞ is

(i) 9, (ii) 1, (iii) 3, (iv) none of these

14 The rational number which equals to the numbr $2\overline{357}$ with recurring decimal is

(a) $\frac{2355}{1001}$, (b) $\frac{2370}{997}$, (c) $\frac{2355}{999}$, (d) none of these

(I I T 83)

15 If $\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c}$, then a, b, c are in

(a) A P, (b) H P, (c) G P,
(d) H P and G P both (M N R 84)

16 (i) $2^{1/4}\, 4^{1/8}\, 8^{1/16}\, 16^{1/32}$ is equal to

(a) 1 (b) 2, (c) [illegible], (d) [illegible]. (M N R 84) (*b*) Let us consider the +ive values between 0 and 2π

$$\tan (A-B)=1=\tan \frac{\pi}{4}$$

$$A-B=n\pi+\frac{\pi}{4}, \quad \therefore \quad A-B=\frac{\pi}{4}, \frac{5\pi}{4} \tag{1}$$

$$\sec (A+B)=\frac{2}{\sqrt{3}} \qquad \cos (A+B)=\frac{\sqrt{3}}{2}=\cos\frac{\pi}{6}$$

$$A+B=2n\pi\pm\frac{\pi}{6}, \qquad A+B=\frac{\pi}{6}, 2\pi-\frac{\pi}{6}=\frac{11\pi}{6} \tag{2}$$

From (1), we observe that $A-B$ is +ive i.e. $A > B$

$$A+B > A-B \tag{3}$$

$$\left.\begin{array}{ll} A+B=\frac{11}{6}\pi & \text{or} \quad A+B=\frac{11}{6}\pi \\ A-B=\frac{\pi}{4} & \text{or} \quad A-B=\frac{5\pi}{4} \end{array}\right\} \tag{4}$$

In both the cases the condition (3) is satisfied

We cannot choose

$$A+B=\frac{\pi}{6}$$

$$A-B=\frac{\pi}{4} \quad \text{or} \quad \frac{5\pi}{4}$$

Because these equations will not satisfy the condition (3)

Solving the equations in (4) we get

$$A=\frac{25}{24}\pi, \quad B=\frac{19\pi}{24}, \quad \text{or} \quad A=\frac{37\pi}{24}, \quad B=\frac{7\pi}{24}$$

General values

$$\tan (A-B)=1=\tan \frac{\pi}{4}$$

$$A-B=m\pi+\frac{\pi}{4} \tag{1}$$

Also

$$\cos (A+B)=\frac{\sqrt{3}}{2}=\cos \frac{\pi}{6}$$

$$A+B=2m\pi\pm\frac{\pi}{6}$$

$$A+B=2n\pi+\frac{\pi}{6} \quad \text{or} \quad 2n\pi-\frac{\pi}{6}$$

Solving both with (1) we get

$$A=(2n+m)\frac{\pi}{2}+\frac{5\pi}{24}, \quad B=(2n-m)\frac{\pi}{2}-\frac{\pi}{24}$$

or $A=(2n+m)\frac{\pi}{2}+\frac{\pi}{24},\ B=(2n-m)\frac{\pi}{2}-\frac{5\pi}{24}$

(c) Proceed as in part (b)

$A=\frac{7\pi}{12},\ B=\frac{\pi}{4}$

$A=(2n+m)\frac{\pi}{2}+\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$

and $B=(m-2n)\frac{\pi}{2}-\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$

or $A=(2n+m)\frac{\pi}{2}-\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$

and $B=(m-2n)\frac{\pi}{2}-\frac{\pi}{6}+(-1)^m\frac{\pi}{12}$

(d) $\cos(2\theta+3\phi)=\frac{1}{2}=\cos\pi/3$

$2\theta+3\phi=2n\pi\pm\pi/3$ (1)

$\cos(3\theta+2\phi)=\sqrt{3}/2=\cos\pi/6$

$3\theta+2\phi=2p\pi\pm\pi/6$ (2)

Multiply (1) by 3 and (2) by 2 and subtract

$\phi=(3n-4p)\frac{\pi}{5}\pm\frac{\pi}{5}\mp\frac{\pi}{15}$

Similarly multiply (1) by 2 and (2) by 3 and subtract

$\theta=(6p-4n)\frac{\pi}{5}\pm\frac{\pi}{10}\mp\frac{2\pi}{15}$

(e) $\theta=(2n+p)\frac{\pi}{2}+(-1)^p\frac{\pi}{8}\pm\frac{\pi}{8}$

$\phi=(p-2n)\frac{\pi}{8}+(-1)^p\frac{\pi}{8}\mp\frac{\pi}{8}$

11 (a) $(\sin^3\theta+\cos^3\theta)-(1-\sin\theta\cos\theta)=0$

or $(\sin\theta+\cos\theta)(\sin^2\theta+\cos^2\theta-\sin\theta\cos\theta)$

$-(1-\sin\theta\cos\theta)=0$

or $(\sin\theta+\cos\theta)(1-\sin\theta\cos\theta)-(1-\sin\theta\cos\theta)=0$

or $(1-\sin\theta\cos\theta)(\sin\theta+\cos\theta-1)=0$

$1-\sin\theta\cos\theta=0$ $\quad 2\sin\theta\cos\theta=2$ or $\sin 2\theta=2$

Above is not possible because $\sin 2\theta$ can never be greater than 1

Again $\sin\theta+\cos\theta=1$ $\quad$ Divide by $\sqrt{(1+1)}=\sqrt{2}$

$\frac{1}{\sqrt{2}}\sin\theta+\frac{1}{\sqrt{2}}\cos\theta=\frac{1}{\sqrt{2}}$

or $\cos\left(\theta-\frac{\pi}{4}\right)=\frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}$ $\qquad \theta-\frac{\pi}{4}=2n\pi\pm\frac{\pi}{4}$

(ii) The value of $(0.2)^{\log_{\sqrt{5}}(1/4+1/8+1/16+\dots\infty)}$ is

(a) 1, (b) 2, (c) 1/2 (d) 4

17 The sum of the series

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+\dots$ to 9 terms is

(a) $-\frac{5}{6}$, (b) $-\frac{1}{2}$, (c) 1, (d) $-\frac{3}{2}$ (M N R 85)

18 (a) If a, b, c are in H P then

$$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$$ (M N R 85)

(b) If a, b, c are in A P then

$\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are

(i) A P (ii) G P (iii) H P (iv) None (M N R 82)

(c) If a, b, c are in A P then prove

$(a-c)^2=4(b^2-ac)$ (Roorkee 75)

19 If $\frac{3+5+7+\dots+n \text{ termrs}}{5+8+11+\dots+10 \text{ terms}}=7$, the value of n is

(a) 35 (b) 36 (c) 37 (d) 40 (M N R 83)

20 The first term of a G P whose second term is 2 and sum to infinity is 8 will be

(a) 6, (b) 3, (c) 4, (d) 1 (M N R 79)

21 The sum of first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+\dots$ is equal

(a) 2^n-n-1, (b) $1-2^{-n}$, (c) $n+2^{-n}-1$ (d) 2^n-1

(I I T 1988)

22 The sum of first n terms of the series $1^2+2\cdot2^2+3^2+2\cdot4^2+5^2+2\cdot6^2+\dots$ is $n(n+1)^2/2$ when n is even When n is odd, the sum is (I I T 88)

23 If the first and $(2n-1)$th terms of an A P, a G P and H P are equal and their n th terms are a, b aud c respectively, then

(a) $a=b=c$ (c) $a+c=b$

(b) $a \quad b \geqslant c$ (d) $ac-b^2=0$ (I I T 88)

24 If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2) \leqslant 0$, then a b, c d are in

(i) A P (ii) G P (iii) H P (iv) $ab=cd$

(v) None of these (I I T 87)

$$\frac{b^2-ac}{B^2-AC}=\left(\frac{a}{A}\right)^2$$

16 If α and β be the roots of $x^2+px-q=0$ and γ, δ the roots of $x^2+px+r=0$, prove that

$$(\alpha-\gamma)(\alpha-\delta)=(\beta-\gamma)(\beta-\delta)=q+r$$

17 If α and β are the roots of $x^2+px+1=0$ and γ, δ are the roots of $x^2+qx+1=0$, show that

$$q^2-p^2=(\alpha-\gamma)(\beta-\gamma)(\alpha+\delta)(\beta+\delta)$$ (I I T 78)

18 If the roots of $px^2+qx+2=0$ are reciprocals of each other, then

(a) $p=0$ (b) $p=-2$ (c) $q=0$ (d) $p=2$

19 Find the equation whose roots are $(\alpha+\beta)^2$ and $(\alpha-\beta)^2$, where α and β are the roots of $2x^2+2(m+n)x+(m^2+n^2)=0$

20 Find the condition that the roots of the equation $ax^2+bx+c=0$ be such that

(a) One root is n times the other

(b) One root is three times the other

(c) Both roots are equal

(d) If the roots of the equation $ax^2+bx+c=0$ are of the form $\frac{k+1}{k}$ and $\frac{k+2}{k+1}$ prove that $(a+b+c)^2=b^2-4ac$

21 (a) If one root of the equation $ax^2+bx+c=0$ be square of the other, then prove that

$$b^3+ac^2+a^2c=3abc$$

(b) If one root of the quadratic equation $ax^2+bx+c=0$ is equal to the n^{th} power of the other root, then show that

$$(ac^n)^{1/(n+1)}+(a^nc)^{1/(n+1)}+b=0$$ (I I T 83)

22 If the roots of the equation $\frac{1}{x+p}+\frac{1}{x+q}=\frac{1}{r}$ are equal in magnitude but opposite in sign show that $p+q=2r$ and that the product of the roots is equal to $-\frac{1}{2}(p^2+q^2)$

23 If the sum of the roots of $ax^2+bx+c=0$ be equal to sum of their squares prove that $2ac=ab+b^2$

24 If the sum of the roots of the equation $ax^2+bx+c=0$ is equal to sum of the squares of their reciprocals, then show that

bc^2, ca^2, ab^2 are in A P (I I T 76)

25 α, β are the roots of the equation $\lambda(x^2-x)+x+5=0$ If λ_1 and λ_2 are the two values of λ for which the roots α, β are

Solutions of Objectives

1 Ans (b) See Q 37 (i) A P

2 Ans (i) Use method of difference

3 (ii)

4 $\alpha, \beta, \gamma, \delta$ being in G P they may be taken as k, kr, kr^2, kr^3

$S=k(1+r)=3 \quad kr^2(1+r)=12 \quad \therefore 3r^2=12$ or $r=2 \quad \therefore k=1$

$P=k^2r=a, k^2r^5=6$ Putting for r and k, $a=2, b=32$

5 Ans (a)

We have, $T_n=444 \ldots 4888 \ldots 89$

It is evident that 4 is repeated n times and 8 is repeated $(n-1)$ times

We can write T_n as

$$T_n=9+(8\times10+8\times10^2+8\times10^3+\ldots+8\times10^{n-1})$$
$$+(4\times10^n+4\times10^{n+1}+4\times10^{n+2}+\ldots+4\times10^{2n-1})$$

$$=9+\frac{80(10^{n-1}-1)}{10-1}+\frac{4\times10^n(10^n-1)}{10-1}$$

$$=\frac{1}{9}[81+80\times10^{n-1}-80+4\times10^{2n}-4\times10^n]$$

$$=\frac{1}{9}[1+2\times4\times10^n-4\times10^n+4\times10^{2n}]$$

$$\frac{1}{9}[1+4\times10^n+4\times10^{2n}]=\left(\frac{1+2\times10^n}{3}\right)^2$$

Hence each T_n is the square of an integer since $2\times10^n+1$ is divisible by 3 for we have

$$1+2\times10^n=1+2(1+9)^n=1+2[1+{}^nc_1\,9+{}^nc_2\,9^2+\ldots+9^n]$$
$$=3+2[{}^nc_1\,9+{}^nc_2\,9^2+\ldots+9^n]$$

For example, $T_1=7^2$, $T_2=\ \ 7^2$, $T_3=667^2$ etc

6 Ans (iv)

$2b=a+c$, $b^2=ac$ so that $\left(\frac{a+c}{2}\right)^2=ac$

Hence $(a+c)^2-4ac=0$ or $(a-c)^2=0 \qquad a=c$

$a=b=c \qquad 2b=2a$

(iv) is correct

7 Arrange as a quadratic in $n=\sqrt{\left(\frac{a}{b}\right)}$ etc. (c) is correct

8 Ans (a) $A>G>H$ or $H<G<A$ ascending

9 (iii)

10 (d), $b^2=ac$ gives $x=-4, -4, -6, -9$

$T_4=-9(3/2)=-13\ 5$

connected by the relation $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$, find the value of $\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}$

26 If the ratio of the roots of the equation, $x^2+px+q=0$ be equal to ratio of the roots of $x^2+lx+m=0$, then prove that $p^2m=l^2q$

27 (a) If the ratio of the roots of the equation $lx^2+nx+n=0$ be $p : q$, then prove that

$$\sqrt{\left(\frac{p}{q}\right)}+\sqrt{\left(\frac{q}{p}\right)}+\sqrt{\left(\frac{n}{l}\right)}=0$$

(b) Find the value of p for which $x+1$ is a factor of $x^4+(p-3)x^3-(3p-5)x^2+(2p-9)x+6$
Find the remaining factors for this value of p (I I T 75)

(c) If x^2-3x+2 is a factor of $x^4-px^2+q=0$, prove $p=5$, $q=4$ (I I T 74)

(d) The roots x_1 and x_2 of the equation $x^2+px+12=0$ possess the property $x_1-x_2=1$ Find the value of p

(e) Knowing that 2 and 3 are the roots of the equation $2x^3+mx^2-13x+n=0$, determine m and n and find the third root of the equation

(f) Find all the roots of the equation $4x^4-24x^3+57x^2+18x-45=0$ if one of them is $3+i\sqrt{6}$

Common Roots

28 Find the condition that in the equations $ax^2+bx+c=0$, $a'x^2+b'x+c'=0$

(a) One root be common

(b) A root of first be reciprocal of a root of the second

(c) Both have the same roots

29 If the equations $x^2+px+q=0$ and $x^2+p'x+q'=0$ have a common root show that it must be equal to

$$\frac{pq'-p'q}{q-q'} \text{ or } \frac{q-q'}{p'-p}$$

30 (a) Find k if the equations $4x^2-11x+2k=0$ and $x^2-3x-k=0$ have a common root and obtain the common root for this value of k (I I.T 1976 71)

(b) Determine a such that $x^2-11x+a$ and $x^2-14x+2a$ may have a common factor (Roorkee 81)

31 (a) If the equations $x^2-x-p=0$ and $x^2+2xp-12=0$ have common root, find it.

(ii) The value of $(0\ 2)^{\log_{\sqrt{5}}(1/4+1/8+1/16+\ \infty)}$ is

(a) 1, (b) 2, (c) 1/2 (d) 4

17 The sum of the series

$\frac{1}{2}+\frac{1}{3}+\frac{1}{6}+$ to 9 terms is

(a) $-\frac{5}{6}$, (b) $-\frac{1}{2}$, (c) 1, (d) $-\frac{3}{2}$ (M N R 85)

18 (a) If a, b, c are in H P then

$$\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)=\frac{4}{ac}-\frac{3}{b^2}$$ (M N R 85)

(b) If a, b, c are in A P then

$\frac{a}{bc}, \frac{1}{c}, \frac{2}{b}$ are

(i) A P (ii) G P (iii) H P (iv) None (M N R 82)

(c) If a, b, c are in A P then prove

$(a-c)^2=4(b^2-ac)$ (Roorkee 75)

19 If $\frac{3+5+7+\ \ +n \text{ termrs}}{5+8+11+\ \ +10 \text{ terms}}=7$, the value of n is

(a) 35 (b) 36 (c) 37 (d) 40 (M N R 83)

20 The first term of a G P whose second term is 2 and sum to infinity is 8 will be

(a) 6, (b) 3, (c) 4, (d) 1 (M N R 79)

21 The sum of first n terms of the series $\frac{1}{2}+\frac{3}{4}+\frac{7}{8}+\frac{15}{16}+$ is equal

(a) 2^n-n-1, (b) $1-2^{-n}$, (c) $n+2^{-n}-1$ (d) 2^n-1 (I I T 1988)

22 The sum of first n terms of the series $1^2+2\ 2^2+3^2+2\ 4^2+5^2+2\ 6^2+$ is $n(n+1)^2/2$ when n is even When n is odd, the sum is (*I I T 88*)

23 If the first and $(2n-1)$th terms of an A P, a G P and H P are equal and their n th terms are a, b and c respectively, then

(a) $a=b=c$ (c) $a+c=b$

(b) $a \quad b \geqslant c$ (d) $ac-b^2=0$ (I I T 88)

24 If a, b, c, d and p are distinct real numbers such that $(a^2+b^2+c^2)p^2-2(ab+bc+cd)p+(b^2+c^2+d^2) \leqslant 0$, then $a, b\ c, d$ are in

(i) A P (ii) G P (iii) H P (iv) $ab=cd$

(v) None of these (I I T 87)

(b) If a, b, c are in G P, then the equations $ax^2+2bx+c=0$ and $dx^2+2ex+f=0$ have a common root if $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in

(A) A P (B) G P (C) H P (D) None of these

(I I T 85)

32 (a) If α, β are the roots of $x^2+px+q=0$ and γ, δ are the roots of $x^2+rx+s=0$, evaluate $(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$, in terms of p, q, r and s

Deduce the condition that the equations have a common root (I I T 79)

(b) Eliminate x from the equations

$$a+c=\frac{b}{x}-dx, \; a-c=\frac{d}{x}-bx$$ (I I T 71)

33 (a) If α, β are the roots of $ax^2+bx+c=0$, $\alpha_1, -\beta$ are the roots of $a_1x^2+b_1x+c_1=0$, show that α, α_1 are the roots of

$$\frac{x^2}{\frac{b}{a}+\frac{b_1}{a_1}}+x+\frac{1}{\frac{b}{c}+\frac{b_1}{c_1}}=0$$

(b) If the equations $x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root, then their other roots are the roots of the equation, $x^2+ax+bc=0$

34 (a) Find the value of a so that the equations

$(2a-5)x^2-4x-15=0$ and $(3a-8)x^2-5x-21=0$

have a common root

(b) If the equations $ax^2+2bx+c=0$, $a'x^2+2b'x+c'=0$ have a common root, prove that the equation

$(b^2-ac)x^2+(2bb'-ac'-a'c)x+(b'^2-a'c')=0$

has equal roots

(c) If the equations $x^2+ax+b=0$ and $x^2+bx+a=0$ have a common root then the numerical value of $a+b$ is (I I T 86)

Nature of the roots

35 If the roots of the equation

$(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$

be equal prove that either $a=0$ or $a^3+b^3+c^3=3abc$

36 For what values of m the roots of the equation

$x^2-2x(1+3m)+7(3+2m)=0$ will be equal

37 Determine the values of m for which the equation,

Solutions of Objectives

1 Ans (b) See Q 37 (i) A P

2 Ans (i) Use method of difference

3 (ii)

4 $\alpha, \beta, \gamma, \delta$ being in G P they may be taken as $k, kr,$
$S=k(1+r)=3,\ kr^2(1+r)=12 \quad \therefore \quad 3r^2=12$ or $r=2$
$P=k^2r=a,\ k^2r^5=6$ Putting for r and k, $a=2,\ b=32$

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$$T_n=9+(8\times10+8\times10^2+8\times10^3+\ \ +8\times10^{n-1})$$
$$+(4\times10^n+4\times10^{n+1}+4\times10^{n+2}+\ \ +4\times10$$

$$=9+\frac{80(10^{n-1}-1)}{10-1}+\frac{4\times10^n(10^n-1)}{10-1}$$

$$=\frac{1}{9}[81+80\times10^{n-1}-80+4\times10^{2n}-4\times10^n]$$

$$=\frac{1}{9}[1+2\times4\times10^n-4\times10^n+4\times10^{2n}]$$

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For example, $T_1=7^2,\ T_2=\ \ 7^2,\ T_3=667^2$ etc

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$(a+c)^2-4ac=0$ or $(a-c)^2=0$

Hence $\qquad a=b=c \qquad 2b=2a \qquad a=c$

(iv) is correct

7 Arrange as a quadratic in $n=\sqrt{\left(\frac{a}{b}\right)}$ etc (c) is correct

8 Ans (a) $\quad A>G>H$ or $H<G<A$ $\quad$ ascending

9 (iii)

10 (d), $b^2=ac$ gives $x=-4, -4, -6, -9$

$T_4=-9(3/2)=-13\ 5$

$5x^2-4x+2+m(4x^2-2x-1)=0$ will have

(a) Equal roots (b) Product of roots as 2

(c) Sum of the roots as 6

38 If the roots of the equation,

$$(b-c)x^2+(c-a)x+(a-b)=0$$

be equal, then prove that, a, b, c are in arithmetical progression

39 If $a(b-c)x^2+b(c-a)x+c(a-b)=0$ has equal roots, prove that a, b, c are in harmonical progression

40 Prove that $(x-a)(x-b)+(x-b)(x-c)+(x-c)(x-a)=0$ has equal roots if and only if $a=b=c$

41 Show that the roots of

$$(a^2+b^2)x^2-2b(a+c)x+(b^2+c^2)=0$$

will be real if a, b, c are in G P and in this case these roots will be equal as well

42 If the roots of the equation

$$(a^2+b^2)x^2-2(bc+ad)x+(c^2+d^2)=0,$$

be real, then prove that they will be equal as well and then

$$\frac{a}{b}=\frac{d}{c}$$

43 Examine whether for real values of a and b
$a(x^2-y^2)-bxy$ has real linear factors (I I T 73)

44 Discuss the nature of the roots of the equations

(a) $(a+c-b)x^2+2cx+(b+c-a)=0$

(b) $(b+c)x^2-(a+b+c)x+a=0$

(c) $2(a^2+b^2)x^2+2(a+b)x+1=0$

4 If the roots of the equation $x^2-2cx+ab=0$ be real and unequal, then prove that the roots of $x^2-2(a+b)x+(a^2+b^2+2c^2)=0$ will be imaginary

46 (i) Show that if p, q, r, s are real numbers and $pr=2(q+s)$ then at least one of the equations $x^2+px+q=0$ $x^2+rx+s=0$ has real roots (I I T 75)

(ii) If $P(x)=ax^2+bx+c$ and $Q(x)=-ax^2+dx+c$ where $ac\neq 0$ then $P(x)\,Q(x)=0$ has at least two real roots (I I T 85)

(iii) If $a<b<c<d$, then the roots of the equation $(x-a)(x-c)+2(x-b)(x-d)=0$ are real and distinct

(a) True, (b) False (I I T 84)

47 (a) If the roots of the equation $x^2+a^2=8x+6a$ be real then prove that a lies between -2 and 8

11 (b)

12 (ii) $b^2=ac$ satisfies (ii)

13 (iii) $(9)^{S_\infty}=9^{1/2}=3$ $\qquad S_\infty=\frac{1/3}{1-1/3}=1/2$

14 (c) Hint 2 357 357 357 $=2+357+000357+0000003$
+ etc

15 Ans (b)

$$\frac{1}{b-a}+\frac{1}{b-c}=\frac{1}{a}+\frac{1}{c} \text{ or } \frac{1}{b-a}-\frac{1}{c}=\frac{1}{a}-\frac{1}{b-c}$$

or $\frac{c-b+a}{c(b-a)}=\frac{b-c-a}{a(b-c)}$

$-\frac{1}{c(b-a)}=\frac{1}{a(b-c)}$ or $ac-bc=ab-ac$

or $b=\frac{2ac}{a+c}$ Hence a, b, c are in H P

16 (i) Ans (b)

The given expression$=2^{1/4+2/8+3/16+4/32}$

Now let $S=\frac{1}{4}+\frac{2}{8}+\frac{3}{16}+\frac{4}{32}+$

$\frac{1}{2}S= \quad \frac{1}{8}+\frac{2}{16}+\frac{3}{32}+$

Subtracting, $\frac{1}{2}S=\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}+ \quad =\frac{\frac{1}{4}}{1-\frac{1}{2}}=\frac{1}{2}$

$S=1$

Hence the given expression$=2^1=2$

(ii) Ans (d)

17 Ans (d)

This is a $\qquad \frac{1}{6}$

Sum$=\frac{9}{2}\{$

3

2

(b) Prove that if the roots of $9x^2+4ax+4=0$ are imaginary then a must lie between -3 and 3

48 (a) If p, q, r are real and $p \neq q$ then roots of the equation $(p-q)x^2-5(p+q)x-2(p-q)=0$ are

(a) Real and equal (b) Complex (c) Real and unequal
(d) None of these (IIT 79)

(b) The roots of $px^2+2qx+r=0$ and $qx^2-2\sqrt{(pr)}\,x+q=0$ are simultaneously real then

(a) $p=q,\ r\neq 0$ (b) $p/q=q/r$ (c) $2q=\pm\sqrt{(pr)}$ (d) none of these

49 Prove that the roots of
$(a-b)^2x^2+2(a+b-2c)x+1=0$ are real or imaginary according as c does not or does lie between a and b, $a<b$

50 If $a=2+i\sqrt{(3)}$ then find the value of

(a) $4x^3+8x+35$

(b) If $2+i\sqrt{3}$ is a root of $x^2+px+q=0$ where p, q are real then $(p, q)=(-,\ )$ (ILT 82)

(c) If $x=1+2i$ then prove that
$$x^3+7x^2-13x+16=-29$$

(d) Find he quadratic equation one of whose roots is $2+\sqrt{3}$ and hence find the value of expression $x^4-7x^2+13x-2$ for $x=2+\sqrt{3}$

Algebraic Expressions

51 (a) Find for what real values of x the expression
(i) x^2-2x-3 (ii) $2x^2+5x-3$ is positive or negative

(b) For real values of x, prove that the value of the expression $\dfrac{11x^2+12x+6}{x^2+4x+2}$ can not lie between -5 and 3

52 (a) Prove that for real values of x the expression $\dfrac{(x-1)(x+3)}{(x-2)(x+4)}$ cannot lie between $\dfrac{4}{9}$ and 1

(b) If x is real, the expression $\dfrac{x^2+2x-11}{x-3}$ takes all values which do not lie between 4 and 12 (IIT 72)

(c) If x is real, show that the expression $(x^2-bc)/(2x-b-c)$ has no real values between b and c

53 If x is real, find maximum and minimum values of
$$\frac{x^2+14x+9}{x^2+2x+3}$$

54 (a) If x is real show that $\dfrac{x^2-x+1}{x^2+x+1}$ takes values from $\frac{1}{3}$ to 3

$=\frac{1}{b^2}-\frac{4}{b^2}+\frac{4}{ac}$ by (I) $=\frac{4}{ac}-\frac{3}{b^2}$

(b) a, b, c are in A P Dividing by bc we get $\frac{a}{bc}, \frac{1}{c}, \frac{1}{b}$ are also in A P Hence either the last term should be $\frac{1}{b}$ in place of $\frac{2}{b}$ in the question and in that Case (i) is correct answer, otherwise (iv) is correct answer

(c) R H S $4b^2-4ac=(2b)^2-4ac=(a+c)^2-4ac$

$=(a-c)^2=$ L H S

19 L H S $\frac{n(n+2)}{5\ 37}=7 \qquad n(n+2)=35\ 37$

Above will hold when $n=35$, (a) is correct

20 $ar=2, \frac{a}{1-r}=\frac{1}{8}$ Eliminate r and we get

$a-8a+16=0 \qquad (a-4)^2=0$ or $a=4$ (c) is correct

21 Ans (c)

Hint $T_n=\frac{2^n-1}{2^n}=1-(\frac{1}{2})^n$

$S_n=n-\sum_{n=1}^{n}(\frac{1}{2})^n$

$=n-\frac{1/2\{1-(1/2)^n\}}{1-1/2}$

$=n+2^{-n}-1$

22 Ans $\frac{(n-1)n^2}{2}+n^2=(n+1)n^2/2$

Hint When n is odd, the last term will be n^2 and the sum of first $(n-1)$ even terms is obtained by replacing n by $n-1$ in the given formula

23 (b) and (d)

Let α be the first and β the $(2n-1)$th terms of an A P, a G P and an H P whose nth terms are a, b and c respectively

For A P $\beta=\alpha+(2n-2)d$ or $d=\frac{\beta-\alpha}{2(n-1)}$

$a=\alpha+(n-1)d=\alpha+\frac{\beta-\alpha}{2}=\frac{\alpha+\beta}{2}$

For G P $\beta=\alpha r^{2(n-1)}$ or $r=(\beta/\alpha)^{1/2(n-1)}$

$b=\alpha r^{n-1}=\alpha(\beta/\alpha)^{1/2}=\sqrt{(\alpha\beta)}$

(b) Show that $\frac{x^2-3x+4}{x^2+3x+4}$ can never be greater than 7 nor less than $1/7$ for real values of x

(c) Find out the range in which the value of the function $\frac{x^2+34x-71}{x^2+2x-7}$ lies for all real values of x Justify your answer (Roorkee 83)

55 (a) Find the value of x for which the following inequality holds $\frac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ (I I T 77)

(b) Find the values of x, which satisfy the inequality

$$\frac{x-2}{x+2} > \frac{2x-3}{4x-1}$$

56 (a) Show that if x is real, the expressio $\frac{x^2-bc}{2x-b-c}$ has no real values between b and c

(b) For real x, the function $\frac{(x-a)(x-b)}{x-c}$ will assume all real values provided

(a) $a > b > c$, (b) $a < b < c$,
(c) $a > c > b$, (d) $a < c < b$ (I I T 84)

57 (a) Let $y=\sqrt{\left(\frac{(x+1)(x-3)}{x-2}\right)}$

Find all the real values of x for which y takes real values (I I T 80)

(b) Find the set of all x for which

$$\frac{2x}{(2x^2+5x+2)} > \frac{1}{(x+1)}$$ (I I T 87)

58 Show that the expression $\frac{px^2+3x-4}{p+3x-4x^2}$ will be capable of all values when x is real, provided that p has any value between 1 and 7

59 If 1, a_1, a_2, , a_{n-1} are the n nth roots of unity, then show that $(1-a_1)(1-a_2)(1-a_3)$ $(1-a_{n-1})=n$ (I I T 84)

60 If α and β be the roots of the equation $x^2-ax+b=0$ and $V_n=\alpha^n+\beta^n$ then show that $V_{n+1}=aV_n-bV_{n-1}$ Hence obtain the value of $\alpha^5+\beta^5$

61 If $f(x)=a_1x^2+b_1x+c_1$ and α, β are the roots of the equation $ax^2+bx+c=0$, prove that $f(\alpha)$ $f(\beta)$

For H P $\frac{1}{\beta}=\frac{1}{\alpha}+(2n-2)\,d$ or $d=\frac{\alpha-\beta}{2\alpha\beta(n-1)}$

$\frac{1}{c}=\frac{1}{\alpha}+(n-1)\,d=\frac{1}{\alpha}+(n-1)\frac{\alpha-\beta}{2\alpha\beta(n-1)}=\frac{\alpha+\beta}{2\alpha\beta}$

$c=2\alpha\beta/(\alpha+\beta)$

Now $a-b=\frac{\alpha+\beta}{2}-\sqrt{(\alpha\beta)}=\frac{1}{2}(\sqrt{\alpha}-\sqrt{\beta})^2\geqslant 0$ $\quad a\geqslant b$

Similarly we can prove that $b-c \geqslant 0$ or $b\geqslant c$

Hence $a\geqslant b \geqslant c$

Also $ac=\frac{\alpha+\beta}{2}\cdot\frac{2\alpha\beta}{\alpha+\beta}=\alpha\beta=b^2$ or $ac-b^2=0$

24 The given relation can be re written as,

$(a^2p^2-2abp+b^2)+(\qquad)+(\qquad)\leqslant 0$

or $(ap-b)^2+(bp-c)^2+(cp-d)^2\leqslant 0$ (I)

Since a b, c, d and p are all real the solution (I) is possible only when each of the factors is zero i e $ap-b=0$ $bp-c=0$, $cp-d=0$

or $p=\frac{b}{a}=\frac{c}{b}=\frac{d}{c}$ or a, b, c, d are in G P (ii) is correct

$$=\frac{(c_1a-ca_1)^2-(ab_1-a_1b)(bc_1-b_1c)}{a^2}$$

62 Prove that if the coefficients of the quadratic

$$ax^2+bx+c=0$$

are odd integers, then prove that the roots of the equation cannot be rational numbers

63 If α, β are the roots of $x^2+px+q=0$ and also of

$$x^{2n}+p^n x^n+q^n=0$$

and if $\frac{\alpha}{\beta}$, $\frac{\beta}{\alpha}$ are the roots of $x^n+1+(x+1)^n=0$, then prove that n must be an even integer

Solutions to Problem Set (A)

1 Since α and β are the roots of $ax^2+bx+c=0$

$$a\alpha^2+b\alpha+c=0 \quad \text{or} \quad a\alpha+b=-\frac{c}{\alpha}$$

$$a\beta^2+b\beta+c=0 \quad \text{or} \quad a\beta+b=-\frac{c}{\beta}$$

Also $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=\frac{c}{a}$

(a) $\frac{1}{a\alpha+b}+\frac{1}{a\beta+b}=-\frac{\alpha}{c}-\frac{\beta}{c}=-\frac{1}{c}(\alpha+\beta)=-\frac{1}{c}\left(-\frac{b}{a}\right)=\frac{b}{ac}$

(b) $\frac{\beta}{a\alpha+b}+\frac{\alpha}{a\beta+b}=-\frac{\alpha\beta}{c}-\frac{\alpha\beta}{c}=-\frac{2}{c}\cdot\frac{c}{a}=-\frac{2}{a}$

(c) $(a\alpha+\beta)^{-3}+(a\beta+b)^{-3}=-\frac{\alpha^3+\beta^3}{c^3}=-\frac{1}{c^3}[(\alpha+\beta)^3-3\beta(\alpha+\beta)]$

$$=\frac{b^3-3abc}{a^3c^3}$$

(d) $(a\alpha+b)^{-2}+(a\beta+b)^{-2}=\frac{\alpha^2+\beta^2}{c^2}=\frac{1}{c^2}[(\alpha+\beta)^2-2\alpha\beta]$

$$=\frac{b^2-2ac}{a^2c^2}$$

2. $\alpha+\beta=-\frac{b}{a}$, $\alpha\beta=-\frac{c}{a}$

(a) Sum $=\frac{1}{\alpha+\beta}+\frac{\alpha+\beta}{\alpha\beta}=-\frac{a}{b}-\frac{b}{c}=-\frac{(ac+b^2)}{bc}$

Product $\frac{1}{\alpha+\beta}\cdot\frac{\alpha+\beta}{\alpha\beta}=\frac{1}{\alpha\beta}=\frac{a}{c}$

Equation is $x^2-xS+P=0$

$$x^2+\frac{ac+b^2}{bc}x+\frac{a}{c}=0 \text{ or } bc\,x^2+(b^2+ac)\,x+ab=0$$

Theory of Quadratic Equations 4

§ 1 Roots of the equation

$$ax^2+bx+c=0$$

Multiplying both sides by $4a$, we get

$$4a^2x^2+4abx=-4ac$$

or $4a^2x^2+4abx+b^2=b^2-4ac$ Add b^2 to both sides

$$(2ax+b)^2=b^2-4ac$$

$$2ax+b=\pm\sqrt{(b^2-4ac)}$$ Take square root

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

Sum and Product of the roots

If α and β be the roots, then

$$\alpha=\frac{-b+\sqrt{(b^2-4ac)}}{2a},\ \beta=\frac{-b-\sqrt{(b^2-4ac)}}{2a}$$

Sum of the roots

$$=\alpha+\beta=-\frac{2b}{2a}=-\frac{b}{a}$$

or $\alpha+\beta=-\frac{b}{a}=-\frac{\text{coeff of } x}{\text{coeff of } x^2}$

Product of roots

$$=\alpha\beta=\frac{(-b)^2-\{\sqrt{(b^2-ac)}\}^2}{4a^2}=\frac{4ac}{4a^2}=\frac{c}{a}$$

or $\alpha\beta=\frac{\text{constant term}}{\text{coeff of } x^2}$

§ 2 To find the equation whose roots are α and β

The required equation will be $(x-\alpha)(x-\beta)=0$

or $x^2-(\alpha+\beta)\,x+\alpha\beta=0$

or $x^2-Sx+P=0$

where S is sum and P is product e g If the roots be 3, −7

then $S=\text{Sum}=3-7=-4$,

$P=\text{Product}=-21$

Equation $x^2-Sx+P=0$

or $x^2-(-4)\,x+(-21)=0$

or $x^2+4x-21=0$

(b) $acx^2-(b^2-2ac)x+ac=0$

(c) $acx^2+b\,(a+c)\,x+(a+c)^2=0$

(d) $S=\alpha^2+\beta^2+\frac{1}{\alpha^2}+\frac{1}{\beta^2}=(\alpha^2+\beta^2)\left(1+\frac{1}{\alpha^2\beta^2}\right)$

$=\frac{b^2-2ac}{a^2}\;\frac{a^2+c^2}{c^2}$ (Page 187)

$P=(\alpha^2+\beta^2)\;\frac{(\alpha^2+\beta^2)}{\alpha^2\beta^2}=\frac{(b^2-2ac)^2}{a^2c^2}$

$a^2c^2x^2-(b^2-2ac)\,(a^2+c^2)\,x+(b^2-2ac)^2=0$

(e) By question (1) the roots are $-\frac{\alpha}{c}$, $-\frac{\beta}{c}$

$ac\,x^2-bx+1=0$

3 By virtue of the given relations we can say that α and β satisfy $x^2=5x-3$ or they are the roots of

$x^2-5x+3=0 \qquad \alpha+\beta=5,\ \alpha\beta=3$

$S=19/3$, $P=1$ and the equation is $3x^2-19x+3=0$

4 The given equation is $x^2-px-(p+c)=0$

$\alpha+\beta=p,\ \alpha\beta=-(p+c)$

$(\alpha+1)\,(\beta+1)=\alpha\beta+(\alpha+\beta)+1=-p-c+p+1=1-c$ (1)

Again $\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}$

$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(1-c)}+\frac{(\beta+1)^2}{(\beta+1)^2-(1-c)}$

$=\frac{(\alpha+1)^2}{(\alpha+1)^2-(\alpha+1)(\beta+1)}+\frac{(\beta+1)^2}{(\beta+1)^2-(\alpha+1)\,(\beta+1)}$ by 1

$\frac{\alpha+1}{\alpha-\beta}+\frac{\beta+1}{\beta-\alpha}=\frac{(\alpha+1)-(\beta+1)}{\alpha-\beta}=1$

5. By given condition

$(x-a)\,(x-b)-k=(x-c)\,(x-d)$

$(x-c)\,(x-d)+k=(x-a)\,(x-b)$

Above shows that the roots of $(x-c)\,(x-d)+k$ are a and b

6 Let the correct equation be $x^2+ax+b=0$ (1)

Roots found by Ramesh are 8, 2 *i e* $S=10$, $P=16$

Equation is $x^2-10x+16=0$ (2)

Since he commited mistake only in constant term $a=-10$

Roots found by Mahesh are -9, -1, $S=-10$, $P=9$

$\therefore$ Equation is $x^2+10x+9=0$

Since he commited mistake in the coefficient of x $b=9$

If $\beta < x < \alpha$, Then $x-\alpha > 0$ and $x-\beta < 0$

so that $(x-\alpha)(x-\beta) < 0$ It follows that the sign of ax^2+bx+c is opposite to that of a in this case

But if $x > \alpha$ or $x < \beta$, then $(x-\alpha)(x-\beta) > 0$ since the factors $x-\alpha$ and $x-\beta$ are either both +ive or both −ive

So in this case the sign of ax^2+bx+c is the same as that of a.

Case III Let the roots α, β be equal Then

$ax^2+bx+c \equiv a(x-\alpha)^2$

and $(x-\alpha)^2$ is positive for all real values of x and therefore ax^2+bx+c has the same sign as a

From cases I, II and III, we concluede

For all real values of x, the expression ax^2+bx+c has the same sign as a except when the roots of the equation $ax^2+bx+c=0$ are real and unequal, and x has a value lying between them

Example (i) The sign of x^2+4x+8 is positive for all real x since $b^2-4ac=16-32 < 0$ and $a=1 > 0$ (See case I of § 4)

(ii) The sign of $-2x^2+7x-11$ is −ve for all real x since $b^2-4ac=49-4(-2)(-11) < 0$ and $a=-2 < 0$ (See case I of § 4)

(iii) The sign of $x^2+5x+6=(x+2)(x+3)$ is +ive if either $x < -3$ or $x > -2$ and −ive if $-3 < x < -2$ (See case II of § 4)

(iv) The sign of $-9x^2+12x-4=-(3x-2)^2$ is −ive for all real x

Problem Set A

1 If α and β are the roots of $ax^2+bx+c=0$, find the following—

(a) $\dfrac{1}{a\alpha+b}+\dfrac{1}{a\beta+b}$ (b) $\dfrac{\beta}{a\alpha+b}+\dfrac{\alpha}{a\beta+b}$

(c) $(a\alpha+b)^{-3}+(a\beta+b)^{-3}$ (d) $(a\alpha+b)^{-4}+(a\beta+b)^{-4}$

2 If α and β are the roots of $ax^2+bx+c=0$ Find the equation whose roots are as given below

(a) $\dfrac{1}{\alpha+\beta}, \dfrac{1}{\alpha}+\dfrac{1}{\beta}$ (b) $\dfrac{\alpha}{\beta}, \dfrac{\beta}{\alpha}$

(c) $\alpha+\dfrac{1}{\beta}$ $\beta+\dfrac{1}{\alpha}$ (d) $\alpha^2+\beta^2, \dfrac{1}{\alpha^2}+\dfrac{1}{\beta^2}$

(e) $\dfrac{1}{a\alpha+b}$ $\dfrac{1}{a\beta+b}$

3 If $\alpha \neq \beta$, but $\alpha^2=5\alpha-3$, $\beta^2=5\beta-3$, find the equation whose roots are α/β and β/α

Having found $a=-10$, $b=9$ the required equation is

$$x^2-10x+9=0 \quad \text{or} \quad (x-9)(x-1)=0$$

i e correct roots are 9, 1

7 Proceed as above $p=7$, $q=12$, roots -3, -4

8 Here $q=-2\times-15=30$, correct value of $p=13$

$x^2+13x+30=0$ or $(x+10)(x+3)=0$ Roots are $-10, -3$

9 $\alpha+\beta=2a$ or $\frac{\alpha+\beta}{2}=a$, $\gamma\delta=a^2$ $\sqrt{(\gamma\delta)}=a$

∴ A M of roots of 1st = G M of roots of 2nd

10 $\alpha+\beta=3/2$, $\alpha\beta=-6/2=-3$

$$S=\alpha^2+\beta^2+4=(\alpha+\beta)^2-2\alpha\beta+4=49/4$$

$$P=\alpha^2\beta^2+2(\alpha^2+\beta^2)+4 \quad \alpha^2\beta^2+4+2[(\alpha+\beta)^2-2\alpha$$

$$=\frac{118}{4}$$

11 (a) $x^2-6x+11=0$, (b) $3x^2-2x+1=0$

12 (a) $4x^2+2x-1=0$ $\alpha+\beta=-\frac{1}{2}$, $\alpha\beta=-\frac{1}{4}$

Also $4\alpha^2+2\alpha-1=0$ as α is a root and we have to pr

that $\beta=4\alpha^3-3\alpha$

Now $4\alpha^3-3\alpha=4\alpha^2\,\alpha-3\alpha=\alpha\,(1-2\alpha)-3\alpha$

$$=-2\alpha^2-2\alpha=-\tfrac{1}{2}[4\alpha^2+4\alpha)=-\tfrac{1}{2}[1-2\alpha+$$

$$=-\tfrac{1}{2}(1+2\alpha)=-\tfrac{1}{2}-\alpha=\beta \quad \alpha+\beta=-\tfrac{1}{2}$$

Hence the other root β is $4\alpha^3-3\alpha$

(b) On rationlizing, the roots are

$$\frac{a}{b}[\sqrt{a}\pm\sqrt{(a-b)}] \qquad S=\frac{2a\sqrt{a}}{b}, P=\frac{a^2}{b}$$

Ans $bx^2-2a\sqrt{a}\,x+a^2=0$

13 $\alpha+\beta=-b/a$ $\alpha\beta=c/a$, $\gamma+\delta=-m/l$ $\gamma\delta=n/l$

$$\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta=\frac{b^2-2ac}{a^2},$$

$$\gamma^2+\delta^2=\frac{m^2-2nl}{l^2}$$

$$S=(\alpha\gamma+\beta\delta)+(\alpha\delta+\beta\gamma)=\alpha(\gamma+\delta)+\beta(\gamma+\delta)$$

$$=(\alpha+\beta)(\gamma+\delta)=\frac{bm}{al}$$

$$P=(\alpha\gamma+\beta\delta)(\alpha\delta+\beta\gamma)$$

$$=\alpha^2\gamma\delta+\alpha\beta\,\delta^2+\alpha\beta\,\gamma^2+\beta^2\,\gamma\delta$$

$$=(\alpha^2+\beta^2)\gamma\delta+\alpha\beta(\gamma^2+\delta^2)$$

4 If α and β are the roots of $x^2-p(x+1)-c=0$, show that $(\alpha+1)(\beta+1)=1-c$. Hence prove that

$$\frac{\alpha^2+2\alpha+1}{\alpha^2+2\alpha+c}+\frac{\beta^2+2\beta+1}{\beta^2+2\beta+c}=1$$

5 If the roots of the equation $(x-a)(x-b)-k=0$ be c and d then prove that the roots of the equation $(x-c)(x-d)+k=0$, are a and b

6 Ramesh and Mahesh solve an equation. In solving Ramesh commits a mistake in constant term and finds the roots 8 and 2. Mahesh commits a mistake in the coefficient of x and finds the roots -9 and -1. Find the correct roots.

7 Two candidates attempt to solve a quadratic of the form $x^2+px+q=0$. One starts with a wrong value of p and finds the roots to be 2 and 6. The other starts with a wrong value of q and finds the roots to be $2, -9$. Find the correct roots.

8 The coefficient of x in the quadratic equation $x^2+px+q=0$ was taken as 17 in place of 13, its roots were found to be -2 and -15. Find the roots of the original equation. (I I T 77)

9 Prove that A.M. of the roots of $x^2-2ax+b^2=0$ is equal to the geometric mean of the roots of the equation, $x^2-2bx+a^2=0$, and vice versa.

10 If α and β are the roots of the equation $2x^2-3x-6=0$, find the equation whose roots are α^2+2, β^2+2

11 If α, and β are the roots of the equation $x^2-2x+3=0$, find the equation whose roots are—

(a) $\alpha+2, \beta+2$, (b) $\frac{\alpha-1}{\alpha+1}, \frac{\beta-1}{\beta+1}$

12 (i) If α be a root of the equation $4x^2+2x-1=0$, prove that $4\alpha^3-3\alpha$ is the other root.

(ii) Form a quadratic equation whose roots are $\frac{a}{\sqrt{a}\pm\sqrt{(a-b)}}$

13 If α, β be the roots of $ax^2+bx+c=0$ and γ, δ those of $lx^2+mx+n=0$, then find the equation whose roots are $\alpha\gamma+\beta\delta$ and $\alpha\delta+\beta\gamma$

14 If α β be the roots of $x^2-px+q=0$ and α', β' be those of $x^2-p'x+q=0$ find the value of

$$(\alpha-\alpha')^2+(\beta-\alpha')^2+(\alpha-\beta')^2+(\beta-\beta')^2$$

15 If α β be the roots of $ax^2+2bx+c=0$ and $\alpha+\delta$, $\beta+\delta$ be those $Ax^2+2Bx+C=0$, then prove that

$$=\frac{n}{l}\left(\frac{b^2-2ac}{a^2}\right)+\frac{c}{a}\left(\frac{m^2-2nl}{l^2}\right)$$
$$=\frac{ln\,(b^2-2ac)+ac\,(m^2-2nl)}{a^2l^2}$$
$$=\frac{b^2nl+m^2ac-4acnl}{a^2l^2}$$

Required equation is $x^2-xS+P=0$

14 On simplification the given expression
$$=2\,(\alpha^2+\beta^2+\alpha'^2+\beta'^2)-2\,(\alpha+\beta)\,(\alpha'+\beta')$$
$$=2\,[p^2-2q+p'^2-2q'-pp']$$

15 Let the roots of the second equation be denoted by α' and β'
$$\alpha'=\alpha+\delta,\ \beta'=\beta+\delta$$
$$\alpha'-\beta'=(\alpha+\delta)-(\beta+\delta)=\alpha-\beta$$
$$(\alpha'-\beta')^2=(\alpha-\beta)^2$$
or $(\alpha'+\beta')^2-4\alpha'\beta'=(\alpha+\beta)^2-4\alpha\beta$

or $\frac{4B^2}{A^2}-4\frac{C}{A}=4\frac{b^2}{a^2}-4\frac{c}{a}$ or $\frac{b^2-ac}{a^2}=\frac{B^2-AC}{A^2}$

or $\frac{b^2-ac}{B^2-AC}=\left(\frac{a}{A}\right)^2$

16 $\alpha+\beta=-p,\ \gamma+\delta=-p,\ \alpha\beta=-q,\ \gamma\delta=r$ $\quad\alpha+\beta=\gamma+\delta$

Hence $(\alpha-\gamma)\,(\alpha-\delta)=\alpha^2-\alpha\,(\gamma+\delta)+\gamma\delta$
$$=\alpha^2-\alpha\,(\alpha+\beta)+\gamma\delta$$
$$=-\alpha\beta+\gamma\delta=q+r$$

Similarly
$$(\beta-\gamma)\,(\beta-\delta)=q+r$$

17 $\alpha+\beta=-p\ \ \alpha\beta=1,\ \gamma+\delta=-q,\ \gamma\delta=1$

R H S $=[\alpha\beta-\gamma\,(\alpha+\beta)+\gamma^2]\,[\alpha\beta+\delta\,(\alpha+\beta)+\delta^2]$
$$=(\gamma^2+p\gamma+1)\,(\delta^2-p\delta+1)$$

But γ and δ are the roots of $x^2+qx+1=0$
$$\gamma^2+q\gamma+1=0\quad\text{or}\quad\gamma^2+1=-q\gamma$$
$$\delta^2+q\delta+1=0\quad\text{or}\quad\delta^2+1=-q\delta$$

R H S $=(p\gamma-q\gamma)\,(-q\delta-p\delta)$
$$=-\gamma\delta\,(p-q)\,(p+q)=-1\,(p^2-q^2)=q^2-p^2$$

18 $\alpha=\frac{1}{\beta}\qquad\alpha\beta=1$ or $\frac{2}{p}=1\qquad p=2$

Hence (d) is the answer

19 $\alpha+\beta=-(m+n),\ \alpha\beta=\frac{m^2+n^2}{2}$
$$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=(m+n)^2-2\,(m^2+n^2)$$

$$\theta = 2n\pi + \frac{\pi}{4} \pm \frac{\pi}{4}, 2n\pi - \frac{\pi}{4} + \frac{\pi}{4}$$

i e $\theta = 2n\pi + \pi/2$, or $2n\pi$

(*b*) $\sqrt{3} \cos \theta - 3 \sin \theta = 2 (\sin 5\theta - \sin \theta)$

$\sqrt{3} \cos \theta - \sin \theta = 2 \sin 5\theta$ Divide by $\sqrt{(3+1)} = 2$

$$\frac{\sqrt{3}}{2} \cos \theta - \frac{1}{2} \sin \theta = \sin 5\theta$$

$$\sin \frac{\pi}{3} \cos \theta - \cos \frac{\pi}{3} \sin \theta = \sin 5\theta$$

$$\sin \left(\frac{\pi}{3} - \theta\right) = \sin 5\theta \quad \text{or} \quad \sin 5\theta = \sin \left(\frac{\pi}{3} - \theta\right)$$

$5\theta = n\pi + (-1)^n (\pi/3 - \theta)$

n even $= 2r$ $\qquad 5\theta = 2r\pi + \pi/3 - \theta$

or $6\theta = 2r\pi + \frac{\pi}{3}$ $\qquad \theta = \frac{r\pi}{3} + \frac{\pi}{18}$

n odd $= 2r+1$ $\qquad 5\theta = (2r+1)\pi - \pi/3 + \theta$

or $4\theta = 2r\pi + \pi - \pi/3 = 2r\pi + 2\pi/3$

$$\theta = \frac{r\pi}{2} + \frac{\pi}{6}$$

(*c*) $\frac{1}{2}\sqrt{3} \sin x = \cos x + \cos^2 x$ Square

$3 (1 - \cos^2 x) = 4 (\cos^2 x + 2 \cos^3 x + \cos^4 x)$

or $4 \cos^4 x + 8 \cos^3 x + 7 \cos^2 x - 3 = 0$

Clearly $\cos x = -1$ satisfies above hence it can be written as

$(\cos x + 1) (4 \cos^3 x + 4 \cos^2 x + 3 \cos x - 3) = 0$

Again $\cos x = \frac{1}{2}$ satisfies the 2nd factor

$(\cos x + 1) (2 \cos x - 1) (2 \cos^2 x + 3 \cos x + 3) = 0$

$\cos x = -1 = \cos \pi$ $\qquad x = 2n\pi + \pi = (2n+1)\pi$

$\cos x = \frac{1}{2} = \cos \pi/3$ $\qquad x = 2n\pi \pm \pi/3$

$2 \cos^2 x + 3 \cos x + 3 = 0$ $\qquad B^2 - 4AC = 9 - 4 \cdot 2 \cdot 3 = -$ive

Hence this factor does not give any real values of $\cos x$

(*d*) $2 (\cos x + 2 \cos^2 x - 1) + 2 \sin x \cos x (1 + 2 \cos x)$
$- 2 \sin x = 0$

or $2 (2 \cos^2 x + \cos x - 1) + 2 \sin x (2 \cos^2 x + \cos x - 1) = 0$

$2(1 + \sin x) (\cos x + 1) (2 \cos x - 1) = 0$

We have to determine values of x s t $-\pi \leqslant x \leqslant -\pi$

$1 + \sin x = 0$ $\qquad \sin x = -1 = \sin (-\pi/2)$

$x = n\pi - (-1)^n \frac{\pi}{2}$ $\qquad x = -\frac{\pi}{2}$

$\cos x = -1 = \cos \pi$ $\qquad x = 2n\pi \pm \pi$, $\quad x = -\pi, \pi$

$2\cos x-1=0 \qquad \cos x=\frac{1}{2}=\cos\frac{\pi}{3} \qquad x=2n\pi\pm\frac{\pi}{3}$

$x=\frac{\pi}{3}, \frac{-\pi}{3}$

Hence the values of x s t $-\pi\leqslant x\leqslant\pi$ are

$-\pi, -\frac{\pi}{2}, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$

(d_1) L H S $=2(\sin x-\cos 2x)-2\sin x\cos x$

$-2\sin x\cos x$

or $(\sin x-\cos 2x)-\sin x\cos x(2\sin x)+2\cos x=0$

or $(\sin x-\cos 2x)-\cos x(\sin x-\cos 2x)-0$

or $(\sin x-\cos 2x)(1-\cos x)=0$

$\cos x=1=\cos 0°$

or $\sin x-\cos 2x=0 \qquad x=2n\pi$

$\cos 2x=\cos(\pi/2-x)$

$2x=2n\pi\pm(\pi/2-x)$

$x=\frac{\pi}{6}\left(4n+1\right)$ or $\frac{\pi}{2}\left(4n-1\right)$

(e) $(y+\cos x)^2=\sin^2 x=1-\cos^2 x$

or $y^2+2y\cos x+2\cos^2 x-1=0$

$2\cos^2 x+2y\cos x+(y^2-1)=0$ (1)

For a real solution the discriminant of the above equation should be $\geqslant 0$

or $4y^2-8(y^2-1)\geqslant 0$

$y^2-2y^2+2\geqslant 0$ or $2-y^2\geqslant 0$

Hence y must lie between $-\sqrt{2}$ and $\sqrt{2}$ and

$-\sqrt{2}\leqslant y\leqslant\sqrt{2}$

If $y=1$ then from (1) we have

$2\cos^2 x+\cos x=0$ or $2\cos x(\cos x+1)=0$

$\cos x=0$ or $x=\frac{\pi}{2}$

$\cos x+1=0$ or $\cos x=-1$

$x=\frac{\pi}{2}\ \pi \qquad x=\pi$

(f) We have to find the values of x s t $0<x\leqslant 2\pi$

$\frac{3+2i\sin\theta}{1-2i\sin\theta}$

$=\frac{3+2i\sin\theta}{1-2i\sin\theta}\times\frac{1+2i\sin\theta}{1+2i\sin\theta}$

$=\frac{(3-4\sin^2\theta)+i(8\sin\theta)}{1+4\sin^2\theta}$

It will be purely real if $I\ P=0$ i e $\sin\theta=0 \qquad \theta=n\pi$

$= -(m^2+n^2-2mn) = -(m-n)^2,$

and $(\alpha+\beta)^2 = (m+n)^2$

$\therefore\ S = (\alpha+\beta)^2 + (\alpha-\beta)^2 = (m+n)^2 - (m-n)^2 = 4mn$

$P = (\alpha+\beta)^2\,(\alpha-\beta)^2 = -(m+n)^2\,(m-n)^2$

$= -(m^2-n^2)^2$

$\therefore$ Required equation is

$x^2 - 4mnx - (m^2-n^2)^2 = 0$

20 (a) Let one root be α then the other root will be $n\alpha$ by given condition

$$\text{Sum} = \alpha + n\alpha = -\frac{b}{a} \quad \text{or} \quad \alpha = -\frac{b}{a\,(1+n)} \qquad (1)$$

$$\text{Product} = n\alpha^2 = \frac{c}{a} \qquad \alpha^2 = \frac{c}{an} \qquad (2)$$

$$\text{Now} \quad \alpha^2 = (\alpha)^2 \quad \therefore \quad \frac{c}{an} = \frac{b^2}{a^2(1+n)^2} \text{ by (1) and (2)}$$

$\therefore\ nb^2 = ac\,(n+1)^2$

(b) Putting $n=3$ the condition is $3b^2 = 16ac$

(c) Putting $n=1$ the condition is

$b^2 = 4ac$ i e $b^2 - 4ac = 0$

(d) We have $\dfrac{k+1}{k} + \dfrac{k+2}{k+1} = \dfrac{-b}{a}$ and $\dfrac{k+1}{k}\cdot\dfrac{k+2}{k+1} = \dfrac{c}{a}$

or $\dfrac{k+2}{k} = \dfrac{c}{a}$ or $\dfrac{2}{k} = \dfrac{c}{a} - 1 = \dfrac{c-a}{a}$ $\quad k = \dfrac{2a}{c-a}$

Now eliminate k

Putting the value of k in Ist relation we get

$$\frac{c+a}{2a} + \frac{2c}{(c+a)} = -\frac{b}{a}$$

or $(a+c)^2 + 4ac = -2b(a+c)$

or $(a+c)^2 + 2b(a+c) = -4ac$ Add b^2 to both sides

$\therefore\ (a+c+b)^2 = b^2 - 4ac$

21 (a) Here $S = \alpha + \alpha^2 = -\dfrac{b}{a}$

$$P = \alpha\,\alpha^2 = \frac{c}{a} \quad \text{or} \quad \alpha^3 = \frac{c}{a}$$

We have to eliminate α to get the required condition Cubing Ist relation we get

$$(\alpha+\alpha^2)^3 = -\frac{b^3}{a^3} \quad \text{or} \quad \alpha^3 + \alpha^6 + 3\alpha\,\alpha^2(\alpha+\alpha^2) = -\frac{b^3}{a^3}$$

or $16(3-y)^2-8(11-y)(3-y) \geqslant 0$ i e +ive

$8(3-y)[6-2y-11+y] \geqslant 0$

$8(3-y)(-5-y) \geqslant 0$

or $8(y-3)(y+5) \geqslant 0$ i e +ive or $8[y-(-5)](y-3)$

Hence arguing as in Q 51 (a), y should not lie between -5 and 3

52 (a) Prooceed as above

(b) Do yourself (c) Do yourself

53 Let $\dfrac{x^2+14x+9}{x^2+2x+3}=y$

$x^2(1-y)+2x(7-y)+3(3-y)=0$ (1)

For real values of x, B^2-4AC of (1) should be $\geqslant 0$

$(7-y)^2-12(1-y)(3-y) \geqslant 0$

or $(49-14y+y^2)-3(3-4y+y^2) \geqslant 0$

or $-2y^2-2y+40 \geqslant 0$

or $-2(y^2+y-20) \geqslant 0$

or $(y+5)(y-4)$ is $\leqslant 0$

or $[y-(-5)](y+4)$ is $-$ive

By § 5 y should lie between -5 and 4

Therefore the maximum value is 4 and minimum is -5

54 (a) Proceeding as above for real values of x we have

$-3y^2+10y-3 \geqslant 0$

or $3y^2-10y+3 \leqslant 0$ i e $-$ive

or $(3y-1)(y-3) \leqslant 0$ i e $-$ive

or $3(y-1/3)(y-3)$ is $-$ive

By § 5 y lies between $\frac{1}{3}$ and 3

(b) Do yourself

(c) The value of the function can be $\leqslant 5$ or $\geqslant 9$

55 (a) $\dfrac{8x^2+16x-51}{(2x-3)(x+4)} > 3$ or $\dfrac{8x^2+16x-51}{2x^2+5x-12}-3>0$

or $\dfrac{2x^2+5x-15}{2x^2+5x-12}>0$ or $\dfrac{(2x-5)(x+3)}{(2x-3)(x+4)}>0$

Put $\dfrac{(2x-5)(x+3)}{(2x-3)(x+4)}=P$ or $\dfrac{2[x-(-3)](x-5/2)}{2[x-(-4)](x-3/2)}$

Above is +ive if both N^r and D are +ive or both $-$ive

Case (I) Both +ive

N^r +ive x does not lie between -3 and $5/2$

D +ive x does not lie between -4 and $3/2$

Hence from both the above we conclude

$x<-4$ or $x>5/2$ (1)

Case 2 Both $-$ive

N^r $-$ive x lies between -3 and $5/2$

D $-$ive x lies between -4 and $3/2$

Hence from above we conclude

x lies between -3 and $3/2$ (2)

i e $-3<x<3/2$

Hence inequality holds when

or $\frac{c}{a}+\frac{c^2}{a^2}+\frac{3c}{a}\left(-\frac{b}{a}\right)=-\frac{b^3}{a^3}$ $\quad\because\ \alpha^3=\frac{c}{a}$

or $b^3+ac^2+a^2c=3abc$ is the required condition

(b) $\alpha+\alpha^n=-b/a,\ \alpha\,\alpha^n=c/a$ $\qquad \alpha=\left(\frac{c}{a}\right)^{1/(n+1)}$

Putting in Ist relation, we get

$$\left(\frac{c}{a}\right)^{1/(n+1)}+\left(\frac{c}{a}\right)^{n/(n+1)}=-\frac{b}{a}$$

or $a\left(\frac{c}{a}\right)^{1/(n+1)}+a\left(\frac{c^n}{a^n}\right)^{1/(n+1)}+b=0$

or $(a^nc)^{1/(n+1)}+(ac^n)^{1/(n+1)}+b=0$

22 On simplification the given equation is

$$x^2+x(p+q-2r)+(pq-pr-qr)=0$$

By given condition $\beta=-\alpha$ or $\alpha+\beta=0$ (1)

$\therefore\ p+q-2r=0$ or $p+q=2r$

Product of roots

$=pq-pr-qr=pq-r\,(p+q)$

$=pq-\frac{p+q}{2}(p+q)=\frac{1}{2}[2pq-(p+q)^2]$ by (1)

$=-\frac{1}{2}[p^2+q^2]$

23 $\alpha+\beta=\alpha^2+\beta^2=(\alpha+\beta)^2-2\alpha\beta$ etc

24 $\alpha+\beta=\frac{1}{\alpha^2}+\frac{1}{\beta^2}=\frac{\alpha^2+\beta^2}{\alpha^2\beta^2}=\frac{(\alpha+\beta)^2-2\alpha\beta}{\alpha^2\beta^2}$

or $\left(-\frac{b}{a}\right)\left(\frac{c^2}{a^2}\right)=\frac{b^2}{a^2}-\frac{2c}{a}$

or $-bc^2=ab^2-2ca^2$

or $2ca^2=ab^2+bc^2$

$\therefore$ $bc^2,\ ca^2,\ ab^2$ are in A P

25 The given equation is

$\lambda x^2-(\lambda-1)\,x+5=0$

$$\alpha+\beta=\frac{\lambda-1}{\lambda},\ \alpha\beta=\frac{5}{\lambda}$$

Also $\frac{\alpha}{\beta}+\frac{\beta}{\alpha}=\frac{4}{5}$ or $5\,(\alpha^2+\beta^2)=4\alpha\beta$

or $5[(\alpha+\beta)^2-2\alpha\beta]=4\alpha\beta$

or $5\left(\frac{\lambda-1}{\lambda}\right)^2=14\frac{5}{\lambda}$

or $\lambda^2-2\lambda+1=14\lambda$ or $\lambda^2-16\lambda+1=0$

Its roots are λ_1, λ_2, $\quad \lambda_1+\lambda_2=16,\ \lambda_1\lambda_2=1$

$x < -4$ or $-3 < x < \frac{3}{2}$ or $x > \frac{5}{2}$

(b) $\frac{x-2}{x+2} - \frac{2x-3}{4x-1} > 0$

or $\frac{2(x^2-5x+4)}{(x+2)(4x-1)} > 0$ or $\frac{2(x-4)(x-1)}{(x+2)(4x-1)} > 0$

or $\frac{2(x-1)(x-4)}{4[x-(-2)](x-\frac{1}{4})} > 0$

Now proceed as in part (a) Then

$x < -2$ or $\frac{1}{4} < x < 1$ or $x > 4$

56 (a) Proceeding as usual

$y^2 - y(b+c) + bc \geqslant 0$

or $(y-b)(y-c) \geqslant 0$ *i e* +ive

Above is possible only when y does not lie between b and c

(b) Ans (c) and (d) are both correct

Let $y = \frac{(x-a)(x-b)}{x-c}$ or $y(x-c) = x^2 - (a+b)x + ab$

or $x^2 - (a+b+y)x + ab + cy = 0$

$\Delta = (a+b+y)^2 - 4(ab+cy) = y^2 + 2y(a+b+2c) + (a-b)^2$

Since x is real and y assumes all real values we must have $\Delta \geqslant 0$ for all, real values of y

This will be so if

$4(a+b-2c)^2 - 4(a-b)^2 < 0$

or $4(a+b-2c+a-b)(a+b-2c-a+b) < 0$

or $16(a-c)(b-c) < 0$ or $16(c-a)(c-b) = -$ive (1)

c lies between a and b *i e* $a < c < b$ (1)

where $a < b$ but if $b < a$ then the above condition will be

$b < c < a$ or $a > c > b$ (2)

Hence from (1) and (2) we observe that both (*c*) and (*d*) are correct answers

57 (a) Let $y = \sqrt{\left(\frac{(x+1)(x-3)}{x-2}\right)}$

If y is real then y^2 is +ive or zero

$\frac{(x+1)(x-3)}{x-2} \geqslant 0$ or $\frac{[x-(-1)](x-3)}{x-2} \geqslant 0$

$x = -1, 3$, for the case $y = 0$ (1)

Above is +ive when both N^r and D^r are +ive or both −ive

Case I Both +ive

N^r = + ive x does not lies between −1 and 3

D^r = + ive $x > 2$

Hence from both the above we conclude that $x > 3$ (2)

$$\frac{\lambda_1}{\lambda_2}+\frac{\lambda_2}{\lambda_1}=\frac{\lambda_1^2+\lambda_2^2}{\lambda_1\lambda_2}$$
$$=\frac{(\lambda_1+\lambda_2)^2-2\lambda_1\lambda_2}{\lambda_1\lambda_2}=256-2=254$$

26 We have $\frac{\alpha}{\beta}=\frac{\gamma}{\delta}$ or $\frac{\alpha}{\gamma}=\frac{\beta}{\delta}=k$, say

or $\alpha=k\gamma,\ \beta=k\delta$ (1)

$\alpha+\beta=-p,\ \alpha\beta=q$

$\gamma+\delta=-l,\ \gamma\delta=m$

$$\frac{(\alpha+\beta)^2}{(\gamma+\delta)^2}=\frac{p^2}{l^2} \text{ or } \frac{k^2(\gamma+\delta)^2}{(\gamma+\delta)^2}=\frac{p^2}{l^2}$$

or $$k^2=\frac{p^2}{l^2}$$

Also from (1) $\alpha\beta=k^2\gamma\delta$ or $\frac{q}{m}=k^2$

$\therefore\ \frac{q}{m}=\frac{p^2}{l^2}$ or $p^2m=l^2q$

is the required condition

27 (a) $\alpha+\beta=-\frac{n}{l},\ \alpha\beta=\frac{n}{l}$ (1)

Also $\frac{\alpha}{\beta}=\frac{p}{q}$ (2)

We have to prove that

$$\sqrt{\left(\frac{p}{q}\right)}+\sqrt{\left(\frac{q}{p}\right)}+\sqrt{\left(\frac{n}{l}\right)}=0$$

Now $$\sqrt{\left(\frac{p}{q}\right)}+\sqrt{\left(\frac{q}{p}\right)}=\sqrt{\left(\frac{\alpha}{\beta}\right)}+\sqrt{\left(\frac{\beta}{\alpha}\right)}$$

$$=\frac{\alpha+\beta}{\sqrt{(\alpha\beta)}}=-\frac{n/l}{\sqrt{(n/l)}}=-\sqrt{\left(\frac{n}{l}\right)}$$

$$\sqrt{\left(\frac{p}{q}\right)}+\sqrt{\left(\frac{q}{p}\right)}+\sqrt{\left(\frac{n}{l}\right)}=0$$

(b) Putting $x=-1$, we get $-6p+24=0$ $p=4$

$x^4+x^3-7x^2-x+6=(x+1)(x^3-7x+6)$

$=(x+1)(x-1)(x^2+x-6)=(x+1)(x-1)(x+3)(x-2)$

The other factors are $x-1,\ x-2,\ x+3$

(c) $x^2-3x+2=(x-2)(x-1)$

Put $x=2$ $q-4p+16=0$

Put $x=1$ $q-p+1=0$ $p=5,\ q=4$

(d) Ans $p=\pm 7$ Hint use $(x_1-x_2)^2=(x_1+x_2)^2-4x_1x_2$

Case II Both −ive

$N' = -$ ive , x lies between -1 and 3 i e -1

$D' = -$ ive $x \nless 2$

From both the above, we conclude that

$$-1 < x < 2 \qquad (3)$$

Hence from (1), (2) and (3) the required limits are

$$-1 < x \leqslant 2 \quad \text{or} \quad x \ngeqslant 3$$

57 (b) The given inequality will hold if on transposing we have

$$-\frac{(3x+2)}{(x+1)(x+2)(2x+1)} > 0 \text{ or } \frac{3x+2}{(x+1)(x+2)(2x+1)} < 0 \quad (I)$$

Now consider the five cases given by values of x in ascending order $-2, -1, -2/3, -1/2$ The five cases are given by (i) $x < -2$, (ii) $-2 < x < -1$, (iii) $-1 < x < -2/3$ (iv) $-2/3 < x < -\frac{1}{2}$ (v) $x > -\frac{1}{2}$

The inequality (1) will hold good in cases II or (iv) Also the inequality (1) holds good for $x = -2$

Hence $-2 \leqslant x < -1$ and $-2/3 < x < -\frac{1}{2}$

58 Let $\frac{px^2+3x \;\; 4}{p+3x-4x^2} = y$ Then $(4y+p)x^2+3x(1-y)-(4+yp) = 0$

Now $b^2-4ac = 9(1-y)^2+4(4y+p)(4+yp)$
$= (16p+9)y^2+2(2p^2+23)y+(16p+9)$

Since y takes all real values for real values of x, we must have $b^2-4ac \;\; 0$ for all real values of y the condition for which by § 4 Case I is

$4(2p^2+23)^2 - 4(16p+9)^2 < 0$ and $16p+9 > 0$

or $16(p+4)^2(p^2-8p+7) < 0$ and $16p+9 > 0$

or $(p+4)^2(p-1)(p-7) < 0$ and $16p+9 > 0$

Both these inequalities are satisfied if $1 < p < 7$

59 We have the identity

$x^n-1 = (x-1)(x-a_1)(x-a_2) \;\; (x-a_{n-1})$

or $(x-a_1)(x-a_2) \;\; (x-a_{n-1}) = \frac{1-x^n}{1-x}$

$$\lim_{x\to 1}(x-a_1)(x-a_2) \;\; (x-a_{n-1})$$

$$= \lim_{x\to 1} \frac{1-x^n}{1-x} = \lim_{x\to 1}(1+x+x^2+ \;\; +x^{n-1}) = n$$

or $(1-a_1)(1-a_2) \;\; (1-a_{n-1}) \;\; n$

60 We have $x^2 = ax - b$

Multiplying by x^{n-1}, we get

$x^{n+1} = ax^n - bx^{n-1}$

In this putting $x = \alpha \;\; \beta$ and adding, we get

$\alpha^{n+1}+\beta^{n+1} = a(\alpha^n+\beta^n) - b(\alpha^{n-1}+\beta^{n-1})$

(e) As 2 and 3 are the roots of the given equation we have $4m+n=10$, and $9m+n=-15$ Solving we get $m=-5$
$\therefore$ $n=30$ Hence the given equation is $2x^3-5x^2-13x+30=0$
Now $\alpha+\beta+\gamma=5/2$ or $2+3+\gamma=5/2$ $\gamma=5/2-5=-5/2$

(f) Ans $3\pm i\sqrt{6}, \pm\frac{\sqrt{3}}{2}$

28 (a) Let α be a common root which will satisfy both the equations

$$a\alpha^2+b\alpha+c=0$$
$$a'\alpha^2+b'\alpha+c'=0$$

Solving by the method of cross multiplication we get

$$\frac{\alpha^2}{bc'-b'c}=\frac{\alpha}{ca'-c'a}=\frac{1}{ab'-a'b}$$

$$\therefore \quad \alpha=\frac{\alpha^2}{\alpha}=\frac{bc'-b'c}{ca'-c'a} \text{ and } \alpha=\frac{\alpha}{1}=\frac{ca'-c'a}{ab'-a'b} \qquad (1)$$

Since $\alpha=\alpha$

$$\therefore \quad \frac{bc'-b'c}{ca'-c'a}=\frac{ca'-c'a}{ab'-a'b}$$

Required condition is

$$(bc'-b'c)(ab'-a'b)=(ca'-c'a)^2$$

The value of the common root is given by (1)

(b) If α be a root of first equation then $1/\alpha$ will be a root of the second $a\alpha^2+b\alpha+c=0$
and $a'(1/\alpha)^2+b'(1/\alpha)+c'=0$ or $c'\alpha^2+b'\alpha+a'=0$

$$\frac{\alpha^2}{ba'-cb'}=\frac{\alpha}{cc'-aa'}=\frac{1}{ab'-bc'}$$

$(cc'-aa')^2=(ba'-cb')(ab'-bc')$ is the required condition

(c) If both the roots are common then their sum and product will be same

$S=-b/a=-b'/a'$ $\quad a/a'=b/b'$
$P=c/a=c'/a'$ $\quad$ or $\quad a/a'=c/c'$

Hence the condition is $a/a'=b/b'=c/c'$

29 Proceed as in Q 28

30 (a) As in Q 28, $\frac{\alpha^2}{17k}=\frac{\alpha}{6k}=\frac{1}{-1}$ (1)

$\therefore$ $(6k)^2=17k(-1)$ or $k(36k+17)=0$
$k=0$ and $-17/36$

Also from (1) $\alpha=\frac{17k}{6k}=\frac{17}{6}$ or $\alpha=\frac{6k}{-1}=0$ or $\frac{17}{6}$

Hence the common root is either zero or 17/6

(b) $a=0, 24$

$\therefore \quad V_{n+1}=a V_n-b V_{n-1}$

Also $V_0=\alpha^0+\beta^0=2$ and $V_1=\alpha+\beta=a$

$V_2=aV_1-bV_0=a\ a-b\ 2=a^2-2b$

Similarly $V_3=aV_2-bV_1=a\ (a^2-2b)-b,a=a^3-3ab$

$V_4=aV_3-bV_2=a\ (a^3-3ab)-b\ (a^2-2b)$

$=a^4-4a^2b+2b^2$

and $V_5=aV_4-bV_3=a(a^4-4a^2b+2b^2)-b\ [a^3-3ab)$

$=a^5-5a^3b+5ab^2$

51 Do yourself

62. We have

$$x=\frac{-b\pm\sqrt{(b^2-4ac)}}{2a}.$$

Since a, b, c are odd integers, we can write

$a=2m+1$, $b=2n+1$, and $c=2p+1$,

where m, n, p are integers Then we have

$b^2-4ac=(2n+1)^2-4\ (2m+1)\ (2p+1)$

$=4n^2+4n-16mp-8m-8p-3$

$=8\left[\frac{n\ (n+1)}{2}-2mp-m-p-1\right]+5$

Since $n\ (n+1)$ is the product of two consecutive integers, it is an even integer and hence $\frac{1}{2}n\ (n+1)$ is an integer Thus we see that if b^2-4ac is a square of an integer, then this whole number must be odd But every odd number can be represented in the form $4q\pm1$ where q is an integer and its square can be written in the form

$(4q\pm1)^2=16q^2\pm8q+1=8\ (2q^2\pm q)+1$

Consequently, the division of the expression $(4q\pm1)^2$ by 8 always leaves remainder 1

But division of b^2-4ac by 8 leaves a remainder 5 Hence the expression b^2-4ac can never be a perfect square It follows that the roots of the given equation cannot be rational

63 We have $\alpha+\beta=-p$ and $\alpha\beta=q$ (1)

Also since α, β are the roots of

$x^{2n}+p^nx^n+q^n=0$

we have

$\alpha^{2n}+p^n\alpha^n+q^n=0$ and $\beta^{2n}+p^n\beta^n+q^n=0$

Subtracting the above relations, we get

$(\alpha^{2n}-\beta^{2n})+p^n\ (\alpha^n-\beta^n)=0 \quad \therefore \quad \alpha^n+\beta^n=-p^n$ (2)

If α/β or β/α is a root of $x^n+1+(x+1)^n=0$, then

31 (a) Common root is 2

(b) Ans (A)

Condition for a common root can be easily found to be

$(cd-af)^2=4(bf-ec)(ae-bd)$

or $c^2d^2+a^2f^2-2cdaf-4bfae+4b^2fd+4e^2ca-4ecbd=0$

or $c^2d^2+a^2f^2-2cdaf-4bfae+4acfd+4e^2b^2-4ecbd=0$

[∵ $b^2=ac$

or $(cd+af)^2-4be(cd+af)+4e^2b^2=0$

or $(cd+af-2be)^2=0$

or $cd+af-2be=0$, dividing by ac

or $\frac{d}{a}+\frac{f}{c}-\frac{2be}{ac}=0$

or $\frac{d}{a}+\frac{f}{c}-\frac{2e}{b}=0$ [$b^2=ac$]

∴ $\frac{d}{a}, \frac{e}{b}, \frac{f}{c}$ are in A.P

32 (a) $\alpha+\beta=-p, \alpha\beta=q, \gamma+\delta=-r, \gamma\delta=s$

$(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)$

$=[\alpha^2-\alpha(\gamma+\delta)+\gamma\delta][\beta^2-\beta(\gamma+\delta)+\gamma\delta]$

$=(\alpha^2+r\alpha+s)(\beta^2+r\beta+s)$ (1

Since α is a root of $x^2+px+q=0$ $\alpha^2+p\alpha+q=0$

or $\alpha^2=-p\alpha-q$ and similarly $\beta^2=-p\beta-q$

Hence from (1) we have to evaluate the value of

$(-p\alpha-q+r\alpha+s)(-p\beta-q+r\beta+s)$

$=[(r-p)\alpha-(q-s)][(r-p)\beta-(q-s)]$

$=(r-p)^2\alpha\beta-(r-p)(q-s)(\alpha+\beta)+(q-s)^2$

$=(r-p)^2q-(r-p)(q-s)(-p)+(q-s)^2$

$=(r-p)[(qr-pq)+(pq-ps)]+(q-s)^2$

$=(q-s)^2-(p-r)(qr-ps)$ (2

In case the equations have a common root then

$\alpha=\gamma$ or $\alpha=\delta$ and in either case $\alpha-\gamma=0$ or $\alpha-\delta=0$

and hence the given expression is zero

Therefore from (2) the required condition is

$(q-s)^2=(p-r)(qr-ps)$

This condition can be obtained directly also as in Q 28

(b) We have $dx^2+(a+c)x-b=0$

and $bx^2+(a-c)x-d=0$

Hence $\frac{x^2}{-d(a+c)+b(a-c)}=\frac{x}{-b^2+d^2}=\frac{1}{d(a-c)-(a+c)}$

$(\alpha/\beta)^n+1+[(\alpha/\beta)+1]^n=0$ or $(\alpha^n+\beta^n)+(\alpha+\beta)^n=0$

or $-p^n+(-p)^n=0$ by (1) and (2)

Above is possible only when n is even

Problem Set B

Objective Questions

1 A quadratic equation with rational coefficients can have
 (a) both roots equal and irrational,
 (b) one root rational and other irrational,
 (c) one root real and other imaginary,
 (d) None of these

2 The number of quadratic equations which are unchang~d by squaring their roots is
 (a) 2, (b) 4,
 (c) 6, (d) None of these

3 If the roots of $ax^2+bx+c=0$ are in the ratio m n then
 (a) $mna^2=(m+n)\ c^2$ (b) $mnb^2=(m+n)\ ac$,
 (c) $mnb^2=(m+n)^2\ ac$, (d) None of these

4 For real x, the expression

$$[(x+m)^2-4mn]/[2\ (x-n)]$$

can have any value except
 (a) between m and $m+n$, (b) more than $m+2n$,
 (c) between $2m$ and $2n$, (d) all values are possible

5 If the expression $y^2+2xy+2x+my-3$ can be resolved into two rational factors, then m must be
 (a) any possible real number,
 (b) any negative real number,
 (c) -2, (d) ?

6 If one root of $5x^2+13x+k=0$ is reciprocal of the other, then
 (a) $k=0$, (b) $k=5$
 (c) $k=\frac{1}{6}$, (d) $k=6$ (M N R 80)

7 The equation $2x^2+3x+1=0$ has an irrational root
 (a) True (b) False (I I T 83)

8 Let $a>0$, $b>0$, $c>0$ then both roots of the equation

$$ax^2+bx+c=0$$

 (a) are real and negative,
 (b) have negative real parts,
 (c) None of these (I I T 80)

9 If one root of the equation $x^2+px+12=0$ is 4, while the equation $x^2+px+q=0$ has equal roots, the value of q is

or $\dfrac{x^2}{c(b+d)-a(b-d)}=\dfrac{x}{b^2-d^2}=\dfrac{1}{c(b+d)+a(b-d)}$

whence eliminating x, we get

$(b^2-d^2)^2=\{c(b+d)-a(b-d)\}\{c(b+d)+a(b-d)\}$

or $(b^2-d^2)^2=c^2(b+d)^2-a^2(b-d)^2$

33 (a) $\alpha+\beta=-\dfrac{b}{a}$ (1)

$\alpha\beta=\dfrac{c}{a}$ (2)

$\alpha_1-\beta=-\dfrac{b_1}{a_1}$ (3)

$-\alpha_1\beta=\dfrac{c_1}{a_1}$ (4)

Adding (1) and (3)

$$\alpha+\alpha_1=-\frac{b}{a}-\frac{b_1}{a_1}=S$$

Dividing (1) by (2) and (3) by (4) we get

$$\frac{1}{\beta}+\frac{1}{\alpha}=-\frac{b}{c} \text{ and } -\frac{1}{\beta}+\frac{1}{\alpha_1}=-\frac{b_1}{c_1}$$

Adding them we get

$$\frac{1}{\alpha}+\frac{1}{\alpha_1}=-\frac{b}{c}-\frac{b_1}{c_1}$$

or $$\frac{\alpha+\alpha_1}{\alpha\alpha_1}=\frac{S}{P}=-\frac{b}{c}-\frac{b_1}{c_1}$$

The equation is $x^2-xS+P=0$

r $$\frac{x^2}{-S}+x-\frac{P}{S}=0$$

(b) Let the roots of the equation be α β, and α, γ as one root is common

$\alpha+\beta=-b$ $\alpha\beta=ca$ (1)

$\alpha+\gamma=-c$, $\alpha\gamma=ab$ (2)

We are to find the equation whose roots are β and γ for which we must know the values of $\beta+\gamma$ and $\beta\gamma$

$x^2+bx+ca=0$ and $x^2+cx+ab=0$ have a common root

$$\frac{x^2}{a(b^2-c^2)}=\frac{x}{a(c-b)}=\frac{1}{(c-b)}$$

or $\dfrac{x^2}{-a(b+c)}=\dfrac{x}{a}=\dfrac{1}{1}$ $\quad {}^2=1\{-a(b+c)\}$

or $a=-(b+c)$ or $a+b+c=0$ is the condition (3)

(a) 49/4, (b) 4/49,
(c) 4, (d) None of these

10 Both the roots of the equation
$(x-b)(x-c)+(x-a)(x-c)+(x-a)(x-b)=0$ are always
(a positive, (b) negative,
(c) real, (d) none of these (I I T 80)

11 If x^2+px+1 is a factor of ax^3+bx+c, then
(a) $a^2+c^2=-ab$,
(b) $a^2-c^2=-ab$,
(c) $a^2-c^2=ab$,
(d) none of these

12 If $x=2+2^{2/3}+2^{1/3}$ then the value of x^3-6x^2+6x is
(i) 3 (ii) 2
(iii) I (iv) None (M N R 85)

13 If the equations
$k(6x^2+3)+rx+2x^2-1=0$ and
$6k(2x^2+1)+px+4x^2-2=0$ have both the roots common, find the value of $2r-p$ (M N R 83)

14 α and β are the roots of $4x^2+3x+7=0$ then the value of

$$\frac{1}{\alpha}+\frac{1}{\beta} \text{ is}$$

(a) $-3/4$ (b) $-3/7$
(c) $3/7$ (d) $7/4$

Solutions

1 Ans (*d*)

Note that imaginary roots occur in conjugate pairs of the form $\alpha \pm \beta i$ and irrational roots occur in conjugate pairs of form $A \pm \sqrt{B}$ in surds

2 Ans (*b*)

Let α, β be the roots of another quadratic and α^2, β^2 the roots of another quadratic, since the quadratics remain unchanged, we have

$$\alpha^2+\beta^2=\alpha+\beta$$

and $$\alpha^2\beta^2=\alpha\beta \quad (2)$$

(2) gives $\alpha\beta(\alpha\beta-1)=0$

$$\alpha=0 \quad \text{or} \quad \beta=0 \quad \text{or} \quad \beta=1$$

when $\alpha=0$, we get from (1), $\beta^2-\beta=0$
which gives $\beta=0$, $\beta=1$

Also the common root $x=a$, i e $\alpha=a$

Now adding relations in (1) and (2) we get

$$(\alpha+\beta)+(\alpha+\gamma)=-b-c$$

$$\beta+\gamma=-b-c-2\alpha=a-2a=-a \text{ by (3) and as } \alpha=a$$

Multiplying relations in (1) and (2), $(\alpha\beta)(\alpha\gamma)=(ca)(ab)$

or $\alpha^2\beta\gamma=a^2bc$ $\beta\gamma=bc,$ $\alpha=a$ (5)

Hence from (4) and (5) the equation whose roots are β and γ is

$$x^2+ax+bc=0$$

34 (a) $a=4$ or 8 Just as in Q 28 or 30

(b) As in Q 28 the condition for common root s

$$(ca'-c'a)^2=4(bc'-b'c)(ab'-a'b) \quad (1)$$

For equality of roots of 2nd equation its discriminant

$$B^2-4AC=0$$

$$(2bb'-ac'-a'c)^2-4(b^2-ac)(b'^2-a'c')=0$$

$$4b^2b'^2+(ac'+a'c)^2-4bb'(ac'+a'c)=4b^2b'^2-4b^2a'c'-4b'^2ac+4aa'cc'$$

$$(ac'+a'c)^2-4aa'cc'=4bb'(ac'+a'c)-4b^2a'c'-4b'^2ac$$

$$(ca'-c'a)^2=4[bc'(ab'-a'b)-b'c(ab'-a'b)]$$

$$=4[(ab'-a'b)(bc'-b'c)]$$

which is true by (1)

(c) Ans -1

35 $(c^2-ab)x^2-2(a^2-bc)x+(b^2-ac)=0$

If the roots be equal, then $B^2-4AC=0$

$$4(a^2-bc)^2-4(c^2-ab)(b^2-ac)=0$$

or $[a^4-2a^2bc+b^2c^2]-[b^2c^2-ab^3-ac^3+a^2bc]=0$

or $a(a^3+b^3+c^3-3abc)=0$

$\therefore$ Either $a=0$ or $a^3+b^3+c^3-3abc=0$

36 As above $m=2$ or $-10/9$

37 The given equation can be put as

$$x^2(5+4m)-x(4+2m)+(2-m)=0$$

(a) Equal $B^2-4AC=0$ $(4+2m)^2-4(5+4m)(2-m)=0$

or $5m^2+m-6=0$ or $(m-1)(5m+6)=0$ $m=1,-6/5$

(b) Product$=2$ $\dfrac{2-m}{5+4m}=2$ $m=-\dfrac{8}{9}$

Sum$=6$ $\dfrac{4+2m}{5+4m}=6$ $m=-\dfrac{13}{11}$

38 $(c-a)^2-4(a-b)(b-c)=0$

$$c^2+a^2-2ca-4ab+4ac+4b^2-4bc=0$$

or $$c^2+a^2+2ca+4b^2-4b(c+a)=0$$

$$(c+a)^2+(2b)^2-2\cdot 2b(c+a)=0$$

Thus we get two sets of values of α and β, namely $\alpha=0,\ \beta=0$ and $\alpha=0,\ \beta=1$

Again taking $\alpha\beta=1$, we get from (1)

$$\alpha^2+\frac{1}{\alpha^2}=\alpha+\frac{1}{\alpha}$$

To solve this, let $\alpha+\frac{1}{\alpha}=u$ Then

$$\alpha^2+\frac{1}{\alpha^2}+2=u^2$$

Hence $u^2-2=u$ or $u^2-u-2=0$]

or $(u-2)(u+1)=0$ so that $u=2, -1$

Taking $u=2$, we get $\alpha+\frac{1}{\alpha}=2$ or $\alpha^2-2\alpha+1=0$

or $(\alpha-1)^2=0$ i e $\alpha=1$

Then $\beta=\frac{1}{\alpha}=1$

Taking $u=-1$, we get $\alpha+\frac{1}{\alpha}=-1$ or $\alpha^2+\alpha+1=0$

This gives $\alpha=\frac{-1\pm i\sqrt{3}}{2}$, that is, $\alpha=\omega, \omega^2$

Then $\beta=\frac{1}{\omega}=\omega^2$ or $\frac{1}{\omega^2}=\omega$

Hence $\alpha=\omega,\ \beta=\omega^2$ or $\alpha=\omega^2,\ \beta=\omega$

From the above discussion, we see that the quadratics which remain unchanged are those having their roots as 0, 0 or 0 1 or 1, 1 or ω, ω^2 Thus there are four such quadratics, namely,

$$x^2=0,\ x^2-x=0,\ x^2-2x+1=0,\ x^2+x+1=0$$

3 Ans (c)

$$\frac{\alpha}{\beta}=\frac{m}{n} \text{ or } \frac{\alpha}{m}=\frac{\beta}{n}=\frac{\alpha+\beta}{m+n}=k$$

$$(\alpha+\beta)^2=k^2(m+n)^2=\frac{\alpha}{m}\ \frac{\beta}{n}(m+n)^2$$

$$mn\left(-\frac{b}{a}\right)^2=(m+n)^2\frac{c}{a} \quad \text{or} \quad mn\,b^2=ac\,(m+n)^2$$

4 Ans (c)

5 Ans (c)

6 Ans (b)

7 Ans (b)

8 Ans (b) Roots are given by $x=-\frac{b}{2a}\pm\frac{\sqrt{(b^2-4ac)}}{2a}$, since

$[(c+a)-2b]^2=0 \quad c+a-2b=0$

or $2b=a+ \quad a, b, c$ are in A.P

39 $b^2 (c-a)^2-4ac (b-c) (a-b)=0$

or $b^2 (c^2+a^2-2ac) - 4ac [ab-ac-b^2+bc]=0$

or $b^2 (c^2+a^2-2ac+4ac)+4a^2c^2-4abc (c+a)=0$

or $[b (c+a)]^2+(2ac)^2-2\ 2ac\ b (c+a)=0$

or $[b (c+a)-2ac]^2=0, \quad b (c+a)=2ac$

or $b=\dfrac{2ac}{a+c}$

b is H M of a and c i e a, b, c are in *H P*

40 The given equation is

$3x^2-2x (a+b+c)+(ab+bc+ca)=0$

Roots Equal $B^2-4AC=0$

or $4 (a+b+c)^2-12 (ab+bc+ca)=0$

or $(a^2+b^2+c^2+2\Sigma ab-3\Sigma ab)=0$

or $a^2+b^2+c^2-ab-bc-ca=0$

or $\frac{1}{2} [2a^2+2b^2+2c^2-2ab-2bc-2ca]=0$

or $(a-b)^2+(b-c)^2+(c-a)^2=0$

Since a, b, c are real therefore each of $(a-b)^2$, $(b-c)^2$ and $(c-a)^2$ is greater than or equal to zero, and so their sum will be zero only when each of them is zero Hence

$a-b=0, b-c=0, c-a=0$ or $a=b=c$

41 $B^2-4AC \geqslant 0$ for roots to be real

Now $B^2-4AC=4b^2 (a+c)^2-4 (a^2+b^2) (b^2+c^2)$

$=4 [b^2a^2+b^2c^2+2b^2 ac-(a^2b^2+a^2c^2+b^4+b^2c^2)]$

$=4 [2b^2ac-a^2c^2-b^4]=-4 [b^4+a^2c^2-2b^2ac]$

$=-4 [b^2-ac]^2$

Now $4 (b^2-ac)^2$ is greater than or equal to zero and hence the discriminant B^2-4AC is always $\leqslant 0$ so that the roots cannot be real unless $b^2-ac=0$ or $b^2=ac$ *i e* a, b, c are in *G P* In this case the discriminant being zero the roots will be equal also

42 Proceeding as above

$B^2-4AC=-(a^2c^2+b^2d^2-2abcd)$

$=-(ac-bd)^2$

Above is clearly $\leqslant 0$ as $(ac-bd)^2$ is $\geqslant 0$ Hence the roots will real if $ac-\ d=0$ or $\dfrac{a}{b}=\dfrac{d}{c}$

Also the roots will bo equal as $B^2-4AC=0$

43 Dividing by y^2, we get

$\left(\dfrac{x}{y}\right)^2-b\ \dfrac{x}{y}\ -a=0$ or $az^2-bz-a=0$ where $z=\dfrac{x}{y}$

It will have real roots if $b^2+4a^2 \geqslant 0$

Above is clearly true for real values of b and a Therefore it can be resolved into linear factors like $a (z-p) (z-q)$

or $a\left(\dfrac{x}{y}-p\right)\left(\dfrac{x}{y}-q\right)$ or $a (x-py) (x-qy)$

$a > 0, b > 0$, we have $-\frac{b}{2a} < 0$ If $b^2-4ac < 0$, then roots are imaginary of which the real part $-\frac{b}{2a}$ is negative If $b^2-4ac > 0$, then roots are real and both negative since $\sqrt{(b^2-4ac)} < b$ Hence in either case, both the roots have negative real parts

9 Ans (a)

Since 4 *is a root of* $x^2+px+12=0$, we have $16+4p+12=0$ or $p=-7$ Further, since the roots of $x^2+px+q=0$ are equal, we have $p^2-4q=0$ or $49-4q=0$

$$\therefore \; q=\frac{49}{4}$$

10 Ans (c) The given equation can be written as

$$3x^2-2x(a+b+c)+bc+ca+ab=0$$

$\Delta=4(a+b+c)^2-12(bc+ca+ab)$

$=4[a^2+b^2+c^2-bc-ca-ab]$

$=2[(b-c)^2+(c-a)^2+(a-b)^2] \geqslant 0$

Hence the roots are real

11 Ans (c) We have the identity

$ax^2+bx+c=(x^2+px+1)(ax+\lambda)$, where λ is a constant

Equating the coeffs of x^2, x and constant term, we get

$a=ap+\lambda,\ b=p\lambda+a,\ c=\lambda \quad \therefore \quad p=-\frac{\lambda}{a}=-\frac{c}{a}$

Hence $b=\left(-\frac{c}{a}\right)c+a$ or $ab=a^2-c^2$

12 $x-2=2^{2/3}+2^{1/3}$ Cube both sides

$(x-2)^3=2^2+2+3\ 2^{2/3}\ 2^{1/3}(x-2)=6+6(x-2)$

or $x^3-6x^2+12x-8=-6+6x$

$x^3-6x^2+6x=2$ (ii) is correct

13 The two equations can be written as

$x^2(6k+2)+rx+(3k-1)=0$ (1)

and $x^2(12k+4)+px+(6k-2)=0$ Divide by 2

$x^2(6k+2)+\frac{p}{2}x+(3k-1) > 0$ (3)

Comparing 1 and 3 we get $r=\frac{p}{2}$ $\quad 2r-p=0$

(b) is correct

44 (a) $B^2-4AC=4(a-b)^2=+$ive and perfect square.
$\therefore$ Roots are real, rational and unequal
(b) $B^2-4AC=(b+c-a)^2$
Roots are real, rational and unequal
(c) $B^2-4AC=-4(a-b)^2=-$ive
Hence roots are imaginary

45 $B^2-4AC>0$ for Ist equation $\therefore$ $4(c^2-ab)>0$
or c^2-ab is $+$ive.
B^2-4AC for the 2nd equation is
$4(a+b)^2-4(a^2+b^2+2c^2)$
$=4[2ab-2c^2]=-8(c^2-ab)=-$ive
as c^2-ab is $+$ive
Hence the roots of the second equation are imaginary

46 (i) Let us assume that neither of the equations has real roots
i e they have imaginary roots
$\therefore$ $p^2-4q<0$ and $r^2-4s<0$ Add the two relations
$\therefore$ $p^2+r^2-4(q+s)<0$ But $2(q+s)=pr$
$\therefore$ $p^2+r^2-2pr<0$ or $(p-r)^2<0$
This is not possible as for real values of p and r, $(p-r)^2\geqslant 0$
Hence our assumption that neither of the equations has real roots is wrong

(ii) Δ_1 of $P(x)=0$ is b^2-4ac
Δ_2 of $Q(x)=0$ is d^2+4ac
Since $ac\neq 0$ $ac=+$ive or $-$ive
If ac is $+$ive then Δ_2 is $+$ive but Δ_1 may be $+$ive or $-$ive so that the roots of $Q(x)=0$ are real
If ac is $-$ive then Δ_1 is $+$ive but Δ_2 may be $+$ive or $-$ive so that the roots of $P(x)=0$ are real
Hence at least two roots are real

(iii) Ans (a) The given equation can be written as
$3x^2-(a+c+2b+2d)x+ac+2bd=0$
$\Delta=(a+c+2b+2d)^2-12(ac+2bd)$
$=[(a+2d)-(c+2b)]^2+4(a+2d)(c+2b)-12(ac+2bd)$
$=[(a+2d)-(c+2b)]^2+8ab+8cd-8ac-8bd$
$=[(a+2d)-(c+2b)]^2+8(c-b)(d-a)>0$
Since $a<b<c<d$ so that $(c-b)(d-a)>0$
Hence the roots are real and distinct

47 (a) B^2-4AC is $4(16+a-a^2)=4(8-a)(2+a)$
If $-2\leqslant a\leqslant 8$, then B^2-4AC is $\geqslant 0$
Hence for roots to be real a should lie between -2 and 8
(b) $B^2-4AC=16a^2-36\times 4=16(a^2-9)=16(a-3)(a+3)<0$
as the roots are imaginary If $a>3$ or <-3 then B^2-4AC $=+\times+=+$ or $-\times-=+$ and in this case the roots will be real Hence for imaginary roots a must lie between -3 and 3

48 (a) $B^2-4AC=25(p+q)^2+8(p-q)^2>0$ for real p and q
Hence roots are real and unequal. (c) is correct answer

5

Logarithms

§ 1 Definitions *Let there be a number* $a > 0$ *and* $a \neq 1$ *A number* x *is called the logarithm of another number* $y > 0$ *to the base* a *if* $a^x = y$

We first observe that the logarithm of a number satisfying the conditions of the definition is unique For if α and β are two distinct logarithms of the number y to base a, then by the above definition, we have

$$a^\alpha = y \text{ and } a^\beta = y$$

whence $a^\alpha = a^\beta$ (1)

But, by the properties of powers with positive base different from 1, we conclude from (1) that $\alpha = \beta$

Thus, if the number y has a logarithm to base a, then this logarithm is unique We denote it by the symbol $\log_a y$ Thus by definition

$$x = \log_a y \text{ if } a^x = y$$

Characteristic and mantissa The integral part of a logarithm is called the characteristic and the decimal part is called the mantissa Logarithms to the base *10 are called common logarithms* The characteristics of common logarithms can be written down by inspection by the *following rule*

Rule The characteristic of the logarithm (base 10) of a number greater than 1 is less by one than the number of digits in the integral part, and is positive The characteristic of the logarithm of a positive decimal fraction less than 1, is greater by unity than the number of consecutive zeros immediately after the decimal point, and is negative,

§ 2 Properties of Logarithms

The students should commit to memory the following

We assume $a > 0, a \neq 1, m > 0, n > 0$

1 $a^x = y$ then $x = \log_a y$

L.H S is called exponential form whereas R H S is called corresponding logarithmic form

(b) $4q^2-4pr \geqslant 0$ and $4pr-q^2 \geqslant 0$
or $q^2-pr \geqslant 0$ and $(q^2-pr) \leqslant 0$
Above is possible only when $q^2-pr=0$ or $p/q=q/r$,
(b) is correct

49 $B^2-4AC=4(a+b-2c)^2-4(a-b)^2$
$=4[(a+b)^2+4c^2-4c(a+b)-(a-b)^2]$
$=4[4ab+4c^2-4c(a+b)]$
$=16[(c-a)(c-b)]$
If c lies between a and b, i e $a < c < b$, then one fact is +ive and other −ive so that B^2-4AC is −ive
Hence the roots will be imaginary
If c does not lie between a and b then either both the factors are −ive or both +ive so that B^2-4AC is +ive
Hence the roots will be real

50 (a) $x=2+i\sqrt{3}$ $(x-2)^2=i^2 3=-3$
or $x^2-4x+7=0$ (1)
Now $4x^3+8x+35=4(x^2-4x+7)+16x-28+8x+35$
$=0+24x+7=24(2+i\sqrt{3})+7$, by (1)
$=55+24\sqrt{3}i$
(b) The equation whose root is $2+i\sqrt{3}$ is by part (a)
$x^2-4x+7=0$ and it is the same as $x^2+px+q=0$
$p=-4$, $q=7$
Alt If $2+i\sqrt{3}$ is a root then $2-i\sqrt{3}$ will also be a root
Sum$=4=-p$ and product $4+3=q$
$p=-4$, $q=7$
(c) $x=1+2i$ $(x-1)^2=-4$
$x^2-2x+5=0$ (1)
Dividing $x^3+7x^2-13x+16$ by x^2-2x+5 we get the quotient as $x+9$ and remainder as -29
$x^3+7x^2-13x+16=(x^2-2x+5)(x+9)-29$
$=0-29=-29$ by (1)
(d) $x=2+\sqrt{3}$, $(x-2)^2=3$
or $x^2-4x+1=0$ (1)
As above $x^3-7x^2+13x-2=(x^2-4x+1)(x-3)+1$
$=0+1=1$ by (1)

51 (a) (i) $x^2-2x-3=(x-3)(x+1)=[x-(-1)](x-3)$
$a=-1$ and $b=3$, $a < b$
By § 5 the above expression is −ive if x does not lie between a and b i e if x does not lie between -1 and 3 Again it wil' be −ive if x lies between a and b i e x lies between -1 and 3
(ii) +ive if x does not lie between -3 and $\frac{1}{2}$
−ive if x lies between -3 and $\frac{1}{2}$
(b) Let $\dfrac{11x^2+12x+6}{x^2+4x+2}=y$
$x^2(11-y)+4x(3-y)+2(3-y)=0$ (1)
For real value of x, B^2-4AC of (1) should be 0

2 $a^1=a$, $b^1=b$ etc $\qquad \log_a a=\log_b b=1$

3 $a^0=1$, $b^0=1$ etc $\qquad \log_a 1=0$, $\log_b 1=0$ etc

4 $\log_b a \log_a b=1$ or $\log_b a=\dfrac{1}{\log_a b}$

Let $\log_b a=x \qquad a=b^x$, $\log_a b=y \qquad b=a^y$

Putting the value of b we get $a=(a^y)^x=a^{xy}$

$$xy=1 \quad \text{or} \quad x=\frac{1}{y}$$

5 Base changing formula

$\log_b a=\log_c a \log_b c$ or $\log_b a=(\log_c a)/(\log_c b)$ by (4)

Let $\log_b a=x \qquad a=b^x$, $\log_c a=y \qquad a=c^y$

and $\log_b c=z \qquad c=b^z \qquad a=(b^z)^y=b^{yz}$

or $b^x=b^{yz} \qquad x=yz$

In general $\log_b a=\log_c a \log_d c \log_e d \qquad \log_b k$

6 $\log_a mn=\log_a m+\log_a n$, $\log_a m/n=\log_a m-\log_a n$

Let $\log_a m=x \qquad m=a^x$, $\log_a n=y \qquad n=a^y$

$\therefore$ $mn=a^x a^y=a^{x+y} \qquad \log_a mn=x+y=\log_a m+\log_a n$

$\dfrac{m}{n}=\dfrac{a^x}{a^y}=a^{x-y} \qquad \log_a\dfrac{m}{n}=x-y=\log_a m-\log_a n$

7 $\log_a m^n=n \log_a m$ or $\log_a a^n=n$ (Imp)

Let $\log_a m=x \qquad m=a^x$ and $m^n=(a^x)^n=a^{nx}$

$\log_a m^n=nx=n \log_a m$ (A)

Particular case Putting $m=a$ in (A) we get

$\log_a a^n=n \log_a a=n \cdot 1=n$

e g $\log_3 3^2=2$, $\log_4 64=\log_4 4^3=3$, $\log_2 \sqrt{(8)}=\log_2 2^{3/2}=3/2$

8 $\log_{a^q} n^p=\dfrac{p}{q}\log_a n$ In particular, $\log_{n^q} n^p=p/q$ **(Important)**

Let $z=\log_{a^q} n^p \qquad (a^q)^z=n^p$ or $a^{qz}=n^p$ (B)

Let $y=\log_a n \qquad a^y=n$ or $(a^y)^p=n^p$ or $a^{yp}=n^p$ (C)

$a^{qz}=a^{yp}$ by (B) and (C) $\qquad qz=yp$ or $z=\dfrac{p}{q}y$

or $\log_{a^q} n^p=p/q \log_a n$

Particular case Putting $a=n$ we get

$\log_{n^q} n^p=\dfrac{p}{q}\log_n n=p/q \cdot 1=p/q$ by (2)

9 $a^{\log_a n}=n$ (V Imp)

Let $p=a^{\log_a n}$

Rewriting the above exponential form to logarithmic form we get $\log_a p=\log_a n \qquad p=n$

It will be purely imaginary if $R\ P=0$ i e $3-4\sin^2\theta=0$

$$\sin\theta=\pm\frac{\sqrt{3}}{2}=\pm\sin\frac{\pi}{3}=\sin\left(\pm\frac{\pi}{3}\right)$$

$$\theta=n\pi\pm\frac{\pi}{3}$$

(g) We know that both $\cos^2 x/2$ and $\sin^2 x$ numerically cannot exceed 1

$$2\cos^2 x/2\ \sin^2 x<2 \qquad (1)$$

Again $(x^2-1)^2$ is + ive i e $\geqslant 0$

$$x^4+1-2x^2\geqslant 0 \text{ or } x^4+1\geqslant 2x^2$$

or
$$x^2+\frac{1}{x^2}\geqslant 2$$

The equation $2\cos^2\frac{x}{2}\sin^2 x=x^2+\frac{1}{x^2}$

in which L H S <2 or R H S $\geqslant 2$ by (1) and (2) has no real solution

12. We re write the equation as

$$\frac{\sin 3x}{\cos 3x}-\frac{\sin 5x}{\cos 5x}=0$$

or
$$\frac{\sin 3x\cos 5x-\sin 5x\cos 3x}{\cos 3x\cos 5x}=0 \text{ or } \frac{\sin 2x}{\cos 3x\cos 5x}=0$$

Now solving the equation $\sin 2x=0$ we get $x=\frac{1}{2}n\pi$ $n\in I$ But we must discard extraneous solutions, that is those for which the denominator $\cos 3x\cos 5x$ vanishes which clearly happens when n is odd Thus the solution of the given equation will be given by $x=\frac{1}{2}n\pi$, where n is even, say $n=2m$, $m\in I$ Hence the required solution is $x=m\pi$, $m\in I$

Quite obviously, it is a serious error to take the solution as $x=\frac{1}{2}n\pi$, $n\in I$

13 The equation can be written as

$$\sin^2 x=\sin^2\pi/8\,[(1-\cos x)^2+\sin^2 x]=\sin^2\pi/8\,(2-2\cos x)$$

or $1-\cos^2 x=(1-\cos\pi/4)(1-\cos x)$

$$[\quad 2\sin^2\pi/8=1-\cos\pi/4]$$

$$(1-\cos x)[(1+\cos x)-(1-\cos\pi/4)]=0$$

or $(1-\cos x)(\cos x+\cos\pi/4)=0$

$$\cos x=1 \quad\text{or}\quad \cos x=-\frac{1}{\sqrt{2}}=-\cos\frac{\pi}{4}=\cos\left(\pi-\frac{\pi}{4}\right)$$

Hence the solution sets are given by

$x=2n\pi$, $x=2m\pi\pm\frac{3}{4}\pi$, $m, n\in I$

$x=2n\pi$ gives no solution between $\frac{5}{2}\pi$ and $\frac{7}{2}\pi$, 16 Eliminating x, y, z we have

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

On expanding this determinant, we get

$7 \sin 3\theta + 14 \cos 2\theta - 14 = 0$

or $(3 \sin \theta - 4 \sin^3 \theta) + 2 (1 - 2 \sin^2 \theta) - 2 = 0$

or $\sin \theta (4 \sin^2 \theta + 4 \sin \theta - 3) = 0$

or $\sin \theta (2 \sin \theta - 1) (2 \sin \theta + 3) = 0$

$\sin \theta = 0$ $\frac{1}{2}$ (not $-\frac{3}{2}$)

$\therefore$ $\theta = n\pi$ or $n\pi + (-1)^n \pi/6$,

17 (a) $2 \cos \frac{x+y}{2} \cos \frac{x-y}{2} = 3/2$ or $2 \cos \frac{\pi}{3} \cos \frac{x-y}{2} = 3/2$

or $\cos \frac{x-y}{2} = 3/2$ This is not possible

Hence ϕ is the answer

(b) We have $(\sin x - 1)(2 \sin x - 1) \geqslant 0$

or $2(\sin x - \frac{1}{2})(\sin x - 1) \geqslant 0$

Now we know that $(x-a)(x-b)$ $(a<b)$ is $\geqslant 0$ if x does not lie between a and b i e $x \leqslant a$ or $x \geqslant b$ i e +ive

Hence $\sin x \leqslant \frac{1}{2}$ or $\sin x \geqslant 1$

Since the values are restricted in the interval $[0\ \pi]$

$\sin x \leqslant \frac{1}{2} \Rightarrow 0 \leqslant x \leqslant \frac{\pi}{6}$ and $\frac{5\pi}{6} \leqslant x \leqslant \pi$

$\sin x \geqslant 1 \Rightarrow x = \pi/2$, in the interval $[0\ \pi]$

Hence the required values are $[0\ \pi/6] \cup [5\pi/6\ \pi] \cup \{\pi/2\}$

18 On multiplying by conjugate of denominator the expression is

$$\frac{\left(\sin \frac{x}{2} + \cos \frac{x}{2} - i \tan x\right)\left(\left(1 - 2i \sin \frac{x}{2}\right)\right.}{1 + 4 \sin^2 \frac{x}{2}}$$

It will be real if imaginary part is zero

$$-2 \sin \frac{x}{2}\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) - \tan x = 0$$

or $2 \sin \frac{x}{2}\left(\sin \frac{x}{2} + \cos \frac{x}{2}\right) \cos x + 2 \sin \frac{x}{2} \cos \frac{x}{2} = 0$

i e $3^{\log_3 5}=5,\ 10^{\log_{10} m}=m,\ 5^{-2\log_5 3}=5^{\log_5 3^{-2}}$

$$=3^{-2}=\frac{1}{9}$$

(10) If $a>1$, then $0<\alpha<\beta \Leftrightarrow \log_a \alpha < \log_a \beta$

(11) If $0<a<1$, then $0<\alpha<\beta \Leftrightarrow \log_a \alpha > \log_a \beta$

(12) If $a>1$, $\alpha>1$, then $\log_a \alpha > 0$

(13) If $0<a<1$, $0<\alpha<1$, then $\log_a \alpha>0$

(14) If $0<a<1$, $\alpha>1$, then $\log_a \alpha<0$

(15) If $a>1$, $0<\alpha<1$, then $\log_a \alpha<0$

(16) If $a>1$, $\alpha>1$, and $\alpha>a$, then $\log_a \alpha>1$

(17) If $a>1$, $\alpha>1$ and $\alpha<a$, then $0<\log_a \alpha<1$

(18) If $0<a<1$ $0<\alpha<1$ and $\alpha>a$, then $0<\log_a \alpha<1$

(19) If $0<a<1$, $0<\alpha<1$ and $\alpha<a$, then $\log_a \alpha>1$

Problem Set (A)

1 Rewrite the following equations in the logarithm form

(i) $4^{3/2}=8$, (ii) $5^0=1$ (iii) $(2\sqrt{2})^{-2/3}=\frac{1}{2}$

2 Rewrite the following equalities in the exponential form

(i) $\log_2 32=5$, (ii) $\log_3 \frac{1}{243}=-5$

(iii) $\log_{5\sqrt{5}} 5=\frac{2}{3}$ (iv) $\log_{100} 0\ 1=-\frac{1}{2}$

3 Using the identity $a^{\log_a n}=n$, find

(i) $3^{-\frac{1}{2}\log 9}$, (ii) $2^{2-\log_2 5}$,

(iii) $(5\ 8)^{\log_{5\ 8} 10+1}$, (iv) $8^{\log_2 \sqrt[3]{(121)}+\frac{1}{3}}$,

(v) $10^{\log_{10} m+\log_{10} n}$

4 Compute without using tables

(i) $\log_\pi \tan(0\ 25\pi)$,

(ii) $\log_2(\log_3 81)$ (iii) $8^{(1/\log_5 3)}+27^{\log_9 36}+3^{4/\log_7 9}$

(iv) $2^{\log_3 5}-5^{\log_3 2}$ (v) $\log_3 5 \log_{25} 27$,

(vi) $\log_9 27-\log_{27} 9$,

(vii) $\log_{10}\tan 40^\circ \log_{10}\tan 41^\circ \ \ldots\ \log_{10}\tan 50^\circ$,

(viii) $\log_{10}\tan 1^\circ+\log_{10}\tan 2^\circ+\ \ldots\ \log_{10}\tan 89^\circ$,

(ix) $\log_3 4 \log_4 5 \log_5 6 \log_6 7 \log_7 8 \log_8 9$,

(x) $\log_3 2 \log_4 3 \log_5 4\ \ldots\ \log_{15} 14 \log_{16} 15$,

(xi) $\sqrt{\left[\left(\frac{1}{\sqrt{27}}\right)^{2-(\log_5 13)/(2\log_5 9)}\right]}$

26 From $a^2+b^2=7ab$, we have

$$(a+b)^2=9ab \text{ or } \left(\frac{a+b}{3}\right)^2=ab$$

Taking logarithm of both sides we get

$$2\log\tfrac{1}{3}(a+b)=\log a+\log b$$

27 $$\text{L.H.S.}=\frac{1}{\log_n a\log_n b}+\frac{1}{\log_n b\log_n c}+\frac{1}{\log_n c\log_n a}$$

$$=\frac{\log_n c+\log_n a+\log_n b}{\log_n a\log_n b\log_n c}=\frac{\log_n abc}{\log_n a\log_n b\log_n c}$$

$$=\frac{\log_a n\log_b n\log_c n}{\log_{abc} n}=\text{R.H.S.}$$

28 Changing all the logarithms to base x ($x>0$, $x\neq 1$) the given equation yields

$$\frac{\log_x a\log_x a}{\log_x b\log_x c}-1+\frac{\log_x b\log_x b}{\log_x c\log_x a}-1$$

$$+\frac{\log_x c\log_x c}{\log_x a\log_x b}-1=0$$

or $$(\log_x a)^3+(\log_x b)^3+(\log_x c)^3$$
$$-3\log_x a\log_x b\log_x c=0$$

Now if we put

$\log_x a=x$, $\log_x b=y$ and $\log_x c=z$,

the above equation becomes

$$x^3+y^3+z^3-3xyz=0$$

or $$(x+y+z)(x^2+y^2+z^2-xy-yz-zx)=0 \quad (1)$$

Since $x\neq y\neq z$ we have

$$x^2+y^2+z^2-xy-yz-zx=\tfrac{1}{2}[(x-y)^2+(y-z)^2+(z-x)^2]\neq 0$$

Hence we conclude from (1) that

$$x+y+z=0$$

that is $$\log_x a+\log_x b+\log_x c=0$$

or $$\log_x abc=0 \text{ or } abc=1$$

29 $$2\log_{10} x-\log_x 0.01$$

$$=2\log_{10} x-\frac{\log_{10} 0.01}{\log_{10} x}$$

$$=2\log_{10} x-\frac{\log_{10} 10^{-2}}{\log_{10} x}=2\log_{10} x+\frac{2}{\log_{10} x}$$

$$=2\left(\log_{10} x+\frac{1}{\log_{10} x}\right)$$

Since $x>1$, $\log_{10} x>0$

5 Compute,

(i) $\log_6 16$ if $\log_{12} 27 = a$

(ii) $\log_{25} 24$ if $\log_6 15 = \alpha$ and $\log_{12} 18 = \beta$

(iii) $\log_{30} 8$ if $\log_{30} 3 = a$ and $\log_{30} 5 = b$

6 If $\log_{12} 18 = \alpha$ and $\log_{24} 54 = \beta$, prove that $\alpha\beta + 5(\alpha - \beta) = 1$

7 Simplify

$$7 \log \frac{16}{15} + 5 \log \frac{25}{24} + 3 \log \frac{81}{80}$$ (Roorkee 80)

8 (*a*) If $\log_{10} 2 = 0\ 30103$, determine how many zeros there are between the decimal point and the first significant digit in $(\frac{1}{2})^{1000}$

(*b*) Given $\log_{10} 2 = 0\ 30103$, $\log_{10} 3 = 0\ 47712$ Find the number of digits in $3^{12} \times 2^8$

(c) Find the least integer n such that $7^n > 10^5$, given that $\log_{10} 343 = 2\ 5353$ (I I T 75)

9 Determine b satisfying

(*a*) $\log_{\sqrt{8}} b = 3\frac{1}{3}$ (Roorkee 85)

(*b*) $\log_e 2 \log_b 625 = \log_{10} 16 \log_e 10$ (Roorkee 86)

10 Without using tables prove

$$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} > 2$$

11 Prove that $\log_2 17 \log_{1/5} 2 \log_3 \frac{1}{5} > 2$

12 Prove that

$$|\log_b a + \log_a b| \geqslant 2$$

where a and b are positive numbers not unity

13 Prove that $\log_{10} 2$ lies between $\frac{1}{4}$ and $\frac{1}{3}$

14 Prove that $\log_2 3$ is an irrational number.

15 Which is greater

(i) $\log_2 3$ or $\log_{1/2} 5$

(ii) $\log_4 5$ or $\log_{1/16} (1/25)$

(iii) $\log_7 11$ or $\log_8 5$

(iv) $\log_2 3$ or $\log_3 11$

(v) $\log_{1/3} \frac{1}{2}$ or $\log_{1/2} \frac{1}{3}$

16 If a, b, c are in G P, prove that $\log_a n, \log_b n, \log_c n$ are in H P

17 Given that $\log_l x$, $\log_m x$ and $\log_n x$ are in arithmetic progression Note that $x \neq 1$ Then prove that

$$n^2 = (ln)^{\log_l m}$$ (Roorkee 82)

18 Show that

$$\log_b a \log_c b \log_d c \log_a d = 1$$

But $\frac{1}{2}\left(\log_{10} x+\frac{1}{\log_{10} x}\right) \geqslant \sqrt{\left[\log_{10} x \times \frac{1}{\log_{10} x}\right]}=1$

[∵ A M $\geqslant$ G M]

$$\log_{10} x+\frac{1}{\log_{10} x} \geqslant 2.$$

Thus $2 \log_{10} x-\log_x 0.01 \geqslant 4$

Hence the minimum value of $2 \log_{10} x-\log_x 0.01$ is 4

30 Since n is a natural number and p_i s are primes, it follows that a_i s are also natural numbers

Taking logarithm of both sides of given relation, we get

$$\log n=\sum_{i=1}^{k} a_i \log p_i \geqslant \sum_{i=1}^{k} a_i \log 2$$

[∵ smallest prime number is 2]

$\geqslant k \log 2$, since each $a_i \geqslant 1$

31 $\text{R H S}=\frac{(1/\log_a n)-(1/\log_b n)}{(1/\log_b n)-(1/\log_c n)}$

$=\frac{\log_n b-\log_n a}{\log_n c-\log_n b} \cdot \frac{\log_n c}{\log_n a}$

$=\frac{\log_n (b/a)}{\log_n (c/b)} \cdot \frac{\log_n c}{\log_n a}=\frac{\log_n c}{\log_n a}$

[∵ $b=\sqrt{ac} \Rightarrow b^2=ac \Rightarrow b/a=c/b$]

32 From the property of right angled triangle we get

$a^2+b^2=c^2$ or $a^2=(c-b)(c+b)$

Hence $2 \log_{c+b} a=\log_{c+b} (c-b)+\log_{(c+b)} (c+b)$

or $2 \log_{c+b} a=\log_{c+b} (c-b)+1$ (1)

and similarly

$2 \log_{c-b} a=\log_{c-b} (c+b)+1$ (2)

Multiplying (1) and (2), we get

$4 \log_{c+b} a \log_{c-b} a=\log_{c+b} (c-b)+\log_{c-b} (c+b)$
$+1+\log_{c+b} (c-b) \log_{c-b} (c+b)$

But $\log_{c-b} (c+b) \log_{c+b} (c-b)=1$

Hence $4 \log_{c+b} a \log_{c-b} a=\log_{c+b} (c-b)$
$+\log_{c-b} (c+b)+2$ (3)

Finally from (1), (2) and (3), we get

$4 \log_{c+b} a \log_{c-b} a=2 \log_{c+b} a-1+2 \log_{c-b} a-1+2$

or $2 \log_{c+b} a \log_{c-b} a=\log_{c+b} a+\log_{c-b} a$

33 Let r be the common ratio of G P and d the common difference of A P

Then $a_n=ar^n$ (1)

and $b_n=b+nd$ (2)

Taking logarithm of both sides of (1) to base β ($\beta \neq 1$, $\beta>0$), we get

19 Show that

$$\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+ \quad +\frac{1}{\log_{43} n}=\frac{1}{\log_{43!} n}$$ (I I T 75)

20 If $n=1983!$, compute the sum

$$\frac{1}{\log_2 n}+\frac{1}{\log_3 n}+\frac{1}{\log_4 n}+ \quad +\frac{1}{\log_{1983} n}$$

21 If $y=a^{\frac{1}{1-\log_a x}}$ and $z=a^{\frac{1}{1-\log_a y}}$,

prove $x=a^{\frac{1}{1-\log_a z}}$

22 If $\frac{\log a}{b-c}=\frac{\log b}{c-a}=\frac{\log c}{a-b}$, prove that $a^a\ b^b\ c^c=1$ (I.I T 74)

23 If $\frac{\log x}{q-r}=\frac{\log y}{r-p}=\frac{\log z}{p-q}$, prove that

$x^{q+r}\ y^{r+p}\ z^{p+q}=x^p\ y^q\ z^r$

24 If $\frac{x(y+z-x)}{\log x}=\frac{y(z+x-y)}{\log y}=\frac{z(x+y-z)}{\log z}$,

prove that $x^y\ y^x=z^y\ y^z=x^z\ z^x$

25 Prove the following identities

(i) $\frac{\log_a n}{\log_{ab} n}=1+\log_a b$,

(ii) $\log_{ab} x=\frac{\log_a x \log_b x}{\log_a x+\log_b x}$

26 If $a^2+b^2=7ab$, prove that

$\log \frac{1}{3}(a+b)=\frac{1}{2}[\log a+\log b]$

27 Prove the identity

$\log_a n \log_b n+\log_b n \log_c n+\log_c n \log_a n$

$=\frac{\log_a n \log_b n \log_c n}{\log_{abc} n}$

28 If a, b, c are distinct positive numbers each different from 1 such that

$(\log_b a \log_c a-\log_a a)+(\log_a b \log_c b-\log_b b)$
$+(\log_a c \log_b c-\log_c c)=0,$ (I I T 76)

then prove that $abc=1$

29 Find the least value of the expression

$2\log_{10} x-\log_x 0.01$ for $x>1$ (I I T 80)

30 If n is a natural number such that

$$n=p_1^{a_1}\ p_2^{a_2}\ p_3^{a_3} \quad p_k^{a_k},$$

$$\log_\beta a_n = \log_\beta a - n \log_\beta r$$

$$\log_\beta a_n - b_n = \log_\beta a + n \log_\beta r - b - nd \quad (3)$$

Now in order that the right hand side of (3) reduces to $\log_\beta a - b$, we must have

$$n \log_\beta r - nd = 0$$

or $\quad \log_\beta r = d$ or $r = \beta^d$

that is, $\quad \beta = r^{1/d}$

Hence there exists a system of logarithms to base $r^{1/d}$ such that

$$\log a_n - b_n = \log_n a - b$$

Problem Set (B)

(Objective Questions)

1 $\quad 16^{\log_4 5}$ equals

(a) 5 (b) 16 (c) 25 (d) none of these

2 $\quad l_n ab - ln\,|\,b\,| =$

(a) $ln a$ (b) $ln\,|\,a\,|$, (c) $-ln a$ (d) none of these

[Note that $ln x$ stands for $\log_e x$]

3 $\quad e^{ln\, ln\, 7} = 7$

(a) True (b) False

4 $\quad 3^{\sqrt{(\log_3 7)}} = 7^{\sqrt{(\log_7 3)}}$

(a) True (b) False

5 $\quad \log_3 5 \log_5 9 \log_3 2$ simplifies to

(a) 2 (b) 1 (c) 5 (d) none of these

6 $\quad$ In the value of $\sqrt{[\log_0 4]}$ is

(a) -2 (b) $\sqrt{(-4)}$ (c) 2 (d) none of these

7 $\quad$ The value of $\frac{1}{\log -} + \frac{1}{\log_3 -}$ is greater than 2

(a) True (b) False

Answers

1 (c) 2 (b) 3 Correct value is $ln\ 7$

4 (a)

Hint Write $\sqrt{(\log_3 7)}$ as $\log_3 7/\sqrt{(\log_3 7)}$

then L H S $= 7^{1/\sqrt{(\log_3 7)}} = 7^{\sqrt{(\log_7 3)}} =$ R H S

5 (b) 6 (c) Hint Since $\log_0 4 = \log_{2^{-1}} 2^2 = -2 \log_2 2 = -2$

we have $\sqrt{(\log_0 4)} = \sqrt{((-2))^2} = \sqrt{(4)} = 2$ $\quad$ 7 (a)

and $p_1, p_2, \ldots p_k$ are distinct primes, then
show that $\log n \geq k \log 2$ **(IIT 84)**

31 If $a > 0$, $c > 0$, $b = \sqrt{ac}$, $a \neq 1$, $c \neq 1$, $ac \neq 1$ and $n > 0$,

prove that $\dfrac{\log_a n}{\log_c n} = \dfrac{\log_a n - \log_b n}{\log_b n - \log_c n}$

32 If a and b are the lengths of the sides and c the length of the hypotenuse of a right angled triangle, $c-b \neq 1$ and $c+b \neq 1$ prove that

$$\log_{c+b} a + \log_{c-b} a = 2 \log_{c+b} a \log_{c-b} a$$

33 Given a geometric progression a, a_1, a_2, a_3
and an arithmetic progression b, b_1, b_2, b_3
with positive terms The common difference of A P and common ratio of G P are both positive Show that there always exists a system of logarithms for which

$$\log a_n - b_n = \log a - b \text{ (for any } n)$$

Find base β of the system

Solutions

1 (i) $\log_4 8 = \frac{3}{2}$ (ii) $\log_5 1 = 0$

(iii) $\log_{2\sqrt{2}} \frac{1}{2} = -\frac{2}{3}$

2 (i) $32 = 2^5$, (ii) $\frac{1}{243} = 3^{-5}$

(iii) $5 = (5\sqrt{5})^{2/3}$, (iv) $0\,1 = 100^{-1}$

3 (i) $3^{-\frac{1}{2}\log_3 9} = 3^{\log_3 9^{-1/2}} = 9^{-1/2} = \frac{1}{3}$

(ii) $2^2\, 2^{-\log_2 5} = 4\, 2^{\log_2 5^{-1}} = 4\, 5^{-1} = \frac{4}{5}$ by (9) of § 2

(iii) $(5\,8)^{\log_{5\,8} 10}\,(5\,8)^1 = 10\,(5\,8) = 58$ by (9) of § 2

(iv) $8^{\log_2 \sqrt[3]{121} + \frac{1}{3}} = 2^3 [\log_2 121^{1/3} + \frac{1}{3}]$

$= 2^{\log_2 121 + 1} = 2^{\log_2 121}\, 2 = 121\, 2 = 242$

(v) $10^{\log_{10} m + \log_{10} n} = 10^{\log_{10} mn} = mn$

4 (i) $\log_\pi \tan (0\,25\pi) = \log_\pi \tan (\frac{1}{4}\pi)$

$= \log_\pi 1 = 0$

(ii) $\log_2 \log_3 81 = \log_2 \log_3 3^4 = \log_2 4$

$= \log 2^2 = 2$ by (7) of § 2

(iii) Ans 890

6

Miscellaneous Equations

Problem Set (A)

Solve the following equations

1 $\frac{x-2}{x-2}+\frac{x+2}{x-2}=\frac{2(x+3)}{x-3}$ 2 $\sqrt{\left(\frac{x}{1-x}\right)}+\sqrt{\left(\frac{1-x}{x}\right)}=2\frac{1}{6}$

3 $(2x-7)(x^2-9)(2x+5)=91$

4 $\frac{3x-2}{2}-\sqrt{(2x^2-5x+3)}=\frac{(x+1)}{3}$

5 (i) $3x^2-4\sqrt{(3x^2-4x+1)}=4x-4$ **(Roorkee 1978)**

(ii) $x^2-\sqrt{(2x^2-8x+12)}=4x+6$

6 (i) $\sqrt{(x+1)}+\sqrt{(x-1)}=1$ (ii) $\sqrt{(x+1)}-\sqrt{(x-1)}=1$

(I I T 1978)

7 (i) $\sqrt{(2x-2)}+\sqrt{(x-3)}=2$ **(Roorkee 1979)**

(ii) $\sqrt{(2x-6)}+\sqrt{(x+4)}=5$

8 $\sqrt{(5x+7)}-\sqrt{(3x+1)}=\sqrt{(x+3)}$

9 (a) $\sqrt{(2x^2-9x+4)}+3\sqrt{(2x-1)}=\sqrt{(2x^2+21x-11)}$

(b) $\sqrt{(5x^2-6x+8)}-\sqrt{(5x^2-6x-7)}=1$ **(Roorkee 1984)**

10 $\frac{x-ab}{a-b}+\frac{x-ac}{a+c}-\frac{x-bc}{b+c}=a+b+c$

11 $\frac{x-a}{bc}+\frac{x-b}{ca}+\frac{x-c}{ab}=2\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

12 $x-1=\sqrt{(a-x)}$

13 $\frac{p+q-x}{r}+\frac{p+r-x}{q}+\frac{q+r-x}{p}+\frac{4x}{p+q+r}=1$

14 $x^4+\frac{8}{9}x^2+1=3x^3+3x$ **(Roorkee 1978)**

15 $(x^4+1)-4x(x+1)-3x=0$

16 $6x^3-11x^2+6x-1=0$

if its roots are in harmonic progression **(Roorkee 1977)**

17 $\frac{a+2x-\sqrt{(a^2-4x^2)}}{a+2x-\sqrt{(a^2-4x^2)}}=\frac{5x}{a}$

18 $\sqrt{(x^2+x)}+\frac{\sqrt{(x-1)}}{\sqrt{(x^3-x)}}=\frac{5}{2}$

(iv) We have $2^{\log_3 5} = 2^{\log_2 5 \log_3 2} = (2^{\log_2 5})^{\log_3 2}$

$= 5^{\log_3 2}$

$2^{\log_3 5} - 5^{\log_3 2} = 5^{\log_3 2} - 5^{\log_3 2} = 0$

(v) $\log_3 5 \log_2 27 = \log_2 5 \log_5 3^3$

$= \log_2 5 \times 3 \log_5 3$ [See (8) of § 2]

$= 3 \log_2 5 \log_5 3 = 3$ [See (4) of § 2]

$\log_9 27 - \log_{27} 9 = \log_{3^2} 3^3 - \log_{3^3} 3^2$

$= \frac{3}{2} \log_3 3 - \frac{2}{3} \log_3 3$ [See (8) of § 2]

$= \frac{3}{2} - \frac{2}{3} = \frac{5}{6}$

(vii) We have $\log_{10} \tan 45° = \log_{10} 1 = 0$

Since $\log_{10} \tan 45°$ is one of the factors of the product, we get $\log_{10} \tan 40° \log \tan_0 41 \ldots \log_{10} 50° = 0$

(viii) $\log_{10} \tan 1° + \log_{10} \tan 2° + \ldots + \log_{10} \tan 89°$

$= \log_{10} (\tan 1° \tan 2° \ldots \tan 44° \tan 45° \tan 46° \ldots \tan 89°)$

$= \log_{10} \{(\tan 1° \tan 89°) (\tan 2° \tan 88°) \ldots (\tan 44° \tan 46°) \tan 45°\}$

$= \log_{10} [(\tan 1° \cot 1°) (\tan 2° \cot 2°) \ldots (\tan 44° \cot 44°) \tan 45°]$

$= \log_{10} 1 = 0$ [$\because$ $\tan 1° \cot 1° = \tan 2° \cot 2° = \ldots = \tan 44° \cot 44° = 1$ and $\tan 45° = 1$]

(ix) The given expression by (5) of § 2 is

$= \log_3 9 = \log_3 3^2 = 2$ by (7) of § 2

(x) The given expression by (5) of § 2 is

$= \log_{16} 2 = \log_{2^4} 2 = \frac{1}{4} \log_2 2 = \frac{1}{4}$ by (8)

(xi) Given expression

$= \sqrt{\left[\left(\frac{1}{\sqrt{(27)}}\right) \times \left(\frac{1}{\sqrt{(27)}}\right)^{-(\log_5 13)/(2 \log_5 9)}\right]}$

$= \frac{1}{\sqrt{(27)}} \times [\sqrt{(27)}]^{\frac{1}{4} \log_9 13}$

$= 3^{-3/2} \times (3^{\frac{3}{8} \log_9 13}) = 3^{-3/2} \times 3^{\frac{3}{16} \log_3 13}$

$= 3^{-3/2} \times (3^{\log_3 13})^{3/16}$

$= 3^{-3/2} \times 13^{3/16}$

5 (i) $\log_6 16 = \log_6 2^4 = 4 \log_6 2 = \frac{4}{\log_2 6}$

19 $x^4-2x^3+x-380=0$

20 $\dfrac{(x-a)(x-b)}{x-a-b}=\dfrac{(x-c)(x-d)}{x-c-d}$

21 (a) $x^4-5x^2-6x-5=0$

(b) $(x^2+2)^2+8x=6x(x^2+2)$ **(Roorkee 1986)**

22 $x^4+3x^2=16x+60$

23 $\sqrt[3]{[6(5x+6)]}-\sqrt[3]{[5(6x-11)]}=1$

24 $(a+x)^{2/3}+4(a-x)^{2/3}=5(a^2-x^2)^{1/3}$

25 $(a+\sqrt{x})^{1/3}+(a-\sqrt{x})^{1/3}=b^{1/3}$

26 $\sqrt{x}+\sqrt{\{x-\sqrt{(1-x)}\}}=1$ **(Roorkee 1981)**

27 $x^3-5x^2-2x+24=0$,
given that the product of two roots is 12 **(Roorkee 1981)**

28 $\sqrt{(4a+b-5x)}+\sqrt{(4b+a-5x)}=3\sqrt{(a+b-2x)}$

29 $\dfrac{(a-x)\sqrt{(a-x)}+(b-x)\sqrt{(x-b)}}{\sqrt{(a-x)}+\sqrt{(x-b)}}=a-b$

30 Evaluate $\sqrt{[6+\sqrt{\{6+\sqrt{(6+\ldots)}\}}]}$ **(I I T 1973)**

31 $3x^3=[x+\sqrt{18x}+\sqrt{32}][x-\sqrt{18x}-\sqrt{32}]-4x^2$
(Roorkee 1988)

Solutions to Problem Set (A)

1 The given equation can be written as

$$\left(\frac{x-2}{x+2}-1\right)+\left(\frac{x+2}{x-2}-1\right)=\frac{2(x+3)}{x-3}-2$$

or $-\dfrac{4}{x+2}+\dfrac{4}{x-2}=\dfrac{12}{x-3}$

or $\dfrac{-x+2+x+2}{(x+2)(x-2)}=\dfrac{3}{x-3}$

or $3(x^2-4)=4x-12$ or $3x^2-4x=0$

$x=0, \frac{4}{3}$

2 $\sqrt{\left(\dfrac{x}{1-x}\right)}+\sqrt{\left(\dfrac{1-x}{x}\right)}=\dfrac{13}{6}$ or $\dfrac{x+1-x}{\sqrt{\{x(1-x)\}}}=\dfrac{13}{6}$

or $169x(1-x)=36$ or $169x^2-169x+36=0$

or $169x^2-117x-52x+36=0$

or $13x(13x-9)-4(13x-9)=0$

or $(13x-9)(13x-4)=0$

$x=\frac{9}{13}, \frac{4}{13}$

3 Do yourself Combine $\{(2x-7)(x+3)$ and $(x-3)(2x+5)\}$

4 Put $(2x^2-5x+3)^{1/2}=y$ and combine the other two terms
Ans $\frac{1}{2}, 2, \frac{1}{4}\{5\pm\sqrt{41}\}$

5 (i) Put $3x^2-4x+1=y$ and combine the other two terms
Ans $0, \frac{4}{3}, \frac{1}{3}\{2\pm\sqrt{10}\}$

(ii) Ans $x=-2, 6$

$$=\frac{4}{\log_2 2+\log_2 3}=\frac{4}{1+\log_2 3} \qquad (1)$$

and $a=\log_{12} 27=\log_{12} 3^3=3\log_{12} 3=\dfrac{3}{\log_3 12}$

$$=\frac{3}{\log_3 3+\log_3 4}=\frac{3}{1+2\log_3 2}$$

∴ $a+2a\log_3 2=3$ or $\log_3 2=\dfrac{3-a}{2a}$

Hence $\log_2 3=\dfrac{2a}{3-a}$

Substituting this value of $\log_2 3$ in (1), we get

$$\therefore \log_6 16=\frac{4}{1+2a/(3-a)}=\frac{4(3-a)}{3+a}$$

(ii) We change all the logarithms to base 5

Thus $\log_{25} 24=\log_{5^2}(2^3\times 3)=\log_{5^2}(2^3)+\log_{5^2}(3)$

$$=\tfrac{3}{2}\log_5 2+\tfrac{1}{2}\log_5 3 \qquad (1)$$

Now $\alpha=\log_6 15=\dfrac{\log_5 15}{\log_5 6}=\dfrac{\log_5 3+1}{\log_5 2+\log_5 3} \qquad (2)$

$\beta=\log_{12} 18=\dfrac{\log_5 18}{\log_5 12}=\dfrac{2\log_5 3+\log_5 2}{2\log_5 2+\log_5 3} \qquad (3)$

Now putting $\log_5 2=x$ and $\log_5 3=y$, we get from (2) and (3)

$$\alpha=\frac{y+1}{x+y} \quad \text{or} \quad \alpha x+(\alpha-1)y-1=0 \qquad (4)$$

$$\beta=\frac{2y+x}{2x+y} \quad \text{or} \quad (2\beta-1)x=(2-\beta)y \qquad (5)$$

Solving (4) and (5) for x and y, we get

$$\log_5 2=x=\frac{2-\beta}{\alpha\beta+\alpha-2\beta+1}$$

and $\log_5 3=y=\dfrac{2\beta-1}{\alpha\beta+\alpha-2\beta-1}$

Substituting these values of $\log_5 2$ and $\log_5 3$, we get

$$\log_{25} 24=\frac{5-\beta}{2\alpha\beta-2\alpha-4\beta-2}$$

(iii) Ans $3(1-a-b)$

Hint $\log_{30} 8=3\log_{30} 2=3\log_{30}\frac{30}{15}=3[1-\log_{30} 3-\log_{30} 5]$
$=3(1-a-b)$

5 $\alpha=\log_{12} 18=\dfrac{\log 18}{\log 12}=\dfrac{\log(3^2\,2)}{\log(3\,2^2)}=\dfrac{2\log 3+\log 2}{\log 3+2\log 2}$ by (5) of § 2

$\beta=\log_{24} 54=\dfrac{\log 54}{\log 24}=\dfrac{\log(3^3\,2)}{\log(3\,2^3)}=\dfrac{3\log 3+\log 2}{\log 3+3\log 2}$

6 (i) Squaring both members of the given equation, we get

$$x+1+x-1+2\sqrt{(x^2-1)}=1$$

or $2\sqrt{(x^2-1)}=1-2x$

or $4(x^2-1)=1-4x+4x^2$ $x=5/4$

Since $x=5/4$ does not satisfy the given equation, the equation has no roots

Important Note Squaring an equation generally leads to an equation not equivalent to the given equation and this new equation in addition to the roots to the given equation may have other roots different from them (the so called extraneous roots) Hence it is necessary to check, by substitution, whether 5/4 is really the root of the original equation The check shows that 5/4 does not satisfy the original equation Another important point to be noted is that only principal values of the roots are to be considered

Thus if n is odd, the symbol $\sqrt[n]{a}$ is understood as the only real number whose nth power is equal to a In this case a may be positive or negative For example, principal value of $\sqrt[3]{8}$ is 2 and the principal value of $\sqrt[3]{(-27)}$ is -3 If n is even the symbol $\sqrt[n]{a}$ is understood as the only positive number of nth power of which is equal to a Here, necessarily $a \geqslant 0$ Under these conditions, for example, we have

$$\sqrt{a^2}=a \text{ if } a>0 \text{ and } \sqrt{a}=-a \text{ if } a<0$$

Thus the principal value of $\sqrt{4}$ is 2, the principal value of $\sqrt[4]{81}$ is 3 and the principal value of $\sqrt{\{(-3)^2\}}$ is $-(-3)=3$ etc

(iii) Proceeding as in part (ii), we find that $x=5/4$ is the root of the equation Here check shows that $x=5/4$ is really a root of the equation

7 (i) $\sqrt{(2x-2)}=2-\sqrt{(x-3)}$

Squaring $2x-2=4+x-3-4\sqrt{(x-3)}$

or $x-3=-4\sqrt{(x-3)}$

Squaring again, $x^2-6x+9=16x-48$

or $x^2-22x+57=0$

or $(x-3)(x-19)=0$ $x=3, 19$

Check shows that $x=3$ satisfies the equation whereas $x=19$ does not satisfy the equation Hence $x=3$ is the only root

(ii) Ans $x=5$

8 $\sqrt{(5x+7)}=\sqrt{(3x+1)}+\sqrt{(x+3)}$

Squaring $5x+7=3x+1+x+3+2\sqrt{[(3x+1)(x+3)]}$

or $x+3=2\sqrt{[(3x+1)(x+3)]}$

If $x=\log 2$ and $y=\log 3$ then

$$\alpha=\frac{x+2y}{2x+y},\ \beta=\frac{x+3y}{3x+y}$$

$$\alpha\beta+5(\alpha-\beta)$$

$$=\frac{(x+2y)(x+3y)}{(2x+y)(3x+y)}+\frac{5\{(x+2y)(3x+y)-(x+3y)(2x+y)\}}{(2x+y)(3x+y)}$$

$$=\frac{(x^2+5xy+6y^2)+5(x^2-y^2)}{6x^2+5xy+y^2}=\frac{6x^2+5xy+y^2}{6x^2+5xy+y^2}$$

$$=1$$

7 Given expression

$$=7\log\left(\frac{2^4}{3\times 5}\right)+5\log\left(\frac{5^2}{3\times 2^3}\right)+3\log\left(\frac{3^4}{5\times 2^4}\right)$$

$$=7[4\log 2-\log 3-\log 5]+5[2\log 5-\log 3-3\log 2]$$
$$+3[4\log 3-\log 5-4\log 2]$$

$$=\log 2$$

8 (a) Let $x=(\frac{1}{2})^{1000}$

Then $\log_{10} x=-1000\log 2=-1000\times 30103$

$$=-301\ 03=\overline{302}\ 97$$

Hence the number of zeros in $(\frac{1}{2})^{1000}$ between the decimal point and the first significant digit is 301 (See the rule in § 1)

(b) Let $x=3^{12}\times 2^8$ Then

$$\log_{10} x=12\log 3+8\log 2=12\times 47712+8\times 30103$$
$$=5\ 72544+2\ 40824=8\ 13368$$

Hence the number of digits in $3^{12}\times 2^8$ is 9

(c) $\log_{10} 343=\log 7^3=3\log 7=2\ 5353$

$\log 7=\frac{1}{3}(2\ 5353)$ Now $7^n>10^5$ if

$n\log 7>5\log_{10} 10$ or $n\ \frac{1}{3}(2\ 5353)>5$

or $n(2\ 5353)>15$ (1)

Clearly there can be infinite values of n satisfying the above inequality but there is only one least value of n which is clearly 6

9 (a) Writing in exponential form we get

$$b=(\sqrt{8})^{10/3}=(2^{3/2})^{10/3}=2^5=32$$

Note According to question given is paper the solution is as follows

$$\log_{\sqrt{8}} b=3^{\frac{1}{3}}\Rightarrow b=(\sqrt{8})^{3^{1/3}}=(\sqrt{2})^{3\times 3^{1/3}}=(\sqrt{2})^{3^{4/3}}$$

(b) Clearly R H S $=\log_a 16$, by (5) of § 2

Hence the given relation can be written as

Squaring again, $x^2+6x+9=4(3x^2-10x+3)$

or $\quad 11x^2-34x+3=0$, or $(x+3)(11x+1)=0$

$\quad x=-3,\ -\frac{1}{11}$

The check shows that $x=-3$ does not satisfy the given equation whereas $x=-1/11$ satisfies the equation Hence $x=-1/11$ is the only root of the equation

9 (a) The equation can be written as

$\sqrt{\{(2x-1)(x-4)\}}+3\sqrt{(2x-1)}=\sqrt{\{(2x-1)(x-11)\}}$

Clearly one root is given by $2x-1=0$, that is, $x=\frac{1}{2}$ is a root

Now cancelling the factor $\sqrt{(2x-1)}$, we have

$$\sqrt{(x-4)}-3=\sqrt{(x-11)}$$

Squaring, $x-4+9-6\sqrt{(x-4)}=x-11$

or $\quad \sqrt{(x-4)}=1$ or $x-4=1$, that is, $x=5$

Since $x=5$ satisfies the original equation, $x=5$ is also a root

Ans $\quad x=\frac{1}{2}, 5$

(b) $L-M=1$, $L^2-M^2=15$ $\quad L+M=15$

Adding $2L=16$ $\quad L^2=64$ or $5x^2-6x+8=64$

or $5x^2-6x-56=0$ $\quad (x-4)(5x+14)=0$ $\quad x=4,\ -14/5$

10 The equation can be written as

$$\left(\frac{x-ab}{a+b}-c\right)+\left(\frac{x-ac}{a+c}-b\right)+\left(\frac{x-bc}{b+c}-a\right)=0$$

or $\dfrac{x-ab-bc-ca}{a+b}+\dfrac{x-ab-bc-ca}{a+c}+\dfrac{x-ab-bc-ca}{b+c}$

$(x-ab-bc-ca)\left(\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}\right)=0$

Assuming $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}\neq 0$ we obtain $x=ab+bc+ca$ as the root of the equation If however $\dfrac{1}{a+b}+\dfrac{1}{b+c}+\dfrac{1}{c+a}=0$ the given equation turns into an identity which holds for every value of x

11 We rewrite the given equation as

$$\left(\frac{x-a}{bc}-\frac{1}{b}-\frac{1}{c}\right)+\left(\frac{x-b}{ac}-\frac{1}{a}-\frac{1}{c}\right)+\left(\frac{x-c}{ab}-\frac{1}{a}-\frac{1}{b}\right)=0$$

or $(x-a-b-c)\left(\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}\right)=0$

or $\quad (x-a-b-c)(a+b+c)=0$

Hence $x=a+b+c$ provided $a+b+c\neq 0$

$\log_b 625 = \log_e 16/\log_e 2 = \log_2 16 = \log_2 2^4 = 4$ by (5) & (7) of § 2

$$625 = b^4 \quad \text{or} \quad 5^4 = b^4 \qquad \therefore b = 5$$

We know that $\pi < 3.2$

Also $\qquad 12 > (3.2)^2$

$$\frac{1}{\log_3 \pi} + \frac{1}{\log_4 \pi} = \log_\pi 3 + \log_\pi 4$$

$= \log_\pi 12 > \log_\pi (3.2)^2$ [See (10) of § 2]

$= 2 \log \log_\pi 3.2 > 2$

[$\because \log_\pi 3.2 > 1$ by (16) of § 2]

Clearly L.H.S. $= \log_3 17 > \log_3 9 = \log_3 3^2 = 2$.

2 Since $a \neq 1$, $b \neq 1$, $a > 0$, $b > 0$, it follows from properties (12) to (16) of § 2 that $\log_b a$ and $\log_a b$ are both positive or both negative. Hence

$$|\log_b a + \log_a b| = |\log_b a| + |\log_a b| \qquad (1)$$

Since A.M. of two positive quantities is greater than their G.M., we get

$$\frac{|\log_b a| + |\log_a b|}{2} > \sqrt{(\log_b a \log_a b)} = 1,$$

by (4) of § 2

Hence from (1) and (2), we obtain

$$|\log_b a + \log_a b| > 2$$

3 $\log_{10} 2 > \frac{1}{4}$ if $2 > 10^{1/4}$

That is, $\log_{10} 2 > \frac{1}{4}$ if $2^4 > 10$ which is true

Again $\log_{10} 2 < \frac{1}{3}$ if $2 < 10^{1/3}$, i.e. if $2^3 < 10$ which is again true

Hence $\qquad \frac{1}{4} < \log_{10} 2 < \frac{1}{3}$

4 Suppose, if possible, that $\log_2 3$ is rational

So we assume, $\qquad \log_2 3 = \dfrac{p}{q}$

where p and q are positive integers having no factor in common. Then

$$3 = 2^{p/q} \quad \text{or} \quad 3^q = 2^p$$

which is impossible since 3^q is odd and 2^p is even. Hence $\log_2 3$ connot be rational, that is it is irrational

15 (i) Since $\log_2 3 > 0$ [See (12) of § 2]

and $\log_{1/2} 5 < 0$ [See (14) of § 2]

we conclude $\log_2 3 > \log_{1/2} 5$

(ii) $\log_{1/16} \dfrac{1}{25} = \log_{4^{-2}} 5^{-2} = (\frac{-2}{-2}) \log_4 5 = \log_4 5$

12 Right hand side is non negative for all (permissible) x, that is, for all x for which $a-x^2 \geqslant 0$, and left hand is non negative for $x \geqslant 1$. Squaring, we get

$$(x-1)^2=a-x^2$$

or $$2x^2-2x+1-a=0$$

or $$x=\frac{2\pm\sqrt{\{4-8(1-a)\}}}{4}=\frac{1\pm\sqrt{(2a-1)}}{2}$$

These values of x are real if $2a-1 \geqslant 0$ i.e. $a \geqslant \frac{1}{2}$.

The root $x=\dfrac{1-\sqrt{(2a-1)}}{2}$ does not satisfy the condition $x \geqslant 1$ and so is not a root of the given equation.

And the root $x=[1+\sqrt{(2a-1)}]/2$ satisfies the equation for $a \geqslant 1$. Hence for $a<1$, the equation has no root and for $a \geqslant 1$, it has the root $x=[1+\sqrt{(2a-1)}]/2$.

13 The given equation can be re written

$$\left(\frac{p+q-x}{r}-1\right)+\left(\frac{p+r-x}{q}-1\right)+\left(\frac{q+r-x}{p}-1\right)$$
$$=4-\frac{4x}{p+q+r}$$

or $$(p+q+r-x)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right)=\frac{4(p+q+r-x)}{p+q+r}$$

or $$(p+q+r-x)\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}-\frac{4}{p+q+r}\right)=0$$

Hence $p+q+r-x=0$ or $x=p+q+r$

14 We rewrite the equation as

$$(x^4+1)-3x(x^2+1)+\tfrac{8}{9}x^2=0$$

Dividing by x^2, we get

$$\left(x^2+\frac{1}{x^2}\right)-3\left(x+\frac{1}{x}\right)+\tfrac{8}{9}=0$$

Now put $x+\dfrac{1}{x}=y$ so that $x^2+\dfrac{1}{x^2}+2=y^2$

Hence $y^2-2-3y+\frac{8}{9}=0$

or $9y^2-27y-10=0$ or $(3y-10)(3y+1)=0$

$y=\frac{10}{3}$ or $-\frac{1}{3}$

Taking $y=\dfrac{10}{3}$, we get $x+\dfrac{1}{x}=\dfrac{10}{3}$

or $3x^2-10x+3=0$, this gives $x=\frac{1}{3}, 3$

Again taking $y=-\frac{1}{3}$, we get

$$x+\frac{1}{x}=-\tfrac{1}{3} \text{ or } 3x^2+x+3=0$$

Hence $\log_4 5$ and $\log_{1/16} \frac{1}{25}$ are equal

(iii) Since $\log_7 11 > 1$ [See (16) of § 2]

and $0 < \log_8 5 < 1$ [See (17) of § 2]

we conclude that

$\log_7 11 > \log_8 5$

(iv) By (16) of § 2, we have

$\log_2 3 > 1$

and $\log_2 3 < \log_2 4 = \log_2 2^2 = 2$

Thus $1 < \log_2 3 < 2$

Again $\log_3 11 > \log_3 9 = \log_3 3^2 = 2$

Since $1 < \log_2 3 < 2$ and $\log_3 11 > 2$

we conculde that $\log_3 11 > \log_2 3$

(v) By (18) of § 2, $0 < \log_{1/2} \frac{1}{2} < 1$

and by (19) of § 2, $\log_{1/2} \frac{1}{3} > 1$

Hence $\log_{1/2} \frac{1}{3} > \log_{1/3} \frac{1}{2}$

16 $\log_a n, \log_b n, \log_c n$ are in H P Taking reciprocals

$\log_n a, \log_n b, \log_n c$ are in A P

or $2 \log_n b = \log_n a + \log_n c$

or $\log_n b^2 = \log_n ac$ $b^2 = ac$

Hence a, b, c are in G P

17 Taking reciprocals we get

$\log_x l, \log_x m, \log_x n$ are in H P

$$\log_x m = \frac{2 \log_x l \log_x n}{\log_x l + \log_x n}$$

or $\log_x m/\log_x l = \log_x n^2/\log_x ln$

or $\log_l m = \log_{ln} n^2$ by (5)

Writing the above in exponential form we get

$n^2 = (ln)^{\log_l m}$

18 L H S $= \log_a a = 1$

19 Do yourself

20 Since $\log_b a = \frac{1}{\log_a b}$, the expression

$= \log_n 2 + \log_n 3 + \log_n 4 + \ldots + \log_n 1983$

$= \log_n (2.3.4 \ldots 1983) = \log_n 1983!$

$= \log_n n = 1$ $\{\because 1983! = n\}$

21 Writing the given relation in logarithmic form

$\log_a y = \frac{1}{1 - \log_a x}$ and $\log_a z = \frac{1}{1 - \log_a y}$

We want a relation in z and x hence eliminate y by puttin[g] the value of $\log_a y$ from Ist in 2nd

Squaring again, $x^2+6x+9=4$ (
or $11x^2-34x-3=0$ or (
$x=-3, -\frac{1}{11}$

The check shows that $x=-3$ d
tion whereas $x=-1/11$ sat
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9 (a) The equation can be writt
$\sqrt{\{(2x-1)(x-4)\}}+3\sqrt{(2x-1)}$
Clearly one root is given by $2x$
Now cancelling the factor $\sqrt{(2}$
$\sqrt{(x-4)}-3=\sqrt{(x}$
Squaring, $x-4+9-6\sqrt{(x-4)}$
or $\sqrt{(x-4)}=1$ or x
Since $x=5$ satisfies the origina
Ans $x=\frac{1}{2}, 5$

(b) $L-M=1, L^2-M^2=15$
Adding $2L=16$ $L^2=64$
or $5x^2-6x-56=0$ $(x-4)($

10 The equation can be written as
$$\left(\frac{x-ab}{a+b}-c\right)+\left(\frac{x-ac}{a+c}-l\right.$$
or $\frac{x-ab-bc-ca}{a-b}-\frac{x-ab-bc-}{a+c}$
$(x-ab-bc-ca)\left(\frac{1}{a-b}\ \frac{1}{b+c}\right.$
Assuming $\frac{1}{a+b}-\frac{1}{b-c}+\frac{1}{c-a}$

the root of the equation If h

the given equation turns in
every value of x

11 We rewrite the given equatio
$$\left(\frac{x-a}{bc}\ \frac{1}{b}-\frac{1}{c}\right)\ \left(\frac{x-}{ac}\right.$$

or $(x-a-b-c)\left(\frac{1}{bc}-\frac{1}{ca}-\frac{1}{c}\right.$
or $(x-a-b-c)(a\ \ b-c$
Hence $x=a-b-c$, provided

$$\log_a z = \frac{1}{1-\frac{1}{1-\log_a x}} = \frac{1-\log_a x}{-\log_a x} = -\frac{1}{\log_a x}+1$$

$$\frac{1}{\log_a x} = 1-\log_a z \quad \text{or} \quad \log_a x = \frac{1}{1-\log_a z}$$

or $x = a^{\frac{1}{1-\log_a z}}$

22 Let $\frac{\log a}{b-c} = \frac{\log b}{c-a} = \frac{\log c}{a-b} = k$

Then $\log a = k(b-c)$, $\log b = k(c-a)$
and $\log c = k(a-b)$
Now $a \log a + b \log b + c \log c$
$= ak(b-c) + bk(c-a) + ck(a-b) = 0$
$\log a^a b^b c^c = 0$
or $a^a b^b c^c = 1$

Note that here it is understood that
$a > 0,\ b > 0,\ c > 0$
$a \neq 1,\ \ b \neq 1,\ \ c \neq 1$

23 Proceed as in problem 22

24 Let

$$\frac{x(y+z-x)}{\log x} = \frac{y(z+x-y)}{\log y} = \frac{z(x+y-z)}{\log z} = \frac{1}{k}$$

Then $\log x = kx(y+z-x)$,
$\log y = ky(z+x-y)$
and $\log z = kz(x+y-z)$
Hence $y \log x + x \log y = 2kxyz$
$y \log z + z \log y = 2kxyz$
and $z \log x + x \log z = 2kxyz$
It follows that
$y \log x + x \log y = y \log z + z \log y = z \log x + x \log z$
or $\log(x^y y^x) = \log(z^y y^z) = \log(x^z z^x)$
$x^y y^x = y^z z^y = z^x x^z$

25 (i) $\frac{\log_a n}{\log_{ab} n} = \frac{\log_n ab}{\log_n a} = \frac{\log_n a + \log_n b}{\log_n a}$

$$= 1 + \frac{\log_n b}{\log_n a} = 1 + \log_a b$$

(ii) $\log_{ab} x = \frac{1}{\log_x ab} = \frac{1}{\log_x a + \log_x b}$

$$= \frac{1}{\frac{1}{\log_a x} + \frac{1}{\log_a x}} = \frac{\log_a x \log_b x}{\log_a x + \log_b x}$$

$$x=\frac{ab\,(c+d)-cd\,(a+b)}{ab-cd}$$

21 (a) The equation can be written as

$$x^4-4x^3-x^2-6x-5=0 \text{ or } x^4-4x+4-(x^2+6x+9)=0$$

or $(x^2-2)^2-(x+3)=0$

or $(x^2+x+1)(x^2-x-5)=0$ by factors of L^2-M^2

[or we may write it as

$$(x^4-x)-5\,(x^2+x+1)=0$$

or $x\,(x-1)\,(x^2+x+1)-5\,(x^2+x+1)=0$ etc.]

$$x=\frac{-1\pm\sqrt{(1-4)}}{2},\ \frac{1\pm\sqrt{(1+20)}}{2}$$

or $x=\dfrac{-1\pm\sqrt{(-3)}}{2},\ \dfrac{1\pm\sqrt{(21)}}{2}$

(b) Dividing by x^2 we get

$$\left(\frac{x^2+2}{x}\right)^2+8=6\left(\frac{x^2+2}{x}\right) \quad \text{Put } \frac{x^2+2}{x}=y$$

$y^2-6y+8=0 \qquad y=2, 4$

Ans $x=1\pm i,\ \ 2\pm\sqrt{2}$

22 Adding x^2+4 to both sides of the given equation, we get

$$x^4+4x^2+4=x^2+16x+64 \text{ or } (x^2+2)^2=(x+8)^2$$

$$x^2+2=\pm(x+8)$$

Taking $x^2+2=x+8$, we get

$$x^2-x-6=0 \text{ or } (x-3)(x+2)=0$$

$x=3, -2$

Again taking $x^2+2=-x-8$, we get

$$x^2+x+10=0$$

$$x=\frac{-1\pm\sqrt{(1-40)}}{2}=\frac{-1\pm\sqrt{(-39)}}{2}$$

Solution set is

$$x=3,\ -2,\ \frac{-1\pm\sqrt{(-39)}}{2}$$

23 Ans $x=6,\ -\dfrac{161}{30}$

24 Dividing both members of the equation by $(a+x)^{1/3}(a-x)^{1/3}$, we get

$$\left(\frac{a+x}{a-x}\right)^{1/3}+4\left(\frac{a-x}{a+x}\right)^{1/3}=5$$

Now put, $\left(\dfrac{a+x}{a-x}\right)^{1/3}=y$ Then $y+\dfrac{4}{y}=5$ or $y^2-5y+4=0$

or $(y-4)(y-1)=0, \quad \therefore y=1, 4$

or $\quad \sin\frac{x}{2}\left[\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\right.$

$$\left.+\cos\frac{x}{2}\right]=0$$

Either $\sin x/2, =0 \qquad x/2, =n\pi$ or $x=2n\pi$

or $\left[\left(\sin\frac{x}{2}+\cos\frac{x}{2}\right)\left(\cos^2\frac{x}{2}-\sin^2\frac{x}{2}\right)\right.$

$$\left.+\cos\frac{x}{2}\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\right)\right]=0$$

Dividing by $\cos^3 x/2$ we get

$\left(\tan\frac{x}{2}+1\right)\left(1-\tan^2\frac{x}{2}\right)+\left(1+\tan^2\frac{x}{2}\right)=0$

or $\quad t^3-t-2=0 \qquad$ where $t=\tan x/2$

$f(t)=t^3-t-2 \qquad f(1)=-2 \qquad f(2)=4$

Since $f(t)$ changes sign for $t=1$ and $t=2$, therefore a root of $f(t)=0$ lies between 1 and 2. Hence $1<k<2$ then

$f(k)=0 \qquad t=k$ or $\tan x/2, =k=\tan\alpha$ say

$x/2, =n\pi+\alpha$ or $x=2n\pi+2\alpha=2n\pi+2\tan^{-1}k$

19 (*a*) The given expression is

$a_1+a_2\cos 2x+a_3\,\frac{1}{2}(1-\cos 2x)=0$

or $\quad (a_1+\frac{1}{2}a_3)+(a_2-\frac{1}{2}a_3)\cos 2x=0 \ \forall\ x$

Above will hold when $a_1+\frac{1}{2}a_3=0$ and $a_2-\frac{1}{2}a_3=0$

Choosing $a_3=k$ we get $a_1=-k/2$ and $a_2=k/2$

Hence the solutions are $(-k/2, k/2, k)$ where k is any real number

Thus the number of solutions is infinite $\qquad$ *d* is correc

(*b*) The given expression $\qquad 1-\cos x+\sin^2 ax=0$

or $\quad 2\sin^2\frac{x}{2}+\sin^2 ax=0 \qquad \sin\frac{x}{2}=0$ and $\sin ax=0$

$\frac{x}{2}=m\pi$ and $ax=n\pi \qquad$ or $\quad x=2m\pi$ and $x=\frac{n\pi}{a}$

$2m\pi=n\pi/a \qquad\qquad n=2am \qquad\qquad (1)$

Now m and n are integers but a is irrational, hence the result (1) will hold good if $m=0$ and $n=0$ and in that case $x=0$ is the only solution

20 First note that $x\neq 0$. Dividing the given equations we get

$14\sin^3 y-39\sin^2 y\cos y+42\sin y\cos^2 y-13\cos^3 y=0$

Dividing throughout by $\cos^3 y$, we get a cubic in $\tan y=t$ as

$14t^3-39t^2+42t-13=0 \qquad =f(t)$ say

$f(0) = -13$, $f(1) = +4$ a root of the equation lies b[etween] 0 and 1 and by trial we find that $t = \frac{1}{2}$ as

$\frac{14}{8} - \frac{39}{4} + \frac{42}{2} - 13 = -\frac{32}{4} + 8 = 0$ or $(2t-1)$ is a factor of

$f(t) = (2t-1)(7t^2 - 16t + 13) = 0$

The second factor gives imaginary values of t as $B^2 - 4AC =$ ...

$\tan y = \frac{1}{2}$ $\sin y = 1/\sqrt{5}$, $\cos y = 2/\sqrt{5}$

$y = \tan^{-1}\frac{1}{2}$ where $2n\pi < y < 2n\pi + \pi/2$ (1st Quad) (1)

or $\sin y = -1/\sqrt{5}$ and $\cos y = -2/\sqrt{5}$

$y = \tan^{-1}\frac{1}{2}$ where $2n\pi + \pi < y < 2n\pi + 3\pi/2$ (3rd Quad) (2)

In the 1st case $x\left(\frac{8}{5\sqrt{5}} + 3\,\frac{2}{5}\ \frac{1}{\sqrt{5}}\right) = 14$ $x = 5\sqrt{5}$ (3)

In the 2nd case $x = -5\sqrt{5}$ as above (4)

(1) and (3), (2) and (4) give the required solutions

$$x=\frac{-1\pm\sqrt{(1-36)}}{6}=\frac{-1\pm\sqrt{(-35)}}{6}$$

Ans $x=\frac{1}{3}, 3, \frac{-1\pm\sqrt{(-35)}}{6}$

15 Proceed as in Ex 14

Ans $x=\frac{5\pm\sqrt{(21)}}{6}, \ \frac{-1\pm\sqrt{(-3)}}{2}$

16 Put $x=1/y$ Then new equation is

$\frac{6}{y^3}-\frac{11}{y^2}+\frac{6}{y}-1=0$ or $y^3-6y^2+11y-6=0$ (1)

Since the roots of the original equation are in H P roots of (1) must be in A P Let the roots of (1) be $\alpha-\delta, \alpha, \alpha+\delta$

Then $(\alpha-\delta)+\alpha+(\alpha+\delta)=6$ or $3\alpha=6$ or $\alpha=2$

and $(\alpha-\delta)\,\alpha\,(\alpha+\delta)=6$ or $(2-\delta)\,2\,(2+\delta)=6$

or $4-\delta^2=3$, that is, $\delta=\pm1$

Hence roots of (1) are 1, 2 and 3,

The roots of the original equation are, $1, \frac{1}{2}$ and $\frac{1}{3}$

17 Ans $x=\pm\frac{2a}{5}$

Hint Use componendo and dividendo

18 Ans $x=\frac{-1\pm\sqrt{(17)}}{2}, \ \frac{-1\pm\sqrt{2}}{2}$

19 The equation can be re written as

$$(x^2-x)^2-(x^2-x)-380=0$$

Now put $x^2-x=y$ Then equation becomes

$y^2-y-380=0$ or $(y-20)(y+19)=0$

$y=20, -19$

Taking $y=20$ we have $x^2-x=20$

or $x^2-x-20=0$ or $(x-5)(x+4)=0$

$x=5, -4$

Again taking $y=-19$ we get $x^2-x+19=0$

$$x=\frac{1\pm\sqrt{(1-76)}}{2}=\frac{1\pm\sqrt{(-75)}}{2}$$

20 Subtracting x from both the members of the given equation we get

$$\frac{(x-a)(x-b)}{x-a-b}-x=\frac{(x-c)(x-d)}{x-c-d}-x$$

or $\frac{ab}{x-a-b}=\frac{cd}{x-c-d}$

or $x(ab-cd)=ab(c+d)-cd(a+b)$

[Hint From second equation, we get

$$\frac{1}{x-b}-\frac{1}{a-b}=\frac{1}{y-a} \quad \text{or} \quad \frac{a-x}{(x-b)(a-b)}=\frac{1}{y-a}$$

or $$\frac{(x-b)(a-b)}{a-x}=y-a$$

or $$y=a+\frac{(x-b)(a-b)}{a-x}=\frac{a^2-ab+b^2-bx}{a-x}$$

Now substitute this value of y in first equation etc

15 From first equation we get

$$\left\{\frac{3(x-y)}{x+y}\right\}^2+\left\{\frac{3(x+y)}{x-y}\right\}^2=82$$

Put $\frac{x+y}{x-y}=z$. We then have

$$9\left(\frac{1}{z^2}+z^2\right)=82$$

or $9z^4-82z^2+9=0$ or $(z^2-9)(9z^2-1)=0$

$z=\pm 3$ or $z=\pm\frac{1}{3}$

Taking $z=3$ we get $\frac{x+y}{x-y}=3$ or $x=2y$ (1)

Also second given equation is,

$3x+7y=26$ (2)

Solving (1) and (2) we get $x=4$, $y=2$

Similarly solving $z=-3$, that is $\frac{x+y}{x-y}=-3$ with (2), we shall get

$$x=\frac{26}{17} \quad y=\frac{52}{17}$$

Again solving $z=\frac{x+y}{x-y}=\frac{1}{3}$ with (2), we shall obtain,

$$x=-52,\ y=26$$

And finally solving $z=\frac{x+y}{x-y}=-\frac{1}{3}$, we shall get

$$x=-\frac{26}{11} \quad y=\frac{52}{11}$$

16 The given system of equations is equivalent to

$$y=2x/(1+x^2),\ 2(x-1)^2+1+y^3=0$$

Since $2x/(1+x^2) \leqslant 1$ for all real x, it follows from the of these equations that $y \leqslant 1$ or $-1 \leqslant y \leqslant 1$.
from the second equation of the system we get

$$y^3=-1-2(x-1)^2 \leqslant -1$$

$$x=\frac{ab(c+d)-cd(a+b)}{ab-cd}$$

21. (a) The equation can be written as

$x^4-4x^2-x^2-6x-5=0$ or $x^4-4x^2+4-(x^2+6x+9)=0$

or $(x^2-2)^2-(x+3)^2=0$

or $(x^2+x+1)(x^2-x-5)=0$ by factors of L^2-M^2

[or we may write it as

$(x^4-x)-5(x^2+x+1)=0$

or $x(x-1)(x^2+x+1)-5(x^2+x+1)=0$ etc.]

$$x=\frac{-1\pm\sqrt{(1-4)}}{2},\ \frac{1\pm\sqrt{(1+20)}}{2}$$

or $$x=\frac{-1\pm\sqrt{(-3)}}{2},\ \frac{1\pm\sqrt{(21)}}{2}$$

(b) Dividing by x^2 we get

$$\left(\frac{x^2+2}{x}\right)^2+8=6\left(\frac{x^2+2}{x}\right)\quad \text{Put}\quad \frac{x^2+2}{x}=y$$

$y^2-6y+8=0$ $\qquad y=2,\ 4$

Ans $x=1\pm i,\ \ 2\pm\sqrt{2}$

22 Adding x^2+4 to both sides of the given equation, we get

$x^4+4x^2+4=x^2+16x+64$ or $(x^2+2)^2=(x+8)^2$

$x^2+2=\pm(x+8)$

Taking $x^2+2=x+8$, we get

$x^2-x-6=0$ or $(x-3)(x+2)=0$

$x=3,\ -2$

Again taking $x^2+2=-x-8$, we get

$x^2+x+10=0,$

$$x=\frac{-1\pm\sqrt{(1-40)}}{2}=\frac{-1\pm\sqrt{(-39)}}{2}$$

Solution set is

$$x=3,\ -2,\ \frac{-1\pm\sqrt{(-39)}}{2}$$

23 Ans $x=6,\ -\dfrac{161}{30}$

24 Dividing both members of the equation by $(a+x)^{1/3}(a-x)^{1/3}$, we get

$$\left(\frac{a+x}{a-x}\right)^{1/3}+4\left(\frac{a-x}{a+x}\right)^{1/3}=5$$

Now put, $\left(\dfrac{a+x}{a-x}\right)^{1/3}=y$ Then $y+\dfrac{4}{y}=5$ or $y^2-5y+4=0$

or $(y-4)(y-1)=0$ $\qquad y=1,\ 4$

Now since $y \leqslant -1$ and $y \geqslant -1$, we must have $y = -1$ then $x=1$. Thus the solution is $x=1$, $y=-1$.

17 The given equations are

$$ax + by = 1 \qquad (1)$$
$$cx^2 + dy^2 = 1 \qquad (2)$$

Eliminating y from (1) and (2) we obtain

$$cx^2 + d\left(\frac{1-ax}{b}\right)^2 = 1$$

or $\quad b^2cx^2 + d - 2adx + a^2dx^2 = b^2$

or $\quad (b^2c + a^2d)x^2 - 2adx + d - b^2 = 0 \qquad (3)$

Since the given equations have only one solution, the roots of (3) must be equal, the condition for which is

$$4a^2d^2 - 4(b^2c + a^2d)(d - b^2) = 0$$

or $\quad a^2d^2 - b^2cd + b^4c - a^2d^2 + a^2b^2d = 0$

or $\quad b^4c + a^2b^2d - b^2cd = 0$

or $\quad b^2c + a^2d - cd = 0$

or $\quad \dfrac{a^2}{c} + \dfrac{b^2}{d} = 1 \qquad (4)$

Under the condition (4), the root of (3) is given by

$$x = \frac{2ad}{2(b^2c + a^2d)} = \frac{2ad}{2cd\left(\frac{b^2}{d} + \frac{a^2}{c}\right)} = \frac{a}{c} \text{ by (4)}$$

Then $y = \dfrac{1-ax}{b} = \dfrac{1 - \frac{a^2}{c}}{b} = \dfrac{\frac{b^2}{d}}{b}$ by (4) $= \dfrac{b}{d}$

Hence the only one solution of the given equations is

$$x = \frac{a}{c}, \quad y = \frac{b}{d}$$

Problem Set (C)

Solve the following equations

1 $3x + y - 2z = 0$, $4x - y - 3z = 0$, $x^3 + y^3 + z^3 = 467$

2 $xz + y = 7z$, $yz + x = 8z$, $x + y + z = 12$

3 $\dfrac{xy}{ay + bx} = c$, $\dfrac{xz}{az + cx} = b$, $\dfrac{yz}{bz + cy} = a$

4 $x + y - 4xy = 0$, $y + z - 6yz = 0$, $z + x - 8zx = 0$ **(I I T 73)**

5 $xy = a(x + y)$, $yz = c(y + z)$, $zx = b(z + x)$

6 $xyz = a^2(y + z) = b^2(z + x) = c^2(x + y)$

Hence $\left(\frac{a+x}{a-x}\right)^{1/3}=1, 4$ or $\frac{a+x}{a-x}=1, 64$

whence $x=0, \frac{63a}{65}$

25 Ans $x=a^2-\frac{(b-2a)^3}{27b}$

[Hint Cube both sides of the given equation]

26 $\sqrt{\{x-\sqrt{(1-x)}\}}=1-\sqrt{x}$

Squaring, $x-\sqrt{(1-x)}=1+x-2\sqrt{x}$

or $\sqrt{(1-x)}=2\sqrt{x}-1$

Squaring again, $1-x=4x+1-4\sqrt{x}$

or $4\sqrt{x}=5x$ or $\sqrt{x}\,(4-5\sqrt{x})=0$

$x=0$ or $\frac{16}{25}$

$x=0$ does not satisfy the given equation

Hence $x=\frac{16}{25}$ is the only root of the equation

27 Let the roots be α, β, γ Then

$\alpha+\beta+\gamma=5, \alpha\beta+\beta\gamma+\gamma\alpha=-2, \alpha\beta\gamma=-24$

Also $\alpha\beta=12$, since the product of two roots is given 12

These relations give, $\gamma=-\frac{24}{12}=-2$

Then $\alpha+\beta+\gamma=5$ gives $\alpha+\beta=7$

But $(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta=49-48=1$

$\alpha-\beta=\pm 1$

Hence $\alpha=4, \beta=3$ or $\alpha=3, \beta=4$

Then roots are 4, 3, -2

28 Ans $x=a, b$

[Note that if the operations with complex numbers are regarded as unknown there will be only one root either a or b]

29 The equation can be written as

$$\frac{(a-x)^{3/2}+(x-b)^{3/2}}{(a-x)^{1/2}+(x-b)^{1/2}}=a-b$$

whence we have

$(a-x)-(a-x)^{1/2}(x-b)^{1/2}+(x-b)=a-b$

or $\sqrt{\{(a-x)(x-b)\}}=0$

Hence $x=a$ or $x=b$

30 Let $x=\sqrt{[6+\sqrt{\{6+\sqrt{(6+\dots)}\}}]}$ Then $x=\sqrt{(6+x)}$

or $x^2-x-6=0$ $x=3, -2$

Since x is clearly positive we have $x=3$, rejecting $x=-2$

31 We have

$3x^3+4x=\{x^2+(\sqrt{18}x+\sqrt{32})\}\{x-(\sqrt{18}x+\sqrt{\sqrt{32}})\}$

7 $y+z-x=\frac{xyz}{a^2},\ z+x-y=\frac{xyz}{b^2},\ x+y-z=\frac{xyz}{c^2}$

8 $x^2+xy+xz=18,\ y^2+yz+yx+12=0,\ z^2+zx+zy=30$

9 $x(x+y+z)=a^2,\ y(x+y+z)=b^2,\ z(x+y+z)=c^2$

10 (a) Solve $x^2-yz=a,\ y^2-zx=b^2,\ z^2-xy=c^2$ **(Roorkee 77)**

(b) If $\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c}$, show that

$(x+y+z)(a+b+c)=ax+by+cz$ **(I I T 71)**

11 $y^2+z^2-(y+z)x=12,\ x^2+z^2-(x+z)y=8,$
$x^2+y^2-(x+y)z=2$

12 $(b+c)(y+z)-ax=b-c,$
$(c+a)(z+x)-by=c-a$
$(a+b)(x+y)-cz=a-b,$
where $a+b+c\neq 0$ **(I I T 77)**

13 $x(y+z)=a^2,\ y(z+x)=b^2,\ z(x+y)=c^2$

14 $x^2+y^2-z^2=21,\ 3xz+3yz-2xy=18,\ x+y-z=5$

15 $2xy-4x+y=17,\ 3yz+y-6z=52,\ 6xz+3z+2x=29$

16 (a) $y+z+yz=a,\ z+x+zx=b\ \ x+y+xy=c$

(b) $xy+x+y=23,\ xz+z+x=41,\ yz+y+z=27$ **(Roorkee 83)**

17 $(y-z)(z+x)=22\ \ (z+x)(x-y)=33,\ (x-y)(y-z)=6$

18 (a) $x^2=a+(y-z)^2,\ y^2=b+(z-x)^2,\ z^2=c+(x-y)^2$

(b) $(x+y)^2-z^2=-9,\ (y+z)^2-x^2=15$
$(z+x)^2-y^2=3$ **(I I T 75)**

19 $x^2+y^2+xy=9,\ z^2+x^2+xz=-4,\ y^2+z^2+yz=1$

20 $x^2+xy+y^2=37,\ y^2+yz+z^2=19,\ z^2+zx+x^2=28$

21 $x+y+z+u=2a,\ x+y-z-u=2b,$
$x-y+z-u=2c,\ x-y-z+u=2d$

22 (i) $z+ay+a^2x+a^3=0\ \ z+by+b^2x+b^3=0,$
$z+cy+c^2x+c^3=0$

(ii) $\frac{x}{a+\lambda}+\frac{y}{b+\lambda}+\frac{z}{c+\lambda}=1,\ \frac{x}{a+\mu}+\frac{y}{b+\mu}+\frac{z}{c+\mu}=1,$
$\frac{x}{a+\nu}+\frac{y}{b+\nu}+\frac{z}{c+\nu}=1$

23 $x+y+z=\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{2}\ \ xyz=1$

24 (i) $x+y+z=12,\ x^2+y^2+z^2=50,\ x^3+y^3+z^3=216$

(iii) $x+y+z=1,\ x^2+y^2+z^2=1,\ x^3+y^3+z^3=1$

25 $yz=a(y+z)+\alpha,\ zx=a(z+x)+\beta\ \ xy=a(x+y)+\gamma$

26 $x^2+(y-z)^2=a^2,\ y^2+(z-x)^2=b^2,\ z^2+(x-y)^2=c^2$

or $3x^3+4x^2=x^4-(\sqrt{18}x+\sqrt{32})^2=x^4-18x^2-48x-32$

or $x^4-3x^3-22x^2-48x-32=0$

Let $f(x)=x^4-3x^3-22x^2-48x-32$

The possible quadratic factors of $f(x)$ are of the form

$$(x^2+px+1)\,(x^2+qx-32),\ (x^2+px+2)\,(x^2+qx-16)$$

and $(x^2+px+4)\,(x^2+qx-8)$,

the first two of which do not give consistent values of p and q on comparison with $f(x)$ and comparing the coefficient of x^3, x^2, x in

$$(x^2+px+b)\,(x^2+qx-8)\equiv f(x)$$

we get $p+q=-3$ (1)

$-4+pq=-22$ (2)

and $-8p+4q=-48$ or $-2p+q=-12$ (3)

Solving (1) and (3) we get $p=3$ and $q=-6$

These values of p and q satisfy the equation (2)

Hence $f(x)=(x^2+3x+4)\,(x^2-6x-8)=0$

$x^2+3x+4=0$ gives $x=\dfrac{-3\pm\sqrt{(9-16)}}{2}=\dfrac{-3\pm\sqrt{7}i}{2}$

and $x^2-6x-8=0$ gives $x=\dfrac{6\pm\sqrt{(36+32)}}{2}=3\pm\sqrt{17}$

Problem Set (B)

Solve the following equations

1 $\sqrt{(x/y)}+\sqrt{(y/x)}=5/2,\ x+y=6$ (Roorkee 1970)

2 $(x^2/y)+(y^2/x)=9/2,\ x+y=3$ (I I T 1976)

3 $2^x+2^y=20,\ x+y=6$ (I I T 1971)

4 Solve the equations

(i) $x^2-xy+y^2=7,\quad x^4+x^2y^2+y^4=133$ (Roorkee 1980)

(ii) $x+y-\sqrt{xy}=6,\quad x^2+y^2+xy=84$

5 $x-y=13-3xy,\ x^2y-y^2x=12$

6 $x+y+xy=11,\ x^2y+xy^2=30$ (Roorkee 1979)

7 $3x^2-4xy-4=0,\ x^2-2y^2-2=0$ (Roorkee 1973)

8 $x^3-y^3=127,\ x^2y-xy^2=42$ (Roorkee 1981)

9 $x^2+y\,(x+1)=17,\ y^2+x\,(y+1)=13$ (I I T 1974)

10 $\dfrac{bx}{y+b}+\dfrac{ay}{x+a}=\dfrac{a+b}{2},\ \dfrac{x}{a}+\dfrac{y}{b}=2$ (Roorkee 1974)

11 (i) $x^4-y^4=15,\ x-y=1$

(ii) $\sqrt[4]{(1+5x)}+\sqrt[4]{(5-y)}=3,\ 5x-y=11$

12 $(x+y)^{2/3}+2\,(x-y)^{2/3}=3\,(x^2-y^2)^{1/3},\ 3x-2y=13$

13 $4x^2+5y=6+20xy-25y^2+2x,\ 7x-11y=17$

27 $x+y+z=a+b+c,\ \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3,$
$ax+by+cz=bc+ca+ab$

28 Find real values of x, y, z satisfying the equations
$x+y=2,\quad xy-z^2=1$ (IIT 72)

29 Given $y^2+z^2=ayz,\ z^2+x^2=bxz,\quad x^2+y^2=cxy$
express $\frac{y^2}{xz}+\frac{xz}{y^2}$ in terms of a b, c (IIT 72)

30
$$y^3-9x^2+27x-27=0$$
$$z^3-9y^2+27y-27=0$$
$$x^3-9z^2+27z-27=0$$

Solutions to Problem Set (C)

1
$$3x+y-2z=0 \quad (1)$$
$$4x-y-3z=0 \quad (2)$$
$$x^3+y^3+z^3=467 \quad (3)$$

Solving (1) and (2) by the method of cross multiplication, we obtain

$$\frac{x}{-3-2}=\frac{y}{-8+9}=\frac{z}{-3-4}$$

or $\frac{x}{5}=\frac{y}{-1}=\frac{z}{7}=k$, say

$$x=5k,\ y=-k,\ z=7k$$

Substituting in (3), we get

$$125k^3-k^3+343k^3=467$$

or $467k^3=467$ or $k=1$

The solution is,

$$x=5,\ y=-1,\ z=7$$

2 Ans $x=4, \frac{60}{7},\ y=6,\ \frac{66}{7},\ z=2,\ -6$

3 Rewriting the given equations, we get

$$\frac{ay+bx}{xy}=\frac{1}{c},\quad \frac{az+cx}{xz}=\frac{1}{b},\quad \frac{bz+cy}{yz}=\frac{1}{a}$$

or $\frac{a}{x}+\frac{b}{y}=\frac{1}{c},\ \frac{a}{x}+\frac{c}{z}=\frac{1}{b},\ \frac{b}{y}+\frac{c}{z}=\frac{1}{a}$

Adding, $2\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

Hence $2\left[\frac{a}{x}+\frac{1}{a}\right]=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$

or $\frac{2a}{x}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}-\frac{2}{a}=\frac{1}{b}+\frac{1}{c}-\frac{1}{a},$

14 $\frac{x-a}{a^2}+\frac{y-b}{b^2}=\frac{1}{x-b}-\frac{1}{y-a}-\frac{1}{a-b}=0$

15 $\left(3-\frac{6y}{x+y}\right)^2+\left(3+\frac{6y}{x-y}\right)^2=82,\ 3x+7y=26$

16 $x^2y^2-2x+y^2=0$
$2x^2-4x+3+y^3=0$

17 If the equations $ax+by+1$, $cx^2+dy^2=1$ have only one solution prove that $\frac{a^2}{c}+\frac{b^2}{d}=1$, and $x=\frac{a}{c}$, $y=\frac{b}{d}$

Solutions to Problem Set (B)

1 The equations can be written as

$$2\,(x+y)=5\sqrt{(xy)} \tag{1}$$
$$x+y=6 \tag{2}$$

These equations give, $xy=\frac{144}{25}$

Since $x+y=6$, $xy=\frac{144}{25}$, we can say that x and y are the roots of $t^2-6t+\frac{144}{25}=0$ or $25t^2-150\,t+144=0$

$(5t-24)\,(5t-6)=0 \qquad t=\frac{24}{5}$ or $\frac{6}{5}$

Hence $x=\frac{24}{5}$, $y=\frac{6}{5}$ or $x=\frac{6}{5}$, $y=\frac{24}{5}$

2 Ans $x=2, y=1$ or $x=1, y=2$

3 Ans $x=2, y=4$ or $x=4, y=2$

4 From 2nd equation we have

$$(x^2+y^2)^2-x^2y^2=133$$

or
$$(x^2+y^2-xy)\,(x^2+y^2+xy)=133$$
$$x^2+y^2+xy=133/7=19$$

Adding and Subtracting we get

$$x^2+y^2=13, \qquad xy=4$$
$$(x+y)^2=13+8=21 \text{ or } x+y=\pm\sqrt{21}$$

and
$$(x-y)^2=13-8=5 \text{ or } x-y=\pm\sqrt{5}$$
$$x=\tfrac{1}{2}\,[\sqrt{21}+\sqrt{5}],\ -\tfrac{1}{2}\,[\sqrt{21}+\sqrt{5}]$$
$$y=\tfrac{1}{2}\,[\sqrt{21}-\sqrt{5}],\ -\tfrac{1}{2}\,[\sqrt{21}-\sqrt{5}]$$

(ii) Proceed as above, Ans $x=8, y=2$ or $x=2, y=8$

5 Ans $x=2+\sqrt{7}, y=-2+\sqrt{7}$
or $x=2-\sqrt{7}, y=-2-\sqrt{7}$

6 Put $x+y=u$ and $xy=v$ Then the given equations can be written as

$$u+v=11,\ uv=30$$
$$u-v=\sqrt{\{(u+v)^2-4uv\}}=\sqrt{(121-120)}=\pm 1$$

Hence $u=6, v=5$ or $u=5, v=6$

Thus $x+y=6, xy=5$ or $x+y=5, xy=6$

$$x=\frac{2abc}{ac+ab-bc}$$

Similarly

$$y=\frac{2ab^2c}{bc+ab-ac} \text{ and } z=\frac{2abc^2}{bc+ac-ab}$$

4 Do as in problem 3

Ans $x=\frac{1}{2}, y=1, z=\frac{1}{3}$

5 Do as in problem 3

Ans $x=\frac{2abc}{bc+ca-ab}$, $y=\frac{2abc}{ab+bc-ca}$, $z=\frac{2abc}{ca+ab-bc}$

6 Ans $x=\pm\frac{\sqrt{\{2(1/b^2+1/c^2-1/a^2)\}}}{\sqrt{\{(1/a^2+1/c^2-1/b^2)(1/a^2+1/b^2-1/c^2)\}}}$ etc

7 One obvious solution is $x=0, y=0, z=0$

Adding the given equations pairwise, we get

$$2z=xyz\left(\frac{1}{a^2}+\frac{1}{b^2}\right) \text{ etc}$$

Hence $\frac{2}{xy}=\frac{1}{a^2}+\frac{1}{b^2}$. Similarly $\frac{2}{yz}=\frac{1}{b^2}+\frac{1}{c^2}$, $\frac{2}{zx}=\frac{1}{a^2}+\frac{1}{c^2}$

Multiplying these, we obtain

$$x^2y^2z^2=\frac{8a^4b^4c^4}{(a^2+b^2)(b^2+c^2)(c^2+a^2)}$$

Hence $xyz=\frac{2\sqrt{2}\,a^2b^2c^2}{\sqrt{[(a^2+b^2)(b^2+c^2)(c^2+a^2)]}}$

Now using the equality $xy=\frac{2a^2b^2}{a^2+b^2}$, we find

$$z=\pm\frac{\sqrt{2}c\sqrt{\{(a^2+b^2)\}}}{\sqrt{\{(b^2+c^2)(c^2+a^2)\}}}$$

Similar expressions for values of x and y can be found

8 Given equations can be rewritten as

$x(x+y+z)=18$ (1)

$y(x+y+z)=12$ (2)

$z(x+y+z)=30$ (3)

Adding (1), (2) and (3) we get

$(x+y+z)^2=36$ or $x+y+z=\pm6$ (4)

Then from (1) and (4) we get $x=\pm3$

Similary from (2) and (4), $y=\pm2$

and from (3) and (4), $z=\pm5$

Hence the solutions are

$x=3, y=2, z=5$

or $x=-3, y=-2, z=-5$,

First we take, $x+y=6$ (1)

and $xy=5$

x and y are the roots of $t^2-6t+5=0$

or $(t-5)(t-1)=0, \quad t=5, 1$

Hence $x=5, y=1$ or $x=1, y=5$

We now take $x+y=5$, $xy=6$

Solving these, we shall get

$x=3, y=2$ or $x=2, y=3$

Hence the solution set is

$(5, 1), (1, 5), (3, 2), (2, 3)$

7 Ans $x=2, y=1$ or $x=-2, y=-1$

8 Putting $y=mx$, the given equations can be written as

$$x^3(1-m^3)=127 \quad (1)$$

and

$$x^3(m-m^2)=42 \quad (2)$$

Dividing, we get

$$\frac{1-m^3}{m-m^2}=\frac{127}{42} \text{ or } \frac{1+m+m^2}{m}=\frac{127}{42}$$

or $42m^2-85m+42=0$ or $42m^2-49m-36m+42=0$

or $(6m-7)(7m-6)=0 \quad m=\frac{7}{6}$ or $\frac{6}{7}$

We first take $m=7/6$ and we get from (1)

$x^3(1-\frac{343}{216})=127$ or $x=-6$

and $y=mx=\frac{7}{6}(-6)=-7$

Similarly taking $m=\frac{6}{7}$ we shall get $x=7, y=6$

Hence solutions are

$x=7, y=6$ or $x=-6, y=-7$

9 The given equations are

$$x^2+xy+y=17 \quad (1)$$

$$y^2+xy+x=13 \quad (2)$$

Adding (1) and (2), we get

$x^2+2xy+y^2+(x+y)=30$

or $(x+y)^2+(x+y)-30=0$

or $(x+y+6)(x+y-5)=0$

$x+y=-6$ (3)

or $x+y=5$ (4)

We have now to solve equations, (1) and (3) and (1) and (4)

Or we may solve (2) and (3), and (2) and (4)]

Eliminating y from (1) and (3), we get, 9 Do as in problem 8

Ans $x=\pm\frac{a^2}{\sqrt{(a^2+b^2-c^2)}},\quad y=\pm\frac{b^2}{\sqrt{(a^2+b^2+c^2)}},$

$z=\pm\frac{c^2}{\sqrt{(a^2+b^2+c^2)}}$

10 (*a*) Multiplying the given equations by y, z, x respectively and adding, we get

$$c^2x+a^2y+b^2z=0 \qquad (1)$$

Again multiplying the given equations by z, x, y respectively and adding, we get

$$b^2x+c^2y+a^2z=0 \qquad (2)$$

Solving (1) and (2) by cross multiplication,

$$\frac{x}{a^4-b^2c^2}=\frac{y}{b^4-c^2a^2}=\frac{z}{c^4-a^2b^2}=k, \text{ say} \qquad (3)$$

Substituting in any one of the given equations, for x, y, z in terms of k from (3), we get

$$k^2(a^6+b^6+c^6-3a^2b^2c^2)=1$$

$$\frac{x}{a^4-b^2c^2}=\frac{y}{b^4-c^2a^2}=\frac{z}{c^4-a^2b^2}$$

$$=\pm\frac{1}{\sqrt{(a^6+b^6+c^6-3a^2b^2c^2)}}$$

(*b*) $$\frac{x^2-yz}{a}=\frac{y^2-zx}{b}=\frac{z^2-xy}{c} \qquad (1)$$

$$=\frac{x^2-yz+y^2-zx+z^2-xy}{a+b+c}$$

$$=\frac{x(x^2-yz)+y(y^2-zx)+z(z^2-xy)}{ax+by+cz}$$

$$=\frac{x^3+y^3+z^3-3xyz}{ax+by+cz}=\frac{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}{ax+by+cz}$$

$$(x+y+z)(a+b+c)=ax+by+cz$$

11 We rewrite the equations as

$$(y^2-zx)+(z^2-xy)=12 \qquad (1)$$

$$(x^2-yz)+(z^2-xy)=8 \qquad (2)$$

$$(x^2-yz)+(y^2-zx)=2 \qquad (3)$$

Adding (1), (2) and (3), we get

$$2(y^2-zx)+2(z^2-xy)+2(x^2-yz)=22$$

or $$(y^2-zx)+(z^2-xy)+(x^2-yz)=11 \qquad (4)$$

From (1) and (4), $x^2-yz=-1$ (5)

Similarly from (2) and (4) $y^2-zx=3$ (6)

and from (3) and (4), $z^2-xy=9$ (7)

$$-6x-x-6=17 \text{ or } x=-\tfrac{23}{7}$$

Then $\quad y=-x-6=\tfrac{23}{7}-6=-\tfrac{19}{7}$

One solution is ,

$$x=-\tfrac{23}{7},\ y=-\tfrac{19}{7}$$

Similarly solving (1) and (4), we shall get

$x=3$ and $y=2$

Hence the solutions are

$$x=-\frac{23}{7},\ y=-\frac{19}{7} \quad \text{or} \quad x=3,\ y=2$$

13 From the second equation, we get

$$\frac{x}{c}-1=1-\frac{y}{b} \quad \text{or} \quad \frac{x-a}{a}=\frac{b-y}{b}=k, \text{ say}$$

Then $\quad x=a\,(1+k)$ and $y=b\,(1-k)$ (1)

Substituting these values of x and y in first equation, we have

$$\frac{ab\,(1+k)}{b\,(2-k)}+\frac{ab\,(1-k)}{a\,(2+k)}=\frac{a+b}{2}$$

or $$\frac{a\,(1+k)}{2-k}+\frac{b\,(1-k)}{2+k}=\frac{a+b}{2}$$

or $$\left[\frac{a\,(1+k)}{2-k}-\frac{a}{2}\right]+\left[\frac{b\,(1-k)}{2+k}-\frac{b}{2}\right]=0$$

or $$\frac{3ak}{2-k}-\frac{3bk}{2+k}=0$$

or $$k\left[\frac{a\,(2+k)-b\,(2-k)}{(2-k)\,(2+k)}\right]=0$$

or $$k\,[(a+b)\,k-2\,(b-a)]=0$$

$$k=0 \quad \text{or} \quad k=\frac{2\,(b-a)}{b+a}$$

Substituting these values of k in (1), we get

$$x=a\,(1+0)=a \text{ and } y=b\,(1-0)=b$$

or $$x=a\left\{1+\frac{2\,(b-a)}{b+a}\right\}=\frac{a\,(3b-a)}{a+b},$$

and $$y=b\left\{1-\frac{2\,(b-a)}{b+a}\right\}=\frac{b\,(3a-b)}{a+b}$$

Hence the solutions are

$$x=a,\ y=b \quad \text{or} \quad x=\frac{a\,(3b-a)}{a+b},\ y=\frac{b\,(3a-b)}{a+b}$$

11 (i) Hint Put $x=u+v$, $y=u-v$

Ans $\quad x=2,\ \dfrac{-1\pm\sqrt{(-31)}}{4},\ y=1,\ \dfrac{-5\pm\sqrt{(-31)}}{4}$

Multiplying (5), (6), (7) by y, z, x respectively and adding, we get

$$-y+3z+9x=0 \quad \text{or} \quad 9x-y+3z=0 \qquad (8)$$

Again multiplying these equations by z, x and y, and adding, we get $\quad -z+3x+9y=0 \quad$ or $\quad 3x+9y-z=0 \qquad (9)$

Solving (8) and (9) by *cross multiplication*, we get

$$\frac{x}{1-27}=\frac{y}{9+9}=\frac{z}{81+3} \quad \text{or} \quad \frac{x}{-26}=\frac{y}{18}=\frac{z}{84}$$

or $\quad \frac{x}{-13}=\frac{y}{9}=\frac{z}{42}=k$ say

Substituting these values in (5), we get

$$169k^2-378k=-1$$

$$209k^2=1 \quad \text{or} \quad k=\pm\frac{1}{\sqrt{(209)}}$$

Hence $\quad x=\mp\frac{13}{\sqrt{(209)}} \quad y=\pm\frac{9}{\sqrt{(209)}}, \; z=\pm\frac{42}{\sqrt{(209)}}$

12 Adding all the three equations we get

$$(x+y+z)(a+b+c)=0,$$

Hence $\quad x+y+z=0$

Then from first of the given equations and this equation we get $\quad (b+c)(-x)-ax=b-c$

or $\quad x=\frac{c-b}{a+b+c} \quad$ Similarly $y=\frac{a-c}{a+b+c}$, $z=\frac{b-a}{a+b+c}$

13 The given system of equations can be written as

$$xy+xz=a^2 \qquad (1)$$
$$yz+yx=b^2 \qquad (2)$$
$$zx+zy=c \qquad (3)$$

Adding,

$$xy+yz+zx=\tfrac{1}{2}(a+b+c^2)$$

Then from (1) and (4), $\quad yz=\frac{1}{2}(b+c-a^2)$

Similarly $\quad zx=\frac{c+a-b^2}{2} \quad$ and $\quad xy=\frac{a^2+b^2-c^2}{2}$

Multiplying these,

$$(xyz)=\frac{(b^2+c-a)(c^2+a-b)(a+b^2-c^2)}{8}$$

$$xyz=\pm\sqrt{[\{\tfrac{1}{8}(b+c^2-a)(c+a^2-b^2)(a+b^2-c^2)\}]}$$

Now we easily get

$$x=\pm\sqrt{\left\{\frac{(c^2+a^2-b^2)(a+b^2-c^2)}{2(b^2+c^2-a)}\right\}}$$

There is only one real solutions $x=2, y=1$

(ii) Ans $x=0, y=-11$ or $x=3, y=4$

Hint Put $\sqrt[4]{(1+5x)}=u$ and $\sqrt[4]{(5-y)}=v$ Then

$$u+v=3,\ u^4+v^4=1$$

Solving these symmetric equations, we shall get

$$u=1, v=2, \quad \text{or} \quad u=2, v=1 \text{ etc}$$

12 Dividing the first equation by $(x^2-y^2)^{1/3}$, we get

$$\left(\frac{x+y}{x-y}\right)^{1/3}+2\left(\frac{x-y}{x+y}\right)^{1/3}=3$$

Put $\left(\frac{x+y}{x-y}\right)^{1/3}=u$ Then $u+\frac{2}{u}=3$ or $u^2-3u+2=0$

or $(u-1)(u-2)=0$ or $u=1$ or 2

Hence $\left(\frac{x+y}{x-y}\right)^{1/3}=1$ or 2

First take $x+y=x-y$, that is, $y=0$

Then from the second equation

$$3x-2y=13 \tag{1}$$

we get $x=\frac{13}{3}$

Again $\left(\frac{x+y}{x-y}\right)^{1/3}=2$ or $x+y=8x-8y$

or $7x-9y=0$ (2)

Solving (1) and (2), we get $x=9, y=7$

Hence the solutions are $x=\frac{13}{3}, y=0$ or $x=9, y=7$

13 First equation can be written as

$$4x^2-20xy+25y^2+(5y-2x)-6=0$$

or $(5y-2x)^2+(5y-2x)-6=0$

or $(5y-2x+3)(5y-2x-2)=0$

Hence $5y-2x+3=0$ (1)

or $5y-2x-2=0$ (2)

And the second given equation is

$$7x-11y=17 \tag{3}$$

Solving (1) and (3), we get $x=4, y=1$

Again solving (2) and (3), we get

$$x=\frac{107}{13},\ y=\frac{48}{13}$$

Hence solutions are $x=4, y=1$ or $x=\frac{107}{13}, y=\frac{48}{13}$

14 Ans $x=\frac{a^2}{b}, \frac{a(2b-a)}{b}, y=\frac{b^2}{a}, \frac{b(2a-b)}{a}$

$$y=\pm\sqrt{\left\{\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)}{2(c^2+a^2-b^2)}\right\}}$$

$$z=\pm\sqrt{\left\{\frac{(c^2+a^2-b^2)(b^2+c^2-a^2)}{2(a^2+b^2-c^2)}\right\}}$$

14 Do yourself

Ans $x=4,\ \frac{2\pm\sqrt{(151)}}{3},\ y=3,\ 4\ \frac{2\mp\sqrt{(151)}}{3}$

and $z=2,\ -\frac{11}{3}$

15 The equations can be rewritten as

$$2x(y-2)+(y-2)=17-2$$

or $(2x+1)(y-2)=15$ (1)

Similarly $(y-2)(3z+1)=50$ (2)

and $(3z+1)(2x+1)=30$ (3)

Multiplying (1) (2) and (3), we get

$$[(2x+1)(y-2)(3z+1)]^2=15\times50\times30$$

$(2x+1)(y-2)(3z+1)=\pm150$ (4)

Now from (1) and (4)

$3z+1=\pm\frac{150}{15}=\pm10$ so that $z=3,\ -\frac{11}{3}$

From (2) and (4)

$2x+1=\pm\frac{150}{50}=\pm3$ or $x=1,-2$

From (3) and (4),

$y-2=\pm\frac{150}{30}=\pm5$ or $y=7,-3$

Hence the solution sets are

$x=1,\ y=7\ z=3$ or $x=-2,\ z=-3\ z=-\frac{11}{3}$

16 (a) Hint—The equations can be rewritten as

$(y+1)(z+1)=a+1,\ (z+1)(x+1)=b+1,$

$(x+1)(y+1)=c+1$

Now proceeding as is problem 15, we shall get

$x=\pm\sqrt{\left\{\frac{(1+b)(1+c)}{1+a}\right\}}-1\quad y=\pm\sqrt{\left\{\frac{(1+c)(1+a)}{1+b}\right\}}-1$

and $z=\pm\sqrt{\left\{\frac{(1+a)(1+b)}{1+c}\right\}}-1$

(b) Ans $x=5,\quad y=3,\quad z=6$

or $x=-7\quad y=-5\quad z=-8$

17 Ans $x=8,\ y=5,\ z=3$ or $x=-8\ y=-5\ z=-3$

[Hint From second equation, we get

$$\frac{1}{x-b}-\frac{1}{a-b}=\frac{1}{y-a} \quad \text{or} \quad \frac{a-x}{(x-b)(a-b)}=\frac{1}{y-a}$$

or $$\frac{(x-b)(a-b)}{a-x}=y-a$$

or $$y=a+\frac{(x-b)(a-b)}{a-x}=\frac{a^2-ab+b^2-bx}{a-x}$$

Now substitute this value of y in first equation etc

15 From first equation, we get

$$\left\{\frac{3(x-y)}{x+y}\right\}^2+\left\{\frac{3(x+y)}{x-y}\right\}^2=82$$

Put $\frac{x+y}{x-y}=z$ We then have

$$9\left(\frac{1}{z^2}+z^2\right)=82$$

or $9z^4-82z^2+9=0$ or $(z^2-9)(9z^2-1)=0$

$z=\pm 3$ or $z=\pm\frac{1}{3}$

Taking $z=3$ we get $\frac{x+y}{x-y}=3$ or $x=2y$ (1)

Also second given equation is,

$3x+7y=26$ (2)

Solving (1) and (2) we get $x=4$, $y=2$

Similarly solving $z=-3$, that is, $\frac{x+y}{x-y}=-3$ with (2) we shall get

$$x=\frac{26}{17} \quad y=\frac{52}{17}$$

Again solving $z=\frac{x+y}{x-y}=\frac{1}{3}$ with (2), we shall obtain,

$$x=-52, \; y=26$$

And finally solving $z=\frac{x+y}{x-y}=-\frac{1}{3}$, we shall get

$$x=-\frac{26}{11}, \; y=\frac{52}{11}$$

16 The given system of equations is equivalent to

$$y^2=2x/(1+x^2), \; 2(x-1)^2+1+y^3=0$$

Since $2x/(1+x^2) \leqslant 1$ for all real x, it follows from the first of these equations that $y^2 \leqslant 1$ or $-1 \leqslant y \leqslant 1$ Again from the second equation of the system, we get

$$y^3=-1-2(x-1)^2 \leqslant -1$$

18 (a) The equations can be rewritten as

$$x^2-(y-z)^2=a \quad \text{or} \quad (x+y-z)(x+z-y)=a$$

Similarly $(y+z-x)(y+x-z)=b$ and $(z+x-y)(y+z-x)=c$

Multiplying and taking square root we get

$$(x+y-z)(x+z-y)(y+z-x)=\pm\sqrt{(abc)}$$

Hence $y+z-x=\pm\sqrt{\left(\frac{bc}{a}\right)}$ $z+x-y=\pm\sqrt{\left(\frac{ca}{b}\right)}$

and $x+y-z=\pm\sqrt{\left(\frac{ab}{c}\right)}$

Adding last two equations we get

$$x=\pm\tfrac{1}{2}\left\{\sqrt{\left(\frac{ca}{b}\right)}+\sqrt{\left(\frac{ab}{c}\right)}\right\}$$

Similarly $y=\pm\tfrac{1}{2}\left\{\sqrt{\left(\frac{bc}{a}\right)}+\sqrt{\left(\frac{ab}{c}\right)}\right\}$

and $z=\pm\tfrac{1}{2}\left\{\sqrt{\left(\frac{bc}{a}\right)}+\sqrt{\left(\frac{ca}{b}\right)}\right\}$

(b) Ans $x=-1, y=-1, z=-3$

$$x^2+y^2+xy=9 \qquad (1)$$
$$z^2+x^2+xz=4 \qquad (2)$$
$$y^2+z^2+yz=1 \qquad (3)$$

Subtracting (2) from (1) and factorizing, we get

$$(y-z)(x+y+z)=5 \qquad (4)$$

Similarly $(x-y)(x+y+z)=3$ (5)

$$(z-x)(x+y+z)=-8 \qquad (6)$$

Again subtracting (5) from (4), we get

$$(2y-x-z)(x+y+z)=2 \qquad (7)$$

Similarly $(2x-y-z)(x+y+z)=11$ (8)

Now putting $x+y+z=t$, we get from (7) and (8)

$$(3y-t)\,t=2 \quad \text{or} \quad y=\frac{t^2+2}{3t},$$

and $(3x-t)\,t=11$ or $x=\frac{t^2+11}{3t}$

Substituting these values of x and y in (1)

$$\frac{(t^2+11)^2}{9t^2}+\frac{(t^2+2)^2}{9t^2}+\frac{(t^2+11)(t^2+2)}{9t^2}=9$$

or $3t^4-42t^2+147=0$ or $t^4-14t^2+49=0$

or $(t^2-7)^2=0$

$$(x+y+z)^2=t^2=7 \quad \text{or} \quad x+y+z=t=\pm\sqrt{7}$$

Hence $x=\pm\frac{7+11}{3\sqrt{7}}=\pm\frac{6}{\sqrt{7}}$ and $y=\pm\frac{7+2}{3\sqrt{7}}=\pm\frac{3}{\sqrt{7}}$

Now since $y \leqslant 1$ and $y \geqslant 1$, we must have $y = 1$ then $x=1$ Thus the solution is $x=1$, $y=1$

17 The given equations are

$$ax+by=1 \qquad (1)$$
$$cx^2+dy^2=1 \qquad (2)$$

Eliminating y from (1) and (2) we obtain

$$cx^2+d\left(\frac{1-ax}{b}\right)^2=1$$

or $\quad b^2cx^2+d-2adx+a^2dx^2=b^2$

or $\quad (b^2c+a^2d)x^2-2adx+d-b^2=0 \qquad (3)$

Since the given equations have only one solution, the roots of (3) must be equal the condition for which is

$$4a^2d^2-4(b^2c+a^2d)(d-b^2)=0$$

or $\quad a^2d^2-b^2cd+b^4c-a^2d^2+a^2b^2d=0$

or $\quad b^4c+a^2b^2d-b^2cd=0$

or $\quad b^2c+a^2d-cd=0$

or $\quad \dfrac{a^2}{c}+\dfrac{b^2}{d}=1 \qquad (4)$

Under the condition (4), the root of (3) is given by

$$x=\frac{2ad}{2(b^2c+a^2d)}=\frac{2ad}{2cd\left(\frac{b^2}{d}+\frac{a^2}{c}\right)}=\frac{a}{c} \text{ by (4)}$$

Then $y=\dfrac{1-ax}{b}=\dfrac{1-\frac{a^2}{c}}{b}=\dfrac{\frac{b^2}{d}}{b}$ by (4) $=\dfrac{b}{d}$

Hence the only one solution of the given equations is

$$x=\frac{a}{c},\quad y=\frac{b}{d}$$

Problem Set (C)

Solve the following equations

1 $3x+y-2z=0,\ 4x-y-3z=0,\ x^3+y^3+z^3=467$

2 $xz+y=7z,\ yz+x=8z,\ x+y+z=12$

3 $\dfrac{xy}{ay+bx}=c,\ \dfrac{xz}{az+cx}=b,\ \dfrac{yz}{bz+cy}=a$

4 $x+y-4xy=0,\ y+z-6yz=0,\ z+x-8zx=0$ (I I T 73)

5 $xy=a(x+y),\ yz=c(y+z),\ zx=b(z+x)$

6 $xyz=a^2(y+z)=b^2(z+x)=c^2(x+y)$

Then from (4), we get

$$\left(\frac{3}{\sqrt{7}}-z\right)\sqrt{7}=5 \quad \text{or} \quad \left(-\frac{3}{\sqrt{7}}-z\right)(-\sqrt{7})=5$$

$$z=-\frac{2}{\sqrt{7}} \quad \text{or} \quad z=\frac{2}{\sqrt{7}}$$

Hence the solution sets are

$$x=\frac{6}{\sqrt{7}}, y=\frac{3}{\sqrt{7}}, z=\frac{2}{\sqrt{7}} \quad \text{or} \quad x=-\frac{6}{\sqrt{7}}, y=-\frac{3}{\sqrt{7}}, z=-\frac{2}{\sqrt{7}}$$

20. Do as in problem 19

Ans $x=4,\ -4,\ \frac{10\sqrt{3}}{3},\ -\frac{10\sqrt{3}}{3}$

$y=3,\ -3,\ \frac{1}{3}\sqrt{3},\ -\frac{1}{3}\sqrt{3},$

$z=2,\ -2,\ -\frac{8}{3}\sqrt{3},\ \frac{8}{3}\sqrt{3}$

21 Adding Ist two and adding last two

$$x+y=a+b,\ x-y=c+d$$

$$x=\tfrac{1}{2}(a+b+c+d),\ y=\tfrac{1}{2}(a+b-c-d)$$

Subtracting Ist two and subtracting last two

$$z+u=a-b \text{ and } z-u=c-d$$

$$z=\tfrac{1}{2}(a-b+c-d),\ u=\tfrac{1}{2}(a-b-c+d)$$

22 (i) The given equations show that the polynomial

$$f(t)=t^3+xt^2+yt+z$$

vanishes at three different values of t, namely at $t=a$, $t=b$ and $t=c$

Set up a difference

$$t^3+xt^2+yt+z=(t-a)(t-b)(t-c)$$

This difference also becomes zero at $t=a, b, c$ Expanding this expression in powers of t, we get

$$(x+a+b+c)\,t^2+(y-ab-bc-ca)\,t+z+abc=0$$

This *second degree* trinomial in t vanishes at *three different* values of t and is therefore identically zero, and so its coefficients must vanish separately Hence

$$x+a+b+c=0,\ y-ab-bc-ca=0$$

and $z+abc=0$

$\therefore\ x=-(a+b+c),\ y=ab+bc+ca$, and $z=-abc$

(ii) Consider the following equation in t,

$$\frac{x}{a+t}+\frac{y}{b+t}+\frac{z}{c+t}=1-\frac{(t-\lambda)(t-\mu)(t-\gamma)}{(t+a)(t+b)(t+c)} \qquad (1)$$

x, y, z being for the present regarded as known quantities

3

Heights and Distances

§ 1 (*a*) Define angle of elevation and depression of a point

Suppose a straight line AX is drawn in the horizontal direction. Then the angle XAP where P is a point above AX is called the angle of elevation of P as seen from A. Similarly the angle XAQ where Q is below AX, is called the angle of depression of some point Q

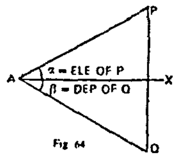


Fig 64

(*b*) **Any perpendicular to a plane is perpendicular to every line lying in the plane**

Place your pen PQ upright on your note book so that its lower end Q is on the note book. Through the point Q draw lines QA QB, QC, , in your note book in different directions and you will observe that each of the angle PQA, PQB, PQC, , is a right angle. In other words PA is perpendicular to each of the lines QA, QB, QC, lying in the plane

(*c*) **To express one side of a right angled triangle in terms of the other side** ($AB=c$ is hypotenuse)

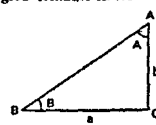


Fig 65

From the figure it is clear that

$b=a \tan B$ or $a=b \tan A$

i e **Any side x=other side× tangent of angle opposite to side x**

Also we can say that

$a=b \cot B$ or $b=a \cot A$

Any side x=other side× co tangent of angle adjacent to side x

The Above rule will be frequently used

(*b*) **Geometrical properties**

(*i*) **In a triangle the internal bisector of the angle divides** the opposite side in the ratio of the arms of the angle

(ii) In a isosceles triangle the median is perpendicular to the base

(iii) In similar triangles the corresponding sides are proportional

(e) Trigonometrical results

$$\tan A \pm \tan B = \frac{\sin (A \pm B)}{\cos A \cos B}$$

$$\cot A \pm \cot B = \frac{\sin (B \pm A)}{\sin A \sin B}$$

$$\sin (\alpha+\beta) \sin (\alpha-\beta) = \sin^2 \alpha - \sin^2 \beta = \cos^2 \beta - \cos^2 \alpha.$$

(f) Sine formula, cosine formula and $m-n$ theorem of trigonometry (Q 52 Chap 5) be remembered

Problem Set

1 PQ is a post of given height a, and AB is a tower at some distance, α and β are the angles of elevation of B, the top of the tower at P and Q respectively Find the height of the tower and its distance from the post

2 The angle of elevation of a cloud at a height h above the level of water in a lake is α and the angle of depression of its image in the lake is β Prove that the height of the cloud above the surface of the lake is

$$\frac{h (\tan \beta+\tan \alpha)}{\tan \beta-\tan \alpha} \quad \text{or} \quad \frac{h \sin (\alpha+\beta)}{\sin (\beta-\alpha)} \quad \text{or} \quad \frac{h (\cot \alpha+\cot \beta)}{\cot \alpha-\cot \beta}$$

3 The angle of elevation of a cloud from a point x feet above a lake is θ and the angle of depression of its reflection in the lake is ϕ, prove that its height is

$$\frac{x \sin (\phi+\theta)}{\sin (\phi-\theta)}$$

(Roorkee 83)

4 The angle of elevation of a stationary cloud from a point 2500 cm above a lake is 15° and the angle of depression of its reflection in the lake is 45° What is the height of the cloud above the lake level

(IIT 76)

5 If the angle of elevation of a cloud from a point h meters above a lake be θ and the angle of depression of its reflection in the lake be ϕ, prove that the distance of the cloud from the point of observation is

$$\frac{2h \cos \phi}{\sin (\phi-\theta)}$$

A tower subtends an angle α at a point A in the plane of its

This equation when cleared of fractions is of second degree in t, and is satisfied for three values $t=\lambda$, $t=\mu$ and $t=\nu$ by virtue of the given equations hence it must be an identity.

To find the value of x, we multiply (1) by $a+t$, and then put $t=-a$. Thus

$$x=\frac{(-a-\lambda)(-a-\mu)(-a-\nu)}{(-a+b)(-a+c)}$$

or $$x=\frac{(a+\lambda)(a+\mu)(a+\nu)}{(a-b)(a-c)}$$

By reason of symmetry, we have

$$y=\frac{(b+\lambda)(b+\mu)(b+\nu)}{(b-c)(b-a)} \text{ and } z=\frac{(c+\lambda)(c+\mu)(c+\nu)}{(c-a)(c-b)}$$

Note. We know that if α, β are the roots of an equation then this equation is $x^2-(\alpha+\beta)x+\alpha\beta=0$

or $x^2-S_1x+S_2=0$ where S_1 is sum of roots taken one at a time and S_2 is sum of products of roots taken two at a time. Similarly if there is an equation whose roots are α, β, γ then $S_1=\alpha+\beta+\gamma$, $S_2=\alpha\beta+\beta\gamma+\gamma\alpha$, $S_3=\alpha\beta\gamma$ and the corresponding equation will be

$$t^3-S_1t^2+S_2t-S_3=0$$

Again we know that if a is a root of the equation $f(x)=0$ then a will satisfy the equation i.e. $f(a)=0$.

Alternative Solution of Q. 22 (i)

Consider the equation

$$z+ty+t^2x+t^3=0$$

It is clear that the above equation is satisfied by $t=a, b, c$ by virtue of the above three given equations $z+ay+a^2x+a^3=0$, which is true by 1st equation and so on. Thus a, b, c are the roots of the equation

$$t^3+t^2x+ty+z=0$$

$$S_1=a+b+c=-x,$$

$$S_2=ab+bc+ca=y, \quad S_3=abc=-z$$

$$x=-(a+b+c),\ y=\Sigma ab,\ z=-abc$$

23. The given equation can be re written as

$$x+y+z=\tfrac{7}{2} \quad (1)$$

$$yz+zx+xy=\tfrac{7}{2} \quad (2)$$

$$xyz=1 \quad (3)$$

From (2) $yz+x(y+z)=\frac{7}{2}$

Hence, from (3) and (1)

$$\frac{1}{x}+x\left(\tfrac{7}{2}-x\right)=\tfrac{7}{2}$$

where $n=0, \pm 1, \pm 2, \pm 3,$

(ii) Hint The equation will reduce to $5^{1+\cos 2x}=5^{\sin 2x}$

Ans $x=n\pi+\pi/4, \quad x=n\pi+\pi/2, \ n \in I$

9 $\log_{10} \log_{10} \log_{10} x=0$

Taking anti logarithm, $\log_{10} \log_{10} x=10^0=1$

Again taking anti- logarithm, $\log_{10} x=10^1=10$

Hence $x=10^{10}$

10 The domain of x is $x>0 \quad x\neq 1$ The given equation is equivalent to

or $\quad 3x+10x=x^3$ or $x^3-3x^2-10x=0$

or $\quad x(x-5)(x+2)=0$

Since $x=0$ and $x=-2$ do not lie in the domain of x, the only solution of the given equation is $x=5$

11 The equation can be written as

$98+\sqrt{(x^3-x-12x+36)}=10=100$

or $\quad \sqrt{(x^3-x^2-12x+36)}=2$

or $\quad x^3-x-12x+36=4$

or $\quad x^3-x-12x+32=0$

or $\quad x^2(x+4)-5x(x+4)+8(x+4)=0$

$(x+4)(x-5x+8)=0$

$x=-4$

or $\quad x=\frac{5\pm\sqrt{(25-32)}}{2}=\frac{5\pm\sqrt{(-7)}}{2}$

Note If we consider only real roots, then the only solution is $x=-4$

12 (*a*) The given equation can be written as

$$\frac{2}{\log_a x}+\frac{1}{\log_a a+\log_a x}+\frac{2}{2\log_a a+\log_a x}=0$$

Now put $\log_a x=y$ Then

$$\frac{2}{y}+\frac{1}{1+y}+\frac{3}{2+y}=0$$

or $\quad 2(1+y)(2+y)+y(2+y)+3y(1+y)=0$

or $\quad 6y+11y+4=0$ or $6y+8y+3y+4=0$

or $\quad (3y+4)(2y+1)=0 \qquad y=-\frac{4}{3}, -\frac{1}{2}$

Hence $\log_a x=-\frac{4}{3}$ or $-\frac{1}{2}$, $\qquad x=a^{-4/3}$ or $a^{-1/2}$

(*b*) Since $\log_9 3=\log_{3^2} 3=\frac{1}{2}$,

$\log_2 4=\log 2=2$ and $\log_x 83=\log_{10} 83 \log_x 10$

The given equation can be written as

$4^{1/2}+9^2=10^{\log_{10} 83 \log_x 10}$ or $83=(10^{\log_{10} 83})^{\log_x 10}$

or $\quad 2x^3-7x+7x-2=0$

or $\quad 2(x^3-1)-7x(x-1)=0$

or $\quad (x-1)(2x^2-5x+2)=0$

or $\quad (x-1)(x-2)(2x-1)=0$

$x=1, 2 \text{ or } \frac{1}{2}$

Taking $x=1$, we have from (1),

$y+z=\frac{5}{2}$, and from (3), $yz=1$

whence $\left.\begin{array}{l} y=2 \\ z=\frac{1}{2} \end{array}\right\}$ or $\left.\begin{array}{l} y=\frac{1}{2} \\ z=2 \end{array}\right\}$

Hence solutions are

$x=1, y=2, z=\frac{1}{2}$ and $x=1, y=\frac{1}{2}, z=2$

Similarly taking $x=2$ and $\frac{1}{2}$, the other sets of solutions are

$x=2, y=\frac{1}{2}, z=1, x=2, y=1, z=\frac{1}{2}$,

$x=\frac{1}{2}, y=2, z=1$ and $x=\frac{1}{2}, y=2$

Thus in all there are six sets of solutions

Alt Let $S_1=x+y+z=7/2, S_2=xy+yz+zx=7/2, S_3=xyz=1$

x, y, z are the roots of

$t^3-S_1t^2+S_2t-S_3=0$ or $t^3-\frac{7}{2}t^2+\frac{7}{2}t-1=0$

or $2t^3-7t^2+7t-2=0$ clearly $t=1$ satisfies it

$(t-1)(2t^2-5t+2)=0$

or $(t-1)(t-2)(2t-1)=0$

$t=1, 2, 1/2$ are the values of x, y, z

Choosing $x=1$ we can say

	x	y	z
	1	2	1/2
or	1	1/2	2

Similarly choosing $x=2$ and then $x=3$ we can have the following four other solution sets

x	y	z	x	y	z
2	1	1/2,	1/2	1	2
2	1/2	1	1/2	2	1

Thus there are six solution sets of the given equations

24 (i) From first two equations we get

$xy+yz+zx=\frac{1}{2}[(x+y+z)^2-(x^2+y^2+z^2)]$

$=\frac{1}{2}[144-50]=47$

Also $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$

or $\quad 216-3xyz=12(50-47)$

$xyz=60$

Hence the given equations reduce to

$x+y+z=12, yz+zx+xy=47, xyz=60$

Now proceed as in problem 23

Solution sets in this case are

(3, 4, 5), (3, 5, 4), (4, 3, 5), (4, 5, 3), (5, 3, 4) and (5,

Alt Here $S_1=12$ Also $x^2+y^2+z^2=50$ gives

$(x+y+z)^2-2(xy+yz+zx)=50$

or $144-2S_2=50$ $2S_2=94$ or $S_2=47$

Again $x^3+y^3+z^3=216$

But $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-\Sigma xy$

or $216-3S_3=12(50-47)=36$

or $216-36=3S_3$ $S_3=60$

$S_1=12,\quad S_2=47\quad S_3=60$

x, y, z are the roots of cubic in t

$t^3-S_1t^2+S_2t-S_3=0$

or $t^3-12t^2+47t-60=0$

By trial $t=3$ satisfies it

$(t-3)(t^2-9t+20)=4$ or $(t-3)(t-4)(t-5)=0$

$t=3, 4, 5$, Hence the solution sets as above are

(3, 4, 5) or (4 3 5) or (5, 3, 4)
(3, 5, 4) or (4 5 3) or (5, 4, 3)

(ii) Ans (1, 0, 0), (0, 1, 0), (0, 0, 1)

25 Hint The equation can be re written as

$y(z-a)-a(z-a)=\alpha+a^2$

or $(y-a)(z-a)=\alpha+a^2$

Similarly $(z-a)(x-a)=\beta+a^2$

and $(x-a)(y-a)=\gamma+a^2$ etc

Ans $x=a\pm\sqrt{\left\{\dfrac{(a^2+\beta)(a^2+\gamma)}{a^2+\alpha}\right\}}$, etc

26 Subtracting the given equations in pairs,

$x^2-y^2+(y-z)^2-(z-x)^2=a^2-b^2$

i e $(x-y)(x+y)+(y-z+z-x)(y-z-z+x)=a^2-b^2$

or $(x-y)\,2z=a^2-b^2$

Similarly, $(y-z)\,2x=b^2-c^2$

and $(z-x)\,2y=c^2-a^2$

Now adding first of the given equations and (2), we get

$x^2+2x(y-z)+(y-z)^2=a^2+b^2-c^2$

or $(x+y-z)^2=a^2+b^2-c^2$

Similarly $(y+z-x)^2=c^2-a^2+b^2$

and $(z+x-y)^2=a^2-b^2+c^2$

Taking square roots and then adding first two, we get

Sum of *a* G P $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \quad \infty = \frac{1}{1-\frac{1}{2}}=2$

Sum of A P $=\frac{n}{2}[a+l]=y^2$ for N^r and $\frac{y}{2}\,(3y+5)$ for D^r

The given equations can be re written as

$\log_{10} x^2=y \quad$ or $\quad 2\log_{10} x=y$ (1)

and $\frac{2y^2}{y\,(3y+5)}=\frac{20}{7\log_{10} x}$ or $\frac{2y}{3y+5}=\frac{40}{7\cdot 2\log_{10} x}$

$=\frac{40}{7y}$ by I,

$7y^2=60y+100 \quad$ or $\quad 7y^2-70y+10y-100=0$

or $(y-10)\,(7y+10)=0$

$y=10, \ -\frac{10}{7}$ Since y is a +ve integer $y=10$

$2\log_{10} x=10$ by (I) or $\log_{10} x=5 \quad x=(10)^5$

18 (a) The given equation can be re written as

$\log_{2^4} x+\log_{2^2} x+\log_2 x=7$

or $\frac{1}{4}\log_2 x+\frac{1}{2}\log_2 x+\log_2 x=7$

or $\frac{7}{4}\log_2 x=7$ or $\log_2 x=4$

Hence $x=2^4=16$

(b) The given equation can be re written as

$\log_{(2x+3)}(2x+3)\,(3x+7)+\log_{(3x+7)}(2x+3)^2=4$

or $\log_{(2x+3)}(2x+3)+\log_{(2x+3)}(3x+7)$

$+2\log_{(3x+7)}(2x+3)=4$

or $y+2\,\frac{1}{y}=3 \quad \because \log_b a=\frac{1}{\log_a b}$

where $y=\log_{(2x+3)}(3x+7)$

or $y^2-3y+2=0 \quad \therefore y=1,\ 2$

$\log_{(2x+3)}(3x+7)=1$ or 2

$3x+7=2x+3$ or $3x+7=(2x+3)^2$

$x=-4$, and $4x^2+9x+2=0$ or $x=-2,\ -\frac{1}{4}$

Thus we have three values of x namely $-4, -2, -\frac{1}{4}$ (1)

Now by difinitions of logarithms

$2x-3>0,\ \ 2x+3\neq 1$ and $3x+7>0$ and $3x+7\neq 1$

$$2y=\pm\sqrt{(a^2+b^2-c^2)}\pm\sqrt{(c^2-a^2+b^2)} \text{ etc}$$

27 We write the equations as

$$(x-a)+(y-b)+(z-c)=0$$

$$\left(\frac{x}{a}-1\right)+\left(\frac{y}{b}-1\right)+\left(\frac{z}{c}-1\right)=0$$

or $\quad bc(x-a)+ca(y-b)+ab(z-c)=0 \qquad (3)$

and $\quad ax+by+cz=bc+ca+ab \qquad (3)$

From (1) and (2), we have by cross multiplication

$$\frac{x-a}{ab-ca}=\frac{y-b}{bc-ab}=\frac{z-c}{ca-bc}=k, \text{ say}$$

Then $x=a+a\,(b-c)\,k,\ y=b+b\,(c-a)\,k,\ z=c+c\,(a-b)\,k$ (4)

Substituting in (3), we get

$$a^2+b^2+c^2+\{a^2\,(b-c)+b^2\,(c-a)+c^2\,(a-b)\}\,k=bc+ca+ab$$

or $\quad a^2+b^2+c^2-(b-c)\,(c-a)\,(a-b)\,k=bc+ca+ab$

(Factorizing the coefficient of k)

or $$k=\frac{a^2+b^2+c^2-bc-ca-ab}{(b-c)\,(c-a)\,(a-b)} \qquad (5)$$

Hence solution is given by (4) where k is given by (5)

28 We have $\quad x+y=2,\ xy=1+z^2$

$\therefore \quad x-y=\pm\sqrt{\{(x+y)^2-4xy\}}=\pm\sqrt{\{4-4-4z^2\}}=\pm\sqrt{(-4z^2)}$

Since we are concerned with real values of x, y, z, we must have $z=0$. Then $x-y=0$ which together with $x+y=2$ gives $x=y=1$

Hence the solution is $x=1,\ y=1,\ z=0$

29 Multiplying Ist and 3rd relations, we get

$$y^2x^2+y^4+z^2x^2+z^2y^2=acy^2xz$$

or $\quad (y^2x^2+y^2z^2)+(y^4+z^2x^2)=acy^2xz$

Dividing by y^2xz, we get

$$\frac{x^2+z^2}{xz}+\frac{y^2}{zx}+\frac{xz}{y^2}=ac \qquad (1)$$

Substituting $\frac{x^2+z^2}{xz}=b$ from 2nd relation in (1), we get

$$\frac{y^2}{zx}+\frac{zx}{y^2}=ac-b$$

Problem Set (D)

Solve the following equations for real roots

1 (a) ...

$$x > -\frac{3}{2},\ x \neq -1,\ x > -\frac{7}{3},\ x \neq -2 \qquad (2)$$

Clearly from (1) and (2) $x=-4$ $x=-2$ are rejected and the only solution left is $x=-1/4$

19 The given equation can be re written as

$$2^{2x}+2^{2x-1}=3^{x-1/2}+3^{x+1/2}$$

$$2^{2x}(1+\tfrac{1}{2})=3^{x}\left(\frac{1}{\sqrt{3}}+\sqrt{3}\right)$$

or $3\ 2^{2x-1}=4\ 3^{x-1/2}$ or $2^{2x-3}=3^{x-3/2}$

Now by trial, $x=\frac{3}{2}$ is one solution

Now taking logarithm to the base 10, we get

$$(2x-3)\log_{10}2=(x-\tfrac{3}{2})\log_{10}3$$

or $(2\log 2-\log_{10}3)\,x=\frac{3}{2}(2\log 2-\log 3)$

This also gives $x=\frac{3}{2}$

Hence $x=\frac{3}{2}$ is the only solution

20 The given equation can be written as

$$x+\log_{10}(1+2^{x})-x\log_{10}5=\log_{10}6$$

or $x\log_{10}10+\log_{10}(1+2^{x})-x\log_{10}5=\log_{10}6$

or $x\log_{10}(10/5)+\log_{10}(1+2^{x})=\log_{10}6$

or $\log_{10}2^{x}+\log_{10}(1+2^{x})=\log_{10}6$

or $\log_{10}2^{x}(1+2^{x})=\log_{10}6$

Hence $2^{x}(1+2^{x})=6$ or $2^{2x}+2^{x}-6=0$

or $(2^{x}+3)(2^{x}-2)=0$

Since $2^{x}>0$, $2^{x}=-3$ is impossible So we have

$$2^{x}=2 \quad \text{or} \quad x=1$$

Hence $x=1$ is the only root of the given equation

21 The roots must satisfy the condition $x>0$, $x\neq 1$ (why ?)

The equation can be written as

$$[\log_a a+\log_a x]\,[\log_x a+\log_x x]=\log_a a^{-1}$$

or $(1+\log_a x)\left(\frac{1}{\log_a x}+1\right)=-\frac{1}{2}$

Now put $\log_a x=y$ Then

$$\frac{(1+y)^2}{y}=-\tfrac{1}{2} \quad \text{or} \quad 2y^2+5y+2=0$$

or $(y+2)(2y+1)=0$

This gives $y=-2$ or $-\frac{1}{2}$ that is, $\log_a x=-2$ or $-\frac{1}{2}$

Hence $x=a^{-2}$ or $a^{-1/2}$

22 Hint $2^{x}\,3^{x}\,(\frac{2}{3})^{y}-3\ 2^{x+y}-8\ 3^{x-y}+24=0$

or $2^{x+y}\,3^{x-y}-3\ 2^{x+y}-8\ 3^{x-y}+24=0$

or $(2^{x+y}-8)(3^{x-y}-3)=0$

Hence $2^{x+y}=2^{3}$ or $3^{x-y}=3^{1}$

$$x+y=3 \qquad (1)$$

(b) $3^{2x}-3^{x+1}-3^{x-1}+1=0$ (I

2. $a^{2x}(a^2+1)=(a^{3x}+a^x)\,a$

3 $3^x\,8^{x(x+2)}=6$

4 $4^x+6^x=9^x$

5 $3^x+4^x=5^x$

6 $2^{2x+3}-6^x-2\times 3^{2x+2}=0$

7 $2\times 81^x=36^x+3\times 16^x$

8 (i) $16^{\sin^2 x}+16^{\cos^2 x}=10$

(ii) $5\left(\frac{1}{25}\right)^{\sin^2 x}+4\times 5^{\cos 2x}=25^{(\sin 2x)/2}$

9 $\log_{10}\log_{10}\log_{10} x=0$

10 $\log_x(3x^2+10x)=3$

11 $\log_{10}[98+\sqrt{(x^3-x^2-12x+36)}]=2$ (Roork

12 (a) For $a>0$, solve for x the equations

$$2\log_x a+\log_{ax} a+3\log_{(a^2x)} a=0$$ (I I T

(b) $4^{\log_9 3}+9^{\log_2 4}=10^{\log_x 83}$ (I I T

13 $\log_{10}\left(\frac{1}{2^x+x-1}\right)=x[\log_{10}5-1]$

14 $\log_{10}(x^2-x-6)-x=\log_{10}(x+2)-4$

15 (i) $x^x=x,$ (ii) $x^{\sqrt{x}}=\sqrt{x^x}$

16 $\log_2 x+\log_{x^2} 3=1$

17 (i) $\log x+\log_x 5=\frac{5}{2},$

(ii) $5^{\log_{10}x}=50-x^{\log_{10}5}$

(iii) $\log_{10}x+\log_{10}x^{1/2}+\log_{10}x^{1/4}+\quad=y$

and $\frac{1+3+5+\qquad+(2y-1)}{4+7+10+\qquad+(3y+1)}=\frac{20}{7\log_{10}x}$ (Roorkee

18 (a) $\log_{16} x+\log_4 x+\log_2 x=7$

(b) $\log_{(2x+3)}(6x^2+23x+21)=4-\log_{(3x+7)}(4x^2+12x$

(I I T 8

19 $4^x-3^{x-\frac{1}{2}}=3^{x+\frac{1}{2}}-2^{2x-1}$ (I I T 7

20 $x+\log_{10}(1+2^x)=x\log_{10}5+\log_{10}6$

21 $\log_a(ax)\log_x(ax)=\log_{a^2}\left(\frac{1}{a}\right)\ (a>0,\ a\neq 1)$

22. $6^x(2/3)^y-3.2^{x+y}-8\ 3^{x-y}+24=0,\ xy=2$ (I I T 7

Sum of a G P $1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+ \ldots \infty = \frac{1}{1-\frac{1}{2}}=2$

Sum of A P $=\frac{n}{2}[a+l]=y^2$ for N^r and $\frac{y}{2}$ $(3y+5)$ for D^r

The given equations can be re-written as

$\log_{10} x^2=y$ or $2\log_{10} x=y$ (I)

and $\frac{2y}{y(3y+5)}=\frac{20}{7\log_{10} x}$ or $\frac{2y}{3y+5}=\frac{40}{7\,2\log_{10} x}$

$=\frac{40}{7y}$ by I,

$7y^2=60y+100$ or $7y^2-70y+10y-100=0$

or $(y-10)(7y+10)=0$

$y=10,\ \frac{-10}{7}$ Since y is a +ve integer $y=10$

$2\log_{10}x=10$ by (I) or $\log_{10}x=5$ $x=(10)^5$

18 (a) The given equation can be re written as

$\log_{2^4}x+\log_{2^2}x+\log_2 x=7$

or $\frac{1}{4}\log_2 x+\frac{1}{2}\log_2 x+\log_2 x=7$

or $\frac{7}{4}\log_2 x=7$ or $\log_2 x=4$

Hence $x=2^4=16$

(b) The given equation can be re written as

$\log_{(2x+3)}(2x+3)(3x+7)+\log_{(3x+7)}(2x+3)=4$

or $\log_{(2x+3)}(2x+3)+\log_{(2x+3)}(3x+7)$

$+2\log_{(3x+7)}(2x+3)=4$

or $1+y+2\frac{1}{y}=3$ $\quad \log_b a=\frac{1}{\log_a b}$

where $y=\log_{(2x+3)}(3x+7)$

or $y^2-3y+2=0$ $\quad y=1, 2$

$\log_{(2x+3)}(3x+7)=1$ or 2

$3x+7=2x+3$ or $3x+7=(2x+3)^2$

$x=-4$, and $4x^2+9x+2=0$ or $x=-2,-\frac{1}{4}$

Thus we have three values of x namely $-4, -2, -\frac{1}{4}$ (1)

Now by difinitions of logarithms

$2x-3>0$, $2x+3\neq 1$ and $3x+7>0$ and $3x+7\neq 1$

23 $3^x 5^y = 75,\ 3^y 5^x = 45$

24 $\log_y x + \log_x y = 2,\ x^2 + y = 12$

25 $\log\ xy = 5,\ \log_{1/2}(x/y) = 1$

26 $5(\log_y x + \log_x y) = 26,\ xy = 64$

27 $2^{x+y} = 6^y,\ 3^x = 3\ 2^{y+1}$

28 $\log_2 x + \log_4 y + \log_4 z = 2$

$\log_3 y + \log_9 z + \log_9 x = 2$

$\log_4 z + \log_{16} x + \log_{16} y = 2$

29 Show that the equation

$$e^{\sin x} - e^{-\sin x} - 4 = 0$$

has no real solution (I I T 82)

30 Find all solutions of $|x^2 - x - 6| = x + 2$ Note that x is a real variable (Roorkee 82)

31 Solve the equation

$$(x+1)(|x| - 1) = -\tfrac{1}{2}$$

32 Find all numbers x such that

$$|x+1| - |x| + 3|x-1| |x-2| = x + 2 \quad \text{(I. I T 76)}$$

33 Solve $x^2 - 2|x| - 3 = 0$

34 Solve the equation $|x^2 - 9| + |x^2 - 4| = 5$

35 Find the solution to the equation

$$2^{|x+2|} - |2^{x+1} - 1| = 2^{x+1} + 1$$

36 Solve the equation

$$\tfrac{3}{2}\log_{1/4}(x+2)^2 - 3 = \log_{1/4}(4-x)^3 + \log_{1/4}(x+6)^3$$

37 Solve the equation

$x|x+1| + a = 0$ for every real number a

38 Solve the system of equations

$$|x^2 - 2x| + y = 1,\ x + |y| = 1$$

39 The equation

$$2\cos^2(\tfrac{1}{2}x)\sin^2 x = x^2 + x^{-2},\ 0 < x \leqslant \pi/2 \text{ has}$$

(a) no real solution, (b) one real solution, (c) more than one real solution (I I T 80)

40 Solve the equations

(i) $\sin x = x^2 + x + 1$, (ii) $7^{6-x} = x + 2$

41 Solve the equation

$$2\cos^2\frac{x^2+x}{2} = 2^x + 2^{-x}$$

42 Solve for x —

(i) $(5+2\sqrt{6})^{x^2-3} + (5-2\sqrt{6})^{x^2-3} = 10$ (I I.T 85)

(ii) $[\sqrt{(5+2\sqrt{6})}]^x + [\sqrt{(5-2\sqrt{6})}]^x = 10$

$$x > -\frac{3}{2},\ x \neq -1,\ x > -\frac{7}{3},\ x \neq -2 \qquad (2)$$

Clearly from (1) and (2) $x=-4$ $x=-2$ are rejected and the only solution left is $x=-1/4$

19 The given equation can be re-written as

$$2^{x}+2^{x-1}=3^{x-1/2}+3^{x+1/2}$$

$$2^{x}(1+\tfrac{1}{2})=3^{x}\left(\frac{1}{\sqrt{3}}+\sqrt{3}\right)$$

or $3 \cdot 2^{2x-1}=4 \cdot 3^{x-1/2}$ or $2^{2x-3}=3^{x-3/2}$

Now by trial, $x=\frac{3}{2}$ is one solution.

Now taking logarithm to the base 10, we get

$$(2x-3)\log_{10} 2=(x-\tfrac{3}{2})\log_{10} 3$$

or $(2\log 2-\log_{10} 3)\,x=\frac{3}{2}(2\log 2-\log 3)$

This also gives $x=\frac{3}{2}$

Hence $x=\frac{3}{2}$ is the only solution

20 The given equation can be written as

$$x+\log_{10}(1+2^{x})-x\log_{10} 5=\log_{10} 6$$

or $x\log_{10} 10+\log_{10}(1+2^{x})-x\log_{10} 5=\log_{10} 6$

or $x\log_{10}(10/5)+\log_{10}(1+2^{x})=\log_{10} 6$

or $\log_{10} 2^{x}+\log_{10}(1+2^{x})=\log_{10} 6$

or $\log_{10} 2^{x}(1+2^{x})=\log_{10} 6$

Hence $2^{x}(1+2^{x})=6$ or $2^{2x}+2^{x}-6=0$

or $(2^{x}+3)(2^{x}-2)=0$

Since $2^{x}>0$, $2^{x}=-3$ is impossible So we have

$$2^{x}=2 \quad \text{or} \quad x=1$$

Hence $x=1$ is the only root of the given equation

21 The roots must satisfy the condition $x>0$, $x\neq 1$ (why ?)

The equation can be written as

$$[\log_a a+\log_a x]\,[\log_x a+\log_a x]=\log_a a^{-1}$$

or $(1+\log_a x)\left(\frac{1}{\log_a x}+1\right)=-\frac{1}{2}$

Now put $\log_a x=y$ Then

$$\frac{(1+y)^2}{y}=-\tfrac{1}{2} \quad \text{or} \quad 2y^2+5y+2=0$$

or $(y+2)(2y+1)=0$

This gives $y=-2$ or $-\frac{1}{2}$, that is, $\log_a x=-2$ or $-\frac{1}{2}$

Hence $x=a^{-2}$ or $a^{-1/2}$

22 Hint $2^{x} 3^{x} (\frac{2}{3})^{y}-3 \cdot 2^{x+y}-8 \cdot 3^{x-y}+24=0$

or $2^{x+y} 3^{x-y}-3 \cdot 2^{x+y}-8 \cdot 3^{x-y}+24=0$

or $(2^{x+y}-8)(3^{x-y}-3)=0$

Hence $2^{x+y}=2^{3}$ or $3^{x-y}=3^{1}$

$$x+y=3 \qquad (1)$$

43 For $a \leqslant 0$ determine all real roots of the equation

$$x^2-2a|x-a|-3a^2=0 \qquad \text{(IIT 86)}$$

44 $$\tfrac{3}{5}\, a^{\log_a x \log_{10} a \log_a 5} - 3^{\log_{10}(x/10)} = 9^{\log_{100} x+\log_4 2} \qquad \text{(Roorkee 88)}$$

45 $$|x^2+4x+3|+2x+5=0 \qquad \text{(IIT 88)}$$

Solutions to Problem Set (D)

1 The given equation can be written as

$$\frac{2^{x^2}}{2^{2x}}=8 \text{ or } 2^{x^2-2x}=2^3$$

Hence $x^2-2x=3$ or $x^2-2x-3=0$

or $(x-3)(x+1)=0$, $x=3, -1$

(b) Ans $x=\pm 1$

2 Ans $x=\pm 1$

3. By trial, we see that $x_1=1$ is a solution of the given equation

Now taking logarithms to base 10, we get

$$x \log_{10} 3+\frac{3x}{x+2}\log_{10} 2=\log_{10} 6$$

or $x^2\log_{10} 3+(3\log_{10} 2+2\log_{10} 3-\log_{10} 6)\,x-2\log_{10} 6=0$

Let x_1, x_2 be its roots Then

$$x_1x_2=-2\log_{10} 6/\log_{10} 3$$

or $1\cdot x_2=-2\log_3 6$

Hence the original equation has two roots $x_1=1$ $x_2=-2\log_3 6$

Remark Students must not think that the solution found by trial $x_1=1$ is the only solution So it is useful to be able to guess a root, but never consider the guessing as the whole solution

4 $4^x+6^x=9^x=3^{2x}$

or $\frac{4^x}{3^{2x}}+\frac{6^x}{3^{2x}}=1$ or $(\frac{2}{3})^{2x}+(\frac{2}{3})^x-1=0$

$$\left(\frac{2}{3}\right)^x=\frac{-1\pm\sqrt{(1+4)}}{2}=\frac{-1\pm\sqrt{5}}{2}$$

Since exponential function is always positive, we have

$$\left(\frac{2}{3}\right)^x=\frac{\sqrt{5}-1}{2}$$

$$x[\log_{10}2-\log_{10}3]=\log_{10}(\sqrt{5}-1)-\log_{10}2$$

or $$x=\frac{\log_{10}(\sqrt{5}-1)-\log_{10}2}{\log_{10}2-\log_{10}3}$$

or $\quad x-y=1$ (2)

Also second given equation is

$$xy=2 \quad (3)$$

Now solve (1) and (3) and (2) and (3)

Ans $\quad x=2,\ y=1$ or $x=1,\ y=2$

or $\quad x=-1,\ y=-2$

23 Dividing the given equations, we get

$$3^{x-y}\,5^{y-x}=\frac{75}{45} \quad \text{or} \quad \left(\frac{3}{5}\right)^{x-y}=\left(\frac{3}{5}\right)^{-1}$$

Hence $\quad x-y=-1$ (1)

Again multiplying the equations, we get

$$(3^x\times5^x)\,(3^y\times5^y)=75\times45$$

or $\quad 15^{x+y}=15^3$

Hence $\quad x+y=3$ (2

Solving (1) and (2), we get

$$x=1,\ y=2$$

24 The first equation can be written as

$$\log_y x+\frac{1}{\log_y x}=2$$

or $\quad (\log_y x)^2-2\log_y x+1=0$

or $\quad (\log_y x-1)^2=0$ *i e* $\log_y x=1$

This gives $x=y$ $(x>0,\ y>0,\ x\neq1,\ y\neq1)$

Then the second given equation gives

$$x^2+x-12=0 \quad \text{or} \quad (x+4)(x-3)=0$$

Since $x=-4$ is not in the domain of x, the only solution is $x=y=3$

25 The equations can be re written as

$$\log_2|x|+\log_2|y|=5$$

$$\text{and } \log_{1/2}|x|-\log_{1/2}|y|=1,$$

or $\quad -\log_2|x|+\log_2|y|=1$

Put $\log_2|x|=u$ and $\log_2|y|=v$

Then $u+v=5$ and $v-u=1$

Solving these we get $u=2$, $v=3$ and so $|x|=2^2=4$

and $|y|=2^3=8$

These relations give $x=\pm4$, $y=\pm8$

Hence the solutions of the original equation are

$$x=4,\ y=8 \quad \text{or} \quad x=-4,\ y=-8$$

Note that $x=4, y=-8$ or $x=-4, y=8$ will not be the solution otherwise $\log_2 xy$ and $\log_{\frac{1}{2}}\left(\frac{x}{y}\right)$ will not be meaningful (why ?)

5 By trial, $x=2$ is a root of the equation Dividing the equation by 5^x, we get

$$\left(\frac{3}{5}\right)^x+\left(\frac{4}{5}\right)^x=1 \tag{1}$$

If $x<2$, then $(3/5)^x > (3/5)^2$ and $(4/5)^x > (4/5)^2$ and so

$$(3/5)^x+(4/5)^x > (3/5)^2+(4/5)^2=1$$

Hence (1) is not satisfied for any value of $x<2$ Similarly, for $x>2$, we will always have the inequality $(3/5)^x+(4/5)^x<1$

Thus $x=2$ is the only root of the given equation

Note that it is immaterial how we found a root, but it is necessary to show that there are no other roots

6 The equation can be written as

$$4\cdot 2^{2x}-2^x 3^x-18\cdot 3^{2x}=0$$

Dividing by $2^x 3^x$, we get $4\left(\frac{2}{3}\right)^x-1-18\left(\frac{3}{2}\right)^x=0$

Put $\left(\frac{3}{2}\right)^x=t$ Then $4/t-1-18t=0$

or $18t^2+t-4=0$ or $(2t+1)(9t-4)=0$

$$t=\left(\frac{3}{2}\right)^x=\left(\frac{4}{9}\right)=\left(\frac{2}{3}\right)^2=\left(\frac{3}{2}\right)^{-2}, \text{ giving } x=-2,$$

Note that here we reject $t=\left(\frac{3}{2}\right)^x=-\frac{1}{2}$, since exponential function can never be negative

Hence the only solution is $x=-2$

7 Do yourself Ans $x=\frac{1}{2}$

8 (i) $16^{\sin^2 x}+16^{\cos^2 x}=10$

or $16^{\sin^2 x}+16^{1-\sin^2 x}=10$ or $16^{\sin^2 x}+16\cdot 16^{-\sin^2 x}=10$

Put $16^{\sin^2 x}=t$ Then $t+16/t=10$ or $t^2-10t+16=0$

or $(t-2)(t-8)=0$, giving $t=2$ or 8

Hence $16^{\sin^2 x}=2$ or $2^{4\sin^2 x}=2$ which gives

$$4\sin^2 x=1 \text{ or } \sin^2 x=\tfrac{1}{4}=\sin^2 \pi/6 \tag{1}$$

Solution of (1) is, $x=n\pi\pm\frac{\pi}{6}$

where $n=0, \pm 1, \pm 2, \pm 3,$

Again $t=8$ gives $2^{4\sin^2 x}=8=2^3$

or $4\sin^2 x=3$ i e $\sin^2 x=\frac{3}{4}=\sin^2\frac{\pi}{3}$ (2)

The solution, of (2) is, $x=n\pi\pm\frac{\pi}{3}$

26 We first observe that $x > 0, y > 0, x \neq 1$ and $y \neq 1$

The first equation can be written as

$$5\left[\log_y x + \frac{1}{\log_y x}\right] = 26$$

Putting $\log_y x = u$, this becomes

$$5u^2 - 26u + 5 = 0 \text{ or } (u-5)(5u-1) = 0$$

Hence $u = 5, 5^{-1}$ or $\log_y x = 5, 5^{-1}$

Hence the solution of the given equation to be found among the system of equations

$$\log_y x = 5, \; xy = 64$$

and of the system

$$\log_y x = \tfrac{1}{5}, \; xy = 64$$

Solving these systems and choosing those solutions which satisyf the conditions $x > 0$, $x \neq 1$ $y > 0$ $y \neq 1$, we obtain the answer

Thus the original system has two solutions

(We have left the details two be supplied by the students)

27 Ans $x = \dfrac{\log_{10} 3}{\log_{10} 3 - \log_{10} 2}, \; y = \dfrac{\log_{10} 2}{\log_{10} 3 - \log_{10} 2}$

28 Hint First equation can be written as

$$\log_2 x + \log_2 y + \log_2 z = 2$$

or $\log x + \frac{1}{2}\log y + \frac{1}{2}\log z = 2$

or $\log x\sqrt{(yz)} = 2$ or $x\sqrt{(yz)} = 2^2 = 4$

or $x^2 yz = 16$ (1)

Similarly $y^2 zx = 81$ (2)

and $z^2 xy = 256$ (3)

The equations (1), (2) and (3) can now be easily solved This is left for the students

Ans $x = 2/3$, $y = 27/8$, $z = 32/3$

29 Put $e^{\sin x} = y$ Then the given equation reduces to

$$y - \frac{1}{y} - 4 = 0 \text{ or } y^2 - 4y - 1 = 0$$

$$y = \frac{4 \pm \sqrt{(16+4)}}{2} = 2 \pm \sqrt{5} \text{ or } e^{\sin x} = 2 \pm \sqrt{5}$$

Since exponential function is always positive, we cannot have

$$e^{\sin x} = 2 - \sqrt{5}$$

And for $e^{\sin x} = 2 + \sqrt{5}$, we have

$\sin x = \log_e (2 + \sqrt{5}) > 1$ [$\because$ $e > 1$ and $2 + \sqrt{5} > e$]

which is impossible

Hence the equation can have no real solution

30 We consider two cases

(i) $x^2-x-6<0$ In this case, the given equation reduces to
$-(x^2-x-6)=x+2$ or $-x^2+x+6=x+2$
which gives $x=\pm 2$ Now we check whether these values of x satisfy the condition (i) Substituting $x=2$ and $x=-2$ in (I) we obtain the inequalities $-4<0$ and $0<0$ The first is valid and second is not So in this case $x=2$ is the only root of the given equation

(ii) $x^2-x-6\geqslant 0$ In this case, we obtain the equation $x^2-x-6=x+2$ or $x^2-2x-9=0$ The roots of this equation are $x=4$, $x=-2$ Since both of these values satisfy condition (ii), both 4 and -2 are the roots of the original equation

Hence the original equation has three roots -2, 2 and 4

31 Ans $\frac{1}{\sqrt{2}}$, $-1\pm\left(\frac{1}{\sqrt{2}}\right)$

32 Consider five case

(a) $x<-1$ (b) $-1\leqslant x<0$
(c) $0\leqslant x<1$, (d) $1\leqslant x<2$ (e) $x\geqslant 2$

In case (a), the given equation reduces to

$$-x-1+x+3\ (1-x)\ (2-x)=x+2$$

or $3x^2-10x+3=0$ or $(x-3)\ (3x-1)=0$

$$x=3 \text{ or } \tfrac{1}{3}$$

Since these values of x do not satisfy condition (a), the original equation has no solution in this case

In cash (b), the original equation reduces to

$$x+1+x+3\ (1-x)\ (2-x)=x+2$$

or $3x^2-8x+5=0$ or $(x-1)\ (3x-5)=0$

$$x=1,\ \tfrac{5}{3}$$

Since these values of x do not satisfy the condition (b), the original equation has no solution in this case also

In case (c), the equation is

$$x+1-x+3\ (1-x)\ (2-x)=x+2$$

or $3x^2-10x+5=0$

$$x=\frac{10\pm\sqrt{(100-60)}}{6}=\frac{10\pm 2\sqrt{(10)}}{6}=\frac{5\pm\sqrt{(10)}}{3}$$

The root $x=\frac{5-\sqrt{(10)}}{3}$ satisfies the condition (c) whereas the root $x=\frac{5+\sqrt{(10)}}{3}$ does not satisfy the condition Hence or $83=83^{\log_x 10}$

Hence $\log_x 10=1$ whence $x=10$

13 The given equation can be written as

$$\log_{10}\left(\frac{1}{2^x+x-1}\right)=x[\log_{10}5-\log_{10}10]$$
$$=x\log_{10}5/10=x\log_{10}(\tfrac{1}{2})$$
$$=\log_{10}2^{-x}$$

Hence $\log_{10}\left(\frac{1}{(2^x+x-1)}\right)-\log_{10}2^{-x}=0$

or $\log_{10}\left(\frac{2^x}{2^x+x-1}\right)=0$

$$\frac{2^x}{2^x+x-1}=1 \quad \text{or} \quad 2^x=2^x+x-1$$

Hence $x=1$ is the solution of the given equation

14 The equation can be written as

$$\log_{10}\left(\frac{x^2-x-6}{x+2}\right)=x-4 \quad \text{or} \quad \log_{10}\frac{(x-3)(x+2)}{(x+2)}=x-4$$

or $\log_{10}(x-3)=x-4$ or $x-3=10^{x-4}$ (1)

By trial, $x=4$ is the solution of (1) which is therefore the solution of the given eqution

Observe that $x>4$ or $x<4$ does not satisfy (1) Hence $x=4$ is the only solution of the original equation

15 (i) $x^x=x$

Taking logarithm to the base 10, we get

$x\log_{10}|x|=\log_{10}|x|$ or $(x-1)\log_{10}|x|=0$

Hence $x=1$

or $\log_{10}|x|=0$ which gives $|x|=1$ or $x=\pm 1$

Hence the solution of the given equation is $x=\pm 1$

(ii) The equation can be re-written as

$$x_{\sqrt{x}}=x^{x/2}$$

By trial, $x=1$ is a solution

Now equating the exponents we have

$$\sqrt{x}=\tfrac{1}{2}x \quad \text{or} \quad x=\tfrac{1}{4}x^2 \quad \text{or} \quad x(x-4)=0$$

Since $x=0$ does not satisfy the given equation, we get $x=4$

Hence the roots of the given equation are $x=1, 4$

16 Ans $x=3$

17 (i) Ans $x=25, \sqrt{5}$ (ii) Ans$=100$

$$5^{\log_{10}x}=5^{\log x\,\log_{10}5}=x^{\log_{10}5}$$

We have $2\,5^{\log_{10}x}=50$ or $5^{\log_{10}x}=5^2$ etc

$\frac{5-\sqrt{(10)}}{3}$ is the only root of the original equation in this case In case (d), the equation becomes

$$x+1-x+3\,(x-1)\,(2-x)=x+2$$

or $$3x^2-8x+7=0$$

which gives imaginary values of x

Hence there is no solution in this case Finally in case (e), the original equation reduces to

$$x+1-x+3\,(x-1)\,(x-2)=x+2$$

$$3x^2-10x+5=0$$

As in case (c) the roots are $x=\frac{5\pm\sqrt{(10)}}{3}$

Since the root $(5-\sqrt{10})/3$ does not satisfy the condition (e) whereas the root $(5+\sqrt{10})/3$ satisfies it the only root of the original equation is at $x=(5+\sqrt{10})/3$

Thus from the above discussion we see that the root of the given equation are at $x=\frac{5\pm\sqrt{(10)}}{3}$

33 We can solve this problem as in problems 30, 31 or 32 by considering two cases (a) $x<0$ (b) $x\geqslant 0$

Another method to solve this equation is as given below

Put $|x|=y$ Then $x^2=|x|^2=y^2$

Hence the give equation can be written as

$$y^2-2y-3=0 \quad \text{or} \quad (y-3)\,(y+1)=0$$

This gives $|x|=y=3$ or -1 of which $|x|=-1$ is meaningless

and $|x|=3$ gives $x=\pm 3$

which is the required solution of the original equation

34 We consider three cases,

(a) $x^2<4$, (b) $4\leqslant x^2\leqslant 9$, (c) $x^2>9$

In case (a), the equation reduces to

$$9-x^2+4-x^2=5 \quad \text{or} \quad x^2=4 \quad \text{or} \quad x=\pm 2$$

Since both these values of x do not satisfy condition (a), there is no solution in this case

In case (b), the equation reduces to

$$9-x^2+x^2-4=5 \quad \text{or} \quad {}^{\prime}=5$$

base and the angle of depression of the foot of the tower at a point b ft just above A is β. Prove that the height of the tower is $b \tan \alpha \cot \beta$

7 The width of a road is b feet, on one side of which there is a window h feet high. A building in front of it subtends an angle θ at it. Prove that the height of the building is

$$\frac{(b^2+h^2) \sin \theta}{b \cos \theta + h \sin \theta}$$

8 (*a*) The angle of elevation of the top of a pillar at any point A on the ground is 15°. On walking 100 ft towards the pillar, the angle becomes 30°. Find the height of the pillar and its distance from A

(*b*) A person standing on the bank of a river observes that the angle subtended by a tree on the opposite bank is 60°, when he retires 40 feet from the bank he finds the angle to be 30°. Find the height of the tree and the breadth of the river (I I T 75)

(*c*) The angle of elevation of the top of a tower from a point on the same level as the foot of the tower is 30°. On advancing 150 meters towards the foot of the tower the angle of elevation increases to 60°. Find the height of the tower

9 A man observes that when he moves up a distance c meters on a slope, the angle of depression of a point on the horizontal plane from the base of the slope is 30°, and when he moves up further a distance c meters the angle of depression of that point is 45°. Obtain the angle of inclination of the slope with the horizontal (Roorkee 86)

10 (*a*) Find the height of a chimney when it is found that on walking towards it 100 ft in a horizontal line through its base the angular elevation of its top changes from 30° to 45°

(*b*) The shadow of a tower standing on a level ground is found to be 60 meters longer when the sun's altitude is 30° than when it is 45°. Find the height of the tower

(*c*) A man in a boat rowed away from a cliff 150 meters high takes 2 minutes to change the angle of elevation of the top of the cliff from 60° to 45°. Find the speed of the boat

(*d*) A man on a cliff observes a boat at an angle of depression

30° which is sailing towards the shore to the point immediately beneath him. Three minutes later the angle of depression of the boat is found to be 60°. Assuming that the boat sails at a uniform speed determine how much more time it will take to reach the shore.

(e) The length of the shadow of a rod inclined at 10° to the vertical towards the sun is 2.05 metres when the elevation of the sun is 38°. Find the length of the rod. **(I I T 67)**

11 On level ground the angle of elevation of the top of a tower is 30°. On moving 20 *m* nearer the angle of elevation is 60°. What is the height of the tower? **(Roorkee 1976)**

12 A person stands at a point A due south of a tower and observes its elevation is 60°. He then walks westwards towards B where the elevation is 45°. At a point C on AB produced he finds it to be 30°. Prove that $AB=BC$. **(I I T 67)**

13 A man walking due North observes that the elevation of a baloon which is due East of him and is sailing towards the north-west is then 60°; after he has walked 400 yards the baloon is vertically over his head. Find its height, supposing it to have always remained the same.

14 A person wishing to ascertain the height of a tower, stations himself on a horizontal plane through its foot at a point at which the elevation of the top is 30°. On walking a distance a in a certain direction he finds that the elevation of the top is the same as before and on walking a distance $\frac{5}{3}a$ at right angles to his former direction he finds that the elevation of the top to be 60°. Prove that the height of the tower is either

$$\sqrt{\left(\frac{5}{6}\right)}\,a \quad \text{or} \quad \sqrt{\left(\frac{85}{48}\right)}\,a$$

From an aeroplane vertically over a straight horizontal road the angles of depression of two consecutive milestones on opposite sides of the aeroplane are observed to be α and β. Show that the height in miles of aeroplane above road is given by

$$\frac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

So in this case the original equation reduces to an identity It means that any value of x which satisfies the condition $4 \leqslant x^2 \leqslant 9$ is a solution of the given equation Now the double inequality $4 \leqslant x^2 \leqslant 9$ is equivalent to two double inequalities $-3 \leqslant x \leqslant -2$ and $2 \leqslant x \leqslant 3$

In case (c), the equation becomes

$$x^2-9+x^2-4=5 \quad \text{or} \quad x^2=9 \quad \text{or} \quad x=\pm 3$$

Since these values of x do not satisfy the condition (c), there is no solution in this case

From the above discussion, we conclude that the solution sets of the given equation are

$$-3 \leqslant x \leqslant -2 \text{ and } 2 \leqslant x \leqslant 3$$

35 We consider two cases

(a) $x+2 \geqslant 0$ In this case, the equation reduces to

$$2^{x+2}-|2^{x+1}-1|=2^{x+1}+1$$

or $\quad 2\,2^{x+1}-2^{x+1}-1=|2^{x+1}-1|$

or $\quad 2^{x+1}-1=|2^{x+1}-1|$

This equation is clearly satisfied for $2^{x+1}-1 \geqslant 0$, which is the same thing as, $x+1 \geqslant 0$, that is, $x \geqslant -1$ These values of x satisfy condition (a) and so they constitute roots of the original equation

(b) $x+2 < 0$ Thus $|x+2|=-x-2$ and $x+1 < -1$ so that $2^{x+1} < \frac{1}{2}$ Hence the given equation in this case reduces to

$2^{-x-2}-(1-2^{x+1})=2^{x+1}+1$ or $2^{-x-2}=2$

Hence $\quad -x-2=1$ or $x=-3$

Since this value of x satisfies (b), it is a root of the given equation Combining the solutions of cases (a) and (b), we get the answer $x=-3$ and $x \geqslant -1$

36 Since $\log_{1/4}(x+2)^2=2\log_{1/4}|x+2|$,

$\log_{1/4}(4-x)^3=3\log_{1/4}(4-x)$ and $\log_{1/4}(x+6)^3=3\log_{1/4}(x+6)$, the given equation can be written as

$3\log_{1/4}|x+2|-3=3\log_{1/4}(4-x)+3\log_{1/4}(x+6)$

or $\quad \log_{1/4}|x+2|=\log_{1/4}4=\log_{1/4}(4-x)(x+6)$

$[\quad \log_{1/4}4=-1]$

or $\quad \log_{1/4}4|x+2|=\log_{1/4}(4-x)(x+6)$

Hence $4|x+2|=(4-x)(x+6)$ (1)

We now consider two cases (a) $x+2<0$ (b) $x+2 \geqslant 0$

where $n! = 1\ 2\ 3 \quad n$

Note that $n! = n(n-1)!$

$= n(n-1)(n-2)!$ etc

(b) **Number of permutations of n dissimilar things taken all at a time**

$$^nP_n = n(n-1)(n-2) \quad \{n-(n-1)\}$$
$$= n(n-1)(n-2) \quad 3\ 2\ 1 = n!$$

(c) **Number of combinations of n dissimilar things taken r at a time**

$$^nC_r = \frac{n!}{(n-r)!\,r!} = \frac{^nP_r}{r!}$$

or $\quad r!\,^nC_r = {}^nP_r$

(d) **Number of combinations of n dissimilar things taken all at a time**

$$^nC_n = \frac{n!}{n!\,(n-n)!} = \frac{1}{0!} = 1 \qquad 0! = 1$$

(e) If out of n things p are exactly alike of one kind, q exactly alike of second kind and r exactly alike of third kind and the rest all different, then the number of permutations of n things taken all at time

$$= \frac{n!}{p!\,q!\,r!}$$

(f) **Number of circular permutations of n different things taken all at a time**

Here having fixed one thing, the remaining $(n-1)$ things can be arranged round the table in $(n-1)!$ ways

(g) **If some or all of n things be taken at a time then the number of combinations will be $2^n - 1$**

(h) $^nC_r = {}^nC_{n-r}$

(i) $^nC_{r_1} = {}^nC_{r_2} \Rightarrow r_1 = r_2$ or $r_1 + r_2 = n$

(j) $^nC_r + {}^nC_{r-1} = {}^{n+1}C_r$ **Important**

(k) Number of combinations of n dissimilar things taken r at a time when p particular things always occur

$$= {}^{n-p}C_{r-p}$$

(l) Number of combinations of n dissimilar things taken r at a time when p particular things never occur

$$= {}^{n-p}C_r$$

(m) Number of permutations of n dissimilar things taken r at a time when p particular things always occur

$$= {}^{n-p}C_{r-p}\ r!$$

In case (a), the equation (1) becomes

$$-4(x+2)=(4-x)(x+6) \quad \text{or} \quad x^2-2x-32=0$$

$$x=\frac{2\pm\sqrt{(4+128)}}{2}=1\pm\sqrt{(33)}$$

The root $1+\sqrt{(33)}$ does not satisfy condition (a) and so it is not a root of (1)
But $1-\sqrt{(33)}$ is a root of (1) since it satisfies condition (a).
In case (b), the equation (1) becomes

$$4(x+2)=(4-x)(x+6)$$

or $\quad x^2+6x-16=0$

or $\quad (x+8)(x-2)=0$

This gives $x=-8, 2$

The root $x=-8$ does not satisfy condition (b), so it is not a root of (1) Hence $x=2$ is the only root of (1) in this case
Combining cases (a) and (b) we see that the roots of equation (a) are

$$x_1=2 \quad \text{and} \quad x_2=1-\sqrt{(33)}$$

We must now check whether these roots satisfy the original equation or not It is easily seen that all the expressions under the sign of logarithm in the given equation for $x=x_1$ and $x=x_2$ are positive and so both these numbers lie in the domain of x and are therefore roots of the given equation
Thus we get the answer

$$x_1=2,\ x_2=1-\sqrt{(33)}$$

37 We consider two cases

(a) $x<-1$ (b) $x\geqslant-1$

In case (a), the equation reduces to the form

$$x(-x-1)+a=0 \quad \text{or} \quad x^2+x-a=0 \qquad (1)$$

We are interested in finding real roots of (1) which satisfy the condition (a)
Now the condition for (1) to have real roots is

$$1+4a\geqslant 0 \quad \text{or} \quad a\geqslant-\tfrac{1}{4}$$

[Note that for real roots, the discriminant of (1) must be non negative]

The roots of (1) are

$$x_1=\frac{-1+\sqrt{(1+4a)}}{2},\ x_2=\frac{-1-\sqrt{(1+4a)}}{2}$$

(n) Number of permutations of n dissimilar things taken r at time when p particular things never occur

$$= {}^{n-p}C_r\, r\,!$$

(o) **Division into groups**

(i) The number of ways in which $m+n$ things can be divided into two groups containing m and n things respectively

$$= \frac{(m+n)\,!}{m\,!\,n\,!}$$

(ii) If $n=m$, the groups are equal, and in this case the number of different ways of subdivision

$$= \frac{2m\,!}{m\,!\,m\,!\,2\,!}\,,$$

for in any one way it is possible to interchange the two groups without obtaining new division

(iii) But if $2m$ things are to be divided equally between two persons, then the number of divisions

$$= \frac{2m\,!}{m\,!\,m\,!}$$

(iv) Similarly the number of divisions of $m+n+p$ things in to groups of m, n and p things respectively

$$= \frac{(m+n+p)\,!}{n\,!\,n\,!\,p\,!}$$

(v) If $3m$ things are divided into three equal groups, then the number of divi ions

$$= \frac{(3m)\,!}{m\,!\,m\,!\,m\,!\,3\,!}$$

(vi) But if $3m$ things are to be divided among three persons, then the number of divisions

$$= \frac{(3m)\,!}{m\,!\,m\,!\,m\,!}$$

(p) Number of permutations of n dissimilar things taken r at a time when each thing can be repeated once, twice, upto r times $= n^r$

(q) Total number of ways in which it is possible to make selection by taking some or all out of $p+q+r$ things, where of

Out of these roots, we have to take those which satisfy th condition $x < -1$

To do this, we have to solve the inequalities

$$\frac{-1+\sqrt{(1+4a)}}{2} < -1$$

and $\frac{-1-\sqrt{(1+4a)}}{2} < -1$

The first inequality reduces to

$$1+\sqrt{(1+4a)} < 0$$

which does not hold for values of $a \geqslant -\frac{1}{4}$

The second inequality reduces to

$$\sqrt{(1+4a)} > 1$$

and is valid for $a > 0$ as can be easily seen

Hence, for $a > 0$, the original equation has one real root at

$$x=\frac{-1-\sqrt{(1+4a)}}{2}$$

that satisfies the condition $x < -1$ and for $a \leqslant 0$, it has no such root

In case (b), the given equation takes the from

$$x^2+x+a=0$$

Discussing as in case (a), we will see that the given equation has two real roots for $0 \leqslant a \leqslant \frac{1}{4}$

$$\frac{-1+\sqrt{(1-4a)}}{2} \text{ and } \frac{-1-\sqrt{(1-4a)}}{2}$$

For $a < 0$, the equation has the root at

$$x=\frac{-1+\sqrt{(1-4a)}}{2}$$

For $a > \frac{1}{4}$, the equation has no roots in the domain $x \geqslant -1$

Combining the cases (a) and (b), we get the final answer

For $a < 0$ $x=\frac{-1+\sqrt{(1-4a)}}{2}$

For $0 \leqslant a \leqslant \frac{1}{4}$, $x=\frac{-1-\sqrt{(1+4a)}}{2}$, $\frac{-1\pm\sqrt{(1-4a)}}{2}$

and for $a > \frac{1}{4}$, $x=\frac{-1-\sqrt{(1+4a)}}{2}$

p are alike of one kind, q alike of a second kind, r alike of a third kind and so on

$$=(p+1)(q+1)(r+1)\ \ -1$$

(r) Number of combinations of n things taken r at a time when each may occur once, twice, thrice etc upto r times in any combination $={}^{n+r-1}C_r$

(s) The greatest value of ${}^nC_r$

(i) When n is even, ${}^nC_r$ is greatest when $r=\frac{n}{2}$

(ii) When n is odd ${}^nC_r$ is greatest when $r=\frac{n-1}{2}$ or $\frac{n+1}{2}$

Problem Set (A)

1 Prove that ${}^nC_r={}^nC_{n-r}$

2 Prove that ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$

3 Find the value of

$${}^{47}C_4+\sum_{r=1}^{5}{}^{(52-r)}C_3 \qquad \text{(I I T 80)}$$

4 Prove with or without the use of the formula

$${}^nP_r={}^{n-1}P_r+r\,{}^{n-1}P_{r-1} \qquad \text{(I I T 71)}$$

5 Prove with or without the use of the formula

$${}^nP_r=n\,{}^{n-1}P_{r-1}$$

6 If ${}^nC_{10}={}^nC_{15}$ find ${}^{27}C_n$

7 If ${}^nC_7={}^nC_4$ find n

8 If ${}^{15}C_{3r}={}^{15}C_{r+3}$, find r (I I T 67)

9 If ${}^nC_{12}={}^nC_8$ find ${}^nC_{17}$ and ${}^{22}C_n$

10 If ${}^8C_r\ \ {}^7C_3-{}^7C_2$ find r

11 If ${}^{15}C_r\ \ {}^{15}C_{r-1}=11\ \ 5$, find r

12 If ${}^9P\ +5\,{}^9P_4={}^{10}P_r$ find r

13 (a) If ${}^nC_2\ \ {}^{n-2}C_2=33\ \ 4$ find n

(b) If ${}^{56}P_{r+6}\ \ {}^{54}P_{r+3}=30800\ \ 1$, find r (Roorkee 83)

(c) If ${}^{n+2}C_8\ \ {}^{n-2}P_4=57\ \ 16$, find n

14 If ${}^{10}P_r=604800$ and ${}^{10}C_r=120$, find r

15 If ${}^{2n+1}P_{n-1}\ \ {}^{2n-1}P_n=3\ \ 5$, find n

16 If ${}^nP_r={}^nP_{r+1}$ and ${}^nC_r={}^nC_{r-1}$, find n and r

17 Prove

$${}^{4n}C_{2n}\ \ {}^{2n}C_n=\{1\ 3\ 5\ \ (4n-1)\}\ \ \{1\ 3\ 5\ \ (2n-1)\}^2$$

1 We consider four cases

(a) $x^2-2x \geqslant 0, y > 0,$ (b) $x^2-2x \geqslant 0, y < 0,$

(c) $x^2-2x < 0, y \geqslant 0,$ (d) $x^2-2x < 0, y < 0$

In case (a), the equations are

$$x^2-2x+y=1,\ x^2+y=1$$

Solving these equations, we get $x=0,\ y=1$

This pair satisfies condition (a) and so is a solution of the given equations

In case (b), the given equations reduce to the form

$$x^2-2x+y=1 \quad (1)$$

$$x^2-y=1 \quad (2)$$

Adding, $2x^2-2x=2$ or $x^2-x-1=0$ (3)

From (3) $x=\dfrac{1\pm\sqrt{(1+4)}}{2}=\dfrac{1\pm\sqrt{5}}{2}$

Now from (2) and (3), we get, $y=x=(1+\sqrt{5})/2$

Hence the solutions in this case are

$$x_1=\frac{1+\sqrt{5}}{2},\ y_1=\frac{1+\sqrt{5}}{2}$$

or $x_2=\dfrac{1-\sqrt{5}}{2},\ y_2=\dfrac{1-\sqrt{5}}{2}$

Since $y_1 > 0$, the pair (x_1, y_1) does not satisfy the condition (b) and it must be rejected Now since $y < 0$ and $x_2^2-2x_2 > 0$, the pair (x_2, y_2) satisfies (b)

Hence in this case, the solution is

$$x=y=(1-\sqrt{5})/2$$

(c) In this case, the given equations take the form

$$-x^2+2x+y=1,\ x^2+y=1$$

Solutions of these equations are easily found to be

$$x_1=0,\ y_1=1 \quad \text{or} \quad x_2=1,\ y_2=0$$

The pair (x_1, y_1) does not satisfy condition (c) and (x_2, y_2) satisfies it

Hence the solution in this case is

$$x=1,\ y=0$$

(d) Here the equations are

$$-x^2+2x+y=1\quad x^2-y=1$$

Adding $2x=2$ or $x=1$ and $y=0$

18 Evaluate ${}^{15}C_8 + {}^{15}C_9 - {}^{15}C_6 - {}^{15}C_7$

19 Prove that ${}^{n}C_r + 2\,{}^{n}C_{r-1} + {}^{n}C_{r-2} = {}^{n+2}C_r$

20 Prove that ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$ if $n > 7$ (I I T 75)

21 If ${}^{28}C_{2r} : {}^{24}C_{2r-4} = 225 : 11$, then

(a) $r=24$, (b) $r=14$,

(c) $r=7$, (d) none of these

Solutions to Problem Set (A)

1 ${}^{n}C_r = {}^{n}C_{n-r}$

The number of combinations of n dissimilar things taken r at a time will be ${}^{n}C_r$. Now if we take out a group of r things, we are left with a group of $(n-r)$ things. Hence the number of combinations of n things taken r at a time is equal to the number of combinations of n things taken $(n-r)$ at a time

$${}^{n}C_r = {}^{n}C_{n-r}$$

Alternative Method

We know that

$${}^{n}C_r = \frac{n!}{r!\,(n-r)!}$$

$${}^{n}C_{n-r} = \frac{n}{(n-r)!\,(n-n+r)!} = \frac{n!}{r!\,(n-r)!}$$

$${}^{n}C_r = {}^{n}C_{n-r}$$

2 ${}^{n}C_{r-1} + {}^{n}C_r = {}^{n+1}C_r$

Without the use of formula

Suppose we have to take r things out of $(n+1)$ things. Then there are ${}^{n+1}C_r$ combinations.

Now to find the combinations in which a particular thing always occurs we shall set aside that particular thing and form the combinations from the remaining n things taking $r-1$ at a time which will be

$${}^{n}C_{r-1} \qquad (1)$$

Now we shall find the combinations in which a particular thing never occurs. We shall suppose that we have only n things from which we have to form combinations of r taken at a time which will be ${}^{n}C_r$ (2)

Clearly the sum of the combinations formed in the above two

Since this pair does not satisfy (d), there is no solution in this case

Combining the cases (a) to (d), the given system of equations has the following three solutions

$$x_1=0,\ y_1=1\ ,\ x_2=y_2=\frac{1\ \ \sqrt{5}}{2}\ ,\ x_3=1,\ y_3=0$$

39 It is evident that the function $2\cos^2(\frac{1}{2}x)\ \sin^2 x$ attains its greatest value at $x=\frac{\pi}{2}$ so that its greatest value is

$$2\cos^2\frac{\pi}{4}\sin\ \frac{\pi}{2}=1$$

Thus we have $2\cos^2\frac{1}{2}x\ \sin^2 x \leqslant 1$

Again since arithmetic mean of two positive numbers greater than or equal to their geometric mean, we have

$$\frac{x^2+x^{-2}}{2}\quad\sqrt{(x^2\,x^{-2})}\ i\,e\ \ x^2+x^{-2}\geqslant 2$$

Thus the left hand member of the given equation is $\leqslant$ whereas the right hand member $\geqslant 2$ It follows that th equation has no real solution Hence alternative (a) is correc

40 (i) We have $x^2+x=x(x+1)>0$ for $x>0$ and for $x<-1$ follows that $x^2+x+1>1$ for $x>0$ and for $x<-1$ Hence for these values of x, we also have

$$x^2+x+1>\sin x$$

Hence given equation has no roots in the intervals

$$x>0 \text{ and } x<-1$$

Also on the remaining interval $-1\leqslant x\leqslant 0$, the inequalities

$$x^2+x+1>0 \text{ and } \sin x\leqslant 0$$

are valid So the given equation has no solution in the interval $-1\leqslant x\leqslant 0$ as well It follows that the given equation has no real solution

(ii) By trial, we see that $x=5$ is a solution There can be no other solution since the function 7^{6-x} decreases monotonically whereas the function $x+2$ increases monotonically so that the graphs of these functions cannot intersect more than once

41 Since the value of $\cos^2\frac{x^2+x}{2}$ cannot exceed 1 for any real x, we have

ways will be ${}^{n+1}C_r$

$$\therefore \quad {}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$$

Alternative Method With the use of formula

$${}^{n+1}C_r=\frac{(n+1)\,!}{r\,!\,(n-r+1)\,!} \tag{1}$$

$${}^nC_{r-1}+{}^nC_r=\frac{n\,!}{(r-1)\,!\,(n-r+1)\,!}+\frac{n\,!}{r\,!\,(n-r)\,!}$$

Now $(n-r+1)\,!=(n-r+1)\,(n-r)\,!$

and $r\,!=r\,(r-1)\,!$

$$\text{R H S}=\frac{n\,!}{(r-1)\,!\,(n-r)\,!}\left[\frac{1}{n-r+1}+\frac{1}{r}\right]$$

$$=\frac{n\,!\,(r+n-r+1)}{(r-1)\,!\,(n-r)\,!\,(n-r+1)\,r}$$

$$=\frac{n\,!\,(n+1)}{r\,(r-1)\,!\,(n-r+1)\,(n-r)\,!}$$

$$=\frac{(n+1)\,!}{r\,!\,(n-r+1)\,!} \tag{2}$$

Hence from (1) and (2), we get

$${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$$

3 The given expression on putting $r=1, 2, 3, 4, 5$ is

$${}^{47}C_4+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3+{}^{47}C_3=A, \text{ say}$$

Combining 1st and last

$${}^{47}C_3+{}^{47}C_4={}^{48}C_4 \qquad \text{by Q 2}$$

$$A={}^{48}C_4+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3+{}^{48}C_3$$

Again combining 1st and last

$$A={}^{49}C_4+{}^{51}C_3+{}^{50}C_3+{}^{49}C_3$$

$$={}^{50}C_4+{}^{51}C_3+{}^{50}C_3={}^{51}C_4+{}^{51}C_3={}^{52}C_4$$

4 With the use of formula

$${}^nP_r=\frac{n\,!}{(n-r)\,!} \tag{1}$$

$${}^{n-1}P_r+r\,{}^{n-1}P_{r-1}=\frac{(n-1)\,!}{(n-r-1)\,!}+r\,\frac{(n-1)\,!}{(n-r)\,!}$$

$$=\frac{(n-1)\,!}{(n-r-1)\,!}\left[1+\frac{r}{n-r}\right]$$

$[\because (n-r)\,!=(n-r)\,(n-r-1)\,!]$

$$=\frac{n\,(n-1)\,!}{(n-r)\,(n-r-1)\,!}=\frac{n\,!}{(n-r)\,!} \tag{2}$$

$$2\cos^2 \frac{x^2+x}{2} \leqslant 2 \quad (1)$$

On the other hand, we have

$$\frac{2^x+2^{-x}}{2} \geqslant \sqrt{(2^x\, 2^{-x})} = 1 \qquad [\text{ A M } > \text{G M}]$$

or $$2^x+2^{-x} \geqslant 2 \quad (2)$$

From (1) and (2) we see that the left and right members of the given equation are equal if and only if they are both equal to 2 In other words the following system of *two* equations in *one* unknown must hold

$$2\cos^2\left(\frac{x^2+x}{2}\right)=2, \quad 2^x+2^{-x}=2$$

The equation $2^x+2^{-x}=2$ has the unique solution $x=0$ This value of x also satisfies

$$2\cos^2\left(\frac{x^2+x}{2}\right)=2$$

Hence the only solution of the given equation is at $x=0$

42. (i) We have

$$5-2\sqrt{6}=\frac{(5+2\sqrt{6})\,(5-2\sqrt{6})}{5+2\sqrt{6}}=\frac{25-24}{5+2\sqrt{6}}=\frac{1}{5+2\sqrt{6}}$$

Now put $(5+2\sqrt{6})^{x^2-3}=y$ Then $(5-2\sqrt{6})^{x^2-3}=1/y$ so that the equation transforms into

$$y+\frac{1}{y}=10 \quad \text{or} \quad y^2-10y+1=0$$

or $$y=\frac{10\pm\sqrt{(96)}}{2}=\frac{10\pm 4\sqrt{6}}{2}=5\pm 2\sqrt{6}$$

or $$(5+2\sqrt{6})^{(x^2-3)}=5\pm 2\sqrt{6}$$

Taking $+$sign, we get $x^2-3=1$ which gives $x=\pm 2$ Taking $-$sign, we get $x^2-3=-1$

$\left[\text{Note that } 5-2\sqrt{6}=\frac{1}{5+2\sqrt{6}}=(5+2\sqrt{6})^{-1}\right]$

This gives $x=\pm\sqrt{2}$.

Hence the solution set is given by

$$x=\pm 2,\ \pm\sqrt{2}.$$

(ii) Ans $x=-2,\ 2$

Hence from (1) and (2)

$$^nP_r = {^{n-1}P_r} + r\,{^{n-1}P_{r-1}}$$

Without the use of formula

As in Q 2 we shall divide the number of permutations of n dissimilar things taken r at a time *i e* ^nP_r into two parts Suppose a particular thing is not to be included, then the number of permutations of the remaining $(n-1)$ things taken r at a time will be $^{n-1}P_r$ Now suppose that a particular thing x is always to be included then the number of permutations of $(r-1)$ things out of the remaining $(n-1)$ things will be $^{n-1}P_{r-1}$ But in permutations we have to note the order of things This particular thing x which is always to be included in each of the $^{n-1}P_{r-1}$ ways can be placed in any of the r places Thus there will be $r\,{^{n-1}P_{r-1}}$ permutations in this case Clearly the sum of the permutations formed in the above two ways is equal to ^nP_r

$$^nP_r = {^{n-1}P_r} + r\,{^{n-1}P_{r-1}}$$

5 $^nP_r = n\,{^{n-1}P_{r-1}}$

$$\text{R H S} = n\frac{(n-1)\,!}{(n-r)\,!} = \frac{n\,!}{(n-r)\,!} = {^nP_r} = \text{L H S}$$

Without the use of formula

Let there be n things $a_1, a_2 \quad a_n$ If we put a_1 in the first place then from the remaining $(n-1)$ things we have to choose $(r-1)$ and the number of permutations will be $^{n-1}P_{r-1}$ Similarly if we put a_2 in the first place then again the number of permutations will be $^{n-1}P_{r-1}$ Since each of the n things can occupy the first place therefore the total number of permutations will be $n\,{^{n-1}P_{r-1}}$ This will be equal to the number of permutations of n things taken r at a time

6 $^nC_{10} = {^nC_{15}}$

We know that

$$^nC_r = {^nC_{n-r}} \quad \text{or} \quad \text{if } {^nC_x} = {^nC_y} \text{ then } x+y=n$$

$$^nC_{10} = {^nC_{n-10}} = {^nC_{15}} \qquad n-10=15$$

or $\quad n=25$

$$^{27}C_n = {^{27}C_5} = \frac{27\,!}{2\,!\;25\,!}$$

$$= \frac{27\times26\times25\,!}{2\,!\;25\,!} = 27\times13 = 351$$

43 Case I Let $x > a$ i e $x-a=+\text{ive}$ then

$x^2-2a(x-a)-3a^2=0$ or $x^2-2ax-a^2=0$

$x=a(1\pm\sqrt{2})$

$x-a=\pm a\sqrt{2}$ But as $a \leqslant 0$, therefore we must choose

$x-a=-a\sqrt{2}=+\text{ive}$

Hence out of two roots $x=a(1-\sqrt{2})$ satisfies the given condition

Case II Let $x<a$ so that $x-a$ is $-\text{ive}$ then the given equation is

$x^2+2a(x-a)-3a^2=0$ or $x^2+2ax-5a^2=0$

$x=a(-1\pm\sqrt{6})$

$x-a=a(-2\pm\sqrt{6})=-\text{ive}$

Since a is $\leqslant 0$, above will hold good if $-2\pm\sqrt{6}$ is $+\text{ive}$ and hence we choose $-2+\sqrt{6}$ which is $+\text{ive}$

Roots are $a(1-\sqrt{2})$ $a(-1+\sqrt{6})$

44 Since $\log_a x \ \log_{10} a=\log_{10} x$ $\log_{10}\left(\frac{x}{10}\right)=\log_{10} x-1$

$\log_{100} x=\frac{1}{2}\log_{10} x$ and $\log_4 2=\frac{1}{2}\log_2 2=\frac{1}{2}$

The given equation can be written as

$$\frac{6}{5}\ a^{\log_{10} x \log_a 5}-3^{(\log_{10} x-1)}=9^{\frac{1}{2}\log_{10} x+\frac{1}{2}}$$

or $$\frac{6}{5}\left(a^{\log_a 5}\right)^{\log_{10} x}=3^{\log_{10} x}\,3^{-1}+3^{\log_{10} x}\,3$$

or $$\frac{6}{5}\ 5^{\log_{10} x}=3^{\log_{10} x}\left(\frac{1}{3}+3\right)=\frac{10}{3}\ 3^{\log_{10} x}$$

$$\frac{1}{5^2}\ 5^{\log_{10} x}=\frac{1}{3^2}\ 3^{\log_{10} x}$$

or $$5^{(\log_{10} x-2)}=3^{(\log_{10} x-2)}$$

Since the basis are different, above will hold only when

$\log_{10} x-2=0$ $x=10^2=100$

45 $x^2+4x+3=(x+3)(x+1)=\{x-(-3)\}\{x-(-1)\}$

The above expression is $+\text{ive}$ when x does not lie between -3 and -1 and is $-\text{ive}$ when x lies between -3 and -1

Case I x does not lie between -3 and -1

$x^2+4x+3+2x+5=0$ or $x^2+6x+8=0$

or $(x+2)(x+4)=0$ $x=-2$ and -4 $x=-4$

7 Ans 11

8 $^{15}C_{2r}={}^{15}C_{r+3}$, $2r+r+3=15$ $r=3$

9 $^{n}C_{12}={}^{n}C_{8}$ $12+8=n$ or $n=20$

$$^{n}C_{17}={}^{20}C_{17}=\frac{20\,!}{3\,!\,17\,!}$$

$$=\frac{20\times19\times18\;17\,!}{3\;2\;1\;17\,!}=20\times19\times3=1140$$

$$^{22}C_{n}={}^{22}C_{20}=\frac{22\,!}{2\,!\,20\,!}=\frac{22\times21\times20\,!}{2\;20\,!}=231$$

10 $^{8}C_{r}-{}^{7}C_{3}={}^{7}C_{2}$

or $^{8}C_{r}={}^{7}C_{2}+{}^{7}C_{3}={}^{8}C_{3}={}^{8}C_{5}$ $r=3$ or 5

11 $\frac{^{15}C_{r}}{^{15}C_{r-1}}=\frac{11}{5}$

$$5\,\frac{(15)\,!}{r\,!\,(15-r)\,!}=11\,\frac{(15)\,!}{(r-1)\,!\,(15-r+1)!}$$

$$\frac{5}{r\,(r-1)\,!\,(15-r)\,!}=\frac{11}{(r-1)\,!\,(15-r+1)\,(15-r)\,!}$$

$5\,(15\quad r+1)=11\,r$

or $80=16r$ $r=5$

12 $^{9}P_{5}+5\,{}^{9}P_{4}={}^{10}P_{r}$

L H S $=\frac{9\,!}{4\,!}+5\,\frac{9\,!}{5\,!}$ but $\frac{5}{5\,!}=\frac{1}{4\,!}$

$$=\frac{2\;9\,!}{4\,!}=\frac{2\times5\;9\,!}{5\times4\,!}=\frac{10\;9\,!}{5\,!}=\frac{10\,!}{5\,!}$$

R H S $=\frac{10\,!}{(10-r)\,!}=\frac{10\,!}{5\,!}=$ L H S

$10-r=5$ or $r=5$

13 (a) $\frac{^{n}C_{6}}{^{n-3}C_{3}}=\frac{33}{4}$

or $$\frac{n\,!}{6\,!\,(n-6)\,!}\;\frac{3\,!\,(n-6)\,!}{(n-3)\,!}=\frac{33}{4}$$

$$\frac{n\,(n-1)\,(n-2)}{6\;5\;4}=\frac{33}{4}$$

or $n\,(n-1)\,(n-2)=30\times33$

$=11\times3\times3\times10$

$=11\times10\times9$

$=11\,(11-1)\,(11-2)$

Case II x lies between -3 and -1

$-(x^2+4x+3)+2x+5=0$ or $x^2+2x-2=0$

$x=-1+\sqrt{2}$ and $-1-\sqrt{2}$

But $-1+\sqrt{2}$ does not lie between -3 and -1 and hence $-1-\sqrt{2}$ is the required root lying between -3 and -1

Ans -4, $(-1-\sqrt{2})$

Problem Set (E)

Objective Questions

1 The solution of the equation

$$2^{3/\log_3 x}=\frac{1}{64} \text{ is}$$

(i) 3, (ii) $\frac{1}{3}$, (iii) $\frac{1}{\sqrt{3}}$, (iv) None of these

2 The solution set of $|3-x|=a$ is
(i) $\{3-a, 3+a\}$ for $a \geqslant 0$ and ϕ for $a<0$, (ii) $[3-a, 3+a]$, (iii) ϕ, (iv) none of these

3 The solution set of the equation $\log_2(3-x)+\log_2(1-x)=3$ is
(i) $\{-1, 5\}$, (ii) $\{-1\}$, (iii) $\{5\}$, (iv) ϕ

4 The solution set of the equation $x^{\log_x(1-x)^2}=9$ is
(i) $\{-2, 4\}$, (ii) $\{4\}$ (iii) $\{0, -2, 4\}$ (iv) none of these

5 The equation $3^{|3x-4|}=9^{2x-2}$ has the solution

6 The equation $3^x+4^x+5^x=6^x$ has the solution set

7 The equation $|x+2|=-2$ has
(i) Only one solution, (ii) Infinite number of solutions
(iii) No solution, (iv) None of these

8 The solution set of the equation $|x-3|=x-3$ is
(i) $]3, \infty[$ (ii) $[3, \infty]$, (iii) ϕ, (iv) The set of R of all real numbers

9 The number of solutions of the equation $|x|^2-3|x|+2=0$ is
(i) 4, (ii) 1, (iii) 3, (iv) 2 (I I T 1982)

10 The equation $|2x-x^2-3|=1$ has
(i) Only one solution, (ii) no solution
(iii) 4 solutions, (iv) infinite no of solutions

11 The solution of the equation

Hence clearly $n=11$

(b) Ans $r=41$

(c) Ans $n=19$

14 We know that

$${}^nP_r = r!\,{}^nC_r$$

Put the values of ${}^nP_r$ and ${}^nC_r$

$$\therefore\ 604800 = r!\ 120$$

$$r! = 5040 = 2! \times 210 = 4! \times 5 \times 42$$
$$= 5!\ 6 \times 7 = 6! \times 7 = 7!$$
$$r = 7$$

15 $5\ {}^{2n+1}P_{n-1} = 3\ {}^{2n-1}P_n$ by given condition

$$5\frac{(2n+1)!}{(n+2)!} = 3\ \frac{(2n-1)!}{(n-1)!}$$

$$5\ \frac{(2n+1)\ 2n\ (2n-1)!}{(n+2)\ (n+1)\ n\ (n-1)!} = 3\ \frac{(2n-1)!}{(n-1)!}$$

or $10\ (2n+1) = 3\ (n+2)\ (n+1)$

or $20n + 10 = 3n^2 + 9n + 6$

or $3n^2 - 11n - 4 = 0$ $(n-4)\ (3n+1) = 0$ $n=4$

16 ${}^nP_r = {}^nP_{r+1}$

$$\frac{n!}{(n-r)!} = \frac{n!}{(n-r-1)!} \text{ or } \frac{1}{(n-r)} = 1 \text{ or } n-r=1 \quad (1)$$

${}^nC_r = {}^nC_{r-1}$ $r + r - 1 = n$ or $2r - n = 1$ (2)

Solving (1) and (2), we get $r=2$ and $n=3$

17 R H S ${}^{4n}C_{2n} - {}^{2n}C_n$

$$= \frac{4n!}{2n!\ 2n!}\ \ \frac{n!\ n!}{2n!} = \frac{4n!}{2n!}\left[\frac{n!}{2n!}\right]^2 \quad (1)$$

or $4n! = 1\ 2\ 3\ 4\ 5\ \ (4n-1)\ (4n)$

$= (2\ 4\ 6\ \ 4n)\ [1\ 3\ 5\ \ (4n-1)]$

$= 2^{2n}\ (1\ 2\ 3\ \ 2n)\ [1\ 3\ 5\ \ (4n-1)]$

$= 2^{2n}\ 2n!\ [1\ 3\ 5\ \ (4n-1)]$

$$\frac{4n!}{2n!} = 2^{2n}\ [1\ 3\ 5\ \ (4n-1)] \quad (2)$$

Now $2n! = 1\ 2\ 3\ \ 2n = (1\ 3\ 5\ \ 2n-1)\ (2\ 4\ 6\ \ 2n)$

$= [1\ 3\ 5\ \ (2n-1)]\ [2^n\ [1\ 2\ 3\ \ n)]$

$$2^n \frac{1}{[1\ 3\ 5\ \ (2n-1)]} = \frac{n!}{2n!} \quad (3)$$

Hence from (1) by the help of (2) and (3),

$$\text{L H S} = 2^{2n}\ [1\ 3\ 5\ \ (4n-1)]\ \frac{1}{2^{2n}\ [1\ 3\ 5\ \ (2n-1)]^2}$$

$|x| - 2|x+1| + 3|x+2| = 0$ is

12 The solution set of the equation
$||x-1| + 2| = 1$ is

Answers

1 (iii) Hint $\frac{3}{\log_1 x} = -6$ 2 (i) 3 (ii) Hint The equation will be meaningful if $x < 1$ The equation reduces to $\log_2 (3-x)(1-x) = 3$ from which we obtain $(3-x)(1-x) = 2^3 = 8$ or $x^2 - 4x - 5 = 0$ Its roots are -1 and 5 The root 5 is rejected since it does not satisfy the condition $x < 1$

4 (ii), 5 8/7 6 {3}, 7 (iii) 8 (ii) 9 (i), 10 (ii)
11 {−2}, 12 ϕ

$$= \frac{[1\ 3\ 5\ \ (4n-1)]}{[1\ 3\ 5\ \ (2n\ \ 1)]^2}$$

18 $({}^{15}C_8 + {}^{1}\ C_9) - ({}^{15}C_8 + {}^{15}C_7)$

$= {}^{16}C_9 - {}^{16}C_7$

$= {}^{16}C_9 - {}^{16}C_{16-7} = {}^{15}C_9 - {}^{16}C_9 = 0, \qquad {}^{n}C_r = {}^{n}C_{n-r}$

19 L H S $= {}^{n}C_r + {}^{n}C_{r-1} + C_{r-1} + {}^{n}C_{r-2}$

$= {}^{n+1}C_r + {}^{n+1}C_{r-1} = {}^{n+2}C_r$ by Q 2

20 ${}^{n-1}C_3 + {}^{n-1}C_4 > {}^{n}C_3$

$\Rightarrow \quad {}^{n}C_4 > {}^{n}C_3$

$\Rightarrow \quad \frac{n\,!}{(n\ \ 4)\,!\,4\,!} > \frac{n\,!}{(n-3)\,!\,3\,!}$

$\Rightarrow \quad \frac{1}{4\ 3\,!\,(n-4)\,!} > \frac{1}{3\,!\,(n-3)\,(n-4)\,!} \Rightarrow \frac{1}{4} > \frac{1}{n-3}$

$\Rightarrow \quad n-3 > 4 \Rightarrow \ n > 7$

21 ${}^{28}C_{2r} - {}^{24}C_{2r-4} = \frac{225}{11}$,

$$\frac{28\,!}{(28-2r)\,!\,2r\,!} \times \frac{(2\ \ -2r)\,!\,(2r-4)\,!}{24\,!} = \frac{225}{11}$$

$$\frac{28 \times 27 \times 26 \times 25}{2r\,(2r-1)\,(2r-2)\,(2r-3)} = \frac{225}{11}$$

$$2r\,(2r-1)\,(2r-2)\,(2r-3) = \frac{11 \times 28 \times 27 \times 26 \times 25}{225}$$

$= 11 \times 28 \times 3 \times 26$

$= 11 \times 14 \times 2 \times 3 \times 13 \times 2 \qquad 25 \times 9 = 225$

$= 11 \times 12 \times 13 \times 14$

$= 14\,(14\ \ 1)\,(14\ \ 2)\,(14-3)$

$2r = 14$ or $r = 7$ Hence (*c*) is the correct answer

Problem Set (B)

Arrangements of Words

Very important question from the point of view of practice

1 How may different words can be formed from the letters of the word GANESHPURI when

(a) All the letters are taken

(b) The letter *G* always occupies the first place

c) The letters *P* and *I* respectively occupy the first and last places

(d) The vowels are always together

(e) The letters *E, H, P* are never together

7
Permutations and Combinations

§ 1 Definitions

Permutation Each of the different arrangements which can be made by taking some or all of a number of things is called a permutation

Combination Each of the different groups or selections which can be made by taking some or all of a number of things (irrespective of order) is called a combination

Illustration Suppose there are 4 questions marked as 1, 2, 3, 4 and out of these you are to select any three You can choose them as 123, 124, 341, 342 *i e* 4 ways of selection But on the other hand if you consider the order also in which you attempt the questions of each selection then it would be permutation

Consider the selection 123 These three question can be arranged as 123 132, 231, 213 312, 321 *i e* 6 arrangements for *one selection Since there are 4 selections, hence the total* number of, arrangements of the three questions will be $4\times6=24$ permutations which is therefore the no of arrangements of 3 things taken out of four

Fundamental Theorem

If there are m ways of doing a thing and for each of the m ways there are associated n ways of doing a second thing then *the total number of ways of doing the two things will be* mn

As an example suppose six subjects are to be taught in four periods For the first period we can put any of the six subjects *i e* there are 6 ways of filling the first period For the second period we are left with remaining subjects and hence there are 5 ways of filling the second period Hecne the number of ways in which first two periods can be filled up is $6\times5=30$ ways

Important results

(a) Number of permutations of n dissimilar things taken r at a time

$${}^nP_r=\frac{n!}{(n-r)!}=n(n-1)(n-2)\quad(n-r+1)$$

The last factor is $\{n-(r-1)\}=n-r+1$

(f) The vowels always occupy even places (*i e* 2nd, 4th etc)

(g) How many words of 5 letters each can be formed each containing 3 consonants and 2 vowels ?

2 (a) Show that the number of permutations of n different things taken all at a time in which p particular things are never together is $n! - (n - p + 1)!\, p!$

(b) How many words can be formed out of the letters of the word ARTICLE so that vowels occupy the even places

3 In how many other ways can the letters of the word SIMPLETON be rearranged

4 How many different words ending and beginning with a consonant can be made out of the letters of the word EQUATION

5 How many different words can be formed with the letters of the word ORDINATE so that

(a) The vowels occupy odd places

(b) Beginning with O

(c) Beginning with O and ending with E

6 How many different words can be formed from the letters of the word INTERMEDIATE ? In how many of them two vowels never come together

7 Find the number of ways of arranging the letters

AAAAA BBB CCC D EE F

in a row if the letters C are separated from one another

8 In how many ways can the letters of the word ARRANGE be arranged so that

(i) The two R's are never together

(ii) The two A s are together but not two R s

(iii) Neither two A s nor the two R s are together

9 (a) How many different arrangements can be made by using all the letters in the word MATHEMATICS ? How many of them begin with C ? How many of them begin with T ?

(b) How many words can be formed by taking 4 letters at a time out of the letters of the word MATHEMATICS

(c) Find the number of ways in which (a) selection (b) an arrangement of 4 letters can be made from the letters of the word PROPORTION

10 How many different permutations can be formed from the letters of the word EXAMINATION taken four at a time

16 An aeroplane flying horizontally 1 km above the ground is observed at an elevation of 60° If after 10 seconds the elevation is observed to be 30°, find the uniform speed per hour of the aeroplane (I I T 65)

17 (*a*) An object is observed from three points A, B, C in the same horizontal line passing through the base of the object The angle of elevation at B is twice and at C thrice that at A If $AB=a$, $BC=b$, *prove that the height* of the object is

$$\frac{a}{2b}\sqrt{\left[(a+b)(3b-a)\right]}$$

(Dhanbad 88, Roorkee 80, I I T 79)

(*b*) A man observes a tower AB of height h from a point P on the ground He moves a distance d towards the foot of the tower and finds that the angle of elevation is doubled He further moves a distance $3d/4$ in the *same direction and finds that the angle of elevation is* three times that of P Prove that $36h^2=35d^2$

(I I T 86)

18 The top of a tower is observed from three points A, B, C on a straight line leading to the tower If the angles of elevation are θ, 2θ 3θ from them prove that

$$AB \quad BC \quad \cot\theta-\cot 2\theta \quad \cot 2\theta-\cot 3\theta$$

If C is found to be at the foot of the tower prove that B trisects AC

19 (*a*) The angles *of elevation of the top* and bottom of a flag staff fixed at the top of a tower at a point distant a ft from the foot of a tower are α and β Prove that the height of the flag staff is $a(\tan\alpha-\tan\beta)$

(*b*) From the top of a cliff 200 ft high the angles of depression of the top and bottom of a tower are observed to be 30° and 60° respectively Find the height of the tower

(*c*) From the top of a spire the angles of depression of the top and bottom of a tower of height h are θ and ϕ Show *that the height of the spire and its horizontal distance* from the tower are respectively

$$h\ \frac{\cos\theta\sin\phi}{\sin(\phi-\theta)} \text{ and } \frac{h\cos\theta\cos\phi}{\sin(\phi-\theta)}$$

20 (*a*) The angular depressions of the top and the foot of a chimney as seen from the top of a second chimney which is

150 meters high and standing on the same level as the first are θ and ϕ respectively. Find the distance between their tops when

$$\tan\theta=\frac{4}{3} \text{ and } \tan\phi=\frac{5}{2}$$

(IIT 65)

20 (b) From the bottom of a pole of height h, the angle of elevation of the top of the tower is α. The pole subtends angle β at the top of the tower. Obtain the height of the tower. (Roorkee 88)

21 AB is vertical pole. The end A is on the level ground. C is the middle point of AB and P is a point on the level ground. The portion CB subtends an angle β at P. If $AP=nAB$, then show that

$$\tan\beta=\frac{n}{2n^2+1}$$

(IIT 80)

22 A baloon moving in a straight line passes vertically above two points A and B on a horizontal plane 1000 ft apart. When above A it has an altitude of 60° as seen from B and when above B it has an altitude of 45° as seen from A. Find the distance from A of the point at which it will touch the plane. (IIT 59)

23 A baloon moving in a straight line passes vertically above two points A and B on a horizontal plane 1000 ft apart. When above A it has an altitude of 60° as seen from B, and when above B, 3 ° as seen from A. Find the distance from A at which it will strike the plane. (IIT 61)

24 (a) A vertical tower stands on a horizontal plane and is surmounted by a vertical flag staff of height h. At a point P on the plane the angle of elevation of the bottom of the flag staff is β and that of the top is α. Prove that the height of the tower is

$$\frac{h\tan\beta}{\tan\alpha-\tan\beta}$$

(b) A chimney 20 meters high standing on the top of a building subtends an angle whose tangent is $\frac{1}{6}$ at a distance 70 meters from the foot of the building. Find the height of the building.

(c) A statue 20 m high standing at the top of a column 150 m high on the bank of a river, subtends at a point on the opposite bank directly facing the column, the same angle as ubtended at the same point by aman 2m high

11 How many different words can be formed out of the letters of the word MORADABAD taken four at a time

12 Find the number of different permutations of the letters of the word BANANA (IIT 63)

13 How many different words can be formed out of the letters of the word ALLAHABAD In how many of them the vowels occupy the even positions

14 How many different words can be formed with the letters of the word HARYANA In how many of these H and N are together and how many of these begin with H and end with N

15 Prove that the number of words which can be formed out of the letters a, b, c, d, e, f taken 3 together, each word containing one vowel at least is 96

16 A person wishes to make up as many different parties as he can out of 20 friends, each party consisting of the same number How many should he invite at a time In how many of these would the same man be found

17 A box contains two white balls, three black balls and four red balls In how many ways can three balls be drawn from the box if at least one black ball is to be included in the draw ? (I.I T 86)

Solution to Problems Set (B)

1 GANESHPURI No of letters = 10, vowels are AEUI i e 4 and consonants 6

(a) When all the letters are taken, then they can be arranged in ${}^{10}P_{10}$ i e 10 ! ways

(b) Having fixed up G at first place we are left with the permutations of remaining 9 letters which can be arranged in 9 ! ways

(c) P occupies Ist and I occupies the last place and so we have to arrange the remaining 8 letters which can be done in 8 ! ways

(d) The four vowels *AEUI* are to be together Take these four letters as one letter and so we have in all $10-4+1=7$ letters which can be arranged in 7 ! ways In each of these 7 ! arrangements the four vowels are together and these can be arranged amongst themselves in 4 ! ways Hence by fundamental theorem the number of word will be 7 ! 4 ! in which the vowels will be together

18 How many natural numbers smaller than 10^8 are there, in the decimal notation of which all the digits are different ?

19 How many seven digit numbers are there the sum of whose digits is even ?

Solution of Problem Set (C)

1 3, 4, 5, 6, 7, 8 $\quad 3 \mid \dfrac{\text{out of } 4}{\text{any two}} \mid 5$

Since the number is to be divisible by 5, it must have 5 in the last place Again as it is to lie between 3000 and 4000 it must have 3 in the first place The number must contain only 4 digits The first and last digits have been fixed by 3 and 5 respectively so we have now to choose 2 digits out of the remaining 4

Hence the number of arrangements is

$$^4P_2=\frac{4!}{2!}=12$$

2 Here we have to use all the six digits 4, 5, 6 7, 8, 9

Number of six digit numbers $=6!=720$

Now 5 numbers are divisible by 5 as 5 is fixed in the last place

Hence the number of numbers which are not divisible by 5 is

$6!-5!=720-120=600$

3 3, 1, 7, 0, 9, 5 *i e* 6 digits

The total numbers of 6 digit numbers $=6!-5!=600$

[Note that 5! numbers are for those having 0 in first place which will be excluded

(i) $5!=120$ as explained above

(ii) Divisible by 5

They will have zero in the last place and hence the remaining 5 can be arranged in $5!=120$ ways

They may have 5 in the last place and as above we will have $5!=120$ ways These will also include numbers which will have zero in the first place Therefore the numbers having zero in 1st and 5 in last place will be $4!$

Therefore 6 digit numbers having 5 in the end will be

$5!-4!=120-24=96$

Therefore the total number of 2 digit numbers divisible by 5 is

$120+96=216$

(e) Take E, H, P as one letter and so the number of letters will be $10-3+1=8$ As in part (*d*), the number of words in which E, H, P are together will be $8!\ 3!$ The total number of arrangements by (a) is $10!$ Hence the number of words when E, H, P are never together is $10!-8!\,3!$

$8!\,[10\times9-6]=84\ 8!$

(f) We have ten places out of which 5 places are odd *i e* Ist, 3rd, 5th, 7th, 9th and five are even *i e* 2nd, fourth, sixth eighth and tenth In the five even places we have to fix up 4 vowels which can be done in ${}^5P_4$ ways Having fixed up the vowels in even places, we will be left with six places namely 5 odd and one even left after fixing the four vowels In these six places we have to fix six consonants which can be done in ${}^6P_6$ *i e* $6!$ ways

Thus the total number of ways is ${}^5P_4\times{}^6P_6$

or $5!\times6!=120\times720=6\ (120)^2$

(g) Each word is to contain 3 consonants out of 6 and 2 vowels out of 4

We can select them in ${}^6C_3$ and ${}^4C_2$ ways Thus the total number of combinations (groups) of 5 letters will be ${}^6C_3\times{}^4C_2$ by fundamental theorem

$$\text{But } {}^6C_3\times{}^4C_2=\frac{6!}{3!\,3!}\times\frac{4!}{2!\,2!}=\frac{6\times5\times4}{3\,2\,1}\ \frac{4\ 3}{1\ 2}=120 \text{ ways}$$

Thus we have 120 groups each containing 5 words *i e* 3 consonants and 2 vowels Now the 5 letters in each group can be arranged amongst themselves in $5!$ ways *i e* 120 ways Hence the total number of different words will be $120\times120=14400$, by fundamental theorem,

2 (a) Refer Q 1 (*a*) and (*e*) Total is $n!$ and when they are together is $(n\ \ p+1)!\ p!$ When p things are never together is

$$n!-(n-p+1)!\,p!$$

(b) See Q 1 (f) seven places 3 even and 4 odd, 3 vow 4 cons

$${}^3P_3\times{}^4P_4=3!\times4!=6\times24=144$$

3 Nine letters Total no of words $9!$

Hence no of other words is $9!-1$

We have excluded one arrangement *i e* Simpleton as we want other ways

(iii) Not divisible by 5

$=$ Total $-$ (divisible by 5)

$=600-216=384$

4 0, 1, 2, ..., 9 Ten digits

Then required number of 4 digit numbers

$={}^{10}P_4-{}^9P_3$

where ${}^9P_3$ corresponds to those numbers which will have zero in the first place

$$=\frac{10\times9\times8\times7\times6\,!}{6\,!}-\frac{9\times8\times7\times6\,!}{6\,!}$$

$=9\times8\times7\,(10-1)=81\times56=4536$

5 ${}^{10}P_9-{}^9P_8=\frac{10\,!}{1\,!}-\frac{9\,!}{1\,!}=9\,!\,(10-1)=9\,(9\,!)$

$=9\times(9\times8\times7\times6\,!)$

$=81\times56\times720=3265920$

Alternative The number is to be of 9 digits The first place can be filled in 9 ways only (as zero can not be in the first place) Having filled up the first place the remaining 8 places can be filled up by the remaining 9 digits in ${}^9P_8=9\,!$ ways Hence the total is

$9\times9\,!$

6 1, 3, 5, 7, 9 five digits

No of numbers $=5\,!=120$

We have to find the sum of these 120 numbers

Suppose 9 is in the first place then the remaining 4 can be arranged in $4\,!=24$ ways Similarly other digits can occupy the first place in 24 ways

Hence sum due to the unit place of all the 120 numbers

$=24\,(9+7+5+3+1)$ units $=24\times25$ units $=600$ units

Again suppose 9 is in the 2nd place *i e* ten s place and it will be so in 24 numbers Similarly each number will be in tens place 24 times Hence the sum of digits due to ten s place of all the 120 numbers is

$24\,(9+7+5+3+1)$ tens $=24\times25$ tens $=600$ tens

$=6000$

Proceeding exactly for hundred, thousands and ten thousands, we have the sum of the numbers

$=600\,(1+10+100+1000+10{,}000)=600\,(11111)$

$=6666600$

4 8 letters *i e* 3 consonants and 5 vowels
The consonants are to occupy 1st and last place and it can be done in ${}^3P_2$ ways We will now be left with 5 vowels and 1 consonant *i e* 6 letters which can be arranged in 6 ! ways Hence the number of words under given condition is

$${}^3P_2 \times 6\,! = 6 \times 720 = 4320$$

5 4 vowels and 4 consonants Total 8 letters
(a) No of words $= 4\,! \times 4\,! = 24 \times 24 = 576$,
because 4 vowels are to be adjusted in 4 odd place and the 4 consonants in the remaining 4 even places
(b) 7 ! ways, *O* being fixed
(c) 6 ! ways *O* fixed in 1st and *E* fixed in last

6 No of letters $= 12$, 6 vowels $(2I, 3E, 1A)$ and 6 consonants $(2T, 1R, 1M, 1N, 1D)$
Since 2 letters are of one kind, 3 of another kind, 2 of another kind and hence by (e) (Page 284) the number of words is

$$\frac{(12)\,!}{2\,!\,3\,!\,2\,!} = 19958400$$

Two vowels do not come together
No of ways of arranging 6 consonants (2 alike) is

$$\frac{6\,!}{2} = 6 \times 5 \times 4 \times 3 = 360 \qquad (1)$$

Let us fix up the consonants and one of the above ways be as under $\times T \times T \times R \times M \times D \times N \times$
in which there are seven blank places $(\times)$ which are to be filled by 6 vowels Now these six places can be selected in ${}^7C_6 = 7$ ways ( )
In each of these ways 6 vowels out of which 2 are alike of one kind and 3 alike of the other kind can be filled up in

$$\frac{6\,!}{3\,!\,2\,!} = \frac{6 \times 5 \times 4}{2\,!} = 60 \text{ ways} \qquad (3)$$

Hence the total number of ways when the two vowels never come together by fundamental theorem is

$$360 \times 7 \times 60 = 151200 \text{ by (1), (2) and ( )}$$

7 As in Q 6 there are 15 letters Let us ignore $3C$'s and thus we have 12 letters $(5A'\ 3B'\ 2E', 1D\ \ 1F)$ and these can be arranged in $\frac{12\,!}{5\,!\,3\,!\,2\,!}$ ways (1)
Now after arranging these 12 letters (No C's) there will be 13 gaps as in Q 6 in which 3 different letters can be arranged in

7 The digits are 1, 2, 3, 4, 5 We have to form numbers greater than 23000

Required number will be

= Total (those beginning with 1) − (those beginning with 21)

$= 5! - 4! - 3! = 120 - (24) - (6) = 90$

8 The digits are 1, 0, 2, 3 We have to form numbers greater than 1000

Required no $4! - 3!$

(for those having 0 in the 1st place)

$= 24 - 6 = 18$

9 There are 4 odd places and there are 4 odd numbers (2 are alike *i e* 1, 1, and 2 are alike *i e* 3, 3) These can be arranged in four places in

$$\frac{4!}{2!\,2!} = \frac{4\times 3}{2} = 6 \text{ ways}$$

There will be 3 even places namely 2nd, 4th and 6th in which 3 even numbers (2 are alike *i e* 2, 2) These can be arranged in

$$\frac{3!}{2!} = \frac{3\times 2}{2} = 3 \text{ ways}$$

Hence the total number of the numbers thus formed is

$6\times 3 = 18$

10 Digits are 2, 3, 7, 0, 8, 6 *i e* six in all

We have to form numbers between 99 and 1000 Clearly they will be of three digits and their number will be

$$^6P_3 = \frac{6!}{3!} = 6\times 5\times 4 = 120$$

Out of these we have to exclude those numbers of 3 digits which have zero in the first place

Their number is $^5P_2 = \frac{5!}{3!} = 5\times 4 = 20$

The required number

$= 120 - 20 = 100$

11 1, 2, 3, 4, 5 5 digits

(i) Even numbers will have 2 in the last place

$4! = 24$

Similarly they will have 4 in the last place

$4! = 24$

$^{13}P_3$ ways But since the 3 letters are alike the number of distinct ways will be

$$\frac{1}{3!}(^{13}P_3)=\frac{1}{6}\ \frac{13!}{10!} \quad (2)$$

∴ The total number of words in which C's are never together is

$$\frac{12!}{5!\times 3!\times 2!}\times\frac{1}{6}\ \frac{13!}{10!}=\frac{12\times 11\times 10\times 9\times 8\times 7\times 6}{6\times 2}$$

$$\times\frac{1}{6}(13\times 12\times 11)=95135040$$

Alternative Method

There are 15 letters 5A's, 3B's 3C's, 2E's, 1D, 1F

Hence the number of different words will be

$$\frac{(15)!}{5!\,3!\,3!\,2!} \quad (1)$$

Now treat C as one word so that there are now 15−3+1=13 words out of which 5A's and 3B's and 2E's are alike and hence the number of words is $\frac{(13)!}{5!\ 3!\,2!}$ in which all the C's are together

Hence the number of ways when C's are not together is

$$\frac{(15)!}{5!\,3!\,3!\,2!}-\frac{(13)!}{5!\,3!\,2!}$$

8 There are 7 words 2A's, 2R's and 3 different

(i) **Two R's never together**

Total number of words will be

$$\frac{7!}{2!\,2!}=\frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{2\times 2}=1260 \quad (1)$$

Now treat the two R's together and the number of words will be 7−2+1=6 out of which 2A's are alike Hence the number of words when 2R's are together will be

$$\frac{6!}{2!}=6\times 5\times 4\times 3=360 \quad (2)$$

Hence from (1) and (2) the number of words when 2R's are never together will be 1260 − 360 = 900

2nd Method

Ignore the 2R's and thus we have to arrange remaining 5 words in which 2A's are alike and we can have

$$\frac{5!}{2!}=5\times 4\times 3=60 \text{ words} \quad (1)$$

(ii) Numbers less than 40,000 will have either 1 or 2 or 3 in the first place and hence as above total of such numbers will be $24+24+24=72$.

12 1, 2, 3, 4, 5, 6, 7 7 digits in all and we have to use 4 digits

(a) ${}^7P_4=\frac{7!}{3!}=7\times6\times5\times4=840$

(b) Numbers greater than 3400 will have, 4 or 5 or 6 or 7 in the first place *i e* there are 4 ways of filling the first place Having filled the first place say by 4 we have to choose 3 digits out of the remaining 6 and the number will be

$${}^6P_3=\frac{6!}{3!}=6\times5\times4=120$$

Therefore total of such numbers by fundamental theorem will be $4\times120=480$ *(1)*
Numbers greater than 3400 can also be those which have 34, 35, 36, 37 in the first two places
Having filled up 34 in the first two places we will have to choose 2 more out of remaining 5 and the number will be

$${}^5P_2=\frac{5!}{3!}=5\times4=20$$

Therefore total as above will be
$20\times4=80$ (2)

Hence all the numbers greater than 3400 will be
$480+80=560$ by (1) and (2)

Alternative Easier Method

Numbers less than 3400 will have 1 or 2 in 1st place or 31, 32 in the first two positions

$${}^6P_3+{}^6P_3=120+120=240$$
$${}^5P_2+{}^5P_2=20+20=40$$

Total which are less than $3400=240+40=280$
Also from part (a) total number of numbers formed is

$${}^7P_4=840$$

Hence numbers greater than 3400 is
$840-280=560$

(c) The numbers will be divisible by 2 if the last digit is divisible by 2 which can be done in 3 ways by fixing 2 or 4 or 6, and the remaining 3 places can be filled up out of remaining 6 digits in ${}^6P_3$ ways Hence the required no $=3\times{}^6P_3=3\times120=360$

After arranging these five letters in a line we have 6 gaps in which 2 different words can be placed in $^6P_2$ ways But these two R' are alike and hence the number of words will be

$$\frac{1}{2!}\,{}^6P_2=\frac{1}{2!}\,\frac{6!}{4!}=15$$

Hence the number of ways when R' are never together is
$60\times15=900$

(ii) **The two A' are together but not two R'**

Treat 2A' together so that we have $7-2+1=6$ words in which $2R'$ are alike and the number of arrangements will be

$$\frac{6!}{2!}=6\times5\times4\times3=360 \qquad (1)$$

In each of the above arrangements there are 6 words ($2A'$ counted as one) Now treat the $2R'$ as one word so that we have now $6-2+1=5$ words only which are all different and these can be arranged in $5!=120$ ways (1) gives A' together $=360$, (2) gives A' together and R' together$=120$ Hence when A' are together but R' are not together is $360-120=240$

2nd Method

Let us ignore the two R' and thus we have $7-2=5$ letters Treat the $2A'$ as one letter $(AA)\ NGE$ thus we have 4 letters which can be arranged in $4!=24$ ways (1)

$$\times AA\times N\times G\times E\times$$

Between these 4 letters (A' together) we have 5 gaps in which 2 different letters can be arranged in $^5P_2$ ways

But here $2R'$ are alike and hence the number of ways will be

$$\frac{1}{2!}\,{}^5P_2=\frac{1}{2}\,\frac{5!}{3!}=10 \text{ ways} \qquad (2)$$

Hence the number of words when A' are together and R' are not together is $24\times10=240$ by fundamental theorem

(iii) **Neither 2A' nor 2R' are together**

From Part (i) $2R'$ are never together in 900 ways

This includes the ways when $2A'$ may be together and may not be together

From Part (ii), no of ways that $2A'$ are together but not $2R'=240$

Hence the number of ways when neither $2A'$ nor $2R'$ are together is $900-240=660$

(d) A number will be divisible by 25 if the last two digits are divisible by 25 and this can be done in two ways for either 25 or 75 can be there and remaining two places out of 5 digits can be filled in $^4P_2$ ways

Hence the required number $=2\times{}^5P_2=2\times20=40$

(e) A number is divisible by 4 if the last two digits are divisible by 4 which can done in 10 ways (12, 16, 24, 32, 36, 52, 56, 64, 72, 76)

Hence number $=10\times{}^5P_2=10\times20=200$

13 0, 1 2, 3, 4, 5 Six numbers 5 are +ive and one zero

We are at liberty to use any number of digits out of the six

Single digit number are clearly 5 which are +ive (

Two digit numbers

$$^6P_2-{}^5P_1=\frac{6!}{4!}-5=30-5=25 \qquad (\text{:}$$

$^5P_1$ corresponds to the two digit numbers having 0 in the fir place

Three digit numbers

$$^6P_3-{}^5P_2=\frac{6!}{3!}-\frac{5!}{3!}=\frac{720-120}{6}=100 \qquad (3$$

Four digit numbers

$$^6P_4-{}^5P_3=\frac{6!}{2!}\ \frac{5!}{2!}=\frac{720-120}{2}=300 \qquad (4$$

5 digit numbers

$$^6P_5-{}^5P_4=6!-5!=720-120=600 \qquad (5)$$

6 digit numbers

$$^6P_6-{}^5P_5=6!-5!=720-120=600 \qquad (6)$$

Total $=600+600+300+100+25+5=1630$

2nd part Numbers greater than 3000

All the five digit numbers 600 and six digit numbers 600 will be greater than 3000 by (5) and (6)

All the four digit numbers with either 3 or 4 or 5 in the first place will be greater than 3000

Numbers of 4 digits having 3 in the first place will be

$$^5P_3=\frac{5!}{2!}=\frac{120}{2}=60$$

Similarly those having 4 and 5 in the first place

Therefore 4 digit numbers greater than 3000 is

9 (a) There are 11 word $2M'$, $2A'$, $2T'$, H, E, I, C, S

(ii) Hence the number of words by taking all at a time

$$=\frac{11!}{2!\,2!\,2!}=\frac{11\times10\times9\times8\times7\times6!}{2\times2\times2}$$

$=990\times7\times720=990\times5040$

$=4989600$

(i) **To Begin with C**

Having fixed C at first place we have 10 letters in which 2 are M', 2 are A' and 2 are T' and rest 4 different

Hence the number of words will be

$$\frac{10!}{2!\,2!\,2!}=\frac{10\times9\times8\times7\times6!}{2\times2\times2}$$

$=90\times7\times720=630\times720=453600$

(iii) **To Begin with T**

Having fixed T in the first place we will have only 10 letters out of which 2 are M' and 2 are A' and rest six are H, E, I, C, S and T

Hence the number of words is

$$\frac{10!}{2!\,2!}=907200 \quad (i\,e \text{ double of part (ii)})$$

(b) We can choose 4 letters from the 11 listed in part (a) as under

(i) **All the four different**

We have 8 different types of letters and out of these 4 can be chosen in $^8P_4=\frac{8!}{4!}=8\times7\times6\times5=1680$ (1)

(ii) **Two different and two alike**

We have 3 pairs of like letters out of which one pair can be chosen in $^3C_1=3$ ways Now we have to choose two out of the remaining 7 different types of letters which can be done in

$$={}^7C_2=\frac{7!}{5!\,2!}=\frac{7\times6}{2}=21 \text{ ways}$$

Hence the total number of groups, of 4 letters in which 2 are different and 2 are alike is $3\times21=63$ groups

Let one such group be M, H, M, I

Each such group has 4 letters out of which 2 are alike and they can be arranged amongst themselves in $\frac{4!}{2!}=12$ ways

Hence the total number of words is $63\times12=756$

Hence total numbers greater than 3000 will be

$$600+600+180=1380$$

Alternative for 4 digit numbers 180 found above

Total number of 4 digit numbers is 300 as found before Out of these we have to exclude those which have either 1 or 2 in the first place If 1 is in the first place we have to choose 3 out of remaining five and hence it will be

$$^5P_3=\frac{5!}{2!}=\frac{120}{2}=60$$

Similarly those which have 2 in the first place will be 60 Hence four digit numbers which do not have either 1 or 2 in the first place *i e* which are greater than 3000 will be

$$300-60-60=180$$

∴ Total is

$$600+600+180=1380$$

14 0, 1, 2, 3, 4 Five numbers

Numbers greater than 1000 and less than or equal to 4000 will be of four digits and will have either 1 (except 1000) or 2 or 3 in the first place or 4 in the first place with 0 in each of the remaining places

Numbers having 1 in first place After fixing 1st place, the second place can be filled by any of the 5 numbers (not 4 numbers because repetition is allowed *i e* 1 can appear again) Similarly 3rd place can be filled up in 5 ways and 4th place can be filled in 5 ways Thus there will be $5\times5\times5=125$ ways in which I will be in the first place But this includes 1000 also which does not satisfy the given condition of being greater than 1000 Hence there will be 124 numbers having 1 in the first place Similarly 125 each when 2 or 3 are in the first place Only one number with 4 in the first place is formed, namely 4000 because of the condition of being less than or equal to 000 Therefore total number of such numbers is $124+125+125+1=375$

15 1, 1 5, 9, 0 Five digits are 0 2 alike and 3 different

Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits

Let 5 be in the first place then we have to fill the remaining 4 places by remaining 4 numbers 1, 1, 9, 0 out of which 2 are alike and hence it will be

$$\frac{4!}{2!}=\frac{24}{2}=12$$

(iii) Two alike of one kind and two alike of other kind

Out of 3 pairs of like letters we can choose 2 pairs in

$^3C_2$ ways $=3$ ways

One such group is $MMAA$

These four letters out of which 2 are alike of one kind and 2 alike of other kind can be arranged in $\frac{4!}{2!2!}=6$ ways

Hence the total number of words of this type is $3\times6=18$

Therefore from (i) (ii) and (iii) the number of 4 letter words is $1680+756+18=2454$

(c) We have got $2P$'s, $2R$'s, $3O$'s, $1I$, $1T$, $1N$ i.e. 6 types of letters. We have to form words of 4 letters

We consider four cases

(i) All 4 different

Selection $^6C_4=15$ Arrangement $15\cdot4!=15\times24=360$

Alt $^6P_4=\frac{6!}{2!}=6\times5\times4\times3=360$

(ii) Two different and two alike

P's, R's and O's in $^3C_1=3$ ways. Having chosen one pair we have to choose 2 different letters out of the remaining 5 different letters in $^5C_2=10$ ways. Hence the number of selections is $10\times3=30$. Each of the above 30 selections has 4 letters out of which 2 are alike and they can be arranged in $\frac{4!}{2!}=12$ ways

Hence the number of arrangements will be $12\times30=360$

(iii) 2 Alike of one kind and 2 of other

Out of three sets of two like letters we can choose 2 sets in $^3C_2=3$ ways. Each such selection will consist of 4 letters out of which 2 are alike of one kind 2 of the other. They can be arranged in $\frac{4!}{2!\,2!}=6$ ways

Hence the number of arrangements is $3\times6=18$

(iv) 3 alike and 1 different

There is only one set consisting of 3 like letters and it can be chosen in 1 way. Now we are left with 5 different letters out of which we can choose 1 in $^5C_1=5$ ways

Hence the number of selections is $1\times5=5$

Each of the selections consists of 4 letters out of which 3 are alike and each of them can be arranged in $\frac{4!}{3!}=4$ ways

Similarly 12 is the number when 9 is in the first place
Therefore total number is $12+12=24$ which are greater than 50000

15 **Repetition allowed** 5, 6, 7, 8, 9, 0 Six digits
The first place can be filled by 6, 7, 8, 9 *i e* in 4 ways as the number is to be greater than 600000 The last place can be filled in by 5, 7, 9 *i e* 3 ways as the number is to be odd Hence the number of ways of filling the Ist and the last place is $4\times3=12$ ways
We have to fill in the remaining 4 places of the six digit number *i e* 2nd, 3rd, 4th and 5th place Since repetition is allowed each place can be filled in 6 ways Hence the 4 places can be filled in $6\times6\times6\times6=1296$ ways
Hence by fundamental theorem the total numbers will be

$$1296\times12=15552$$

Repetition not allowed

1st Place	Last (Number to be odd)
6	5, 7, 9 = 3 ways
8	5, 7, 9 = 3 ways
7	5, 9 = 2 ways
9	5, 7 = 2 ways

Total number of ways of filling the first and the last pla under given condition is $3+3+2+2=10$ ways
(It was 12 when repetition was allowed)
Having filled in the first and the last places in 10 ways, w can fill in the remaining 4 places out of 4 numbers (repetitio not allowed) in

$$4\,!=24 \text{ ways}$$

∴ Total number of ways is

$$10\times24=240$$

17 After factorizing, we write the number a as $a=2^3\,3^2\,7^2\,11$
Hence by § 1 *q* (iii), the total 9 number of divisions

$$=(3+1)\,(2+1)\,(2+1)\,(1+1)=72$$

But this includes the divisions 1 and a
Hence required number of divisions $=72-2=70$

18 **Ans** 5274
[Hint The required natural numbers consist of 4 digits, 3 digits, 2 digits and one digit so that

Hence the number of arrangements is $5\times4=20$

From (a), (b) and (c), we get

Number of Selections $=15+30+3+5=53$

Number of Arrangements $=360+360+18+20=758$

10 Exactly as Q 9 (b) Ans 2454

11 In MORADABAD, we have 6 different types of letters $3A$'s, $2D$'s and rest four different

We have to form words of 4 letters

(i) All different $^6P_4=6\times5\times4\times3=360$

(ii) Two different two alike $^2C_1\times{}^5C_2\times\frac{4!}{2!}=240$

(iii) 3 alike 1 different $^1C_1\times{}^5C_1\times\frac{4!}{3!}=20$

(iv) 2 alike of one type and 2 alike of other type

$$^2C_2\times\frac{4!}{2!\,2!}=6$$

Total number of words $=360+240+20+6=626$

12 BANANA

$3A$'s, $2N$'s B i e 6 letters, 3 alike of one type and 2 of another type Number of words taken all at a time is

$$\frac{6!}{3!\,2!}=\frac{6\times5\times4}{2}=60$$

13 ALLAHABAD

$4A$'s $2L$'s, H B D, i e 9 letters

Number of words $\frac{9}{4!\,2!}=\frac{9\times8\times7\times6\times5}{2}=72\times105=7560$

There are 4 vowels, and all are alike i e $4A$'s

There are 4 even places i e 2nd, 4th, 6th, and 8th

These 4 even positions can be filled by 4 vowels in $\frac{4!}{4!}=1$ way

Now we are left with 5 places in which 5 letters out of which $2L$'s are alike and rest different can be filled in

$$\frac{5!}{2!}=5\times4\times3=60 \text{ ways}$$

Hence the total number of words is $60\times1=60$

14 HARYANA 7 letters

$3A$'s, H, R, Y, N

19 Suppose $\alpha_1\alpha_2\alpha_3\alpha_4\alpha_5\alpha_6\alpha_7$ represents any seven digit number so that α_1 takes the value 1, 2, 9 and $\alpha_2, \alpha_3,$, α_7 all take values 0, 1, 2, 9

If keep $\alpha_1, \alpha_2,$, α_6 fixed, then the sum $\alpha_1+\alpha_2+$ $+\alpha_6$ is either even or odd Since α_7 takes 10 values 0, 1, 2, , 9, 5 of the 10 numbers so formed will be even and 5 odd

Hence the required number of numbers

$=9\ 10\ 10\ 10\ 10\ 10\ 5=45\ 0000$

Problem Set (D)

Committees and seating arrangements

1 How many committees of 5 members each can be formed from 8 official and 4 non official members in the following cases

(a) Each consisting of 3 official and 2 non official members

(b) Each contains at least two non official members

(c) A particular official member is never included

(d) A particular non official member is always included

2 There are 10 professors and 20 students out of whom a committee of 2 professors and 3 students is to be formed Find the number of ways in which this can be done Further find in how many of these committees (i) A particular professor is included (ii) A particular student is included (iii) A particular student is excluded

3 (i) From 6 gentlemen and 4 ladies a committee of 5 is to be formed In how many ways can this be done if

(a) The committee is to include at least one lady

(b) There is no restriction about its formation (I.I.T 68)

(ii) From 4 officers and 8 jawans in how many ways can 6 be chosen (a) to include exactly one officer (ii) to include at least one officer (Roorkee 1985)

4 A candidate is required to answer 6 out of 10 questions which are divided into two groups each containing 5 questions and he is not permitted to attempt more than 4 from each group In how many ways can he make up his choice (I I T 58)

5 (a) A candidate is required to answer 7 questions out of 12 questions which are divided into two groups, each containing 6 questions He is not permitted to attempt more than 5 questions from either group In how many different ways can he choose the 7 questions (I I T 66)

(i) The number of words $=\frac{7!}{3!}=7\times6\times5\times4=840$

(ii) Treating H and N together we have $7-2+1=6$ letters out of which three are alike i e A' and hence they can be arranged in $\frac{6!}{3!}=120$ ways

But H and N can be arranged amongst themselves in $2!=2$ ways

Hence the number of ways is $120\times2=240$

(iii) Fix up H in first and N in last, we have 5 letters out of these three are alike i e A' and hence the number of words is

$$\frac{5!}{3!}=5\times4=20$$

15 Hint: Reqd no of words

$$={}^3C_1\times{}^4C_2+{}^3C_2\times{}^4C_1)\ 3!=(12+4)\times6=96$$

16 Since $n=20$ is even, ${}^{20}C_r$ is greatest when $r=\frac{20}{2}=10$ Hence the maximum number of parties $={}^{20}C_{10}$ Thus he should invite 10 friends at a time in order to form the max number of parties Also the same man will be found in ${}^{19}C_9$ parties

17 The required number of ways

$$={}^3C_1\times{}^6C_2+{}^3C_2\times{}^6C_1+{}^3C_3=64$$

Problem Set (C)

Arrangement of Numbers

1 How many numbers divisible by 5 and lying between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number ?

2 How many different numbers of six digits each (without repetition of digits) can be formed from the digits 4, 5, 6, 7, 8, 9 ? How many of these are not divisible by 5

3 How many different numbers of six digits (without repetition of digits) can be formed from the digits 3, 1, 7, 0, 9 5 ?

(i) How many of them will have 0 in the unit place ?

(ii) How many of them are divisible by 5 ?

(iii) How many of them are not divisible by 5 ?

4 How many different numbers of 4 digits can be formed from the ten digits 0, 1, 2, 9, no digit being repeated in any number

(b) A question paper consists of two sections having respectively 3 and 4 quest ons The following note is given on the paper "It is not necessary to attempt all the questions One question from each section is compulsory" In how many ways can a candidate select the questions

(c) In how many ways can clear and overcast days occur in one week

(d) A student is allowed to select at most n books from a collection of $(2n+1)$ books If the total number of ways in which he can select one book is 63, find the valu· of n (I I T 87)

6 At an election, three wards of a town are canvassed by 4, 5 and 8 men respectively If 20 men volunteer in, how many ways can they be alloted to the different wards (I I T 56)

7 A committee of 6 is chosen from 10 men and 7 women so as to contain at least 3 m n and 2 women In how many different ways can this be done if two particular women refuse to serve on the same committee (I I T 57)

8 A guard of 12 men is formed from a group of n soldiers in all possible ways Find

(a) The number of times two particular soldiers A and B are together on guard

(b) The number of times three particular soldiers C, D E are together on guard

(c) Also find n if it is found that A and B are three times as often together on guard as C, D, E are

9 (a) There are 16 vacancies for clerks in a certain office 20 applications are received In how many ways can the clerks be appointed ? How many times may a particular candidate be selected

(b) To fill 12 vacancies there are 25 candidates of which 5 are from scheduled castes If 3 of the vacancies are reserved for scheduled cast candidates while the rest are open to all, find the number of ways in which the selection can be made (Roorkee 81)

10 Find the number of ways of selecting 10 clerks from 22 male 17 female applicants if the selection is to consist of either all males or all females

5 Find the total number of 9 digits numbers which have all different digits (I I T 82)

6 How many different numbers (without repetition of digits) can be formed from the digits 1, 3 5, 7 9 when taken all at a time and what is their sum ?

7 How many numbers greater than 23000 can be formed from the digits 1 2 3 4, 5 ?

8 Prove that only 18 numbers greater than 1000 can be formed from the digits 1, 0 2 3 ?

9 Prove that only 18 numbers can be formed by using all the digits 1 2 3, 4, 3, 2 1 so that the odd digits always occupy the odd places (*I I T 69*)

10 How many numbers lying between 99 and 1000 can be formed from the digits 2, 3, 7 0 8, 6 ?

11 How many numbers which are
(i) Even
(ii) Less than 40,000
can be formed by taking all the digits 1, 2, 3, 4, 5 ?

12 A number of 4 different digits is formed by using the digits 1, 2, 3, 4, 5, 6, 7 in all possible ways Find
(a) How many such numbers can be formed ?
(b) How many of them are greater than 3400 ?
(c) How many of them are exactly divisible by 2 ?
(d) How many of these are exactly divisible by 25 ?
(e) How many of these are exactly divisible by 4 ?

13 Find the number of +ive integers which can be formed by using any number of digits from 0 1 2, 3, 4, 5, but using each digit not more than once in each number How many of these integers are greater than 3000 ? (I I T 76)

14 How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4, repetition of digits being allowed ? (I I T 76)

15 How many different numbers greater than 50000 can be formed with the digits 1, 1, 5, 9, 0 ?

16 How many odd numbers, greater than 600000 can be formed from the digits 5 6, 7 8, 9, 0 if
(a) Repetitions are allowed
(b) Reptetions are not allowed

17 How many divisions are there of the numbers $a=38808$ exclusive of the divisor 1 and a

11 How many different groups can be selected for playing tennis out of 4 ladies and 3 gentlemen there being one lady and one gentleman on each side

12 In how many ways can a mixed doubles tennis game be arranged from 7 married couples, if no husband and wife play in the same game

13 Out of 8 sailors on a boat, 3 can work only on one side and 2 only on the other side In how many ways can the sailors be arranged on the boat

14 (a) Two men enter a railway compartment having 6 seats unoccupied In how many ways can they be seated

(b) In how many ways can n men be seated around a round table when in no two ways a man have the same neighbours ?

15 (a) In how many ways may 6 Hindus and 6 Muslims sit around a round table so that two Hindus may never sit together ?

(b) m men and n women are to be seated in a row so that no two women sit together If $m > n$, then show that the number of ways in which they can be seated is $\frac{m!\,(m+1)}{(m-n+1)}$

(I I T 83)

(c) Five boys and five girls form a line with the boys and girls alternating Find the number of ways of making the line In how many different ways could they form a circle such that the boys and girls alternate

(I I T 75)

16 In how many ways can 7 boys be seated at a round table so that two particular boys are (i) next to each other (ii) separated

17 A round table conference is to be held between 20 delegates of 20 countries In how many ways can they be seated if two particular delegates are (i) always to sit together (ii) or never to sit together (I I T 70)

18 20 persons were invited for a party In how many ways can they and the host be seated at a circular table ? In how many of these ways will two particular persons be seated on either side of the host (I I T 77)

standing at the base of the column Find the breadth of the river (Roorkee 77)

(*d*) A tower 51 m high has a mark at a height of 25 m from the ground Find at what distance the two parts subtend equal angles to an eye at the height of 5m from the ground

25 A ladder 20 ft long reaches a point 20 ft below the top of a flag The angle of elevation of the top of the flag at the foot of the ladder is 60° Find the height of the flag

26 The angle of elevation of the top of a tower (which is yet incomplete) at a point 120 ft from its base is 45° How much higher should it be raised so that the elevation at the same point may become 60° ?

27 A vertical pole (more than 100 ft high) consists of two portions the lower being $\frac{1}{3}$ rd of the whole If the upper portion subtends an angle $\tan^{-1}\frac{1}{2}$ at a point in a horizontal plane through the foot of the pole and distar 40 ft from it, find the height of the pole (I I T 1964)

28 On the bank of a river there is a column 200 ft high supporting a statue of 30 ft height the statue to an observer on the opposite bank subtends an equal angle with a man 6 ft high standing on the base of the column Find the breadth of the river neglecting the height of the observer's eye from the ground

29 The angles of elevation of the top of a tower standing on a horizontal plane from two points on a line passing through the foot of the tower at distances a and b respectively are complementary angles Prove that the height of the tower is $\sqrt{(ab)}$. If the line joining the two points subtends an angle θ at the top of the tower, show that

$$\sin\theta=\frac{a-b}{a+b}$$

30 A tower is observed from two stations A and B where B is East of A at a distance 100 meters The tower is due north of A and due north west of B The angles of elevation of the tower from A and B are complementary Find the height of the tower (I I T 78)

31 (*a*) At the foot of a mountain the angle of elevation of its summit is found to be α After ascending a ft towards the mountain up a slope of inclination β the angle of

elevation is found to be γ Show that the height of the mountain is

$$\frac{a \sin \alpha \sin (\gamma-\beta)}{\sin (\gamma-\alpha)} \text{ feet}$$

(b) A vertical tree stands on a hill side that makes an angle α with the horizontal From a point directly up the hill from the tree the angle of elevation of the tree top is β From a point m cm further up the hill, the angle of depression of the tree top is γ If the tree is H cm tall express H in terms of α β γ (Roorkee 1973)

(c) A and B are two points on one bank of a straight river, C and D are two points on the other bank The direction from C to D is the same as from A to B If $AB=a$, $\angle CAD=\alpha$, $\angle DAB=\beta$ $\angle CBA=\gamma$, prove that

$$CD=\frac{a \sin \alpha \sin \gamma}{\sin \beta \sin (\alpha+\beta+\gamma)}$$

32 (a) A vertical tower stands on a declivity which is inclined at 15° to the horizon From the foot of the tower a man ascends the declivity for 80 feet and then finds that the tower subtends an angle of 30° Prove that the height of the tower is

$$40(\sqrt{6}-\sqrt{2}) \text{ feet}$$

(b) A person observes the top of a vertical tower of height h from a station S_1 and finds that β_1 is the angle of elevation He moves in a horizontal plane to second station S_2 and finds that $\angle PS_2S_1$ is γ_1 and the angle subtended by S_2S_1 at P is δ_1 and the angle of elevation is β_2 He moves again to a third station S_3 such that $S_3S_2=S_2S_1$ $\angle PS_3S_2=\gamma_2$ and the angle subtended by S_3S_1 at P is δ_2 Show that

$$\frac{\sin \gamma_1 \sin \beta_1}{\sin \delta_1}=\frac{\sin \gamma_2 \sin \beta_2}{\sin \delta_2}=\frac{h}{S_1S_2}$$

(Roorkee 85)

33 A ladder leaning against a vertical wall is inclined at an angle α to the horizontal The top of the ladder touches the parapet (मुंडेरी) On moving its foot a feet away from the wall the ladder now stands inclined at an angle β to the horizon and its top now touching a window Prove that the distance of the parapet from the window is $a \cot \frac{\alpha+\beta}{2}$

34 A tower is b ft high having a flag staff at its top The tower and the flag staff subtend equal angles at a point distant a

19 A business man hosts a dinner to 21 guests. He is having 2 round tables which can accomodate 15 and 6 persons each. In how many ways can he arrange the guests?

20 (a) A tea party is arranged for 16 people along two sides of a long table with 8 chairs on each side. Four men wish to sit on one particular side and two on the other side. In how many ways can they be seated?

(I I T 82)

(b) Eight chairs are numbered 1 to 8. Two women and three men wish to occupy one chair each. First the women choose the chairs from amongst the chairs marked 1 to 4, and then the men select the chairs from amongst the remaining. The number of possible arrangements is

(i) $^6C_2 \times {}^4C_3$	(ii) $^4C_2 \times {}^4P_3$
(iii) $^4P_2 \times {}^4P_3$	(iv) None of these

(I I T 81)

21 Find the number of ways in which ten candidates $A_1, A_2, A_3, \ldots, A_{10}$ can be ranked (i) if A_1 and A_2 are next to each other, and (ii) if A_1 is always above A_2.

(I I T 72)

22 (a) In a class of 10 students there are 3 girls, *A, B, C*. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive? **(I I T 61)**

(b) The total number of ways in which six '+' and four '−' signs can be arranged in a line such that two '−' signs occur together is **(I I T 88)**

23 There are six students *A, B, C, D, E, F*.

(a) In how many ways can they be seated in a line so that *C* and *D* do not sit together?

(b) In how many ways can a committee of four be formed so as always to include *C*?

(c) In how many ways can a committee of four be formed so as to always include *C* but exclude *D*? **(I I T 67)**

24 (a) A family consists of a grand father, 6 sons and daughters and 4 grand children. They are to be seated in a row for dinner. The grand children wish to occupy the two seats at each end and the grand father refuses to have a grand child on either side of him. In how many ways can the seating arrangements be made for the dinner?

∴ The total nnmber of seating arrangements is

$$\frac{(10)!}{6!\,4!}\times 8!\times 8!$$

$$=\frac{(10)!}{6!\,4!}\times 8\times 7\,6!\times 8\times 7\times 6\times 5\,4!$$

$$30\times 8^2\times 7^2\times 10!=341397504000$$

Alternative Method

The four persons who wish to occupy side A can be acco modated on eight chairs in

$$^8P_4=\frac{8!}{4!}=8\times 7\times 6\times 5 \text{ ways}$$

The two persons who wish to occupy side B can be accomo dated on eight chairs in

$$^8P_6=\frac{8!}{6!}=8\times 7 \text{ ways}$$

Having seated 6 persons on 6 chairs we are left with 10 per sons on 10 chairs on both sides aad they can be seated in 10! ways

The number of arrangements is

$(10)!\times(8\times 7\times 6\times 5)\times(8\times 7)=(10)!\times 30\times 8^2\times 7^2$ etc

(b) Ans (iv) The required no of arrangements$={}^4P_2\times{}^5P_3$

21 Regard $A_1\ A_2$, as one group so that there are now $10-2+1=9$, candidates which can be arranged amongst themselves in 9! ways

But these two A_1 and A_2 can be arranged amongst themselves in $2!=2$ ways

Hence the number of ways when A_1 and A_2 are together is

$$2\,9!=725760$$

(ii) Total number of arrangements for 10 persons is 10!

In any arrangement A_1 can be above A_2 or A_2 can be above.

Hence in half of the these A_1 is above A_2

$$=\tfrac{1}{2}\,(10)!=1814400$$

22. (a) After arranging boys in 7! ways we will have 8 place in which we can arrange the girls in $^8P_3$ ways

Hence by fundamental theorem the number of arrangements is

$$7!\times{}^8P_3=7!\times\frac{8!}{5!}=7!\times 8\times 7\times 6$$

$$=336\times 7!$$

(b) There are 6 professors of whom two are from Science, 2 from Arts and the remaining two from Commerce They have to stand in a line so that the two Science teachers two Arts teachers and also the two Commerce teachers are together Find the number of ways in which they can do so

(c) A man has 7 relatives, 4 of them are ladies and 3 gentlemen, his wife has 7 relatives, and 3 of them are ladies and 4 gentlemen In how many ways can they invite a dinner party of 3 ladies and 3 gentlemen so that there are 3 of man's relatives and 3 of wife's relatives (I.I T 85)

Miscellaneous Types

25 Show that the number of diagonals of a polygon of n sides is

$$\frac{n(n-3)}{2}$$

26 Find the number of (i) diagonals (ii) triangles formed in a decagon

27 (a) Out of 18 points in a plane, no three are in the same straight line except five points which are collinear How many (i) straight lines (ii) triangles can be formed by joining them

(b) There are n points in a plane, no three of which are in the same straight line with the exception of p, which are all in the same straight line, find the number (1) of straight lines, (2) of triangles which result from joining them

(c) There are n points in space, no four of which are in the same plane with the exception of p which are all in the same plane Find how many planes there are each containing three of the points

28 There are 12 points in a plane of which 5 are collinear Find (i) the number of triangles that can be formed with vertices at these points (ii) the number of straight lines obtained by joining these points in pairs

29 Six X's have to be placed in the squares of the figure on next page such that each row contains at least one X In how many different ways can this be done? (I I T 78)

22 (b) By Combenation There will be seven gaps in between six + signs in which we have to arrange 4 − signs Thus we have to make a relection of 4 out of seven $^7C_4=35$

By Permutation Six different Things could be arranged in 6 ! ways but as all the six are identical therefore the number of ways of arranging the six identical signs of + is $\frac{6!}{6!}=1$

Now we are left with seven places in which 4 different things can b arraged is $7P_4$ ways but as all the four— ive signs are identical therefore the number ofways is $\frac{7P_4}{4!}=\frac{7!}{4!3!}=35$

23 (a) It is exactly Q 16

Total no of ways$=6!=720$ Treat C and D as one and we have $6-2+1=5$ units which can be arranged in $5!=120$ ways But C and D can be arranged in $2!=2$ ways Hence the no of ways when they are together is $120\times2=240$

Hence the arrangements when C and D will not be together is $720-240=480$

(b) Ans $^5C_3=\frac{5!}{3!2!}=\frac{5\times4}{2}=10$

(c) Since C is always to be included so we have to select only 3 out of 5 and hence $^5C_3=\frac{5!}{3!2!}=10$ ways

In the above 10 ways C is always there but D may be or may not be there If both C and D are included then out of 4 we have to choose only 2 and this can be done in

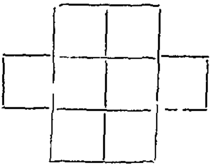
$$^4C_2=\frac{4!}{2!2!}=\frac{24}{2\cdot2}=6 \text{ ways}$$

Hence the number of ways when C is included but D is excluded is $10-6=4$

Alternative for (c)

Exclude D so that out of remaining 5 we have to select 4 Since C is to be included always, we are left with only 4 students out of which we have to choose 3 (the committee to consist of only 4 students out of which C is a must)

Hence the required number of committees is $^4C_3=4$ in which D is excluded but C is included



30 How many different signals can be given by using any number of flags from six flags of different colours

31 Three men have 4 coats, 5 waist coats and 16 caps In how many ways can they wear them ?

32 In how many was can three letters be posted in four letter boxes in a village ? If all the three letters are not posted in the same letter box, find the corresponding number of ways of posting

33. If there are n students and r prizes $(r < n)$ Prove that they can be given away (a) in n^r ways when a student can receive any number of prizes (b) in $n^r - n$ ways when a student cannot receive all the prizes

34 There are 3 copies each of 4 different books In how many ways can they be arranged in a shelf (I I T 64)

35 Find the number of ways of dividing 15 things into groups of 8, 4 and 3 respectively

36 In how many ways can a pack of 52 cards be

(a) divided equally among four players in order

(b) formed into 4 groups of 13 cards each

(c) In 4 sets three of them having 17 cards each and the fourth just 1 card (I I T 79)

37 In how many ways can 10 balls be divided between two boys, one receiving two and the other eight balls

38 If some or all of n things be taken at a time, prove that the number of combinations is $2^n - 1$

39 Find the number of ways in which 5 identical balls can be distributed among 10 identical boxes if not more than one ball can go into a box (I I T 73)

40 Given 5 different green dyes, four different blue dyes and three different red dyes, how many combination of dyes can be chosen taking at least one green and one blue dye (I I T 74)

41 Show that the total number of permutations of n different

∴ The total nnmber of seating arrangements is

$$\frac{(10)!}{6!\,4!}\times 8!\times 8!$$

$$=\frac{(10)!}{6!\,4!}\times 8\times 7\ 6!\times 8\times 7\times 6\times 5\ 4!$$

$$30\times 8^2\times 7^2\times 10!=341397504000$$

Alternative Method

The four persons who wish to occupy side A can be acco modated on eight chairs in

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$$^8P_6=\frac{8!}{6!}=8\times 7 \text{ ways}$$

Having seated 6 persons on 6 chairs we are left with 10 per sons on 10 chairs on both sides aad they can be seated in $10!$ ways

The number of arrangements is

$(10)!\times(8\times 7\times 6\times 5)\times(8\times 7)=(10)!\times 30\times 8^2\times 7^2$ etc

(b) Ans (iv) The required no of arrangements$={}^4P_2\times{}^6P_6$

21 Regard $A_1\ A_2$, as one group so that there are now $10-2+1=9$, candidates which can be arranged amongst themselves in $9!$ ways

But these two A_1 and A_2 can be arranged amongst themselves in $2!=2$ ways

Hence the number of ways when A_1 and A_2 are together is

$$2\ 9!=725760$$

(ii) Total number of arrangements for 10 persons is $10!$

In any arrangement A_1 can be above A_2 or A_2 can be above A_1

Hence in half of the these A_1 is above A_2

$$=\tfrac{1}{2}(10)!=1814400$$

22. (a) After arranging boys in $7!$ ways we will have 8 places in which we can arrange the girls in $^8P_3$ ways

Hence by fundamental theorem the number of arrangements is

$$7!\times{}^8P_3=7!\times\frac{8!}{5!}=7!\times 8\times 7\times 6$$

$$=336\times 7!$$

22 (b) By Combenation There will be seven gaps in between six + signs in which we have to arrange 4 − signs Thus we have to make a relection of 4 out of seven $^7C_4=35$

By Permutation. Six different Things could be arranged in 6 ! ways but as all the six are identical therefore the number of ways of arranging the six identical signs of + is $\frac{6!}{6!}=1$

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23 (a) It is exactly Q 16

Total no of ways$=6!=720$ Treat C and D as one and we have $6-2+1=5$ units which can be arranged in $5!=120$ ways But C and D can be arranged in $2!=2$ ways Hence the no of ways when they are together is $120\times2=240$

Hence the arrangements when C and D will not be together is $720-240=480$

(b) Ans $^5C_2=\frac{5!}{3!2!}=\frac{5\times4}{2}=10$

(c) Since C is always to be included so we have to select only 3 out of 5 and hence $^5C_3=\frac{5!}{3!2!}=10$ ways

In the above 10 ways C is always there but D may be or may not be there If both C and D are included then out of 4 we have to choose only 2 and this can be done in

$^4C_2=\frac{4!}{2!2!}=\frac{24}{2\ 2}=6$ ways

Hence the number of ways when C is included but D is excluded is $10-6=4$

Alternative for (c)

Exclude D so that out of remaining 5 we have to select 4 Since C is to be included always, we are left with only 4 students out of which we have to choose 3 (the committee to consist of only 4 students out of which C is a must)

Hence the required number of committees is $^4C_3=4$ in which D is excluded but C is included

Solutions to Problem Set (D)

1 In this question we must bear in mind that we have only to form committees We are not concerned with the arrangement of officials or non officials

8 officials 4 non officials

(a) **3 officials and 2 non officials**

3 officials out of 8 can be selected in

$$^8C_3=\frac{8!}{5!3!}=\frac{8\times7\times6}{6}=56 \text{ ways}$$

2 non officials out of 4 can be selected in

$$^4C_2=\frac{4!}{2!2}=6 \text{ ways}$$

∴ By fundamental theorem the number of ways in which the committee can be formed is

$$56\times6=336$$

(b) **At least two non official members**

Two non officials and 3 officials *i e*

$$^4C_2\times{}^8C_3=6\times56=336$$

Three non officials and 2 officials

$$^4C_3\times{}^8C_2=4\times28=112$$

Four non officials and 1 official

$$^4C_4\times{}^8C_1=1\times8=8$$

Total $336+112+8=456$

(c) **A particular official never included**

Required no of ways

$$={}^{12-1}C_5={}^{11}C_5=\frac{11\ 10\ 9\ 8\ 7}{1\ 2\ 3\ 4\ 5}=462$$

[See (*l*) page 284

Here $p=1$ $r=5$ and $n=12$]

(d) **A particular Non official is to be always included**

The required number of ways $={}^{11}C_4=\frac{11\ 10\ 9\ 8}{1\ 2\ 3\ 4}=330$

[See (*k*) page 276 with $p=1$, $r=5$ and $n=12$]

2 10 Professors and 20 Students

(a) $2P$ and $3S$

$$^{10}C_2\times{}^{20}C_3=45\times(20\times19\times3)=51300$$

(b) **Particular professor is included**

$$^9C_1\times{}^{20}C_3=9\times(20\times19\times3)=180\times57=10260$$

(c) **Particular Student included**

$$^{10}C_2\times{}^{19}C_2=45\times(19\times9)=7695$$

24 (a) There are 6 adults, 4 children and 1 grand father Let us mark the seats for 11 persons from 1 to 11

C C   C C
1 2 3 4 5 6 7 8 9 10 11

Seats no 1, 2 and 10, 11 at the ends are to be occupied by 4 grand children and it can be done in ${}^4P_4=4!=24$ ways *Now we will seat the grand father who cannot occupy seat no* 3 or seat no 9 because he does not want to have a child by his side Hence he has to choose any of five seats 4, 5, 6, 7, 8 *i e* he can seat himself in 5 ways

Now there are six seats left on which six adults can be arranged in ${}^6P_6=6!=720$ ways

Hence by fundamental theorem, total number of seating arrangement for the family is $24\times5\times720=86400$

(b) Treat them as 3 units which can be arranged in $3!=6$ ways The two persons corresponding to each of these can be arranged in $2!=2$ ways

Hence the total number is $6\times2\times2\times3=48$

(c) There are four possibilities

(i) 3 ladies from husbands side and 3 gentlemen from wife's side

No of ways in this case$={}^4C_3\times{}^4C_3=16$

(ii) 3 gentlemen from husband s side and 3 ladies from wife's side

No of ways in this case$={}^3C_3\times{}^3C_3=1$

(iii) 2 ladies and one gentlemen from husband s side and one lady and 2 gentlemen from wife's side

No of ways in this case$=({}^4C_2\times{}^3C_1)\times({}^3C_1\times{}^4C_2)=324$

(iv) One lady and 2 gentlemen from husband s side and 2 ladies and one gentlemen from wife's side

No of ways in this case$=({}^4C_1\times{}^3C_2)\times({}^3C_2\times{}^4C_1)=144$

Hence the total no of ways$=16+1+324+144=485$

25 (a) The number of lines each joining 2 out of the n points is

${}^nC_2=\frac{n!}{2!(n-2)!}=\frac{n(n-1)}{2}$ But these will include the n sides of the polygon

Hence the number of diagonals is $\frac{n(n-1)}{2}-n=\frac{n(n-3)}{2}$

(d) **Particular Student excluded**

$^{10}C_2 \times {}^{19}C_3 = 45 \times (57 \times 17) = 43605$

3 (i) 6 Gentlemen, 4 Ladies, Committee of 5

(a) At least one lady to be included

(1*L*, 4*G*), or (2*L*, 3*G*) or (3*L*, 2*G*), or (4*L*, 1*G*)

$^4C_1 \times {}^6C_4 + {}^4C_2 \times {}^6C_3 + {}^4C_3 \times {}^6C_2 + {}^4C_4\, {}^6C_1$

$= 60 + 120 + 60 + 6 = 245$

(b) No restriction on the formation, then out of 10 persons we have to choose 5

$$^{10}C = \frac{10 \times 9 \times 8 \times 7 \times 6\ 5\,!}{5\,!\ 5\,!} = 63 \times 4 = 252$$

(ii) (a) Ans $^4C_1 \times {}^8C_4 = 224$

(b) Ans $^4C_1 \times {}^8C_5 + {}^4C_2 \times {}^8C_4 + {}^4C_3 \times {}^8C_3 + {}^4C_4 \times {}^8C_2$

$= 224 + 420 + 224 + 28$

$= 896$

4 Group *A* 5, Group *B* 5 Questions to be attempted 6 but not more than 4 from any group

(4*A* 2*B*), (3*A*, 3*B*), (2*A*, 4*B*)

$^5C_4 \times {}^5C_2 + {}^5C_3 \times {}^5C_3 + {}^5C_2 \times {}^5C_4$

$= 50 + 100 + 50 = 200$

5 (a) Exactly as above

$^6C_5 \times {}^6C_2 + {}^6C_4 \times {}^6C_3 + {}^6C_3 \times {}^6C_4 + {}^6C_2 \times {}^6C_5$

$= 6 \times 15 + 15 \times 20 + 20 \times 15 + 15 \times 6$

$= 90 + 300 + 300 + 90 = 780$

(b) Now if some or all of *n* things be taken at a time, then the number of combinations will be $2^n - 1$, as explained below Each of the things can be taken or left *i e* each gives 2 combinations So the combinations of *n* things will be 2^n But this includes the case when all have been left Hence the number of combinations will be $2^n - 1$ Section A 3 questions Section B 4 questions and one question from each section is compulsory *i e* all of a particular section can not be left

$2^3 - 1 = 7, \quad 2^4 - 1 = 15$

Total is

$7 \times 15 = 105$ by fundamental theorem

(c) Ans $2^7 = 128$

Since the student is allowed to select at the most *n* books out of $(2n+1)$ books, therefore he can select, one book, two

26 (a) $^{10}C_2-10=45-10=35$ an in Q 25

(b) $^{10}C_3=\frac{10!}{3!7!}=\frac{10\times9\times8}{6}=120$

27 (a) *18 points, 5 collinear*

(i) No of lines$=^{18}C_2-^{5}C_2+1=153-10+1=144$

(ii) No of $\Delta' s=^{18}C_3-^{5}C_3=816-10=806$

(b) Ans (1) $^{n}C_2-^{p}C_2+1$, (9) $^{n}C_2-^{p}C_2$

(c) Ans $^{n}C_3-^{p}C_3+1$

28 (a) No of lines$=^{12}C_2-^{5}C_2+1=66-10+1=57$

(b) No of $\Delta' s=^{12}C_3-^{5}C_3=220-10=210$

29 In all we have 8 squares in which 6X's have to be placed and it can be done in $^{8}C_6=\frac{8!}{6!2!}=\frac{8\times7}{2}=28$ ways

But this includes the possibility that either the top horizontal row does not have any *X* or the bottom horizontal row has no *X*. Since we want that each row must have at least one *X*, these two possibilities are to be excluded. Therefore the required number is $28-2=26$

30 By alteration in the arrangement of flags the signals will change. So we have to find permutations *i e* arrangements of flags. We are at liberty to use any number of flags at a time. Therefore the required number of signals is

$$^{6}P_1+^{6}P_2+^{6}P_3+^{6}P_4+^{6}P_5+^{6}P_6$$

$$=6+30+120+360+720+720=1956$$

31 $^{4}P_3\times^{5}P_3\times^{16}P_3=172800$

32 We can post the first letter in 4 ways and 2nd too in 4 ways as well as the third. Therefore the total number of ways is

$$4\times4\times4=64$$

All the three letters can be posted in any of the four letter boxes in 4 ways

Hence the corresponding number when all are not posted in the same letter box is $64-4=60$

33 The answer is self explanatory

34 $\frac{12!}{(3!)^4}$ since there are 4 sets of 3 alike books

35 Ans $\frac{15!}{8!4!3!}$

books or at the most n books Hence the number of selecting at least one book is

$$^{2n+1}C_1+^{2n+1}C_2+ \dots +^{2n+1}C_n=S=63$$

Again we know that

$$^{2n+1}C_0+^{2n+1}C_1+^{2n+1}C_2+ \dots +^{2n+1}C_{2n}+^{2n+1}C_{2n+1}=2^{2n+1}$$

Now $^{2n+1}C_0={}^{2n+1}C_{2n+1}=1$, $^{2n+1}C_1={}^{2n+1}C_{2n}$ etc

Hence we have

$1+1+2S=2^{2n+1}$ or $2+2\cdot 63=2^{2n+1}$

or $128=2^{2n+1}$ or $2^7=2^{2n+1}$ $2n+1=7$

or $2n=6$ $n=3$

6 Ward A Ward B Ward C

4 5 8

For ward A we have to choose 4 out of 20 and then for ward B, 5 out of remaining 16 and then for ward C 8 out of remaining 11

$^{20}C_4 \times {}^{16}C_5 \times {}^{11}C_8$

$$=\frac{20!}{16!\,4!}\times\frac{16!}{11!\times 5!}\times\frac{11!}{8!\times 3!}$$

$$=\frac{20\times 19\times 18\times 17}{24}\times\frac{16\times 15\times 14\times 13\times 12}{120}\times\frac{11\times 10\times 9}{6}$$

$=4845\times 4368\times 165=3491888400$

7 $10M$, $7W$, Committee of 6

At least 3 Men and 2 Women

Now let us consider the case when 2 particular women are always there in the same committee

In this case we have to make a selection of 4 persons out of $10M$ and $5W$ to form a committee of six The restriction of at least 3 men and 2 women stands

$(4M$, No $W)$ $(3M, 1W)$

$^{10}C_4+{}^{10}C_3\times {}^5C_1$

$$=\frac{10!}{6!\,4!}+\frac{10!}{7!\,3!}\times\frac{5!}{4!\,1!}=\frac{10\times 9\times 8\times 7}{24}+\frac{10\times 9\times 8}{6}\times 5$$

$=210+600=810$

Hence the number of committees when two particular women are never together is

$8610-810=7800$

8, A and B are together and so out of remaing $n-2$ we have to choose 10

$$^{n-2}C_{10}=\frac{(n-2)!}{(10)!\,(n-12)!}$$

36 (a) $\frac{52!}{(13!)^4}$ (b) $\frac{52!}{(13!)^4 4!}$

(c) $\frac{52!}{(17!)^3 1! 1! 3!}$ [See (O) P 285]

37 When A receives 2 and B gets 8, then $\frac{(10!)}{2!8!}=45$,

When A receives 8 and B gets 2, then also $\frac{(10!)}{2!8!}=45$

Total is 45+45=90

38 Each of the things can be taken or left out i e each gives two ways So the combinations of n things will be $2\times2\times2\times \ldots n$ factors$=2^n$ But this includes the case when all have been left out So the number of combinations is 2^n-1

39 Out of 10 boxes we have to choose only 5 boxes (select) Because the balls are identical and the boxes are also identical, the required number of ways

$$={}^{10}C_5=\frac{(10!)}{5!5!}=252$$

40 At least one green dye can be selected out of 5 green dyes in 2^5-1 i e in 31 ways Similarly at least one blue dye can be selected out of 4 in 2^4-1 i e in 15 ways And at least one red or no red dye can be selected in 2^3 i e in 8 ways Hence the required number of ways

$$31\times15\times8=3720$$

41 Here the restriction is taken not more than r at a time and repetition is allowed

Arranging 1 at a time will be n

Now suppose we take 2 at a time The first place can b filled in n ways and then the second place can again be filled in n ways as repetition is allowed Hence the number of ways when taken 2 at a time will be $n\times n=n^2$

Similarly when taken 3 at time will be $n\times n\times n=n^3$,

and when taken r at a time will be $n\times n\times n\times \ldots r$ time $=n^r$

Hence the total number of ways$=n+n^2+\ldots+n^r$

$$=\frac{n(n^r-1)}{n-1} \quad \text{Sum of G P whose } a=n \text{ and } r=n$$

42 First of all, we have to divide 5 balls into three groups (i) containing 3 balls, one ball and one ball or (ii) containing 2

(b) C, D, E are together and so out of remaining $n-3$ we have to choose 9

$$^{n-3}C_9=\frac{(n-3)!}{9!(n-12)!} \quad (2)$$

(c) By given condition

$$\frac{(n-2)!}{(10)!(n-12)!}=3\frac{(n-3)!}{9!(n-12)!}$$

or $$\frac{(n-2)(n-3)!}{10\times 9!}=3\frac{(n-3)!}{9!}$$

or $$n-2=30 \quad \text{or} \quad n=32$$

9 (a) Ans $^{20}C_{16}=4845$, $^{19}C_{15}=3876$

(b) Ans $^{6}C_{3}\times ^{22}C_{9}=4974200$

Males Females

10 $^{22}C_{10}$ + $^{17}C_{10}$

11 $4L$ $3G$

A B

$1L, 1G$ $1L, 1G$

Selection of side A

$$^{4}C_1\times ^{3}C_1=4\times 3=12$$

After selecting side A we are left with $3L$ and $2G$ from which one each is to be chosen for side B

Selection of side B

$$^{3}C_1\times ^{2}C_1=3\times 2=6$$

Hence the number of ways of selection for the team

$$=12\times 6=72.$$

12 $7H$ $7W$

Let us choose two husbands for each side A and B and it can be done in $^{7}C_2=\frac{7!}{5!2!}=21$ ways

Having chosen two husbands, we have to exclude their wives from the selection because no husband and wife can play in the same game Therefore now we have to select 2 ladies out of 5 which can be done in

$$^{5}C_2=\frac{5!}{3!2!}=10 \text{ ways}$$

Again the two ladies can be placed on the two sides A and B in two ways *i e* Kamal in side A, Vijay in side B or Vijay in side A and Kamal in side B

Hence the total number of selections is

$$21\times 10\times 2=420$$

balls, 2 balls and one ball (Note that these are the only two types of divisions of 5 balls since no box remains empty)

Now the number of divisions corresponding to (i)

$$=\frac{5!}{3!1!1!1!2!}=10 \qquad \text{[See } O \text{ (ii) P 285]}$$

and the number of divisions corresponding to (ii)

$$=\frac{5!}{2!2!1!1!2!}=15 \qquad \text{[See } O \text{ (ii) P 285]}$$

Finally corresponding to each division there are 3 ! i e 6 arrangements Hence the required number of ways

$=6\ (10+15)=150$

43 (i) Four first prizes can be given in 20^4 ways since first prize of Mathematics can be given in 20 ways, first prize of Physics also in 20 ways, similarly first prizes of Chemistry and English can be given in 20 ways each (Note that a boy can stand first in all the four subjects) Then two second prizes can be given in 19^2 ways since a boy cannot get both the first and second prizes

Hence the required number of ways

$=20^4\times 19^2=57760000$

(ii) Ans $16^3+15^2\times 14=12902400$

44 Ans $3^{12}=531441$

45 Ans $12^3-1=1727$

46 Number of ways of getting $2m$ marks

$=\text{coeff}^t$ of x^{2m} in $(x^0+x^1+x^2+x^3+\ \ +x^m)^4$

$=\text{coeff}^t$ of x^{2m} in $(1+x+x^2+\ \ +x^m)^4$

$=\text{coeff}^t$ of x^{2m} in $\left(\frac{1-x^{m+1}}{1-x}\right)^4$

$=\text{coeff}^t$ of x^{2m} in $(1-x^{m+1})^4\,(1-x)^{-4}$

$=\text{coeff}^t$ of x^{2m} in $(1-4x^{m+1}+6x^{2m+2}-4x^{3m+3}+x^{4m+4})$

$$\times\left[\ 1+4x+\ \ +\frac{(r+1)(r+2)(r+3)}{3!}x^r+\ \ \right]$$

$$=1\times\frac{(2m+1)(2m+2)(2m+3)}{6}-4\times\frac{m(m+1)(m+2)}{6}$$

$$=\frac{(m+1)}{3}\,[(2m+1)\,(2m+3)-2m\,(m+2)]$$

$=\frac{1}{3}\,(m+1)\,(2m^2+4m+3)$

13 Note here that we have to first make a selection and then arrange the sailors, 4 on each side Let us call the two sides as A and B Three are to be fixed on side A and two on side B Having fixed these five we are left with 3 sailors out of 8 From these 3 we choose 1 for side A to make four in A, and this can be done in ${}^3C_1=3$ ways Now we are left with 2 sailors and only two are needed for side B and this can be done is ${}^2C_2=1$ way

Hence the number of selections is

$$3\times 1=3$$

Now the sailors on side A can be arranged amongst them selves in 4 ! ways *i e* 24 ways and so also the sailors on side B can be arranged is 24 ways Hence the total number of arrangements is

$$3\times 24\times 24=3\times 576=1728$$

14 (a) The first man can occupy the seat in 6 ways and now the second man has only 5 ways left for choosing a seat

Hence the number of ways for the two to occupy two seats is $6\times 5=30$

(b) For seating on a round table we have to fix up one man at a place then there are $n-1$ persons left which can be arranged in $(n-1)!$ ways Now suppose Mr Khanna is seated first and in any arrangement he has Mr Lal and Mr Sehgal on right and left or on left and right In the two ways a man has the same neighbours Hence the required number of arrrngements is

$\frac{1}{2}(n-1)!$ as these are to be counted as one

15 (a) Fix up a Muslim and the remaining 5 Muslims can be sea ted in 5 ! ways Now no two Hindus are to sit together and as such the 6 Hindus are to be arranged in six empty seats between two consecutive Muslims and number of of arrangements will be 6 !,

Hence by fundamental theorem the total number of ways is $5!\times 6!=120\times 720=86400$

(b) m men can be arranged in m ! ways in a row and we have now $(m+1)$ places in which n women can be arranged in

$${}^{m+1}P_n=\frac{(m+1)!}{(m+1-n)!} \text{ ways}$$

47 (i) If the order of the three words in any arrangement does not change, the vowels can be regarded as like letters, since in that case if we choose *any three* positions for them, then there is *only one* way of putting them in those three positions, namely, *u* in the first of those positions, *i* in the scond, *e* in the third *i e* in the order *u i e* in which they originally occur

Hence, the total no of arrangements of the letters in this case

$$=\frac{8!}{3!\,2!}=8\ 7\ 6\ 5\ 2=3360$$

(ii) Since vowels remain fixed in their positions, it is clear that the total no of arrangements of the letters is, the same as the no of arrangements of the other 5 letters in the remaining 5 positions and therefore

$$=\frac{5!}{2!}=60$$

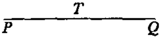
(iii) Since the relative order of vowels and consonants rema ins the same, they are to be arranged in their respective positions Hence no of arrangements in this case

$$=\frac{5!}{2!}\times 3!=360$$

48 Ans ${}^{3+6-1}C_6={}^8C_6=\frac{8\times 7}{1\times 2}=28$

[Hint See part (*r*) § P 285]

49 Suppose *PQ* is any one of the *n* straight lines and let it be



intersected by some other straight line *RS* at the point *T* Then, it is evident that *PQ* contains $(n-1)$ of the points of intersection, since it is intersected by the remaining $(n-1)$ straight lines in $(n-1)$ different points

Similarly, each of the other straight lines contains $(n-1)$ points It follows that the total no of points contained in the *n* straight lines is $n(n-1)$ But in making up this total, each point has evidently been counted twice, *e g*, the point *T* has been counted once among the points situated on *PQ* and again among those on *RS* Hence, the actual no of points is $\frac{1}{2}n(n-1)$ Now we find the no of new lines formed by joining these points

Hence by fundamental Theorem the number of ways is as given

(c) Ans $5!\,6! = 86400$, $4!\,5! = 2880$

16 The boys can be seated on a round table in $6! = 720$ ways as we have to fix one boy first

Now treat the two boys together and treat them as one unit So we have six boys and having fixed one we will have

$$5! = 120 \text{ ways}$$

But the two particular boys can be arranged amongst themselves in $2! = 2$ ways Therefore the total number of arrangements when the two particular boy are next to each other is

$$120 \times 2 = 240$$

Hence when the two particular boys are never together is

$$720 - 240 = 480$$

17 Exactly as above

(i) $(18)! \times 2! = 2\,(18)!$

(ii) $(19)! - 2\,(18)!$

8 20 guests + 1 host = 21 persons They can be seated in $(20)!$ ways Treat the host and two particular persons as one unit So we have now $21 - 3 + 1 = 19$ and the number of arrangements will be $(18)!$ But these two persons can be arranged on either side of the host in 2 ways Hence there will be $2\,(18)!$ ways

19 Out of 21 we can choose 15 for one table in ${}^{21}C_{15}$ ways and for each selection we are left with 6 guests for the second table having 6 seats Now 15 for table *A* can be arranged in $(14)!$ ways and 6 for table *B* can be arranged in $5!$ ways Hence the total number is

$${}^{21}C_{15} \times 14)! \times 5!$$

20 It is exactly same as Q 13

Having seated 4 on side A and 2 on side B we are left with 10 persons Out of which we choose 4 for side *A* in ${}^{10}C_4$ ways and now *for side B we are left with 6 persons and 6 have to* be seated so that they can be seated in ${}^{6}C_6 = 1$ way

Hence the number *of selections for the two sides is*

$${}^{10}C_4 \times 1 = \frac{(10)!}{6!\,4!}$$

Again 8 persons on each side can be arranged amongst them selves in $8!$ ways, Now the no of *new* lines passing through T is equal to the no of points lying outside the lines PQ and RS, because we get a *new* line by joining T with each of those points only But, each of the lines PQ and RS contains $n-2$ points besides the point T

Hence the no of points situated on PQ and RS
$=2(n-2)+1=2n-3$

Therefore the no of points outside PQ and RS is

$$\frac{1}{2}n(n-1)-(2n-3)$$

The no of *new* lines passing through T, and similarly through each of other points is $\frac{1}{2}n(n-1)-(2n-3)$, and so the total no of new lines passing through the points is $\frac{1}{2}n(n-1)\{\frac{1}{2}n(n-1)-(2n-3)\}$

But in making up this aggregate, every new line is counted twice, for example, if M be a point outside PQ and RS, the line TM is counted once among the lines passing through T and again, among those passing M

Hence the actual no of new lines formed

$$=\frac{1}{2}[\frac{1}{2}n(n-1)\{\frac{1}{2}n(n-1)-(2n-3)\}]$$
$$=\frac{1}{4}n(n-1)\frac{1}{2}(n^2-5n+6)$$
$$=\frac{1}{8}n(n-1)(n-2)(n-3)$$

Problem Set (E)

(Objective Questions)

1 If ${}^nC_{r-1}=36$, ${}^nC_r=84$ and ${}^nC_{r+1}=126$, then r is equal to
(a) 1, (b) 2, (c) 3, (d) None of these (I I T 79)

2 $\sum_{r=0}^{m} {}^{n+r}C_n$ is equal to
(a) ${}^{n+m+1}C_{n+1}$ (b) ${}^{n+m+2}C_n$
(c) ${}^{n+m+3}C_{n-1}$ (d) None of these

3 A polygon has 44 diagonals, then the number of its sides are
(i) 11, (ii) 7, (iii) 8, (iv) none of these

4 If 7 points out of 12 are in the same straight line, then the number of triangles formed is
(i) 19, (ii) 185, (iii) 201, (iv) none of these

5 Ten different letters of an alphabet are given Words with five letters are formed from these given letters Then the number of words which have at least one letter repeated is
(a) 69760, (b) 30240, (c) 99748, (d) none of these
(I I T 80)

feet from the foot of the tower Show that the length of the flag staff is

$$\frac{b(a^2+b^2)}{a^2-b^2}$$

35 A wireless pole $2m$ meters high is fixed on the top of a verandah of a house which is 15 meters high At a point R on the ground directly opposite, the wireless pole and verandah subtend equal angles Find the distance of R from the verandah

36 A tower has a flag staff at its top which subtend equal angles α at points distant 9 yds and 11 yds from the foot of the tower If $\tan\alpha = 1/10$, find the height of the tower and the flag staff

37 A vertical tower 50 ft high stands on a sloping ground The foot of the tower is at the same level as the middle point of a vertical flag pole From the top of the tower the angle of depression of the top and the bottom of the pole are 15° and 45° respectively. Find the length of the pole (I I T 69)

38 The elevation of a tower due north of a station A is α or (at a place due south of it is α) and at another station B due west of A it is β Prove that the height of the tower is

$$\frac{AB}{\sqrt{(\cot^2\beta-\cot^2\alpha)}} \text{ or } \frac{AB\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha-\sin^2\beta)}}$$

$$\frac{AB\sin\alpha\sin\beta}{[\sin(\alpha+\beta)\sin(\alpha-\beta)]^{1/2}} \text{ or } \frac{AB\tan\alpha\tan\beta}{\sqrt{(\tan^2\alpha-\tan^2\beta)}}$$

(M N R 86 Roorkee 79)

Note Tower being north of station A is the same as the station A being south of the tower

39 (*a*) The angle of elevation of the top of a tower from a point A due south of the tower is α and from B due east of the tower is β If $AB = d$, show that the height of the tower is

$$\frac{d}{\sqrt{(\cot^2\alpha+\cot^2\beta)}}$$

(*b*) An observer at an anti aircraft post A identifies an enemy aircraft due east of his post at an angle of elevation of 60° At the same instant a detection post D situated 4 kms south of A reports the aircraft at an elevation of 30° Calculate the altitude at which the plane is flying

(I I T 1972)

(c) A man standing south of a lamp post observes his shadow on the horizontal plane to be 24 feet long. On walking eastwards, 300 feet he finds his shadow as 30 feet. If his height is 6 ft, obtain the height of the lamp above the plane. (IIT 66)

40 Four ships A, B, C and D are at sea in the following relative positions. B is on the straight line segment AC, B is due north of D and D is due west of C. The distance between B and D is 2 km. $\angle BDA=40°$, $\angle BCD=25°$. What is the distance between A and D? (Take $\sin 25°=0.423$) (IIT 1983)

41 An isosceles triangle of wood is placed in a vertical plane, vertex upwards, and faces the sun. If $2a$ be the base of the triangle, h its height and $30°$ the altitude of the sun, prove that the tangent of the angle at the apex of the shadow is

$$\frac{2ah\sqrt{3}}{3h^2-a^2}$$

42 The angle of elevation of a tower at a point A due north of it is $30°$ and at another point due east of A is $18°$. If $AB=a$, show that the height of the tower is

$$\frac{a}{\sqrt{(2+2\sqrt{5})}}$$

43 PQ is a tower standing on a horizontal plane, Q being its foot. Two points A and B are taken on the plane such that $AB=32$ and $\angle QAB$ is a right angle. It is found that $\cot PAQ=\frac{2}{5}$ and $\cot PBQ=\frac{3}{5}$, find the height of the tower. (IIT 73)

44 A flag staff PN stands upright on level ground. A base AB is measured at right angles to AN such that points A, B, N lie in the same horizontal plane. If $\angle PAN=\alpha$ and $\angle PBN=\beta$, prove that the height of the flag staff is

$$\frac{AB \sin\alpha \sin\beta}{\sqrt{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}}}$$

(Roorkee 79)

45 (a) The angle of elevation of a certain peak when observed from each end of a horizontal base line of length $2a$ is found to be θ. When observed from the mid-point of the base the angle of elevation is ϕ. Prove that the height of the peak is

$$\frac{a \sin\theta \sin\phi}{\sqrt{\{\sin(\phi+\theta)\sin(\phi-\theta)\}}}$$

6 In a certain test, a_i students give wrong answers to at least i questions where $i=1, 2, 3, \ldots k$. No student gave more than k wrong answers. The total number of wrong answers given is **(IIT 82)**

7 If $n_1, n_2, \ldots, n_p$ are p positive integers, whose sum is an even number, then the number of odd integers among them is odd

(a) True (b) False **(IIT 85)**

8 The product of any r consecutive natural numbers is always divisible by $r!$

(a) True (b) False **(IIT 85)**

9 *The number of ordered triples of positive integers which are* solutions of the equation

$$x+y+z=100 \text{ is}$$

(a) 5081, (b) 6005 (c) 4851, (d) none of these

10 The number ways in which 16 sovereigns can be distributed among four applicants, each applicant receiving not less than three is

11 In a football championship there were played 153 matches. Every two teams played one match with each other. The number of teams, participating in the championship is

12 In a class tournament where the participants were to play one game with another two class players fell ill, having played 3 games each. *If the total number of games played is 84, the number of participants at the beginning was*

Solutions

1 Ans (c) We have

$$\frac{n-r+1}{r}=\frac{84}{36}=\frac{7}{3} \quad \text{and} \quad \frac{n-r}{r+1}=\frac{126}{84}=\frac{3}{2}$$

$$\frac{7}{3}r-1=n-r=\frac{3}{2}(r+1)$$

or $\quad 14r-6=9r+9 \quad$ or $\quad r=3$

2 Ans (a) Since ${}^nC_r={}^nC_{n-r}$ and ${}^nC_{r-1}+{}^nC_r={}^{n+1}C_r$, we have

$$\sum_{r=0}^{m} {}^{n+r}C_n=\sum_{r=0}^{m} {}^{n+r}C_r={}^nC_0+{}^{n+1}C_1+{}^{n+2}C_2+\ldots+{}^{n+m}C_m$$

$$=[1+(n+1)]+{}^{n+2}C_2+{}^{n+3}C_3+\ldots+{}^{n+m}C_m$$

$$=({}^{n+2}C_1+{}^{n+2}C_2)+{}^{n+3}C_3+\ldots+{}^{n+m}C_m$$

$$n+2={}^{(n+2)}C_1 \quad \text{or} \quad {}^nC_1=n$$

$$\frac{P(m+1)}{24}=25k+1 \in N,$$

$P(n)$ is true for $n=m+1$

Also when $n=1$, $5^{2\cdot 1}-1=25-1=24$

when $n=2$, $5^{2\cdot 2}-1=625-1=624=24$ (26)

Above relation shows that $P(n)$ is divisible by 24 for $n=1, 2$

Hence $P(n)$ is universally true

8 Proceed as above

9 Let $P(n)$ be true for $n=m$, that is, we assume that

$$\frac{a^m-b^m}{a-b} \text{ is divisible by } a-b$$

$$\therefore \quad \frac{a^m-b^m}{a-b}=k, \text{ say}$$

or $a^m-b^m=k(a-b)$ or $a^m=b^m+k(a-b)$ (1)

Now $P(m+1)=a^{m+1}-b^{m+1}=a^m a-b^m b$

$$=a[b^m+k(a-b)]-b^m.b$$

$$=b^m(a-b)+ak(a-b)=(a-b)(b^m+ak)$$

$\therefore$ $a^{m+1}-b^{m+1}$ is also divisible by $a-b$

Hence $P(n)$ *is true for* $n=m+1$

Also when $n=1$, we get $a-b$ which is divisible by $a-b$,

when $n=2$ we get a^2-b^2 which is also divisible by $a-b$

Hence $P(n)$ is universally true

10 Let $P(n)$ be true for $n=m$

$\therefore$ $\log x^m=m\log x$

$$P(m+1)=\log x^{m+1}=\log x^m x=\log x^m+\log x$$

$$=m\log x+\log x=(m+1)\log x$$

Above relation shows that $P(n)$ is true for $n=m+1$

Now when $n=1$, $\log x^1=1\log x$

$n=2$, $\log x^2=\log x\, x=\log x+\log x=2\log x$

Above relations show that $P(n)$ is true for $n=1, 2$

Hence $P(n)$ is universally true

11 Let $P(n)$ be true for $n=m$

$$(1+x)^m > 1+mx \qquad (1$$

Since $x>-1$, multiplying by $x+1$,

$$P(m+1)=(1+x)^{m+1}=(1+x)^m(1+x)$$

$$>(1+mx)(1+x) \text{ by } (1)$$

$$=1+(m+1)x+mx^2$$

$$>1+(m+1)x \quad \because mx^2>0$$

$= ({}^{n+3}C + {}^{n+3}C_3) + \quad + {}^{n+m}C_m$

$= ({}^{n+4}C_3 + {}^{n+4}C_4) + \quad + {}^{n+m}C_m$

$= {}^{n+m}C_{m-1} + {}^{n+m}C_m$

$= {}^{n+m+1}C_m$

$= {}^{n+m+1}C_{n+1} \qquad [\quad {}^nC_r = {}^nC_{n-r}]$

3 Ans (i) We have

$44 = {}^nC_2 - n = \frac{1}{2} n(n-1) - n$

or $n^2 - 3n - 88 = 0$ or $(n-11)(n+8) = 0$

$n = 11$, since $n \neq -8$

4 Ans 185

No of $\Delta^s = {}^{12}C_3 - {}^7C_3 = \frac{12\ 11\ 10}{1\ 2\ 3} - \frac{7\ 6\ 5}{1\ 2\ 3}$

$= 220 - 35 = 185$

5 Ans (a)

No of words in which all the 5 letters are repeated $= 10^5 = 100000$

and the no of words in which no letter is repeated

$= {}^{10}P_5 = 10\ 9\ 8\ 7\ 6 = 30240$

Hence the no of words which have at least one letter repeated $= 100000 - 30240 = 69760$

6 Hint Total no of wrong answers

$= 1\,(a_1 - a_2) + 2\,(a_2 - a_3) + \quad + (k-1)\,(a_{k-1} - a_k) + k\,a_k$ (why ?)

$= a_1 + a_2 + a_3 + \quad + a_k$

7 Ans (b)

For if the no of odd integers is odd, the sum of these odd integers will be odd whereas the sum of remaining even integers will be even so that the sum of all integers will be odd which is false since the sum is given to be even

8 Ans (a) We prove it by induction

(i) The product of any two consecutive natural numbers is even and so it is divisible by 2!

(ii) Now assume that the product of any $(r-1)$ consecutive natural numbers is divisible by $(r-1)$!

Let P_n stand for the product of r consecutive integers, the least of which is n, then

Above relation shows that $P(n)$ is true for $n=m+1$

Also when $n=1$, $(1+x)^1=1+1\,x$ So in this case, inequality does not hold

But when $n=2$, $(1+x)^2=1+2x+x^2 > 1+2x$

Thus $P(n)$ is true for $n=2$

Hence $P(n)$ is true universally if $n \geqslant 2$

12 Let $P(n)$ be true for $n=m$

$$(\cos\theta+i\sin\theta)^m=\cos m\theta+i\sin m\theta$$

$$\begin{aligned}P(m+1)&=(\cos\theta+i\sin\theta)^{m+1}\\&=(\cos\theta+i\sin\theta)^m(\cos\theta+i\sin\theta)\\&=(\cos m\theta+i\sin m\theta)(\cos\theta+i\sin\theta)\\&=(\cos m\theta\cos\theta-\sin m\theta\sin\theta)\\&\qquad+i(\sin m\theta\cos\theta+\cos m\theta\sin\theta)\\&=\cos(m+1)\theta+i\sin(m+1)\theta\end{aligned}$$

Above relation shows that $P(n)$ is true for $n=m+1$

Again when $n=1$, $(\cos\theta+i\sin\theta)^1=\cos 1\,\theta+i\sin 1\,\theta$

when $n=2$, $(\cos\theta+i\sin\theta)^2$

$$\begin{aligned}&=\cos^2\theta+i^2\sin^2\theta+2i\sin\theta\cos\theta\\&=(\cos^2\theta-\sin^2\theta)+i(2\sin\theta\cos\theta)\\&=\cos 2\theta+i\sin 2\theta\end{aligned}$$

Above relation shows that $P(n)$ is true for $n=1, 2$

Hence $P(n)$ is universally true

13 Let $P(n)=7^{2n}+2^{3n-3}\,3^{n-1}$

For $n=1$, $P(1)=7^2+2^0\,3^0=49+1=50=25\times 2$

Thus $P(1)$ is divisible by 25

Now assume $P(m)$ is divisible by 25 so that we may write

$$7^{2m}+2^{3m-3}\,3^{m-1}=25k \qquad (1)$$

when k is a positive integer

Then
$$\begin{aligned}P(m+1)&=7^{2(m+1)}+2^{3(m+1)-3}\,3^{(m+1)-1}\\&=7^{2m}\,7^2+2^{3m-3}\,2^3\,3^{m-1}\,3\\&=49\;7^{2m}+24\;2^{3m-3}\,3^{m-1}\\&=(50-1)\,7^{2m}+(25-1)2^{3m-3}\,3^{m-1}\\&=25\,(2\;7^{2m}+2^{3m-3}\,3^{m-1})-(7^{2m}+2^{3m-3}\,3^{m-1})\\&=25\,(2\;7^{2m}+2^{3m-3}\,3^{m-1})-25k \qquad \text{by (1)}\end{aligned}$$

Hence $P(m+1)$ is divisible by 25

It follows by mathematical induction that $P(n)$ is divisible by 25 for all positive integers n

14 $x^2-y^2=(x+y)(x-y)$

Hence x^2-y^2 is divisible by $x+y$

$P_n = n(n+1)(n+2) \ldots (n+r-1)$

and $P_{n+1} = (n+1)(n+2)(n+3) \ldots (n+r)$

$n P_{n+1} = (n+r) P_n = n P_n + r P_n$

$P_{n+1} - P_n = \frac{P_n}{n} \times r$

$= r$ times the product of $(r-1)$ consecutive numbers

Now by our assumption, the product of $(r-1)$ consecutive natural numbers is divisible by $(r-1)!$, we must have

$\frac{P_n}{n} = k(r-1)!,\ k \in N$

$P_{n+1} - P_n = rk(r-1)! = k\ r!$ (1)

or $P_{n+1} = P_n + k\ r!$

Now $P_1 = r!$ and so by (1),

$P_2 = r!\ (1+k)$ Then P_2 is a multiple of $r!$ Similarly

$P_3 = P_2 + k\ r! = (1+2k)\ r!$ etc

Hence $P_1, P_2, P_3, P_4, \ldots$ are all divisible by $r!$

We have thus proved that if the product of $(r-1)$ consecutive natural numbers is divisible by $(r-1)!$, then the product of r consecutive natural numbers is divisible by $r!$

Hence by induction it follows that the product of any r consecutive natural numbers is divisible by $r!$

9 Ans (c)

The no of triples of positive integers which are solutions of $x+y+z=100$

$=$ Coeff^t of x^{100} in $(x+x^2+x^3+\ldots)^3$

$=$ Coeff^t of x^{100} in $x^3\ (1-x)^{-3}$ summing the infinite G P

$=$ Coeff^t of x^{100} in $x^3 \left[1+3x+6x^2+\ldots+\frac{(n+1)(n+2)}{2}x^n + \ldots \right]$

$= \frac{(97+1)(97+2)}{2} = 49 \times 99 = 4851$

10 Ans 35

Required no of ways

$=$ Coeff^t of x^{18} in $(x^3+x^4+x^5+\ldots+x^{16})^4$

$=$ Coeff^t of x^{18} in $x^{12}(1+x+x^2+\ldots+x^{13})^4$

$=$ Coeff^t of x^{18} in $x^{12}(1-x^{14})^4(1-x)^{-4}$

$=$ Coeff^t of x^4 in $(1-x^{14})^4(1-x)^{-4}$

Thus the given statement holds for $n=2$

Now assume $x^m - y^m = (x+y) f(x, y)$ (1)

where $f(x, y)$ is a polynomial of degree $m-1$ in x and y and m is an even integer

Then $x^{m+2}-y^{m+2}=x^{m+2}-x^m y^2+x^m y^2-y^{m+2}$

$$=x^m (x^2-y^2)+y^2 (x^m-y^m)$$
$$=(x+y)(x-y)x^m+y^2 (x+y) f(x, y) \text{ by (1)}$$
$$=(x+y) [(x-y) x^m+y^2 f(x, y)]$$

Hence $x^{m+2}-y^{m+2}$ is divisible by $x+y$

Thus we have shown that x^n-y^n is divisible by $x+y$ for $n=2$ and whenever it is divisible by $x+y$ for any even integer, it is divisible by $x+y$ for the next event integer Hence by mathematical induction x^n-y^n is divisible by $x+y$ when n is any even interger

15 The inequality clearly holds for $n=1$ since $2 > 1$

Now assume $2^m > m$

But $2^m > 1$ for all positive integer m

Hence $2^m+2^m > m+1$ i e $2 \cdot 2^m > m+1$

or $2^{m+1} > m+1$

Hence the inequality holds for $n=m+1$

Therefore the inequality $2^n > n$ holds for all natural numbers n by mathematical induction

16 The inequality obviously does not hold for $n=1$ and 2

But the inequality holds for $n=3$, for we have

$$2^3 > 2 \cdot 3+1$$

Now assume

$$2^m > 2m+1$$

where m is a natural number >3

But $2^m > 2$ for all $m > 1$

Adding the two inequalities we obtain for $m > 3$,

$$2^m+2^m > 2m+1+2$$

that is, $2^{m+1} > 2(m+1)+1$

Hence the inequality holds for $n=m+1$

It follows by mathematical induction that the inequality $2^n > 2n+1$ holds for all natural numbers $n \geqslant 3$

17 The inequality holds for $n=1$, since $2^1 > 1^2$

The inequality does not hold for $n=2$, since $2^2=2^2$

The inequality does not hold for $n=3$, since $2^3 < 3^2$

The inequality does not hold for $n=4$, since $2^4=4^2$

$$= \text{Coeff}^t \text{ of } x^4 \text{ in } (1-4x^{11}+\ \ )\left[1+4x+ +\frac{(r+1)(r+2)(r+3)}{3!}x^r+\ \ \right]$$

$$\frac{(4+1)(4+2)(4+3)}{3!}=35$$

11 Ans 18

If n is the number of teams, then

${}^nC_2=153$ or $n^2-n-306=0$

or $(n-18)(n+17)=0$

$n=18$ [$n\neq 17$]

12 Ans 15

Had the two players not fallen ill, they would have participa ted in $(n-1)+(n-1)-1$ *i e* $2n-3$ games, where n was the no of the participants in the beginning

Hence ${}^nC_2-(2n-3)+6=84$

or $n^2-5n-150=0$ or $(n-15)(n+10)=0$

$n=15$ [$n\neq -10$]

The inequality holds for $n=5$, since $2^5 > 5^2$
The inequality holds for $n=6$, since $2^6 > 6^2$
Now assume

$$2^m > m^2$$

where m is a natural number $\geqslant 5$ (1)
By problem 16 we know that

$$2^m > 2m+1 \text{ for } m \geqslant 3 \qquad (2)$$

Adding the inequalities (1) and (2), we obtain

$$2^m+2^m > m^2+2m+1$$

that is, $2^{m+1} > (m+1)^2$
Hence the inequality holds for $n=m+1$
It follows by mathematical induction that the inequality $2^n > n^2$ holds for $n=1$ and for all natural numbers $n \geqslant 5$

3 Let $S_n = \frac{1}{n+1}+\frac{1}{n+2}+ \quad +\frac{1}{2n}$
For $n=2$, we have

$$\frac{1}{3}+\frac{1}{4}=\frac{7}{12}=\frac{14}{24}>\frac{13}{24}$$

Hence the inequality holds for $n=2$
Now assume $S_m > \frac{13}{24}$ for some positive integer $m > 1$
We have

$$S_m = \frac{1}{m+1}+\frac{1}{m+2}+ \quad +\frac{1}{2m}$$

and $S_{m+1} = \frac{1}{m+2}+\frac{1}{m+3}+ \quad +\frac{1}{2m}+\frac{1}{2m+1}+\frac{1}{2m+2}$

$$S_{m+1}-S_m = \frac{1}{2m+1}+\frac{1}{2m+2}-\frac{1}{m+1}$$

that is $S_{m+1}-S_m = \frac{1}{2(m+1)(2m+1)}$

But $\frac{1}{2(m+1)(2m+1)} > 0$ for any natural number m It follows that

$$S_{m+1}-S_m > 0, \qquad \text{that is } S_{m+1} > S_m$$

But $S_m > 13/24$ by our assumption $\quad S_{m+1} > 13/24$
Hence by mathematical induction the given inequality holds for all natural numbers $n > 1$

19 (*i*) The identity is valid for $n=1$, for we have

$$\cos\alpha = \frac{\sin 2\alpha}{2\sin\alpha} = \frac{2\sin\alpha\cos\alpha}{2\sin\alpha} = \cos\alpha$$

8
Mathematical Induction

§ 1 Principle of Mathematical Induction

Let $n \in N$ and $P(n)$ denote a certain statement or formula or theorem Then $P(n)$ holds for every natural number n if

(1) it holds for $n=1$

and (2) it holds for $n=m+1$ whenever it holds for $n=m$

Remark We emphasize that proof by mathematical induction requires the fulfilment of both the conditions (1) and (2) as stated above Even if we prove a certain statement for larger number of values of n, say $n=1, 2, \ldots, 100$, we can not say that the statement is true for all n unless we establish condion (2) For example consider trinomial $f(n)=n^2+n+41$ Substituting 1, 2, 3 4, 5, 6 7, 8, 9, 10 for n in turn we obtain 43, 47, 53, 61, 71, 83, 97, 113, 131, 151 which are all prime numbers On the basis of these results, we assert that the substitution of any positive integer for n in $f(n)$ will always yield a prime number But this reasoning is fallacious In fact $f(n)$ yields a prime number for $n=1, 2, \ldots, 39$, but for $n=40$, we have

$$f(40)=40^2+40+41=40^2+2 \cdot 40+1=(40+1)^2$$
$$=41\times 41 \text{ which is a composite number}$$

This example shows that we cannot make general assertion with respect to any n unless we prove condition (2)

We now show that the condition (1) can not be omitted either

For example, we make the following assertion

Every natural number is equal to the next natural number

To prove this, we assume

$$m=m+1 \tag{1}$$

where m is a natural number On the basis of this, we prove

$$m+1=m+2 \tag{2}$$

In fact, adding 1 to each side of (1), we obtain the equation (2)

This shows that if our assertion is valid for $n=m$, then it is also valid for $n=m+1$ Hence we conclude that the assertion

Now let the identity hold for $n=m$, that is, let

$$\cos\alpha\cos 2\alpha \quad \cos 2^{m-1}\alpha=\frac{\sin 2^m\alpha}{2^m\sin\alpha}$$

Multiplying both sides of this identity by $\cos 2^m\alpha$, we obtain

$$\cos\alpha\cos 2\alpha \quad \cos 2^{m-1}\alpha\cos 2^m\alpha=\frac{\sin 2^m\alpha}{2^m\sin\alpha}\cos 2^m\alpha$$

$$=\frac{2\sin 2^m\alpha\cos 2^m\alpha}{2^{m+1}\sin\alpha}=\frac{\sin 2^{m+1}\alpha}{2^{m+1}\sin\alpha}$$

Thus the identity holds for $n=m+1$

Hence the identity is proved for all positive integers n by mathematical induction

19 (ii)

$P_1=\frac{\sqrt{3}}{2}$, $\qquad P_2=\sqrt{3}$ etc

$$P_{n+1}=P_n+\sin(n+1)\frac{\pi}{3}$$

$$2\sin\frac{n\pi}{6}\sin(n+1)\frac{\pi}{6}+2\sin(n+1)\frac{\pi}{6}\cos(n+1)\frac{\pi}{6}$$

$$2\sin(n+1)\frac{\pi}{6}\left[\sin\frac{n\pi}{6}+\sin\left(\frac{\pi}{2}-(n+1)\frac{\pi}{6}\right)\right]$$

$$=2\sin(n+1)\frac{\pi}{6}\left\{\sin\frac{n\pi}{6}+\sin\left(\frac{\pi}{3}-\frac{n\pi}{6}\right)\right\}$$

$$=2\sin(n+1)\frac{\pi}{6}\left\{2\sin\frac{\pi}{6}\cos\tfrac{1}{2}\left(\frac{2n\pi}{6}-\frac{\pi}{3}\right)\right\}$$

$$=2\sin(n+1)\frac{\pi}{6}\quad 1\cos\left\{\frac{n\pi}{6}-\frac{\pi}{6}\right\}$$

$$=2\sin(n+1)\frac{\pi}{6}\sin\left\{\frac{\pi}{2}+\left(\frac{n\pi}{6}-\frac{\pi}{6}\right)\right\}$$

$$=2\sin(n+1)\frac{\pi}{6}\sin\left(\frac{\pi}{3}+\frac{n\pi}{6}\right)$$

$$=2\sin(n+1)\frac{\pi}{6}\sin(n+2)\frac{\pi}{6}$$

20 The statement is valid for $n=1$ and $n=2$

Now let $A_{m-2}=\cos(m-2)\theta$, $A_{m-1}=\cos(m-1)\theta$

Then $A_m=2\cos\theta\, A_{m-1}-A_{m-2}$

$$2\cos\theta\cos(m-1)\theta-\cos(m-2)\theta$$
$$=\cos m\theta+\cos(m-2)\theta-\cos(m-2)\theta$$
$$=\cos m\theta$$

holds for all natural numbers n. But this argument is again fallacious since we have drawn the conclusion proving the condition (2) only and omitted condition (1). But for $n=1$, the statement is clearly false since $1\neq 2$.

The above remarks show that in order to prove a certain statement for all natural numbers n, it is essential to establish both the conditions (1) and (2).

As a matter of fact, condition (1) creates the basis for carrying out induction and condition (2) gives us the right of an unlimited automatic extension of this basis.

Problem Set

Prove by the method of induction that

1. $2+5+8+11+\ldots+(3n-1)=\frac{1}{2}n(3n+1),\ n\in N$
2. $1\cdot3+2\cdot4+3\cdot5+\ldots+n(n+2)=\frac{1}{6}n(n+1)(2n+7),\ n\in N$
3. $1\cdot2\cdot3+2\cdot3\cdot4+3\cdot4\cdot5+\ldots+n(n+1)(n+2)$
$$=\frac{1}{4}n(n+1)(n+2)(n+3)$$
4. (i) $1\cdot6+2\cdot9+3\cdot12+\ldots+n(3n+3)=n(n+1)(n+2)$

   (ii) $7+77+777+\ldots+\underbrace{777\ldots7}_{n\text{ digits}}$
$$=\frac{7}{81}(10^{n+1}-9n-10)$$
5. $\frac{1}{2\cdot5}+\frac{1}{5\cdot8}+\frac{1}{8\cdot11}+\ldots+\frac{1}{(3n-1)(3n+2)}=\frac{n}{6n+4}$
6. $\frac{1}{3\cdot7}+\frac{1}{7\cdot11}+\frac{1}{11\cdot15}+\ldots+\frac{1}{(4n-1)(4n+3)}=\frac{n}{3(4n+3)}$
7. $5^{2n}-1$ is divisible by 24 where n is a +ive integer
8. $3^{2n}+7$ is divisible by 8 where n is a +ive integer
9. a^n-b^n is divisible by $a-b$ where n is a +ive integer
10. Assuming that $\log(mn)=\log m+\log n$ prove that
$$\log x^n=n\log x$$
11. $(1+x)^n>1+nx$ for $n\geqslant 2,\ n\in N,\ x>-1,\ x\neq 0$
12. $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$
13. Prove that $7^{2n}+(2^{3n-3})\,3^{n-1}$ is divisible by 25, $n\in N$ **(I I T 82)**
14. Prove that x^n-y^n is divisible by $x+y$ when n is even

It follows by mathematical induction that

$A_n = \cos n\theta$ for all positive integers n

21 Do yourself

22 Do yourself

23 For $n=2$, the inequality is valid since

$2! < \left(\frac{2+1}{2}\right)^2$ i e $2 < \frac{9}{4}$

Now suppose $m! < \left(\frac{m+1}{2}\right)^m$ (1)

We shall prove that $(m+1)! < \left(\frac{m+2}{2}\right)^{m+1}$

We have, $(m+1)! = (m+1)\, m! < (m+1)\left(\frac{m+1}{2}\right)^m$ by (1) (2)

We now prove that

$$(m+1)\left(\frac{m+1}{2}\right)^m < \left(\frac{m+2}{2}\right)^{m+1} \quad (3)$$

Inequality (3) can clearly be rewritten as

$\frac{2^{m+1}}{2^m} < \left(\frac{m+2}{m+1}\right)^{m+1}$ or $2 < \left(1+\frac{1}{m+1}\right)^{m+1}$

But by the binomial theorem

$$\left(1+\frac{1}{m+1}\right)^{m+1} = 1+(m+1)\frac{1}{m+1}+ \quad > 2$$

So the inequality (3) holds It now follows from (2) and (3) that

$$(m+1)! < \left(\frac{m+2}{2}\right)^{m+1}$$

The inequality thus holds for $n=m+1$

Hence the inequality $n! < \left(\frac{n+1}{2}\right)^n$ holds for all natural numbers $n > 1$

(ii) Proceed as in part (i)

(iii) For $n=1$ both L H S or R H S are each 1/2

Let us assume the inequality holds for $n=m$

$$\frac{(2m)!}{2^{2m}\,(m!)^2} \leqslant \frac{1}{(3m+1)^{1/2}}\,(m \geqslant 1) \quad (1)$$

Now we shall prove that it holds for $n=m+1$ for which we will have to establish that

$$\frac{(2m+2)!}{2^{2m+2}\{(m+1)!\}^2} \leqslant \frac{1}{(3m+4)^{1/2}} \quad (2)$$

15 Prove that $2^n > n$ for all natural numbers n

16 For what natural numbers n the inequality $2^n > 2n+1$ is valid ?

17 For what natural numbers n is the inequality $2^n > n^2$ valid ?

18 For any natural number $n > 1$, prove

$$\frac{1}{n+1}+\frac{1}{n+2}+\quad+\frac{1}{2n}>\frac{13}{24}$$

19 Prove the identities

(i) $\cos\alpha\cos 2\alpha=\cos 4\alpha\cos\; 2^{n-1}\alpha=\dfrac{\sin 2^n\alpha}{2^n\sin\alpha}$

(ii) $\sin\pi/3+\sin 2\pi/3+\sin 3\pi/3+\quad+\sin(n\pi/3)$

$$=2\sin\frac{n\pi}{6}\sin\frac{n+1}{6}\pi$$

20 Prove that $A_n=\cos n\theta$ if it is known that

$$A_1=\cos\theta,\ A_2=\cos 2\theta$$

and for every natural number $m > 2$, the relations

$$A_m=2A_{m-1}\cos\theta-A_{m-2}$$

hold

21 Prove that

$$(1+i)^n=2^{n/2}\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$$

22 Prove that $4^n+15n-1$ is divisible by 9 for all natural numbers n

23 If $n > 1$, prove that

(i) $n! < \left(\dfrac{n+1}{2}\right)^n$ (I I T 1981)

(ii) $\dfrac{2n!}{(n!)^2}>\dfrac{4n}{2n+1}$

(iii) Prove by mathematical Induction that

$$\frac{(2n)!}{2^{2n}(n!)^2}\le\frac{1}{(3n+1)^{1/2}}$$

for all +ive integers n (I I T 87)

24 Prove that at any time, the total number of persons on the earth who shake hands an odd number of times is even

25 Explain the method of mathematical induction and use it to show that

$$11^{n+2}+12^{2n+1}$$

$$\text{LHS} = \frac{(2m+2)(2m+1)(2m)!}{4(m+1)^2 2^{2m}(m!)^2} \leqslant \frac{(2m+2)(2m+1)}{4(m+1)^2} \times \frac{1}{(3m+1)^{1/2}} \quad \text{by (1)}$$

$$= \frac{2m+1}{2(m+1)} \frac{1}{(3m+1)^{1/2}} \leqslant \frac{1}{(3m+4)^{1/2}}$$

[To be proved by (2)]

Above will be proved if on squaring we establish that

$$(3m+4)(2m+1)^2 \leqslant 4(m+1)^2(3m+1) \qquad (3)$$

It should be noted that all, the factors are +ive

$3m\{(2m+1)^2 - 4(m+1)^2\} + 4\{(2m+1)^2 - (m+1)^2\} \leqslant 0$

or $3m\{-4m-3\} + 4\{3m^2+2m\} \leqslant 0$ or $-m \leqslant 0$

above is true since $m \geqslant 1$ Hence (2) is proved

24 To prove the assertion, we first assign to each hand shake a number in natural order Then our assertion is equivalent to the following "For every n, after a handshake with number n, the number of people who have made an odd number of handshakes is even" This statement depends on n and will be proved by induction For convenience, we call the people who have made an odd number of handshakes type A and the rest type B, that is, type B are those people who had made an even number of handshakes

After the handshake with number 1, we have two people of type A, an even number After the mth handshake, let the number of people of type A be even and let the handshake number $m+1$ take place Three cases arise the handshake number $m+1$ will occur between (a) two people of type A (b) two people of type B, (c) a person of type A and a person of type B

In case (a), two persons of type A add one handshake to their odd number of handshakes and becomes of type B, in case (b) two persons of type B becomes of type A and in case (c) a person of type A becomes of type B and a person of type B is changed into type A Thus the number of people of type A either decreases by two or increases by two or remains unchanged In any case the number remains even and the proof is complete

25 Let $A_n = 11^{n+2} + 12^{2n+1}$

The assertion is valid for $n=1$, since

$$A_1 = 11^{1+2} + (12)^{2+1} = 3059 = 133 \times 23$$

where n is natural number is divisible by 133

(Roorkee 1982)

26 If p is a prime number, prove that $n^p - n$ is divisible by p when n is a natural number greater than 1

27 (a) Prove that $5^{2n+2} - 24n - 25$ is divisible by 576

(b) Use mathematical induction to prove that $2 \cdot 7^n + 3 \cdot 5^n - 5$ is divisible by 24 for all $n > 0$ (I I T 1985)

28 If p be a natural number, then prove that $p^{n+1} + (p+1)^{2n-1}$ is divisible by $p^2 + p + 1$ for every positive integer n

(I I T 1984)

29 For any natural number n, prove the inequality

$$|\sin nx| \leqslant n\,|\sin x|,$$

30 Given $u_{n+1} = 3u_n - 2u_{n-1}$ and $u_0 = 2$, $u_1 = 3$
Prove that $u_n = 2^n + 1$ for all positive integers n

31 Let $v_{n+1} = 3v_n - 2v_{n-1}$ and $v_0 = 0$, $v_1 = 1$ Prove that $v_n = 2^n - 1$

32 Let $u_1 = 1$, $u_2 = 1$ and $u_{n+2} = u_{n+1} + u_n$ for $n \geqslant 1$ Use mathematical induction to show that

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geqslant 1$ (I I T 1981)

33 Prove that

$$x\,(x^{n-1} - na^{n-1}) + a^n\,(n-1)$$

is divisible by $(x-a)^2$ for all positive integers n greater than 1 (I I T 1977)

34 Prove by the method of induction that every even power of every odd number greater than 1 when divided by 8 leaves 1 for a remainder

35 Prove that the sum of the cubes of three' successive natural numbers is divisible by 9

36 Prove that n distinct straight lines drawn in a plane through a point divide the plane into $2n$ parts

37 Use mathematical induction to prove —
If n is any odd positive integer then $n\,(n^2-1)$ is divisible by 24 (I I T 83)

38 If $x^3 = x + 1$, then show that

$$x^{3n} = a_n x + b_n + c_n x^{-1}$$

where $a_{n+1} = a_n + b_n$, $b_{n+1} = a_n + b_n + c_n$, $c_{n+1} = a_n + c_n$

Assume that the assertion holds for $n=m$, that is, let

$$A_m = 11^{m+2} + 12^{2m+1} = 133k$$

where k is a positive integer

Then
$$\begin{aligned} A_{m+1} &= 11^{m+3} + 12^{2(m+1)+1} = 11^{m+3} + 12^{2m+3} \\ &= 11 \cdot 11^{m+2} + 144 \cdot 12^{2m+1} \\ &= 11 \cdot 11^{m+2} + 133 \cdot 12^{2m+1} + 11 \cdot 12^{2m+1} \\ &= 11\,(11^{m+2} + 12^{2m+1}) + 133 \cdot 12^{2m+1} \\ &= 11A_m + 133 \cdot 12^{2m+1} \\ &= 11 \cdot 133k + 133 \cdot 12^{2m+1} \\ &= 133\,(11k + 12^{2m+1}) \end{aligned}$$

This shows that A_{m+1} is divisible by 133. Hence by induction, A_n is divisible by 133 for all natural numbers n.

26. Let $A_n = n^p - n$

For $n=2$, we have

$$\begin{aligned} A_2 &= 2^p - 2 = (1+1)^p - 2 \\ &= (1 + {}^pC_1 + {}^pC_2 + \dots + {}^pC_r + \dots + {}^pC_{p-1} + 1) - 2 \\ &= {}^pC_1 + {}^pC_2 + \dots + {}^pC_r + \dots + {}^pC_{p-1} \qquad (1) \end{aligned}$$

Now
$${}^pC_r = \frac{p\,(p-1)\,(p-2)\dots(p-r+1)}{r\,!}$$

where r is a positive integer $\leqslant p-1$

Since p is prime, no factor of $r\,!$ can divide p. Also since $p > r$, p cannot divide any factor of $r\,!$

But ${}^pC_r$ is a positive integer

It follows that $(p-1)\,(p-2)\dots(p-r+1)$ must be divisible by $r\,!$

Hence ${}^pC_r$ is divisible by p for all positive integers $r \leqslant p-1$

It then follows from (1) that A_2 is divisible by p

Now assume A_m is divisible by p. We have

$$\begin{aligned} A_{m+1} - A_m &= (m+1)^p - (m+1) - (m^p - m) \\ &= pm^{p-1} + \frac{p\,(p-1)}{2\,!}\,m^{p-2} + \dots + pm \end{aligned}$$

Arguing as above, the expression on the right hand side is a multiple of p since p is prime

Thus $A_{m+1} = A_m + a$ multiple of p

$= a$ multiple of p by induction hypothesis on A_m

Finally we conclude that $n^p - n$ is divisible by p.

Remark This problem provides a proof of Fermat's theorem which states

If p is a prime number and n is prime to p, then $n^{p-1} - 1$ is divisible by p

39 If $x_1 x_2 x_3 \ldots x_n = 1$ ($x_i > 0$, $i=1, 2, \ldots n$), prove that
$x_1+x_2+\ldots+x_n \geqslant n$ ($n \geqslant 2$)

40 Given $a_1=\frac{1}{2}\left(a_0+\frac{A}{a_0}\right)$, $a_2=\frac{1}{2}\left(a_1+\frac{A}{a_1}\right)$ and $a_{n+1}=\frac{1}{2}\left(a_n+\frac{A}{a_n}\right)$
for $n \geqslant 2$ where $a > 0$, $A > 0$, prove that

$$\frac{a_n-\sqrt{A}}{a_n+\sqrt{A}}=\left(\frac{a_1-\sqrt{A}}{a_1+\sqrt{A}}\right)^{2^{n-1}},$$

using mathematical induction

41. Prove that $(x+a_1)(x+a_2)(x+a_3)\ldots(x+a_n)$
$=x^n+P_1 x^{n-1}+P_2 x^{n-2}+\ldots+P_{n-1} x+P_n$,
where $P_1=\Sigma a_i$, $P_2=\Sigma a_i a_j$, $P_3=\Sigma a_i a_j a_k$,
$1 \leqslant i \leqslant n \quad 1 \leqslant i < j \leqslant n \quad 1 \leqslant i < j < k \leqslant n$
$P_n=a_1 a_2 a_3 \ldots a_n$

42 Prove the binomial theorem
$(x+a)^n=x^n+{}^nC_1 x^{n-1} a+{}^nC_2 x^{n-2} a^2+\ldots{}^nC_r x^{n-r} a^r+\ldots+a^n$

43 Use mathematical induction to prove that

$$\sum_{k=0}^{n} k^2\, {}^nC_k=n(n+1)\, 2^{n-2} \text{ for } n \geqslant 1$$

(I I T 1986)

Solutions

1 Let $P(n)$ be true for $n=m$, that is, we suppose that
$P(m)=2+5+8+11+\ldots+(3m-1)=\frac{1}{2}m(3m+1)$
Now $P(m+1)=P(m)+T_{m+1}$
$=\frac{1}{2}m(3m+1)+\{3(m+1)-1\}$
$=\frac{1}{2}[3m^2+m+6m+6-2]=\frac{1}{2}[3m^2+7m+4]$
$=\frac{1}{2}(m+1)(3m+4)=\frac{1}{2}(m+1)[3(m+1)+1]$
Above relation shows that $P(n)$ is true for $n=m+1$
Also when $n=1$, $P(n)=2=\frac{1}{2}(3\cdot1+1)$,
$n=2$, $P(n)=2+5=7=\frac{1}{2}\,2\,(3\cdot2+1)$
Above relation shows that $P(n)$ is true for $n=1, 2$ etc.
Hence $P(n)$ is universally true by mathematical induction

2 Let $P(n)$ be true for $n=m$
$1\cdot3+2\cdot4+3\cdot5+\ldots+m(m+2)=\frac{1}{6}m(m+1)(2m+7)$
$P(m+1)=P(m)+T_{m+1}$
$=\frac{1}{6}m(m+1)(2m+7)+(m+1)(m+1+2)$
$=\frac{1}{6}(m+1)[m(2m+7)+6(m+3)]$
$=\frac{1}{6}(m+1)[2m^2+13m+18]$
$=\frac{1}{6}(m+1)(m+2)(2m+9)$

For if n is prime to p, it follows from the above problem that $n^{p-1}-1$ is a multiple of p

27 (a) Let $\qquad A_n = 5^{2n+2} - 24n - 25$

Then $\qquad A_1 = 5^4 - 24 - 25 = 576$

Thus A_1 is divisible by 576

Now assume that A_m is divisible by 576, that is, assume

$$A_m = 5^{2m+2} - 24m - 5 = 576k \qquad (1)$$

where k is a positive integer

Then
$$\begin{aligned} A_{m+1} &= 5^{2(m+1)+2} - 24\,(m+1) - 25 \\ &= 5^2\cdot 5^{2m+2} - 24m - 49 \\ &= 25\,[5^{2m+2} - 24m - 25] + 25\cdot 24m + 25\cdot 25 - 24m - 49 \\ &= 25A_m + 576m + 576 \\ &= 25\cdot 576k + 576\,(m+1) \qquad \text{by (1)} \\ &= 576\,(25k + m + 1) \end{aligned}$$

Hence A_{m+1} is divisible by 576

It follows by methematical induction that A_n is divisible by 576 for all positive integers

(b) Let $\quad A_n = 2\cdot 7^n + 3\cdot 5^n - 5$

Then $\quad A_1 = 2\cdot 7 + 3\cdot 5 - 5 = 14 + 15 - 5 = 24$

Hence A_1 is divisible by 24

Now assume that A_m is divisible by 24 so that we may write $\qquad A_m = 2\cdot 7^m + 3\cdot 5^m - 5 = 24k,\ k \in \mathrm{N} \qquad$ 1)

Then
$$\begin{aligned} A_{m+1} - A_m &= 2\,(7^{m+1} - 7^m) + 3\,(5^{m+1} - 5^m) - 5 + 5 \\ &= 2\cdot 7^m\,(7-1) + 3\cdot 5^m\,(5-1) \\ &= 12\,(7^m + 5^m) \end{aligned}$$

Since 7^m and 5^m are odd integers for all $m \in \mathrm{N}$, their sum must be an even integer, say

$$7^m + 6^m = 2p \quad p \in \mathrm{N}$$

Hence $A_{m+1} - A_m = 12\cdot 2p = 24p$

or $\qquad A_{m+1} = A_m + 24p = 24k + 24p$, by (1)

Hence A_{m+1} is divisible by 24

It follows by mathematical induction that A_n is divisible by 24 for all $n \in \mathrm{N}$

28 Let $f(n) = p^{n+1} + (p+1)^{2n-1}$

We have $f(1) = p^2 + p + 1$ so that $f(1)$ is divisible by p^2+p+1

Now assume that $f(m)$ is divisible by p^2+p+1 *i e* we assume that

$$p^{m+1} + (p+1)^{2m-1} = k\,(p^2 + p + 1) \qquad (1)$$

$=\frac{1}{6}(m+1)\{(m+1)+1\}[2(m+1)+7]$

Above relation shows that $P(n)$ is true for $n=m+1$

Now consider $n=1$, $P(1)=1\cdot 3=\frac{1}{6}\cdot 1(1+1)(2.1+7)$

When $n=2$, $P(n)=1\cdot 3+2\cdot 4=3+8=11$

$=\frac{1}{6}\cdot 2(2+1)(2\cdot 2+7)$

Above relation shows that $P(n)$ is true for $n=1, 2$

Hence $P(n)$ is universally true

3 Proceed as above

4 Proceed as above

5 Let $P(n)$ be true for $n=m$

$$\frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}+\frac{1}{8\cdot 11}+\cdots+\frac{1}{(3m-1)(3m+2)}=\frac{m}{6m+4}$$

$$P(m+1)=P(m)+T_{m+1}$$

$$=\frac{m}{6m+4}+\frac{1}{[3(m+1)-1][3(m+1)+2]}$$

$$=\frac{m}{2(3m+2)}+\frac{1}{(3m+2)(3m+5)}$$

$$=\frac{m(3m+5)+2}{2(3m+2)(3m+5)}=\frac{3m^2+5m+2}{2(3m+2)(3m+5)}$$

$$=\frac{(3m+2)(m+1)}{2(3m+2)(3m+5)}=\frac{m+1}{6m+10}$$

$$=\frac{m+1}{6(m+1)+4}$$

Above relation show that $P(n)$ is true for $n=m+1$

Also when $n=1$, $P(n)=\frac{1}{2\cdot 5}=\frac{1}{10}=\frac{1}{6\cdot 1+4}$

when $n=2$, $P(n)\ \frac{1}{2\cdot 5}+\frac{1}{5\cdot 8}=\frac{5}{40}=\frac{1}{8}=\frac{2}{16}=\frac{2}{6\cdot 2+4}$

Above relation shows that $P(n)$ is true for $n=1, 2$

Hence $P(n)$ is universally true

6 Proceed as above

7 Let $P(n)$ be true for $n=m$, that is, we assume that $5^{2m}-1$ is divisible by 24 so that

$$\frac{5^{2m}-1}{24}=k, \text{ say, where } k\in N$$

$$5^{2m}=24k+1 \qquad (1)$$

Now $P(m+1)=5^{2(m+1)}-1=5^{2m}\,5^2-1$

$=(24k+1)\,25-1$ by (1)

$=(24)(25)k+(25-1)=24[25k+1]$

Now $f(m+1) = p^{m+2} + (p+1)^{2m+2-1} = p^{m+2} + (p+1)^{2m-1}(p+1)^2$

$$= p^{m+2} + \{k(p^2+p+1) - p^{m+1}\}(p+1)^2 \quad \text{by (1)}$$
$$= p^{m+2} - (p+1)^2 p^{m+1} + k(p+1)^2(p^2+p+1)$$
$$= p^{m+1}(p - p^2 - 2p - 1) + k(p+1)^2(p^2+p+1)$$
$$= (p^2+p+1)\{k(p+1)^2 - p^{m+1}\}$$

Hence $f(m+1)$ is divisible by p^2+p+1

By induction, $f(x)$ is divisible by p^2+p+1 for all $n \in N$

29 The inequality clearly holds for $n=1$. We now assume

$$|\sin mx| \leqslant m\,|\sin x| \qquad (1)$$

Now $|\sin(m+1)x| = |\sin mx \cos x + \cos mx \sin x|$

$$\leqslant |\sin mx|\,|\cos x| + |\cos mx|\,|\sin x|$$
$$\leqslant |\sin mx| + |\sin x|$$
$$[\because \ |\cos x| \leqslant 1 \text{ and } |\cos mx| \leqslant 1]$$
$$\leqslant m\,|\sin x| + |\sin x| \qquad \text{by (1)}$$

Thus $|\sin(m+1)x| \leqslant (m+1)\,|\sin x|$

Hence by induction, the required inequality holds for every positive integer n

30 Putting $n=1$ in the basic formula, we get

$$u_2 = 3u_1 - 2u_0 = 3 \cdot 3 - 2 \cdot 2 = 5 = 2^2+1$$

Now assume $u_k = 2^k+1$ $(k=1, 2, 3, \ldots, m)$ and let us prove

$$u_{m+1} = 2^{m+1}+1$$

Indeed, we have

$$u_{m+1} = 3u_m - 2u_{m-1} = 3(2^m+1) - 2(2^{m-1}+1)$$
$$= 2^m(3-1)+1 = 2^{m+1}+1$$

Hence by induction, $u_n = 2^n+1$ holds for all positive integers n

31 Do yourself

32 We have to prove

$$u_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geqslant 1$

We obviously have

$$u_1 = 1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$$

and
$$u_2 = 1 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$

Hence (1) holds for $n=1$ and $n=2$

Now assume

$$u_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$
$$(k=1, 2, 3, \ldots, m)$$

(*b*) A man moves along the bank of a canal and observes a tower on the other bank He finds that the angle of elevation of the top of the tower from each of the two points A and B at a distance $6d$ apart, is α From a third point C, between A and B at a distance $2d$ from A, the angle of elevation is found to be β Find the length of the tower and the width of the canal (Roorkee 87)

(*c*) A 2 meter long object is fired vertically upwards from the midpoint of two locations A and B, 8 meters apart The speed of the object after t seconds is given by $\frac{ds}{dt}=2t+1$ meters per second Let α and β be the angles subtended by the object at A and B, respectively after one and two seconds Find the value of $\cos(\alpha-\beta)$ (I I T 87)

(*d*) A vertical pole stands at a point O on a horizontal ground A and B are points on the ground d meters apart The pole subtends angles α and β at A and B respectively AB subtends an angle γ at O Find the height of the pole (I I T 82)

(*e*) Two pillars of height a and b subtend the same angle α at a point on the line joining their feet If the pillars subtend angles β and γ at another point in the horizontal plane at which the line joining their feet subtends a right angle, prove that

$$(a+b)^2 \cot^2 \alpha = a^2 \cot^2 \beta + b^2 \cos^2 \gamma$$

46 A flag staff on the top of a house subtends the same angle β at two points distant a and b from the house and on the same side of it Prove that the length of the flag staff is

$$(a+b) \tan \beta$$

47 Due south of a tower which is leaning towards north, there are two stations at distances x, y respectively from its foot If α, β respectively be the angles of elevation of the top of the tower at these stations, show that the inclination of the tower to the horizon is

$$\cot^{-1} \frac{y \cot \alpha - x \cot \beta}{y-x}$$ (I.I T 74)

48 The extremity of the shadow of a flag staff, which is h meter high and stands on the top of a pyramid on a square base just reaches the side of the base, and is distant x meter and y meter respectively from the ends of that side prove that the height of the pyramid is

$$\sqrt{\left(\frac{x^2+y^2}{2}\right)}\tan\alpha-h$$

where α is the elevation of the sun

49 A tree standing on a horizontal plane is leaning towards east At two points situated at distances a and b exactly due west of it the angles of elevation of the top are respectively α and β Prove that the height of the top from the ground is

$$\frac{(b-a)\tan\alpha\tan\beta}{\tan\alpha-\tan\beta}$$

50 (a) A chimney leans towards north At equal distances due north and south of it in a horizontal plane the elevations of the top are α β Show that inclination of the chimney to the vertical is

$$\tan^{-1}\left[\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}\right]$$

(b) A flag leaning towards east is inclined at an angle θ to the level ground. A man walks a distance l from the foot of the flag towards west, and observes the angle of elevation of the top of the flag to be α On walking further a distance l_1 in the same direction the angle of elevation decreases by β Show that

$$\tan\theta=\frac{l}{(l+l_1)\cot\alpha-l\cot(\alpha-\beta)}$$

51 A man observes two objects in a straight line in the west On walking a distance c to the north the objects subtend an angle α in front of him and on walking a further distance c to the north they subtend an angle β Prove that the distance between the objects is

$$\frac{3c}{2\cot\beta-\cot\alpha}$$

52 A round baloon of radius r subtends an angle α at the eye of the observer, while the angle of elevation of its centre is β Find the height of the centre of baloon (I I T 70)

Now $u_{m+2} = u_{m+1} + u_m$ for $m \geqslant 1$

$\Rightarrow u_{m+1} = u_m + u_{m-1}$ for $m \geqslant 2$

Hence by induction hypothesis on u_k, we have

$$u_{m+1} = u_m + u_{m-1}$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^m - \left(\frac{1-\sqrt{5}}{2}\right)^m\right] + \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left\{\frac{1+\sqrt{5}}{2}+1\right\} - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left\{\frac{1-\sqrt{5}}{2}+1\right\}\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{6+2\sqrt{5}}{4}\right) - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{6-2\sqrt{5}}{4}\right)\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m-1}\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^{m-1}\left(\frac{1-\sqrt{5}}{2}\right)^2\right]$$
$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{m+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{m+1}\right]$$

Thus the formula (1) holds for $n=m+1$

Hence (1) holds for all positive integers n by induction.

33 Let $A_n = x\,(x^{n-1} - na^{n-1}) + a^n\,(n-1)$ (1)

For $n=2$, $A_2 = x\,(x-2a) + a^2 = x^2 - 2ax + a^2 = (x-a)^2$

Thus A_2 is divisible by $(x-a)^2$

Now assume that A_m is divisible by $(x-a)^2$ for $m \geqslant 2$, that is, assume

$$A_m = x\,(x^{m-1} - ma^{m-1}) + a^m\,(m-1) = (x-a)^2 f(x)$$

so that $x^m = mxa^{m-1} - a^m\,(m-1) + (x-a)^2 f(x)$ (2)

We then have

$$A_{m+1} = x\,\{x^m - (m+1)\,a^m\} + a^{m+1}\,m$$
$$= x\,x^m - (m+1)\,xa^m + ma^{m+1}$$
$$= x\,[mxa^{m-1} - a^m\,(m-1) + (x-a)^2 f(x)] - (m+1)\,xa^m + ma^{m+1} \text{ by (2)}$$
$$= ma^{m-1}\,[(x^2 - 2xa + a^2) + x\,(x-a)^2 f(x)] \text{ after simplification}$$
$$= ma^{m-1}\,(x-a)^2 + x\,(x-a)^2 f(x)$$
$$= (x-a)^2\,[ma^{m-1} + xf(x)]$$

10 Find the coefficient of x^r in the exansions of $\left(x+\frac{1}{x}\right)^n$ if it occurs

11 If x^{4r} occurs in the expansion of $\left(x+\frac{1}{x^2}\right)^{4n}$ prove that its coefficient is

$$\frac{(4n)\,!}{\{\frac{4}{3}(n-r)\}\,!\,\{\frac{4}{3}(2n+r)\}\,!}$$

12 If x^p occurs in the expansion of $\left(x^2+\frac{1}{x}\right)^{2n}$ prove that its coefficient is

$$\frac{(2n)\,!}{\{\frac{1}{3}(4n-p)\}\,!\,\{\frac{1}{3}(2n+p)\}\,!}$$

13 Prove that the coefficient of $(r+1)$th term in the expansion of $(1+x)^{n+1}$ is equal to the sum of the coefficients of rth and $(r+1)$th terms in the expansion of $(1+x)^n$

14 Prove that the coefficient of x^n in the expansion of $(1+x)^{2n}$ is double the coefficient of x^n in the expansion of $(1+x)^{2n-1}$

15 Show that the coefficient of x^5 in the expansion of

$$(1+x^2)^5\,(1+x)^4 \text{ is } 60$$

16 Find the coefficient of x^4 in the expansion of

(a) $(1+x+x^2+x^3)^{11}$

(b) $(2-x+3x^2)^6$,

(c) $(1+x-2x^2)^6$ and *if the complete expansion of the expression is*

$$1+a_1x+a_2x^2+\quad +a_{12}x^{12},$$

prove that

$$a_2+a_4+a_6+\quad +a_{12}=31$$

17 Find the coefficient of the middle term in the expansion of $(1+x)^n$ when n is even and the coefficients of middle terms if n is odd Your answer must not be in factorial notations

18 (a) Prove that the middle term in the expansion of $(1+x)^{2n}$ is

$$\frac{1\;3\;5\quad(2n-1)}{n\,!}\,2^n.x^n$$

(b) Show that the term independent of x in the expansion of

$$\left(x+\frac{1}{x}\right)^{2n} \text{ is } \frac{1\;3\;5\quad(2n-1)}{n\,!}\;2^n$$

(c) Which term in the expansion of the binomial

This shows that A_{m+1} is divisible by $(x-a)^2$

Hence by induction, A_n is divisible by $(x-a)^2$ for all positive integers $n > 1$

34 We first prove that the square of every odd number greater than 1 when divided by 8 leaves 1 for a remainder

First odd integer greater than 1 is 3 and $3^2=9=8\cdot1+1$

Thus the square of 3 when divided by 8 leaves 1 as remainder

Now assume $(2m+1)^2=8k+1$ (1)

where k is a positive integer

We then have

$$(2m+3)^2-(2m+1)^2=8\,(m+1)$$

so that
$$(2m+3)^2=(2m+1)^2+8\,(m+1)$$
$$=8k+1+8\,(m+1) \quad \text{by (1)}$$
$$=8\,(k+m+1)+1$$

Hence $(2m+3)^2$ when divided by 8 leaves 1 as remainder It follows by mathematical induction that for all n, $(2n+1)^2$ when divided by 8 leaves 1 as a remainder, that is, we have proved that $(2n+1)^2=8k+1$ (2)

Let us now assume that $(2n+1)^{2m}$, where m is any positive integer, when divided by 8 leaves 1 as remainder, that is, assume $(2n+1)^{2m}=8p+1$ (3)

where p is positive integer

Then
$$(2n+1)^{2m+2}=(2n+1)^{2m}\,(2n+1)^2$$
$$=(8p+1)\,(8k+1) \quad \text{by (2) and (3)}$$
$$=8\,(8pk+p+k)+1$$

This shows that $(2n+1)^{2m+2}$ when divided by 8 leaves 1 as remainder Hence by mathematical induction the required assertion is proved

35 The sum $1^3+2^3+3^3$ *i e* 36 is divisible by 9 Hence the assertion is valid when the first of three successive numbers is 1

Now let the sum

$$m^3+(m+1)^3+(m+2)^3$$

where m is some natural number, be divisible by 9, that is, a sume

$$m^3+(m+1)^3+(m+2)^3=9k \qquad (1)$$

where k is a positive integer

We now have

$$(m+1)^3+(m+2)^3+(m+3)^3$$
$$=(m+1)^3+(m+2)^3+m^3+9m^2+27m+27$$
$$=m^3+(m+1)^3+(m+2)^3+9\,(m^2+3m+3)$$

$\left[\sqrt[3]{\left(\frac{a}{\sqrt{b}}\right)}+\sqrt{\left(\frac{b}{\sqrt[3]{a}}\right)}\right]^n$ contains a and b to one and the same power

19 In the expansion of $(x+a)^n$ if the sum of odd terms be P and sum of even terms be Q, prove that

(a) $P^2-Q^2=(x^2-a^2)^n$, (Roorkee 86)

(b) $4PQ=(x+a)^{2n}-(x-a)^{2n}$

20 If a_1, a_2, a_3, a_4 are the coefficients of any four consecutive terms in the expansion of $(1+x)^n$, prove that

$$\frac{a_1}{a_1+a_2}+\frac{a_3}{a_3+a_4}=\frac{2a_2}{a_2+a_3}$$ (I I T 75)

21 If the 3rd, 4th, 5th and 6th terms in the expansion of $(x+a)^n$ be respectively a, b, c and d, prove that

$$\frac{b^2-ac}{c^2-bd}=\frac{5a}{3c}$$

22 The 3rd, 4th and 5th terms in the expansion of $(x+a)^n$ are respectively 84 280 and 560, find the velues of x, a and n

23 If the 2nd 3rd and 4th terms in the expansion of $(x+a)^n$ are 240, 720 and 1080 respectively, find x, a, and n

24 If the coefficients of second, third and fourth terms in the expansion of $(1+x)^{2n}$ are in A P , show that

$$2n^2-9n+7=0$$ (I I T 68)

25 (a) The coefficients of the second, third and fourth terms in the expansion of $(1+x)^n$ are in A P , find n

(Roorkee 51)

(b) Determine at what x the 6th term in the expansion of the binomial

$$[\sqrt{\{2^{\log(10-3^x)}\}}+\sqrt[5]{\{2^{(x-2)\log 3}\}}]^m$$

is equal to 21 if it is known that the binomial coefficients of the 2nd 3rd and 4th terms in the expansion represent respectively the first, third and fifth of an A P (the symbol log stands for logarithm to the base 10)

(c) Find n in the binomial $\left(\sqrt[3]{2}+\frac{1}{\sqrt[3]{3}}\right)$ if the ratio of 7th term from the beginning to the 7th term from the end is 1/6

26 The coefficients of 5th, 6th and 7th terms in the expansion of $(1+x)^n$ are in A P , find n

$=9k+9\,(m^2+3m+3)$ by (1)
$=9\,(k+m^2+3m+3)$

Hence by induction, the required assertion is proved

36 The assertion given in the problem is obviously true for $n=1$ since a single straight line divides the plane into 2 parts Now assume the assertion for $n=m$, that is, assume that m straight lines, divided the plane into $2m$ angles The $(m+1)$th line will clearly bisect two vertical angles, *i e*, will increases the number of parts, into which the plane is divided, by two Therefore the $(m+1)$ straight lines divide the plane into $2m+2$ parts The assertion of the problem is then proved by induction

37 Let $f(n)=n\,(n^2-1)$ Then $f(1)=0$, $f(3)=24$
Hence $f(1)$ and $f(3)$ are divisible by 24 Now assume that $f(2m+1)$ is divisible by 24, that is, we assume that

$$f(2m+1)=(2m+1)\ \{(2m+1)^2-1\}$$
$$=8m^3+12m^2+4m=24k \quad (1)$$

Now $f(2m+3)=(2m+3)\ [(2m+3)^2-1]$

$$=8m^3+36m^2+52m+24$$
$$=24k-12m^2-4m+36m^2+52m+24 \text{ from (1)}$$
$$=24k+24m^2+48m+24$$

This shows that $f(2m+3)$ is divisible by 24
It follows by induction that $f(x)$ is divisible by 24 for all odd integers n

38 For $n=1$, we have

$$x^2=x+1=1\ x+1+0\ x^{-1} \quad (4)$$

so that $a_1=1,\ b_1=1,\ c_1=0$

For $n=2,\ x^4=(x^2)^2=(x+1)^2$

$$=x^2+2x+1=\frac{x^3}{x}+2x+1$$

$$=\frac{x+1}{x}+2x+1=2x+2+1.x^{-1} \quad \text{by (1)}$$

so that $a_2=2,\ b_2=2,\ c_2=1$

Here $a_1+b_1=1+1=2=a_2,$
$a_1+b_1+c_1=1+1+0=2=b_2$

and $a_1+c_1=1+0=c_2$

Hence the assertion is valid for $n=2$.
Let us now assume that

27 In the expansion of $(1+x)^n$ the binomial coefficients of three consecutive terms are respectively 220, 495 and 792 find the value of n

28 If the coefficients of rth, $(r+1)$th and $(r+2)$th terms in the expansion of $(1+x)^{14}$ are in A P, find r

29 Find the value of

$$\frac{(18^3+7^3+3 \cdot 18 \cdot 7 \cdot 25)}{3^6+6 \cdot 243 \cdot 2+15 \cdot 81 \cdot 4+20 \cdot 27 \cdot 8+15 \cdot 9 \cdot 16+6 \cdot 3 \cdot 32+64}$$

(I I T 60)

30 Use the Binomial theorem to find 999^3

31 Evaluate $(0.99)^{15}$ and $(1.0025)^{10}$ correct to four decimal places

32 (a) Show that

$$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6=198$$

Hence show that the integral part of $(\sqrt{2}+1)^6$ is 197

(I I T 69)

(b) Evaluate

$$\{x+\sqrt{(x^2-1)}\}^6+\{x-\sqrt{(x^2-1)}\}^6$$

33 Larger of $99^{50}+100^{50}$ and 101^{50} is (I I T 82)

34 If $x=\frac{1}{4}$, find the greatest term in the expansion of $(1+4x)^8$

35 Find the greatest term in the expansion of $(2+3x)^9$ when $x=\frac{3}{2}$

36 (a) Find the value of greatest term in the expansion of

$$\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)^{20}$$

(I I T 66)

(b) Find numerically the greatest term in the expansion of

$$(2+3x)^{12} \text{ when } x=\frac{8}{5}$$

37 Find numerically the greatest term in the expansion of

$$(3-5x)^{15} \text{ when } x=\frac{1}{5}$$

38 Find the two consecutive terms in the expansion of $(3+2x)^{74}$ whose coefficients are equal

39 Show that

(a) $(1+x)^n-nx-1$ is divisible by x^2 when $n \in N$

(b) $2^{3n}-7n-1$ is divisible by 49 Hence show that

$$2^{3n+3}-7n-8$$

is divisible by 49, $n \in N$

(c) $3^{2n+2}-8n-9$ is divisible by 64, $n \in N$ (I I T 77)

(d) Show that $2^{4n}-15n-1$ is divisible by 225

40 (a) If $(2+\sqrt{3})^n=I+f$ where I and n are +ive integers and

$$x^{2m} = a_m x + b_m + c_m x^{-1}$$

where $a_{m+1} = a_m + b_m$, $b_{m+1} = a_m + b_m + c_m$,

$c_{m+1} = a_m + c_m$ Then

$$x^{2(m+1)} = x^{2m} x^2 = (a_m x + b_m + c_m x^{-1})(x+1)$$
$$= a_m x^2 + b_m x + c_m + a_m x + b_m + c_m x^{-1}$$
$$= a_m \frac{x^2}{x} + (a_m + b_m) x + b_m + c_m + c_m x^{-1}$$
$$= a_m \frac{(x+1)}{x} + a_{m+1} x + b_m + c_m + c_m x^{-1}$$
$$= a_m + a_m x^{-1} + a_{m+1} x + b_m + c_m + c_m x^{-1}$$
$$= a_{m+1} x + (a_m + b_m + c_m) + (a_m + c_m) x^{-1}$$
$$= a_{m+1} x + b_{m+1} + c_{m+1} x^{-1}$$

Hence assertion holds for

$$n = m+1$$

Therefore by mathematical induction, the assertion holds for all natural numbers n

39 If $x_1 x_2 = 1$, then

$$x_1 + x_2 = (\sqrt{x_1} - \sqrt{x_2})^2 + 2\sqrt{(x_1 x_2)}$$
$$\geqslant 2\sqrt{(x_1 x_2)} = 2,$$

so that the inequality holds true for $n=2$

Now assume that the sum of any m positive numbers whose product is 1 is greater than or equal to m and let $x_1, x_2, \ldots, x_m, x_{m+1}$ be $(m+1)$ positive integers such that

$x_1 x_2 \ldots x_m x_{m+1} = 1$

We shall prove

$x_1 + x_2 + \ldots + x_m + x_{m+1} \geqslant m+1$

If each x_i $(i = 1, 2, \ldots, m+1)$ is 1, then

$x_1 + x_2 + \ldots + x_m + x_{m+1} = m+1$ so that in this case the inequality holds true

If x_i's are not all 1, then among them there will be a number greater than 1 and a number less than 1 Let $x_m > 1$ and $x_{m+1} < 1$ We have $x_1 x_2 \ldots x_{m-1} (x_m x_{m+1}) = 1$

This is a product of m numbers and so by our induction hypothesis, we can say that $x_1 + x_2 + \ldots + x_{m-1} + x_m x_{m+1} \geqslant m$

But then

$$x_1 + x_2 + \ldots + x_{m-1} + x_m + x_{m+1} \geqslant m - x_m x_{m+1} + x_m + x_{m+1}$$
$$= m + 1 + (x_m - 1)(1 - x_{m+1})$$
$$> m+1,$$

$0 < f < 1$, show that I is an odd integer and

$$(1 \quad f)(I+f)=1$$

(b) If $(5+2\sqrt{6})^n=I+f$ where I and n are +ive integers and f is a +ive fraction less than one, show that I is an odd integer and

$$(I+f)(1-f)=1$$

41 (a) Show that the integer next above $(\sqrt{3}+1)^{2m}$ contains 2^{m+1} as a factor

(b) If $(6\sqrt{6}+14)^{2n+1}=P$, prove that the integral part of P is an even integer and $PF=20^{2n+1}$ where F is the fractional part of P

(c) Let $R=(5\sqrt{5}+11)^{2n+1}$ and $f=R-[R]$ where [ ] denotes the greatest integer function, prove that $RF=4^{2n+1}$

(I I T 88)

(d) If n be any positive integer, show that the integral part of $(7+4\sqrt{3})^n$ is an odd number Also if $(7+4\sqrt{3})^n=I+f$ where I is a +ive integer and f is a proper function, show that $(1-f)(I+f)=1$ **(Dhanbad 88)**

42 (a) The sum of the coefficients of the polynomial

$(1+x-3x^2)^{2143}$ is **(I I T 82)**

(b) Find out the sum of the coefficient in the expansion of the binomial $(5p-4q)^n$, where n is a +ive integer

(Roorkee 87)

42 Using Binomial theorem, prove the inequality

$$n^{n+1} > (n+1)^n, \quad n \geqslant 3, \quad n \in N$$

Solutions to Problem Set (A)

1 (a) Let T_{r+1} be independent of x i e index of x is zero

$$\left(3x-\frac{2}{x^2}\right)^{15}$$

$$T_{r+1}={}^{15}C_r\,(3x)^{15-r}\left(-\frac{2}{x^2}\right)^r$$

$$=(-1)^r\,{}^{15}C_r\,3^{15-r}\,2^r\,x^{15-r-2r} \quad (1)$$

The index of x is $15-3r=0$ $\quad r=5$ Hence the 6th term is the required term Putting $r=5$ in (1), we get

$$T_6=(-1)^5\,{}^{15}C_5\,3^{10}\,2^5\,x^0=-3^{10}\,2^5\,\frac{15\,!}{5\,!\,(10)\,!}$$

$$=-3^{10}\,2^5\,\frac{15\times14\times13\times12\times11}{5\;4\;3\;2\;1}$$

$$=-(3003)\,3^{10}\,2^5$$

since $x_m - 1 > 0$ and $1 - x_{m+1} > 0$

so that $(x_m - 1)(1 - x_{m+1}) > 0$

This completes the proof

40 We have to prove

$$\frac{a_n - \sqrt{A}}{a_n + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{n-1}} \quad \text{for all } n \in \mathrm{N}$$

Clearly for $n = 1$, equality (1) holds

Now from the given relation $a_2 = \frac{1}{2}\left(a_1 + \frac{A}{a_1}\right)$, we have

$$\frac{a_2}{\sqrt{A}} = \frac{1}{2\sqrt{A}}\left(a_1 + \frac{A}{a_1}\right) = \frac{a_1^2 + A}{2a_1\sqrt{A}}$$

Using componendo and dividendo, this gives

$$\frac{a_2 - \sqrt{A}}{a_2 + \sqrt{A}} = \frac{a_1^2 + A - 2a_1\sqrt{A}}{a_1^2 + A + 2a_1\sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^2$$

Hence the equality (1) holds for $n = 2$

Now assume that (1) holds for $n = m$ $(m \geqslant 2)$, that is, w assume

$$\frac{a_m - \sqrt{A}}{a_m + \sqrt{A}} = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^{m-1}} \tag{2}$$

Then by the given relation $a_{n+1} = \frac{1}{2}\left(a_n + \frac{A}{a_n}\right)$, we have

$$\frac{a_{m+1}}{\sqrt{A}} = \frac{1}{2}\left(a_m + \frac{A}{a_m}\right)\Big/\sqrt{A} \ (m \geqslant 2) \text{ so that}$$

$$\frac{a_{m+1} - \sqrt{A}}{a_{m+1} + \sqrt{A}} = \frac{\frac{1}{2}\left(a_m + \frac{A}{a_m}\right) - \sqrt{A}}{\frac{1}{2}\left(a_m + \frac{A}{a_m}\right) + \sqrt{A}} = \frac{a_m^2 - 2a_m\sqrt{A} + A}{a_m^2 + 2a_m\sqrt{A} + A}$$

$$= \left(\frac{a_m - \sqrt{A}}{a_m + \sqrt{A}}\right)^2 = \left(\frac{a_1 - \sqrt{A}}{a_1 + \sqrt{A}}\right)^{2^m} \text{ by (2)}$$

Hence the equality (1) holds for $n = m + 1$

Hence it holds for all $n \in \mathrm{N}$

41 Prove yourself

42 Prove yourself

43 Let $P(n) = \sum_{k=0}^{n} k^2 \, {}^nC_k$

(b) Here $r=6$ and $T_7=7/18$

(c) Here $r=2$ and $T_3=5/4$

(d) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)$

$$=(1+x+2x^3)\left[\left(\frac{3}{2}x^2\right)^9-9C_1\left(\frac{3}{2}x^2\right)^8\frac{1}{3x}\right.$$

$$\left.+{}^9C_6\left(\frac{3}{2}x^2\right)^3\left(\frac{1}{3x}\right)^6-9C_7\left(\frac{3}{2}x^2\right)^2\left(\frac{1}{3x}\right)^7\right]$$

In the second bracket we have to search out terms of x^0 and $1/x^3$ which when multiplied with the terms 1 and $2x^3$ in the first bracket will give a term independent of x The term containing $1/x$ will not occur in the 2nd bracket

$$1\left[{}^9C_6\,\frac{3^3}{2^3}\,\frac{1}{3^6}\right]-2x^3\left[{}^9C_7\,\frac{3^2}{2^2}\,\frac{1}{3^7}\,\frac{1}{x^3}\right]$$

$$=\left[\frac{9\,8\,7}{1\,2\,3}\,\frac{1}{8\,27}\right]-2\left[\frac{9\times8}{1\,2}\,\frac{1}{4\,243}\right]$$

$$=\frac{7}{18}-\frac{2}{27}=\frac{17}{54}$$

(e) $T_6=7$

2 $T_{r+1}={}^{20}C_r\,(2x^2)^{20-r}\,(-1/x)^r=(-1)^r\,{}^{20}C_r\,2^{20-r}\,x^{40-2r-r}$

$40-3r=10$ or 9 $\qquad 3r=30$ or 31

$r=10$, the other value does not give integral value of r so that there will be no term of x^9 Putting $r=10$

$$T_{11}=(-1)^{10}\,{}^{20}C_{10}\,2^{20-10}\,x^{40-30}=\frac{(20)\,!}{(10)\,!\,(10)\,!}\,2^{10}\,x^{10}$$

Hence the required coefficient is

$$\frac{(20)\,!}{(10)\,!\,(10)\,!}\,2^{10}$$

3 $T_{r+1}={}^{10}C_r\,3^{10-r}\,(-b)^r\,x^{20-2r-3r}$

$20-5r=5$ or -15 $\qquad r=3$ or $r=7$

∴ Coeff of x^5 is $-9720b^3$ and that of x^{-15} is $-\frac{40}{27}b^7$

4 Coefficient of x^{10} will be 6th term

$=-{}^{10}C_5$

The term independent of x in the 2nd binomial will be 6th term $\quad=-2^5\,{}^{10}C_5$

Hence their ratio is

1 2^5 or 1 32

Then $P(1)=\sum_{k=0}^{1} k^2\,{}^1C_k=0+1^2\,{}^1C_1=1$

and for $n=1$, $n(n+1)\,2^{n-2}=1\cdot 2\cdot 2^{-1}=1$

Hence the given statement holds for $n=1$

Now assume that the statement holds for $n=m$, that is, assume that

$$P(m)=\sum_{k=0}^{m} k^2\,{}^mC_k=m(m+1)\,2^{m-2},\ m\geqslant 1 \tag{1}$$

$$\text{Now } P(m+1)=\sum_{k=0}^{m+1} k^2\,{}^{m+1}C_k=\sum_{k=0}^{m+1} k^2\,({}^mC_{k-1}+{}^mC_k)$$

$$=\sum_{k=0}^{m+1} k^2\,{}^mC_{k-1}+\sum_{k=0}^{m+1} k^2\,{}^mC_k$$

$$=\sum_{k=1}^{m+1} k^2\,{}^mC_{k-1}+\sum_{k=0}^{m} k^2\,{}^mC_k$$

[First summation becomes meaningless for $k=0$ and second for $k=m+1$]

Now we can write $\sum_{k=1}^{m+1} k^2\,{}^mC_{k-1}$ as $\sum_{k=0}^{m} (k+1)^2\,{}^mC_k$

$$\text{Hence } P(m+1)=\sum_{k=0}^{m} (k+1)^2\,{}^mC_k \quad \sum_{k=0}^{n} k^2\,{}^mC_k$$

$$=\sum_{k=0}^{m} (k^2+2k+1)\,{}^mC_k+\sum_{k=0}^{m} k^2\,{}^mC_k$$

$$=2\sum_{k=0}^{m} k^2\,{}^mC_k+2\sum_{k=0}^{m} k\,{}^mC_k+\sum_{k=0}^{m} {}^mC_k$$

$$=2\sum_{k=0}^{m} k^2\,{}^mC_k+2\sum_{k=1}^{m} m\,{}^{m-1}C_{k-1}+\sum_{k=0}^{m} {}^mC_k$$

[$k\,{}^mC_k=k\,\dfrac{m!}{k!\,(m-k)!}=\dfrac{m\,(m-1)!}{(k-1)!\,(m-k)!}$

$=m\,{}^{m-1}C_{k-1}$],

$=2m\,(m+1)\,2^{m-2}+2m\,2^{m-1}+2^m$

5 The term of x^9 in $\left(x^3-\frac{1}{3x}\right)^9$ will occur in 4th term i e $r=3$ and its coefficient will be ${}^9C_3\,(x^3)^6\left(-\frac{1}{3x}\right)^3$

$$=-\frac{9!}{6!\,3!}\cdot\frac{1}{3^3}=-\frac{9\times8\times7}{3\,2\,1}\times\frac{1}{27}$$

$$=-\frac{28}{9}$$

Similarly term independent of x will be 7th i e $r=6$ and

$$T_7=\frac{18}{243}$$

(c) 10th term i e $r=9$ Value $={}^{15}C_9\,(-3)^6$

6 Coefficient of x^7 in $\left(ax^2+\frac{1}{bx}\right)^{11}$ is ${}^{11}C_5\,\frac{a^6}{b^5}$

Coefficient of x^{-7} in $\left(ax-\frac{1}{bx^2}\right)^{11}$ is ${}^{11}C_6\,\frac{a^5}{b^6}$

In case these coefficients are equal then

$$\frac{a^6}{b^5}=\frac{a^5}{b^7}\ \text{ or }\ a=\frac{1}{b}\ \text{ or }\ ab=1,\qquad {}^{11}C_5={}^{11}C_6$$

7 (i) $\left(\frac{a}{x}+bx\right)^{12}$ will have $12+1=13$ terms and middle term will be $\frac{12}{2}+1$ i e 7th $\therefore$ $T_7=924\,a^6b^6$

(ii) Middle term will be

$$\frac{10}{2}+1 \text{ i e 6th and } T_6=-252$$

(iii) Here $n=9$ the number of terms will be 10 and there will be two middle terms

$$\frac{n+1}{2}\text{ and }\frac{n+3}{2}\text{ i e 5th and 6th}$$

$$T_6=\frac{189}{8}\,x^{17},\ T_5=-\frac{21}{16}\,x^{19}$$

8 (a) $T_{r-1}={}^nC_{r-2}\,x^{r-2}$ Coeff is ${}^nC_{r-2}$

$T_{2r+3}={}^nC_{2r+2}\,x^{2r+2}$ Coeff is ${}^nC_{2r+2}$

Now ${}^nC_{r-2}={}^nC_{2r+2}$ But if ${}^nC_p={}^nC_q$ then $p+q=n$

$\therefore$ $(r-2)+(2r+2)=n=15$

or $3r=15$ $\therefore$ $r=5$

(b) ${}^{43}C_{2r}={}^{43}C_{r+1}$ $2r+r+1=43$

or $3r=42$ $r=14$

[Using (1), the result $\sum_{k=0}^{m} {}^mC_k = 2^m$ and

$$\sum_{k=1}^{m} {}^{m-1}C_{k-1} = 2^{m-1}]$$

$= 2^{m-1}(m^2+m+2m+2)$

$= (m+1)(m+2)\,2^{m-1}$

The statement holds for $n=m+1$

Hence by mathematical induction, the result holds for all $n \geqslant 1$

9 $r=6$

10 $T_{p+1}={}^nC_p\, x^{n-p}\left(\frac{1}{x}\right)^p$

Index of x is

$$n-p-p=r \qquad p=\frac{n-r}{2}$$

Coeff of x^r is

$${}^nC_{(n-r)/2}=\frac{n\,!}{(n-r)/2\,!\,(n+r)/2\,!}$$

11 Proceed as above

12 $T_{r+1}={}^{2n}C_r\,(x^2)^{2n-r}\left(\frac{1}{x}\right)^r$

Index of x is $4n-2r-r=p$

or $$r=\frac{4n-p}{3}$$

which should be an integer

$\therefore$ Coefficient is

$${}^{2n}C_{(4n-p)/3}=\frac{(2n)\,!}{(2n+p)/3\,!\,(4n-p)/3\,!}$$

13 In other words we have to prove that

$${}^{n+1}C_r={}^nC_{r-1}+{}^nC_r$$

which we have proved already in the chapter of permutations and combinations

14 x^n will occur in $(n+1)$th term and their coefficients will be ${}^{2n}C_n$ and ${}^{2n-1}C_n$ and we have to find their ratio

$$\frac{(2n)\,!}{n\,!\,n\,!}\cdot\frac{n\,!\,(n-1)\,!}{(2n-1)\,!}=\frac{2n\,(2n-1)\,!}{n\,(n-1)\,!}\cdot\frac{(n-1)\,!}{(2n-1)\,!}=2$$

Hence first coefficient is double the other

15 We have to find the coefficient of x^5 and hence in the expansion terms of x^6, x^7, etc be rejected

$$(1+x^2)^5\,(1+x)^4$$
$$=(1+{}^5C_1x^2+{}^5C_2x^4+\ )\,(1+{}^4C_1x+{}^4C_2x^2+{}^4C_3x^3+{}^4C_4x^4)$$
$$=(1+5x^2+10x^4+\ )\,(1+4x+6x^2+4x^3+x^4)$$

The term giving x^5 in the above product is

$$(5x^2)\,(4x^3)+(10x^4)\,(4x)=(20+40)\,x^5=60x^5$$

Hence the coefficient is 60

16 $1+x+x^2+x^3=(1+x)+x^2\,(1+x)$
$$=(1+x)\,(1+x^2)$$
$\therefore$ $(1+x+x^2+x^3)^{11}=(1+x)^{11}\,(1+x^2)^{11}$

9
Binomial Theorem

§ 1 Statement of binomial theorem for positive integral index

$$(x+a)^n = x^n + {}^nC_1\, x^{n-1} a + {}^nC_2\, x^{n-2} a^2 + \quad {}^nC_r\, x^{n-r} a^r + $$
$$+ {}^nC_{n-1}\, x a^{n-1} + {}^nC_n\, a^n$$

If the binomial be $x - a$ then the terms in the expansion of $(x-a)^n$ will be alternately +ive and − ive

2 General term $T_{r+1} = {}^nC_r\, x^{n-r} a^r$ or ${}^nC_r\, x^{n-r} (-a)^r$

The index of x is $n - r$ and that of a is r *i e* sum of the indices of x and a in each term is same *i e* n

3 Binomial coefficients of terms equidistant from the beginning and the end are equal

Since ${}^nC_r = {}^nC_{n-r}$,

${}^nC_0 = 1 = {}^nC_n$, ${}^nC_1 = {}^nC_{n-1}$, ${}^nC_2 = {}^nC_{n-2}$ etc

4 Number of terms and middle term

The number of terms in the expansion of $(x+a)^n$ is $n+1$

If $n = 6$ the number of terms will be $6+1=7$ and the middle term will be only one *i e* 4th *i e* $\frac{6}{2}+1=4$

If $n=7$ the number of terms will be $7+1=8$ and in this case there will be 2 middle terms *i e* 4th and 5th

$$\frac{7+1}{2}=4 \quad \text{and} \quad \frac{7+3}{2}=5$$

Hence if n is even there will be only one middle term

i e $\left(\frac{n}{2}+1\right)$ th

If n is odd then there will be two middle terms

i e $\left(\frac{n+1}{2}\right)$th and $\left(\frac{n+3}{2}\right)$th

5 Values of Binomial Coefficients

$${}^nC_0 = 1,\ {}^nC_1 = \frac{n\,!}{(n-1)\,!\,1\,!} = n$$

$${}^nC_2 = \frac{n\,!}{(n-2)\,!\,2\,!} = \frac{n\,(n-1)}{2},\ {}^nC_3 = \frac{n\,(n-1)\,(n-2)}{3\,!}$$

and in general

We want the coeff of x^4

$$=(1+{}^{11}C_1x+{}^{11}C_2x^2+{}^{11}C_3x^3+{}^{11}C_4x^4+\ \ )$$
$$\times(1+{}^{11}C_1x^2+{}^{11}C_2x^4+\ \ )$$

Collect terms which give x^4

$1\ {}^{11}C_2x^4+{}^{11}C_1\ {}^{11}C_2x^4+1\ {}^{11}C_4x^4$

Hence the coefficient of x^4 is

${}^{11}C_2+{}^{11}C_2\times{}^{11}C_1+{}^{11}C_4$

$$=\frac{11\times10}{1\ 2}+\frac{11\times10}{1\ 2}\times\frac{11}{1}+\frac{11\times10\times9\times8}{4\ 3\ 2\ 1}$$

$=55+605+330=990$

(b) $(2-x+3x^2)^6=[2-x(1-3x)]^6$

$=2^6-{}^6C_1\ 2^5\ x\ (1-3x)+{}^6C_2\ 2^4x^2\ (1-3x)^2$

$-{}^6C_3\ 2^3x^3\ (1-3x)^3+{}^6C_4\ 2^2x^4\ (1-3x)^4$

Only 3rd, 4th and 5th term will give the term of x^4

In 3rd it is $\quad {}^6C_2\ 2^4\ x^2\ (9x^2) \qquad {}^6C_2=\frac{6\ 5}{1\ 2}=15$

In 4th it is $\quad -{}^6C_3\ 2^3\ x^3\ (-3\ 3x)\ {}^6C_3=\frac{6\ 5\ 4}{1\ 2\ 3}=20$

In 5th it is $\quad {}^6C_4\ 2^2\ x^4\ 1 \qquad {}^6C_4={}^6C_2=15$

Hence the total coefficient of x^4 is

$(15\times16\times9+20\times8\times9+15\times4)$

$=2160+1440+60)=3660.$

(c) $(1-x-2x^2)^6=(1-x)^6\ (1+2x)^6$

$=(1-6x+15x^2-20x^3+15x^4\ \ )\times$

$[1+6\ (2x)+15\ (2x)^2+20\ (2x)^3+15\ (2x)^4+\ \ ]$

${}^6C_1=6,\ {}^6C_2=15,\ {}^6C_3=20,\ {}^6C_4=15$ etc

Multiply and the term of x^4 will be

$[1\ 15.2^4-6\ 20\ 2^3+15\ 15\ 2^2-20\ 6\ 2+15\ 1]\ x^4$

Hence the coefficient of x^4, is

$240-960+900\ \ 240+15=-(0+15=-45$

$(1+x-2x^2)^6=1+a_1x+a_2x^2+\quad +a_{12}x^{12}$

Putting $x=1$, we get

$0=1+a_1+a_2+\quad +a_{12}$ (1)

Putting $x=-1$ we get

$64=1-a_1+a_2-\quad +a_{12}$ (2)

Adding (1) and (2) we get

$64=2\ (1+a_2+a_4+\quad)$

$a_2+a_4+a_6+\quad +a_{12}=31$

$$^nC_r=\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$$

$$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+\ldots+{}^nC_rx^r+\ldots{}^nC_nx^n$$

6 Term containing x^r will occur in T_{r+1} for $(1+x)^n$ and it will be $^nC_r\,x^r$

7 Properties of Binomial Coefficients

The binomial coefficients are generally written as $C_0, C_1, C_2 \ldots C_n$ instead of $^nC_0, {}^nC_1, {}^nC_2 \ldots$

(a) Sum of the binomial coefficients $=2^n$ (Roorkee 78)

We know that

$$(1+x)^n=C_0+C_1x+C_2x^2+\ldots C_nx^n$$

Putting $x=1$ in the above we get

$$2^n=C_0+C_1+C_2+\ldots+C_n \quad (1)$$

or $\quad C_1+C_2+C_3+\ldots+C_n=2^n-1 \qquad C_0={}^nC_0=1$

Again putting $x=-1$ in the above we get

$$0=C_0-C_1+C_2-C_3+C_4-C_5\ldots$$

or $\quad C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots$

i e $\quad A=B$

Also from (1) $A+B=2^n \qquad 2A=2^n$ or $A=2^{n-1}=B$

Hence $C_0+C_2+C_4+\ldots=C_1+C_3+C_5+\ldots=2^{n-1}$

8 Greatest term in $(1+x)^n$, $x>0$

$$\frac{T_{r+1}}{T_r}=\frac{^nC_r\,x^r}{^nC_{r-1}\,x^{r-1}}=\frac{n!}{r!\,(n-r)!}\cdot\frac{(r-1)!\,(n-r+1)!}{n!}x$$

$$=\frac{(r-1)!\,(n-r+1)\,(n-r)!}{r\,(r-1)!\,(n-r)!}x$$

$$=\frac{(n-r+1)}{r}x$$

For a given value of x we find as to when $\frac{T_{r+1}}{T_r}\geqslant 1$

or $\quad T_{r+1}\geqslant T_r$ where r is to be a positive integer

This inequality will reduce either to the form

$$r\leqslant m+k \quad \text{or} \quad r\leqslant m$$

where m is a +ive integer and $0\leqslant k<1$ In the first case T_{m+1} is the greatest term and in the second case both T_m and T_{m+1} are *numerically* greatest terms

9 Term independent of x in the expansion of $(x+a)^n$

Let T_{r+1} be the term independent of x Equate to zero the index of x and you will find the value of r

17 If n is even then middle term is $\left(\frac{n}{2}+1\right)$th Hence

$$T_{n/2+1}={}^nC_{n/2}\,x^{n/2}$$

∴ Coefficient is ${}^nC_{n/2}=\frac{n!}{(n/2)!\,(n/2)!}$ (n is even)

$$\frac{n(n-1)(n-2)\ldots 3\,2\,1}{(n/2)!\,(n/2)!}$$

$[n(n-2)\ldots 4\,2]\,[(n-1)(n-3)\ldots 3\,1]\div(n/2)!\,(n/2)!$
Even Num   Odd Num

Each bracket contains $n/2$ numbers and the first contains all even numbers and 2nd all odd You can take 2 common from each of the $n/2$ even numbers

$$=2^{n/2}\left[\frac{n}{2}\left(\frac{n}{2}-1\right)\ldots 2\;1\right][(n-1)(n-3)\ldots 5\,3\,1]\div(n/2)!\,(n/2)!$$

$$=2^{n/2}(n/2)!\,[(n-1)(n-3)\ldots 5\,3\,1]\div(n/2)!\,(n/2)!$$

$$=2^{n/2}\frac{[(n-1)(n-3)\ldots 5\,3\,1]}{[1\,2\,3\ldots n/2]}$$

2nd Part If n is odd then the two middle terms are

$$\left(\frac{n+1}{2}\right)\text{th and}\left(\frac{n+3}{2}\right)\text{th}$$

$$T_{(n+1)/2}=T_{(n-1)/2+1}={}^nC_{(n-1)/2}\,x^{(n-1)/2}$$
$$T_{(n+3)/2}=T_{(n+1)/2+1}={}^nC_{(n+1)/2}\,x^{(n+1)/2}$$

∴ The coefficients are

${}^nC_{(n-1)/2}$ and ${}^nC_{(n+1)/2}$

But as ${}^nC_r={}^nC_{n-r}$ therefore the above two coefficients are equal It should be noted that when n is odd then both

$\frac{n-1}{2}$ and $\frac{n+1}{2}$ are integers

Now proceeding as above the coefficient is

$$2^{(n-1)/2}\frac{1\,3\,5\ldots n}{1\,2\ldots\frac{n+1}{2}}$$

18 (a) The middle term is $\frac{2n}{2}+1$ i e T_{n+1} in the expansion of $(1+x)^{2n}$

$$T_{n+1}={}^{2n}C_n\;x^n=\frac{(2n)!}{n!\,n!}x^n$$

$=x^n\,[2n(2n-1)(2n-2)(2n-3)\ldots 5\,4\,3\,2.1]\div n!\,n!$

Problem Set (A)

1 Find the term independent of x in the expansion of

(a) $\left(3x-\frac{2}{x^2}\right)^{15}$ (b) $\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$

(c) $\left(\sqrt{\frac{x}{3}}+\frac{1}{2x^2}\right)^{10}$ (I I T 65)

(d) $(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9$ (I I T 62)

(e) $(\frac{1}{2}x^{1/3}+x^{-1/5})^8$ (Roorkee 1985)

2 Find the coefficient of x^{10} and x^9 in the expansion of

$$\left(2x^2-\frac{1}{x}\right)^{20}$$

3 Find the coefficient of x^5 and x^{-15} in the expansion of

$$\left(3x^2-\frac{b}{3x^3}\right)^{10}$$

4 Prove that the ratio of the coefficient of x^{10} in $(1-x^2)^{10}$ and the term independent of x in

$$\left(x-\frac{2}{x}\right)^{10} \text{ is } 1 \quad 32$$

5 Find the coefficient of (a) x^9, (b) the term indepenuent of x in the expansion of

$$\left(x^2-\frac{1}{3x}\right)^9$$ (Roorkee 81)

(c) Independent of x in $\left(x^3-\frac{3}{x^2}\right)^{15}$ (I I T 77)

6 Find the coefficient of x^7 in

$$\left(ax^2+\frac{1}{bx}\right)^{11} \text{ and } x^{-7} \text{ in } \left(ax-\frac{1}{bx^2}\right)^{11}$$

and *find the relation between* a *and* b *so that those coefficients* are equal (I I T 67)

7 ind the number of terms and the middle term or terms in the expansion of

(a) $\left(\frac{a}{x}+bx\right)^{12}$ (b) $\left(\frac{x}{a}-\frac{a}{x}\right)^{10}$ (c) $\left(3x-\frac{x^3}{6}\right)^9$

8 (a) For what value of r the coefficients of $(r-1)$th and $(2r+3)$th terms in the expansion of $(1+x)^{15}$ are equal

(b) If the coefficients of $(2r+1)$th term and $(r+2)$th term in the expansion of $(1+x)^{43}$ are equal, find r

9 If the coefficients of $(2r+4)$th and $(r-2)$th terms in the expansion of $(1+x)^{18}$ are equal, find r (I I T 65)

$= x^n [2n(2n-2) \ldots 4\ 2]\ [(2n-1)(2n-3) \ldots 5\ 3\ 1] - n!\ n!$
even odd

$= x^n 2^n n! [(2n-1)(2n-3) \ldots 5\ 3\ 1] \times n!\ n!$

$= 2^n [1\ 3\ 5 \ldots (2n-1)]\ x^n \times n!$

(b) The term independent of x will be T_{n+1}

$$T_{n+1} = {}^{2n}C_n\, x^n \frac{1}{x^n} = {}^{2n}C_n$$

which can be shown as given by part (a)

(c) On simplifications, we easily get

$$T_{r+1} = {}^{21}C_r\, a^{7-\frac{1}{2}r}\, b^{\frac{2}{3}r-\frac{7}{2}}$$

Since the powers of a and b are the same, we have

$$7 - \frac{1}{2}r = \frac{2}{3}r - \frac{7}{2} \quad \text{or } r = 9$$

Hence $(9+1)^{th}$, that is, 10^{th} term is the required term

19 $(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + {}^nC_3 x^{n-3} a^3 + \ldots$

$= (x^n + {}^nC_2 x^{n-2} a^2 \ldots) + ({}^nC_1 x^{n-1} a + {}^nC_3 x^{n-3} a^3 + \ldots)$

$= P + Q$

$(x-a)^n = P - Q$, as the terms are alt +ive and −ive

$P^2 - Q^2 = (P+Q)(P-Q) = (x+a)^n (x-a)^n = (x^2 - a^2)^n$

$4PQ = (P+Q)^2 - (P-Q)^2 = (x+a)^{2n} - (x-a)^{2n}$

20 Let the coefficients of $T_{r+1}, T_{r+2}, T_{r+3}, T_{r+4}$ be a_1, a_2, a_3, a_4 respectively in the expansion of $(1+x)^n$

$T_{r+1} = {}^nC_r\, x^r$ coefficient $a_1 = {}^nC_r$

$a_1 = {}^nC_r,\ a_2 = {}^nC_{r+1},\ a_3 = {}^nC_{r+2},\ a_4 = {}^nC_{r+3}$

$a_1 + a_2 = {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}$ (1)

$a_2 + a_3 = {}^nC_{r+1} + {}^nC_{r+2} = {}^{n+1}C_{r+2}$ (2)

$a_3 + a_4 = {}^nC_{r+2} + {}^nC_{r+3} = {}^{n+1}C_{r+3}$ (3)

$$\frac{a_1}{a_1+a_2} = {}^nC_r \times {}^{n+1}C_{r+1}$$

$$= \frac{n!}{r!\,(n-r)!}\,\frac{(n-r)!\,(r+1)!}{(n+1)!} = \frac{r+1}{n+1}$$

Similarly $\dfrac{a_2}{a_2+a_3} = \dfrac{r+2}{n+1}$ and $\dfrac{a_3}{a_3+a_4} = \dfrac{r+3}{n+1}$

$$\frac{a_1}{a_1+a_2} + \frac{a_3}{a_3+a_4} = \frac{r+1+r+3}{n+1} = \frac{2(r+2)}{n+1} = \frac{2a_2}{a_2+a_3} \quad \text{Proved}$$

21 In $(x+a)^n$, $T_3 = a$, $T_4 = b$, $T_5 = c$, $T_6 = d$

$$T_3 = {}^nC_2\, x^{n-2}\, a^2 = \frac{n(n-1)}{1\ 2}\, x^{n-2} a^2 = a \quad (1)$$

53 A circular plate of radius a touches a vertical wall The plate is fixed horizontally at a height b above the ground A lighted candle of length c stands vertically, at the centre of the plate Prove that the breadth of the shadow thrown on the wall where it meets the horizontal ground is

$$\frac{2a}{c}\sqrt{(b^2+2bc)}$$ (I.I T 71)

54 PQ is a vertical tower P is the foot, Q the top of the tower A,B,C are three points in the horizontal plane through P The angles of elevation of Q from A B, C are equal and each is equal to θ The sides of the triangle ABC are a,b,c and the area of the triangle ABC is Δ Show that the height of the tower is $\dfrac{abc\tan\theta}{4\Delta}$ (I I T 80)

55 A flag staff stands in the middle of a square tower A man on the ground, opposite the middle of one face and distant 100 meter, just sees the flag, on his receding another 100 meter the tangents of elevation of the top of the tower and top of the flag staff are found to be $\frac{1}{2}$ and $\frac{5}{9}$ Find the dimensions of the tower and the height of the flag staff, the ground being horizontal

56 (*a*) A right circular cylindrical tower of height h and radius r stands on a horizontal plane Let A be a point in the horizontal plane and PQR be the semi circular edge of the top of the tower such that Q is the point in it nearest to A The angles of elevation of the points P and Q from A are 45° and 60° respectively Show that

$$\frac{h}{r}=\frac{\sqrt{3}(1+\sqrt{5})}{2}$$ (I I T 78)

(*b*) A sign post in the form of an isoceles triangle ABC is mounted on a pole of height h fixed to the ground The base BC of the triangle is parallel to the ground A man standing on the ground at a distance d from the sign-post finds that the top vertex A of the triangle subtends an angle β and either of the other two vertices subtends the same angle α at his feet Find the area of the triangle (I I T 1988)

57 A square tower stands upon a horizontal plane From a point in this plane from which three of its upper corners are

visible, their angular elevations are 45°, 60° and 45°. If h is the height of the tower and a the breadth of one of its sides then show that $\frac{h}{a} = \frac{\sqrt{6}(\sqrt{5}+1)}{4}$

58 A, B, C are three consecutive milestones on a straight road from each of which a distant spire is visible. The spire is observed to bear north east at A, east at B and 60° east of south at C. Prove that the shortest distance of the spire from the road is $\frac{7+5\sqrt{3}}{13}$ miles

59 A person stands in the diagonal produced of the square base of a church tower at a distance $2a$ from it and observes the angles of elevation of each of the two outer corners of the top of the tower to be 30° while that of the nearest corner is 45°. Prove that the breadth of the tower is $a(\sqrt{10}-\sqrt{2})$

60 A flag staff on the top of a tower is observed to subtend the same angle α at two points on a horizontal plane, which lie on a line passing through the centre of the base of the tower and whose distance from one another is $2a$, and an angle β at a point half way between them. Prove that the height of the flag staff is $a \sin \alpha \sqrt{\left(\frac{2 \sin \beta}{\cos \alpha \sin (\beta-\alpha)}\right)}$ (M N R 82)

61 A person moving towards a house observes that a flag staff on the top of the house subtends the greatest angle θ when his distance from the house is d. Find the heights of the flag staff and the house

62 (a) A person walks along a straight road and observes that the greatest angle subtended by two objects is α, from the point where this greatest angle is subtended he walks distance c along the road, and finds that the two objects are now in a straight line which makes an angle β with the road, prove that the distance between the objects is $c \sin \alpha \sin \beta \sec \frac{\alpha+\beta}{2} \sec \frac{\alpha-\beta}{2}$ or $\frac{2c \sin \alpha \sin \beta}{\cos \alpha + \cos \beta}$

(b) A person walking on a straight road towards a tower observes that the angle of elevation of the top of a flag staff on the tower is α, after going a distance a towards the tower he finds that the flag staff subtends the greatest

$$T_4 = {}^nC_3\, x^{n-3}\, a^3 = \frac{n(n-1)(n-2)}{1\,2\,3}\, x^{n-3}\, a^3 = b \tag{2}$$

$$T_5 = {}^nC_4\, x^{n-4}\, a^4 = \frac{n(n-1)(n-2)(n-3)}{1\,2\,3\,4}\, x^{n-4}\, a^4 = c \tag{3}$$

$$T_6 = {}^nC_5\, x^{n-5}\, a^5 = \frac{n(n-1)(n-2)(n-3)(n-4)}{1\,2\,3\,4\,5}\, x^{n-5}\, a^5 = d \tag{4}$$

Dividing (2) by (1) (3) by (2), and (4) by (3), in succession, we get

$$\frac{n-2}{3}\ \frac{a}{x} = \frac{b}{a} \tag{5}$$

$$\frac{n-3}{4}\ \frac{a}{x} = \frac{c}{b} \tag{6}$$

$$\frac{n-4}{5}\ \frac{a}{x} = \frac{d}{c} \tag{7}$$

Now dividing (5) by (6) and (6) by (7) respectively, we get

$$\frac{4(n-2)}{3(n-3)} = \frac{b^2}{ac} \tag{8}$$

$$\frac{5(n-3)}{4(n-4)} = \frac{c^2}{bd} \tag{9}$$

Subtracting (1) from both sides of (8) and (9), we get

$$\frac{n+1}{3(n-3)} = \frac{b^2-ac}{ac} \text{ and } \frac{n+1}{4(n-4)} = \frac{c^2-bd}{bd},$$

whence by division $\frac{4(n-4)}{3(n-3)} = \frac{b^2-ac}{ac} \times \frac{bd}{c^2-bd}$ (10)

Finally multiply (9) and (10),

$$\frac{5}{3} = \frac{b^2-ac}{c^2-bd}\ \frac{c}{a}$$

or

$$\frac{b^2\ \ ac}{c^2-bd} = \frac{5a}{3c}$$

22 As in Q 21

$$T_3 = \frac{n(n-1)}{1\,2}\, x^{n-2}\, a^2 = 84 \tag{1}$$

$$T_4 = \frac{n(n-1)(n-2)}{1.2.3}\, x^{n-3}\, a^3 = 280 \tag{2}$$

$$T_5 = \frac{n(n-1)(n-2)(n-3)}{1.2.3\,4}\, x^{n-4}\, a^4 = 560$$

4 (a) In the usual notations prove that

$$C_1+2C_2x+3C_3x^2+\cdots+nC_nx^{n-1}=n(1+x)^{n-1}$$

(b) $1^2C_1+2^2C_2+3^2C_3+\cdots+n^2C_n=n(n+1)2^{n-2}$

5 If n is an integer greater than 1, show that

$$a-{}^nC_1(a-1)+{}^nC_2(a-2)-\cdots+(-1)^n(a-n)=0$$

(I I T 72)

6 $C_0+2C_1+3C_2+\cdots+(n+1)C_n=2^n+n\,2^{n-1}=(n+2)\,2^{n-1}$

7 $C_0+3C_1+5C_2+\cdots+(2n+1)C_n=(n+1)\,2^n$

8 $\dfrac{C_1}{C_0}+2\dfrac{C_2}{C_1}+3\dfrac{C_3}{C_2}+\cdots+n\dfrac{C_n}{C_{n-1}}=\dfrac{n(n+1)}{2}$

9 If $C_0, C_1, C_2, \ldots, C_{15}$ are the binomial coefficients in the expansion of $(1+x)^{15}$, prove that

$$\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\cdots+15\frac{C_{15}}{C_{14}}=120$$

(I I T 62)

10 $\dfrac{C_0}{1}+\dfrac{C_2}{3}+\dfrac{C_4}{5}+\dfrac{C_6}{7}+\cdots=\dfrac{2^n}{n+1}$

11 $C_0+\dfrac{C_1}{2}+\dfrac{C_2}{3}+\cdots+\dfrac{C_n}{n+1}=\dfrac{2^{n+1}-1}{n+1}$ (I. I T 75)

12 $C_0-\dfrac{C_1}{2}+\dfrac{C_2}{3}-\dfrac{C_3}{4}+\cdots=\dfrac{1}{n+1}$

13 (a) $2C_0+2^2\dfrac{C_1}{2}+2^3\dfrac{C_2}{3}+2^4\dfrac{C_3}{4}+\cdots+2^{n+1}\dfrac{C_n}{n+1}=\dfrac{3^{n+1}-1}{n+1}$

(b) $\dfrac{1}{1!\,(n-1)!}+\dfrac{1}{3!\,(n-3)!}+\dfrac{1}{5!\,(n-5)!}+\cdots=\dfrac{2^{n-1}}{n!}$

14 In the usual notation prove that

$$2C_0+\frac{2^2}{2}C_1+\frac{2^3}{3}C_2+\cdots+\frac{2^{11}}{11}C_{10}=\frac{3^{11}-1}{11}$$

(I I T 70)

15 In the usual notation prove that

$$C_0+\frac{C_1}{2}x+\frac{C_2}{3}x^2+\cdots+\frac{C_n}{n+1}x^n=\frac{(1+x)^{n+1}-1}{(n+1)x}$$

16 Prove that

$$\frac{C_0}{2}+\frac{C_1}{3}+\frac{C_2}{4}+\cdots+\frac{C_n}{n+2}=\frac{1+n\,2^{n+1}}{(n+1)(n+2)}$$

17 (a) $C_0^2+C_1^2+C_2^2+\cdots+C_n^2=\dfrac{(2n)!}{n!\,n!}$

(I I T 73, Roorkee 83)

(b) If $(1+x)^n=C_0+C_1x+C_2x^2+\cdots+C_nx^n$, then show that the sum of the products of the C_i' taken two at a time, represented by $\sum\sum_{0\leqslant i<j\leqslant n} C_iC_j$ is equal to

$$\frac{T_3\times T_5}{T_4^2}=\frac{n^2(n-1)^2(n-2)(n-3)}{(1\,4)\,3\,4}$$

$$\frac{n^2(n-1)^2(n-2)^2}{1^2\,2^2\,3^2}=\frac{84\times 560}{280\times 280}$$

The terms of x and a cancel

$$\frac{n^2(n-1)^2(n-2)(n-3)}{1\,4\,3\,4}\quad\frac{1\,4\,9}{n^2(n-1)^2(n-2)^2}$$

$$=\frac{84\times 2}{280}=\frac{3\times 2}{10}=\frac{3}{5}$$

or $\frac{n-3}{n-2}\cdot\frac{3}{4}=\frac{3}{5}$ or $5(n-3)=4(n-2)$

or $n=7$

$$\frac{T_4}{T_3}=\frac{n-2}{3}\cdot\frac{a}{x}=\frac{280}{84}\text{ or }\frac{5}{3}\cdot\frac{a}{x}=\frac{10}{3}\quad\therefore\ x=\frac{a}{2}$$

Hence $T_3=\frac{7\,6}{1\,2}x^5a^2=84$ $\quad n=7$

or $\left(\frac{a}{2}\right)^5a^2=84\times\frac{2}{7\,6}=4$

or $a^7=2^2\cdot 2^5=2^7$ $\quad a=2$

Hence $x=\frac{a}{2}=1$ $\quad x=1,\ a=2\ n=7$

23 Proceed as above $n=5$, $x=2$ and $a=3$

24 $T_2={}^{2n}C_1\,x$, $T_3={}^{2n}C_2\,x^2$, $T_4={}^{2n}C_3\,x^3$

The coefficients are given to be in A P

$$2\ {}^{2n}C_2={}^{2n}C_1+{}^{2n}C_3$$

$$2\ \frac{(2n)!}{(2n-2)!\,2!}=\frac{(2n)}{(2n-1)!\,1!}+\frac{(2n)!}{(2n-3)!\,3!}$$

$$\frac{1}{(2n-2)}=\frac{1}{(2n-1)(2n-2)}+\frac{1}{6}$$

or $\quad 6(2n-1)=6+(2n-1)(2n-2)$

$12n-6=6+4n^2-6n+2$ or $4n^2-18n+14=0$

or $\quad 2n^2-9n+7=0$

25 (a) $n=7$ or 2 But if $n=2$ then there will be only 3 terms and hence $n=7$

(b) Since ${}^mC_1$, ${}^mC_2$ and ${}^mC_3$ are the first, third and fifth terms of an A P, we may write

$${}^mC_1=a,\ {}^mC_2=a+2d\text{ and }{}^mC_3=a+4d$$

Hence $2\,{}^mC_2={}^mC_1+{}^mC_3$

$$2^{n-1}-\frac{2n!}{2.(n!)^2}$$ (I I T 83)

18 $C_0C_1+C_1C_2+\quad +C_{n-1}C_n=\frac{(2n)!}{(n-1)!(n+1)!}$

19 $C_0C_2+C_1C_3+C_2C_4+\quad +C_{n-2}C_n=\frac{(2n)!}{(n-2)!(n+2)!}$

20 $C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\quad +C_{n-r}C_n$ (I I T 74)

$$=C_0C_r+C_1C_{r+1}+C_2C_{r+2}+\quad +C_rC_0$$

$$=\frac{(2n)!}{(n-r)!(n+r)!}$$

21 $C_0C_n+C_1C_{n-1}+\quad +C_nC_0=\frac{(2n)!}{n!n!}$

22 Prove that according as n is odd or even the value of

$$C_0^2-C_1^2+C_2^2+\quad +(-1)^n C_n^2=0$$

or $$(-1)^{n/2}\frac{n!}{(n/2)!(n/2)!}$$

23 $({}^{2n}C_0)^2-({}^{2n}C_1)^2+({}^{2n}C_2)^2+\quad +({}^{2n}C_{2n})^2=(-1)^n\ 2\ C_n$

(I I T 78)

24 Evaluate $C_0^2-C_1^2+C_2^2+\quad +(-1)^n C_n^2$ for $n=10$ and $n=11$

(I I T 73)

25 Prove that

${}^{m+n}C_r={}^mC_r+{}^mC_{r-1}\,{}^nC_1+{}^mC_{r-2}\,{}^nC_2+\quad +{}^nC_r$

where $r<m$, n, $r<n$ and m, n, r are +ive integers

26 Prove that

$$C_1^2-2.C_2^2+3\ C_3^2\quad -2nC_{2n}^2=(-1)^{n-1}\,n\,C_n$$

(I I T 79)

27 If $(1+x)^{15}=C_0+C_1x+C_2x^2+\quad +C_{15}x^{15}$,
find the value of

$$C_2+2C_3+3C_4+\quad +14C_{15}$$ (I I T 66)

28 In the usual notations prove that

$$(C_0+C_1)\,(C_1+C_2)\quad (C_{n-1}+C_n)$$

$$=\frac{(n+1)^n}{n!}C_1C_2C_3\quad C_n$$

29 If $(1+x)^n=\sum_{r=0}^{n} C_r\,x^r$ prove that

or $m(m-1)=m+\frac{m(m-1)(m-2)}{1\ 2\ 3}$

or $6m-6=6+m^2-3m+2$

[$m\neq 0$]

or $m^2-9m+14=0$

or $(m-2)(m-7)=0$

Since 6th term is 21, $m=2$ is ruled out and we have $m=7$ and

$$21={}^7C_5\times\left[\sqrt{\left\{2^{\log(10-3^x)}\right\}}\right]^{7-5}\times\left[\sqrt[5]{\left\{2^{(x-2)\log 3}\right\}}\right]^5$$

$$=\frac{7\ 6}{1\ 2}\ 2^{\log(10-3^x)}\ 2^{(x-2)\log 3}$$

$$=21\ 2^{\log(10-3^x)+\log 3^{x-2}}$$

Hence $2^{\log\{(10-3^x)\,3^{x-2}\}}=1$

or $\log\{(10-3^x)\,3^{x-2}\}=0$

or $(10-3^x)\,3^{x-2}=1$

or $3^{2x-2}-10\ 3^{x-2}+1=0$

or $3^{2x}-10\ 3^x+9=0$

or $(3^x-1)(3^x-9)=0$

$3^x-1=0$ which gives $x=0$

or $3^x=9=3^2$ which gives $x=2$

Hence $x=0$ or 2

(c) Ans $n=9$

Hint $\frac{1}{6}=\frac{{}^nC_8\,(2^{1/3})^{n-8}\,(3^{-1/3})^8}{{}^nC_{n-8}\,(2^{1/3})^8\,(3^{-1/3})^{n-8}}$

or $6^{-1}=6^{-4}\ 6^{n/3}=6^{n/3-4}$

$n/3-4=-1$ giving $n=9$

26 $n=7$ or 14

27 Let the successive terms be T_r, T_{r+1}, T_{r+2} so that the coefficients

${}^nC_{r-1}=220$, ${}^nC_r=495$, ${}^nC_{r+1}=792$

Now ${}^nC_{r-1}\div{}^nC_r=\frac{220}{495}=\frac{4}{9}$

or $\frac{n!}{(r-1)!\,(n-r+1)!}\cdot\frac{(n-r)!\,r!}{n!}=\frac{4}{9}$

$$\frac{2^2 C_0}{1\,2}+\frac{2^3 C_1}{2.3}+\quad+\frac{2^{n+2} C_n}{(n+1)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

(I. I T 76)

30 $\frac{C_1}{1}-\frac{C_2}{2}+\frac{C_3}{3}-\frac{C_4}{4}+\quad+\frac{(-1)^{n-1}}{n}C_n$

$$=1+\frac{1}{2}+\frac{1}{3}+\quad+\frac{1}{n}$$

31 Given $s_n=1+q+q^2+\quad+q^n$,

$$S_n=1+\frac{q+1}{2}+\left(\frac{q+1}{2}\right)^2+\quad+\left(\frac{q+1}{2}\right)^n,\ q\neq1$$

Prove that

${}^{n+1}C_1+{}^{n+1}C_2\,s_1+{}^{n+1}C_3 s_2+\quad+{}^{n+1}C_{n+1}\,s_n=2^n S_n$ (I I T 84)

32. Find the sum of the series—

$$\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\quad\text{upto } m \text{ terms}\right]$$

(I I T 85)

33 Prove that $\int_0^1(1-x^3)^n\,dx=\frac{C_0}{1}-\frac{C_1}{4}+\frac{C_2}{7}-\frac{C_3}{10}$

$$+\quad+\frac{(-1)^n C_n}{3n+1}$$

Hence or otherwise prove that

$$\frac{C_0}{1}-\frac{C_1}{4}+\frac{C_2}{7}-\frac{C_3}{10}+\quad+\frac{(-1)^n C^n}{3n+1}=\frac{3^n\,n\,!}{1\,4\,7\ \ (3n+1)}$$

Find the sum of

34 $3\,{}^nC_0-8\,{}^nC_1+13\,{}^nC_2-18\,{}^nC_3+\quad+$upto $(n+1)$ terms

(Roorkee 88)

Solutions to Problam Set (B)

1 $C_1+2C_2+3C_3+\quad+n\,C_n$

$$=n+2\ \frac{n\,(n-1)}{2\,!}+3\ \frac{n\,(n-1)\,(n-2)}{3\,!}+\quad+n\,1$$

$$=n\left[\ 1+\frac{n-1}{1\,!}+\frac{(n-1)\,(n-2)}{2\,!}+\quad 1\ \right]$$

Put $n-1=N$

$$\text{R H S}=n\left[1+N+\frac{N\,(n-1)}{2\,!}+\quad 1\right]$$

$$=n\,[{}^NC_0+{}^NC_1+{}^NC_2+\quad+{}^NC_N]$$

$=n\,2^N=n\,2^{n-1}$ See Q 4 for easier method or $\dfrac{r\,(r-1)\,!\,(n-r)\,!}{(r-1)\,!\,(n-r+1)(n-r)\,!}=\dfrac{4}{9}$

$\dfrac{r}{n-r+1}=\dfrac{4}{9}$

Putting $r+1$ for r in above we get

$$\frac{^nC_r}{^nC_{r+1}}=\frac{495}{792} \text{ or } \frac{r+1}{n-r}=\frac{5}{8} \qquad (2)$$

From (1), $13r=4n+4$ From (2) $13r=5n-8$

$4n+4=5n-8$ or $n=12$ and $r=4$

28 Here $^nC_{r-1}$, ^nC_r and $^nC_{r+1}$ for $n=14$ are in A.P

$2\,(^nC_r)={^nC_{r-1}}+{^nC_{r+1}}$

$$2\,\frac{n\,!}{(n-r)\,!\,r\,!}=\frac{n\,!}{(n-r+1)\,!\,(r-1)\,!}+\frac{n\,!}{(n-r-1)\,!\,(r+1)\,!}$$

$$\frac{2}{(n-r)\,(n-r-1)!\,r\,(r-1)\,!}=\frac{1}{(n-r+1)(n-r)\,(n-r-1)!\,(r-1)!}$$

$$+\frac{1}{(n-r-1)\,!\,(r+1)\,r\,(r-1)\,!}$$

or $\dfrac{2}{r\,(14-r)}=\dfrac{1}{(15-r)\,(14-r)}+\dfrac{1}{(r+1)\,r}$ $\qquad n=14$

Multiplying by $r\,(r+1)\,(14-r)\,(15-r)$

$2\,(r+1)\,(15-r)=r\,(r+1)+(14-r)\,(15-r)$

$2\,[-r^2+14r+15]=r^2+r+(210-29r+r^2]$

or $4r^2-56r+180=0$ or $r^2-14r+45=0$

$(r-5)\,(r-9)=0$ $\qquad r=5$ or 9

The two answers are justified because $^{14}C_r={^{14}C_{14-r}}$

29 The numerator is of the form

$a^3+b^3+3ab\,(a+b)=(a+b)^3$ where $a=18$ and $b=7$

$N^r=(18+7)^3=25^3$

For D^r, $3^1=9$, $3^2=9$, $3^3=27$, $3^4=81$, $3^5=243$

$^6C_1=6$, $^6C_2=15$, $^6C_3=20$, $^6C_4={^6C_2}=15$, $^6C_5={^6C_1}=6$, $^6C_6=1$

$D^r=3^6+{^6C_1}\,3^5\,2^1+{^6C_2}\,3^4\,2^2+{^6C_3}\,3^3\,2^3+{^6C_4}\,3^2\,2^4$

$+{^6C_5}\,3\,2^5+{^6C_6}\,2^6$

This is clearly the expansion of $(3+2)^6=5^6=(25)^3$

$$\frac{N^r}{D^r}=\frac{(25)^3}{(25)^3}=1$$

30 $(999)^3=(10^3-1)^3=(10^9-1)-3\;10^3\;1\;(10^3-1)$

$(a-b)^3=[a^3-b^3-3ab\,(a-b)]$

2 On putting for C_1, C_2 etc

$$\text{L H S}=n-2\,\frac{n\,(n-1)}{2\,!}+3\,\frac{n\,(n-1)\,(n-2)}{3\,!}+$$

$$=n\left[1-(n-1)+\frac{(n-1)\,(n-2)}{2\,!}+\quad\right]$$

$$n=\left[1-N+\frac{N\,(N-1)}{2\,!}+\quad\right] \text{ where } -n1=N$$

$$=n\,[{}^NC_0-{}^NC_1+{}^NC_2-\quad]$$

$$=n\;0=0 \qquad C_0-C_1+C_2-C_3+\quad=0$$

See Q 4 also for an easier method

3 $(1+x)^{\,n}=1+{}^nC_1\,x+{}^nC_2\,x^2+\quad+{}^{2n}C_{2n}\,x^{2n}$ (1)

Differentiate both sides w r t x

$2n\,(1+x)^{2n-1}=C_1+2C_2x+3C_3x^2+\quad+2nC_{2n}\,x^r$ (2)

where $C_r={}^{2n}C_r=\frac{2n\,!}{r\,!\,(2n-r)\,!}$ and $r=0, 1, 2, \quad n$

4 (a) $(1+x)^n=C_0+C_1x+C_2x^2+C_3x^3\quad C_nx^n$

Differentiate both sides w r t x

$n\,(1+x)^{n+1}=C_1+2C_2\,x+3C_3\,x^2+\quad+nC_n\,x^{n-1}$ (1)

This proves the result

Putting $x=1$, we get

$n\,2^{n-1}=C_1+2C_2+3C_3+\quad+nC_n$ This is Q 1 P 381

Putting $x=-1$ we get

$0=C_1-2C_2+3C_3-$ This is Q 92 P 381

(b) Differentiating the expansion of $(1+x)^n$ we get

$n\,(1+x)^{n-1}=C_1+2C_2\,x+3C_3\,x^2+\quad nC_n\,x^{n-1}$ (1)

Keeping in view the form of question we multiply both sides of (1) by x

$nx\,(1+x)^{n-1}=C_1x+2C_2x^2+3C_3\,x^3+\quad+nC_nx^{n-1}$

Now differentiate w r t x

$$n\,[1\,(1+x)^{n-1}+x\,(n-1)\,(1+x)^{n-2}]$$

$$=C_1+2^2\,C\,x+3^2\,C_3\,x^2+\quad+n^2\,C_n\,x^{n-1}$$

Now put $x=1$

$$n\,[2^{n-1}+(n-1)\,(2^{n-2})]=C_1+2^2C_2+3^2C_3+\quad+n^2C_n$$

$$=n\,2^{n-2}\,[2+n-1]=n\,(n+1)\,2^{n-2}$$ Proved

5 $a-C_1\,(a-1)+C_2\,(a-2)-C_3\,(a-3)+$

$+(-1)^n\,(a-n)=0$

$=10^3\,[10^6-3\,(10^3-1)]-1=1000\ \ [1000000-2997]-1$
$=1000\times 997003-1$
$=997002999$

31. $(\ 99)^{15}=(1-\ 01)^{15}$

$1-{}^{15}C_1\,(\ 01)+{}^{15}C_2\,(\ 01)^2-{}^{15}C_3\,(\ 01)^3+$

We want the answer correct to only 4 decimal places and as such we have left further expansion

$$(\ 99)^{15}=1-15(\ 01)+\frac{15\ 14}{1\ 2}(\ 0001)-\frac{15\ 14\ 13}{1\ 2\ 3}(\ 000001)$$

$=1-\ 15+\ 0105-\ 000455$
$=1\ 0105-(\ 150455)=1\ 0105-\ 1504$
$=\ 8601$

correct to four places of decimal

32 (*a*) $(x+a)^n+(x-a)^n=2\,(x^n+{}^nC_2\,x^{n-2}\,a^2$
$+{}^nC_4\,x^{n-4}a^4+{}^nC_6\,x^{n-6}\,a^6+\ \ )$

Here $n=6$, ${}^6C_2=15$, ${}^6C_4=15$, ${}^6C_6=1$, $x=\sqrt{2}$, $a=1$

$(\sqrt{2}+1)^6+(\sqrt{2}-1)^6=2\,[(\sqrt{2})^6+15(\sqrt{2})^4\ 1$
$+15(\sqrt{2})^2\ 1+1\ 1]$

$=2\,[8+15\ 4+15\ 2+1]=2\ (99)=198$

$(\sqrt{2}+1)^6=198-(\sqrt{2}-1)^6$ (*1*)

Now $\sqrt{2}-1=1\ 4-\ 1=\ 4<1$

$(\sqrt{2}-1)^6<1$, and it is certainly +ive

$0<(\sqrt{2}-1)^6<1$

Multiplying by — and changing the sign of inequality

$0>-(\sqrt{2}-1)^6>-1$

or $-1<-(\sqrt{2}-1)^6<0$ Add 198

$198-1<198-(\sqrt{2}-1)^6<198$

or $197<(\sqrt{2}+1)^6<198$, by (1)

Hence $(\sqrt{2}+1)^6$ lies between 197 and 198 so that its integral part is 197

(b) As above $x=x$, $a=\sqrt{(x^2-1)}$

$\{(x+\sqrt{(x^2-1)}\}^6+\{(x-\sqrt{(x^2-1)}\}^6$

$=2\,[x^6+15x^4\sqrt{(x^2-1)}^2+15x^2\sqrt{(x^2-1)}^4+\sqrt{(x^2-1)}^6]$

$=2\,[x^6+15x^4\,(x^2-1)+15x^2\,(x^2-1)^2+(x^2-1)^3]$

$=2\,[x^6+15\,(x^6-x^4)+15\,(x^6-2x^4+x^2)+(x^6-3x^4+3x^2-1)]$

$=64x^6-96x^4+36x^2-2$

Now $C_0=1$, $C_n=1$ Collect terms of a in one bracket and without a in the other

$$\text{L H S} = a\,[C_0-C_1+C_2-C_3 \quad +(-1)^n C_n]$$
$$+[C_1-2C_2+3C_3 \quad +(-1)^{n-1} nC_n]$$
$$-(-1)^n=(-1)(-1)(-1)^{n-1}=(-1)^{n-1}$$

Now $(1+x)^n=C_0+C_1x+C_2x^2+ \quad +C_n x^n$

Putting $x=-1$, we get $C_0-C_1+C_2+ \quad (-1)^n C_n=0$ (1)

Differentiate w r t x

$$n\,(1+x)^{n-1}=C_1+2C_2x+3C_3x^2+ \quad +nC_n x^{n-1}$$

Putting $x=-1$

$$C_1-2C_2+3C_3 \quad +(-1)^{n-1} nC_n=0 \qquad (2)$$

L H S $=a\,0+0=0$ by (1) and (2)

6 $C_0+2C_1+3C_2+ \quad +(n+1)\,C_n$

$$=(C_0+C_1+C+ \quad +C_n)+(C_1+2C_2+ \quad +n\,C_n)$$
$$=2^n+n\,2^{n-1}=2^{n-1}\,2+n\,2^{n-1} \qquad \text{by Q (1)}$$
$$=2^{n-1}\,(n+2)$$

7 $C_0+3C_1+5C_2+ \quad +(2n+1)\,C_n$

$$=(C_0+C_1+C_2+ \quad +C_n)+2\,(C_1+2C_2+ \quad +n\,C_n),$$
$$=2^n+2\,(n\,2^{n-1})=2^n+n\,2^n=2^n\,(n+1) \qquad \text{by Q (1)}$$

8 $\dfrac{C_1}{C_0}+2\dfrac{C_2}{C_1}+3\dfrac{C_3}{C_2}+ \quad +n\dfrac{C_n}{C_{n-1}}$

$$=\frac{n}{1}+2\,\frac{n\,(n-1)/(1\,2)}{n}+3\,\frac{n\,(n-1)\,(n-2)/(1\,2\,3)}{n\,(n-1)/(1\,2)}+ \quad +n\,\frac{1}{n}$$
$$=n+(n-1)+(n-2)+ \quad +1=\Sigma n=\frac{n\,(n+1)}{2}$$

9 Putting $n=15$ in Q 4 we get

$$\text{L H S}=\frac{15\,16}{2}=120=\text{R H S}$$

10 Putting the values of C_0, C_2, C_4 we get

$$\text{L H S}=1+\frac{n\,(n-1)}{3\,2\,1}+\frac{n\,(n-1)\,(n-2)\,(n-3)}{5\,4\,1}+$$
$$=\frac{1}{n+1}\left[(n+1)+\frac{(n+1)\,n\,(n-1)}{3\,!}\right.$$
$$\left.+\frac{(n+1)\,n\,(n-1)\,(n-2)\,(n-3)}{5\,!}+ \quad \right]$$

Put $n+1=N$

$$=\frac{1}{N}\left[N+\frac{N\,(N-1)\,(N-2)}{3\,!}\right.$$

33 We have

$101^{50}=(100+1)^{50}$

$=100^{50}+50\ 100^{49}+\frac{50\ 49}{1\ 2}\ 100^{48}+$

and $99^{50}=(100-1)^{50}$

$=100^{50}-50\ 100^{49}+\frac{50\ 49}{1\ 2}\ 100^{48}-$

Subtracting, we get

$101^{50}-99^{50}=2\ [50\ 100^{49}+\frac{50\ 49\ 48}{1\ 2\ 3}\ 100^{47}+$

$=100^{50}+2\ \frac{50\ 49\ 48}{1\ 2\ 3}\ 100^{47}+$

$>100^{50}$

Hence $101^{50}>99^{50}+100^{50}$

34 In the expansion of $(1+x)^n$

$\frac{T_{r+1}}{T_r}=\frac{{}^nC_r}{{}^nC_{r-1}}\frac{x^r}{x^{r-1}}=\frac{n\,!}{(n-r)\,!\ r\,!}\quad\frac{(r-1)\,!\ (n-r+1)\,!}{n\,!}\ x$

or $\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\ x$ Here $n=8,\ x=4x=4\ \frac{1}{3}$

$\frac{T_{r+1}}{T_r}=\frac{9-r}{r}\ \frac{4}{3}$ $T_{r+1}\geqslant T_r$ if $36-4r\geqslant 3r$

or $36\geqslant 7r$ or $r\leqslant 5\frac{1}{7}$

Now if r is less than or equal to $5\frac{1}{7}$, T_{r+1} is the greatest term But r is an integer Hence if $r=5$, T_{r+1} i e, 6th term is greatest

$T_6={}^8C_5(4x)^5=\frac{8\,!}{5\,!\ 3\,!}\ (4/3)^5=\frac{8\times7\times6}{6}(4/3)^5=56\ (4/3)^5$

35 Proceed as in Q 34 $(2+3x)^9=2^9\left(1+\frac{3x}{2}\right)^9$

$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\ x=\frac{10-r}{r}\ \left(\frac{3x}{2}\right)$ $n=9$

$=\frac{10-r}{r}\ \frac{3}{2}\ \frac{3}{2}=\frac{10-r}{r}\ \frac{9}{4}$

$T_{r+1}\geqslant T_r$ if $90-9r\geqslant 4r$ or $90>13r$

or $13r<90$ or $r\leqslant 6\frac{12}{13}$ $r=6$

$$+\frac{N(N-1)(N-2)(N-3)(N-4)}{5!}+\ \Big]$$

$$=\frac{1}{N}[{}^NC_1+{}^NC_3+{}^NC_5+\ \ ]$$

$$=\frac{1}{N}2^{N-1}=\frac{1}{n+1}2^n \qquad N=n+1$$

11 Proceeding as above and putting $n+1=N$

$$\text{L H S}=\frac{1}{N}[{}^NC_1+{}^NC_2+{}^NC_3+\ \ ]$$

$$=\frac{1}{N}(2^N-1)=\frac{1}{n+1}(2^{n+1}-1) \qquad N=n+1$$

12. Proceeding as in Q 6 and putting $n+1=N$

$$\text{L.H S}=\frac{1}{N}[{}^NC_1-{}^NC_2+{}^NC_3-{}^NC_4+\ \ ]$$

$$=\frac{1}{N}[{}^NC_0-\{{}^NC_0-{}^NC_1+{}^NC_2-\ \ \}]$$

$$=\frac{1}{N}[1-0]=\frac{1}{N}=\frac{1}{n+1} \qquad N=n+1$$

Alternative method for Q 10 11, 12

$(1+x)^n=C_0+C_1x+C_2x^2+\ \ +C_n x^n$

Integrating w r t x

$$\frac{(1+x)^{n+1}}{n+1}=C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+\ \ +C_n\frac{x^{n+1}}{n+1}+A$$

where A is a constant of integration and on putting $x=0$ in both sides we get

$$A=-\frac{1}{n+1}$$

$$\frac{(1+x)^{n+1}}{n+1}-\frac{1}{n+1}=C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+ \qquad (1)$$

Now putting $x=1$ and -1 in the above we get

$$\frac{2^{n+1}-1}{n+1}=C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ \ +\frac{C_n}{n+1},\ (x=1) \quad \text{This is Q 11}$$

and $$-\frac{1}{n+1}=-C_0+\frac{C_1}{2}-\frac{C_2}{3}\ \ ,(x=-1)$$

or $$\frac{1}{1+n}=C_0-\frac{C_1}{2}+\frac{C_2}{3}-\frac{C_3}{4}+ \qquad \text{This is Q 12}$$

Adding the two results of Q 11 and Q 12 we get

$$\frac{2^{n+1}-1+1}{n+1}=2\left(C_0+\frac{C_2}{3}+\frac{C_4}{5}+\ \ \right)$$

Hence $T_{r+1}=T_7=2^9\,{}^9C_6\left(\frac{3x}{2}\right)^6=2^9\,\frac{9!}{3!\,6!}\,\frac{3^{12}}{2^{12}}\qquad x=\frac{3}{2}$

$=\frac{9\times8\times7}{1\;2\;3}\,\frac{3^{12}}{2^3}=\frac{3^{13}\,7}{2}$

36 (a) $T_8=\frac{25840}{9}$

(b) $T_8={}^{12}C_7\,2^7\left(\frac{5}{2}\right)^5$

37 $\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\left(\frac{5}{3}\,x\right)\qquad(3-5x)^{15}=3^{15}\left(1-\frac{5}{3}\,x\right)^{15}$

$=\frac{15-r+1}{r}\left(\frac{5}{3}\;\frac{1}{5}\right)\geqslant1\qquad x=\frac{1}{5}$

$16-r>3r\qquad 4r\leqslant16\qquad r\leqslant4$

$r=3,\ r=4$

T_4 and T_5 are numerically equal to each other and greater than any other term

$T_4=3^{15}\,{}^{15}C_3\left(-\frac{5}{3}\;\frac{1}{5}\right)^3=3^{15}\,\frac{(15)!}{(12)!\,3!}\;\frac{1}{3^3}$ numerically

$=3^{12}\,\frac{15\times14\times13}{1\;2\;3}=455\,(3^{12})$

$T_5=3^{15}\,{}^{15}C_4\,(-5/3\;1/5)^4=3^{15}\,\frac{(15)!}{(11)!\,4!}\;\frac{1}{3^4}$

$=3^{11}\times\frac{15\times14\times13\times12}{1\;2\;3\;4}=3^{12}\times5\times7\times13=455\times(3)^{12}$

38 $(3+2x)^{74}=3^{74}\left(1+\frac{2x}{3}\right)^{74}$

$\frac{T_{r+1}}{T_r}=\frac{n-r+1}{r}\left(\frac{2x}{3}\right)$

Coeff $=\frac{n-r+1}{r}\;\frac{2}{3}=1$ by given condition

or $\quad 2\,(74-r+1)=3r\qquad$ or $\qquad 150=5r$

$r=30$

Hence 30th and 31st terms will have their coefficients equal

39 (a) $(1+x)^n=1+nx+\frac{n\,(n-1)}{1\;2}\;x^2+\frac{n\,(n-1)\,(n-2)}{1\;2\;3}\,x^3+$

$(1+x)^n-nx-1=x^2\left[\frac{n\,(n-1)}{1\;2}+\frac{n\,(n-1)\,(n-2)}{1\;2\;3}\;x+\quad\right]$

or $\frac{2^n}{n+1}=C_0+\frac{C_2}{3}+\frac{C_4}{5}+$

This is Q 1

13 (a) Putting for C_0, C_1, C and multiplying and dividing by $n+1$ and putting $n+1=N$ we get

$$L H S=\frac{1}{n+1}[{}^NC_1\,2+{}^NC_2\,2^2+{}^NC_3\,2^3+\quad]$$

Now $(1+x)^N=1+{}^NC_1\,x+{}^NC_2\,x^2+{}^NC_3\,x^3$

$$(1+2)^N-1={}^NC_1\,2+{}^NC_2\,2^2+{}^NC_3\,2^3+$$

$$L H S=\frac{3^N-1}{n+1}=\frac{3^{n+1}-1}{n+1}$$

(b) Multiply both sides by $n!$ the question reduces to

$$\frac{n!}{1!(n-1)!}+\frac{1}{3!}\frac{n!}{(n-3)}+\frac{1}{5!}\frac{n!}{(n-5)!}+\quad=2^{n-1}$$

or $n+\frac{n(n-1)(n-2)}{3!}+\frac{n(n-1)(n-2)(n-3)(n-4)}{5!}+$

$=2^{n-1}$

or $C_1+C_3+C_5+\quad=2^{n-1}$, which is true

14 $(1+x)^n=C_0+C_1\,x+C_2\,x^2+C_3\,x^3+\quad+C_n\,x^n$ (1)

we have to prove that

$$2C_0+\frac{2^2}{2}C_1+\frac{2^3}{3}C_2+\quad+\frac{2^{11}}{11}C_{10}=\frac{3^{11}-1}{11}$$

Integrating both sides of (1) w r t x

$$\frac{(1+x)^{n+1}}{n+1}=C_0\,x+C_1\,\frac{x^2}{2}+C\,\frac{x^3}{3}+\quad+C_n\frac{x^{n+1}}{n+1}+C \tag{2}$$

where C is constant of integration

Putting $x=0$ we get $\frac{1}{n+1}=C$

Now putting $x=2$ and $n=10$, in (2) we get $C=\frac{1}{11}$

$$\frac{3^{11}}{11}=2C_0+\frac{2^2}{2}C_1+\frac{2^3}{3}C_2+\quad+\frac{2^{11}}{11}C_{10}+\quad+\frac{1}{11}$$

Transfer $\frac{1}{11}$ on the other side and we get the result

15 If we take x from R H S to L H S the question reduces to

$$C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+\quad+C_n\frac{x^{n+1}}{n+1}=\frac{(1+x)^{n+1}-1}{n+1}$$

This we have proved in result 2 of Q 14 on putting the value of $C=\frac{1}{n+1}$

From above it is clear that $(1+x)^n-nx-1$ is divisible by x^2 (1)

(b) We have to prove that $2^{3n}-7n-1$ is divisible by 49

or $(1+7)^n-n\,7-1$ is divisible by 7^2

Choosing $x=7$ in (1) we prove the result

Again $2^{3n+3}-7n-8=8\,2^{3n}-56n-8+49n$

$=8\,(2^{3n}-7n-1)+49n$

which is divisible by 49 by the above case

(c) $(1+x)^n-nx-1$, is divisible by x^2

Choose $x=8$

9^n-8n-1 is divisible by 64

Choosing $n+1$ in place of n we can say

$9^{n+1}-8\,(n+1)-1$ is divisible by 64

or $3^{2n+2}-8n-9$ is divisible by 64

(d) $2^{4n}=(2^4)^n=(16)^n=(1+15)^n$

$2^{4n}=1+{}^nC_1\,15+{}^nC_2\,15^2+{}^nC_3\,15^3+$

$2^{4n}-1-15n=15^2\,[{}^nC_2+{}^nC_3\,15+\quad]$

$=225K$

where K is an integer

Hence $2^{4n}-15n-1$ is divisible by 225

40 (a) $(2+\sqrt{3})^n=I+f$

or $I+f=2^n+{}^nC_1\,2^{n-1}\,\sqrt{3}+{}^nC_2\,2^{n-2}\,(\sqrt{3})^2$

$+{}^nC_3\,2^{n-3}\,(\sqrt{3})^3+$ (1)

Now $0<2-\sqrt{3}<1$ $\quad 0<(2-\sqrt{3})^n<1$

Let $(2-\sqrt{3})^n=f'$ where $0<f'<1$

$f'=2^n-{}^nC_1\,2^{n-1}\,\sqrt{3}+{}^nC_2\,2^{n-2}\,(\sqrt{3})^2-{}^nC_3\,2^{n-3}\,(\sqrt{3})^3$ (2)

Adding (1) and (2)

$I+f+f'=2\,[2^n+{}^nC_2\,2^{n-2}\,3+\quad]$

or $I+f+f$ $=$even integer (3)

Now $0<f<1$ and $0<f'<1$

$0<f+f\;<2$

Hence from (3) we conclude that $f+f'$ is an integer between 0 and 2

$f+f'=1$ $\quad f'=1-f$ (4)

From (3) and (4) we get $I+1=$even integer

I is an odd integer

Now $I+f=(2+\sqrt{3})^n$, $f'=1-f=(2-\sqrt{3})^n$

$(I+f)\,(1-f)=[(2+\sqrt{3})\,(2-\sqrt{3})]^n=(4-3)^n=1$

16 $(1+x)^n = C_0 + C_1 x + C \; x^2 + C_3 x^3 +$

Multiply both sides by x

$$x(1+x)^n = C_0 x + C_1 x^2 + C_2 x^3 + C_3 x^4 +$$

Now integrate w r t x

$$x \frac{(1+x)^{n+1}}{n+1} - \int 1 \; \frac{(1+x)^{n+1}}{n+1} \, dx$$

$$= \frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)(n+2)} = C_0 \frac{x^2}{2} + C_1 \frac{x^3}{3} + C_2 \frac{x^4}{4} + \quad + A$$

Putting $x=0$ we get $A = \dfrac{-1}{(n+1)(n+2)}$

Put the value of A and transfer to the other side

$$\frac{x(1+x)^{n+1}}{n+1} - \frac{(1+x)^{n+2}}{(n+1)\;(n+2)} + \frac{1}{(n+1)\;(n+2)}$$

$$= C_0 \frac{x^2}{2} + C_1 \frac{x^3}{3} + C_2 \frac{x^4}{4} +$$

Now put $x=1$ we get

$$\frac{2^{n+1}}{n+1} - \frac{2\;2^{n+1}}{(n+1)(n+2)} + \frac{1}{(n+1)\;(n+2)}$$

$$= \frac{C_0}{2} + \frac{C_1}{3} + \frac{C}{4} + \frac{C_3}{5} + \quad + \frac{C_n}{n+2}$$

$$\text{L H S} = \frac{2^{n+1}\;(n+2-2)}{(n+1)\;(n+2)} + \frac{1}{(n+1)\;(n+2)} = \frac{n\;2^{n+1}+1}{(n+1)\;(n+2)}$$

This proves the result

17 (a) $(1+x)^n = C_0 + C_1 x + C_2 x^2 + \quad + C_n x^n$ (1)

$$\left(1+\frac{1}{x}\right)^n = C_0 + C_1 \frac{1}{x} + C_2 \frac{1}{x^2} + \quad + C_n \frac{1}{x^n} \qquad (2)$$

If we multiply (1) and (2) then

$$C_0^2 + C_1^2 + C_2^2 + \quad + C_n^2$$

is the term independent of x and hence it is equal to the term independent of x in the product

$$(1+x)^n \left(1+\frac{1}{x}\right)^n$$

or in $\dfrac{1}{x^n}(1+x)^{2n}$ or the term containing x^n in $(1+x)^{2n}$

Clearly the coefficient of x^n in $(1+x)^{2n}$ is

$(I+f)(1-f)=1$

(b) Proceed exactly as above

41 (a) Let $(\sqrt{3}+1)^{2m}=I+f$ and $(\sqrt{3}-1)^{2m}=f'$ where $0<f<1$ and $0<f'<1$ and I is an integer

Thus $I+f+f'=(\sqrt{3}+1)^{2m}+(\sqrt{3}-1)^{2m}$

$=(4+2\sqrt{3})^m+(4-2\sqrt{3})^m$

$=2^m[(2+\sqrt{3})^m+(2-\sqrt{3})^m]$

$=2^{m+1}[2^m+{}^mC_2\,2^{m-2}(\sqrt{3})^2+\ldots]$ (1)

Now as in Q 40 $f+f'=1$ and so $I+f+f'$ is an integer next above $(\sqrt{3}+1)^{2m}$ which by (1) contains 2^{m+1} as a factor

(b) Let $(6\sqrt{6}+14)^{2n+1}=P=I+F$

where I is a positive integer and $0<F<1$

clearly $(6\sqrt{6}-14)^{2n+1}=F'$ where $0<F'<1$

Subtracting we get

$(6\sqrt{6}+14)^{2n+1}-(6\sqrt{6}-14)^{2n+1}=I+F-F'$

or $2[{}^{2n+1}C_1(6\sqrt{6})^{2n}\times 14+{}^{2n+1}C_3(6\sqrt{6})^{2n-2}\times 14^3+\ldots]$

$=I+F-F'$

$I+F-F'=$ an even integer

Since I is an integer $F-F'$ must also be an integer

But since $0<F<1$ and $0<F'<1$, the only possibility is that $F-F'=0$ or $F=F'$

Hence I is an even integer

Also $PF=PF'=(6\sqrt{6}+14)^{2n+1}(6\sqrt{6}-14)^{2n+1}$

$=(216-196)^{2n+1}=20^{2n+1}$,

(c) Let I and F denotes respectively the integral and the factional part of R Then by definition of $[R]$,

$R-[R]$ is the fractional part of R

$F=f$

$R=I+f=(5\sqrt{5}+11)^{2n+1}$ $0<f<1$

$f'=(5\sqrt{5}-11)^{2n+1}$ $0<f'<1$

Subtracting $I+f-f'=2[{}^{2n+1}C_1(5\sqrt{5})^{2n}\times 11+{}^{2n+1}C_3$

$(5\sqrt{5})^{2n-2}\,11^3+\ldots]$

$=$ an even integer

Since I is an integer $f-f'$ must also be an integer But $0<f<1, 0<f'<1$ it follows that

$f-f'=0$ or $f=f'$

Now $Rf=Rf'=(5\sqrt{5}+11)^{2n+1}(5\sqrt{5}-11)^{2n+1}$

$=(125-121)^{2n+1}=4^{2n+1}$

(d) Proceed as above in part (c)

$$^{2n}C_n = \frac{(2n)!}{n!\,n!}$$

$$C_0^2 + C_1^2 + C_2^2 + \quad + C_n^2 = \frac{(2n)!}{n!\,n!}$$

(b) $\sum\sum_{0 \leqslant i < j \leqslant n} C_i C_j = \frac{1}{2}[(\Sigma C_i)^2 - \Sigma C_i^2]$

$$= \frac{1}{2}\left[(2^n)^2 - \frac{2n!}{(n!)^2}\right] \text{ by part (a)}$$

$$= 2^{2n-1} - \frac{2n!}{2(n!)^2}$$

18, 19, 20 Multiplying (1) and (2) of Q 17 (a) L H S are respectively the coefficients of x, x^2 and x^r in $\frac{1}{x^n}$ $(1+x)^{2n}$ or the conflicients of x^{n+1}, x^{n+2}, x^{n+r} in $(1+x)^{2n}$ These are $^{2n}C_{n+1}$, $^{2n}C_{n+2}$, $^{2n}C_{n+r}$ respectively

or $\frac{(2n)!}{(n-1)!\,(n+1)!}$, $\frac{(2n)!}{(n-2)!\,(n+2)!}$ and $\frac{(2n)!}{(n-r)!\,(n+r)!}$

21 We know that $C_n = C_0$, $C_{n-1} = C_1$, $C_{n-2} = C_2$
Hence the question reduces to
$C_0^2 + C_1^2 + C_2^2 + \quad + C_n^2$ i e Q 17 (a), and its value is

$$\frac{(2n)!}{n!\,n!}$$

22 $(x+1)^n = C_0 x^n + C_1 x^{n-1} + C_2 x^{n-2} \quad + C_n$ (1)

$(1-x)^n = C_0 - C_1 x + C_2 x^2 + \quad + (-1)^n C_n x^n$ (2)

$C_0^2 - C_1^2 + C_2^2 \quad + (-1)^n C_n^2$ is the coefficient of x^n in the product of (1) and (2) or the coefficient of x^n in the product $(x+1)^n (1-x)^n$ or in $(1-x^2)^n$

Now $x^n = (x^2)^{n/2}$, so that the term containing x^n in $(1-x^2)^n$ will be $(n/2+1)$th

$T_{n/2+1}$ of $(1-x^2)^n = {}^nC_{n/2}\,(1-x^2)^{n/2}$

$$= (-1)^{n/2} \frac{n!}{(n/2)!\,(n/2)!}\, x^n$$

Hence the coefficient is $(-1)^{n/2} \frac{n!}{(n/2)!\,(n/2)!}$

But the index of x^2 can be $n/2$ only when $n/2$ is an integer i e when n is even and hence from (3) we get the coefficient

$$(-1)^{n/2} \frac{n!}{(n/2)!\,(n/2)!}.$$

42. (a) Putting $x=1$ in $(1+x-3x^2)^{2143}$ we get sum of the coefficients as

$$(1+1-3)^{2143}=(-1)^{2143}=-1$$

(b) Sum of the coefficient is obtained by putting $p=q=1$
Required sum $=(5-4)^n=1^n=1$

43 We have to prove $n>\left(\frac{n+1}{n}\right)^n$ or $\left(1+\frac{1}{n}\right)^n<n$ for $n\geqslant 3$

Now $\left(1+\frac{1}{n}\right)^n=1+n\,\frac{1}{n}+\frac{n(n-1)}{2!}\,\frac{1}{n^2}$

$$+\frac{n(n-1)(n-2)}{3!}\,\frac{1}{n^3}+\dots$$

$$+\frac{n(n-1)(n-2)\dots\{n-(n-1)\}}{n!}\,\frac{1}{n^n}$$

$$=1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right)+\frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)$$

$$+\frac{1}{4!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\left(1-\frac{3}{n}\right)$$

$$+\dots+\frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right)$$

$$<1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}$$

$$<1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\dots+\frac{1}{2^{n-1}}$$

$$=1+1\,\frac{\{1-(1/2)^n\}}{1-1/2}=1+2\left\{1-\left(\frac{1}{2}\right)^n\right\}$$

$$=3-\frac{1}{2^{n-1}}$$

$$\left(1+\frac{1}{n}\right)^n<3$$

Since $n\geqslant 3$, we have $\left(1+\frac{1}{n}\right)^n<n$ as required

Problem Set (B)

Prove that

1 $C_1+2C_2+3C_3+\dots+n\,C_n=n\,2^{n-1}$ **(Roorkee 61)**

2 $C_1-2C_2+3C_3-4C_4+\dots=0$

3 $C_1+2C_2\,x+3C_3\,x^2+\dots+2nC_{2n}\,x^{2n-1}=2n\,(1+x)^{2n-1}$

where $C_r=\frac{(2n)!}{r!\,(2n-r)!}$, $r=0, 1, 2, 3\ \dots\ 2n$

But if n is odd then $n/2$ is not an integer and as such there will be no term like $(-x^2)^{n/2}$ or $(-1)^{n/2} x^n$ In other words the term containing x^n will not occur and hence in this case the answer is zero

23 This is just like Q 22 if we take into consideration $(1+x)^{2n}$ and $(x-1)^{2n}$ and multiply their corresponding expansions and equate the coefficient of x^{2n} on both sides

24 Proceed as in Q 22
For $n=10$, the value

$$=(-1)^{10/2}\ {}^{10}C_5=(-1)^5 \frac{(10)!}{5!\,5!}=(-1)^5 \frac{10\times9\times8\times7\times6}{5\,4\,3\,2\,1}$$
$$=-252$$

And for $n=11$, the value$=0$ as $11/2$ is not an integer

25 Multiply $(1+x)^m$ and $(1+x)^n$ and the expression on R H S is coefficient of x^r in the product
Now coeff of x^r in $(1+x)^{m+n}$ is ${}^{m+n}C_r$

26 Differentiate the expansion of $(1+x)^{2n}$ we get

$$2n\,(1+x)^{2n-1}=C_1+2C_2x+3C_3\,x^2+\quad+2n\,C_{2n}\,x^{2n} \qquad (2)$$
$$(1-x)^{2n}=C_0-C_1x+C_2x^2-\quad+(-1)^r\,C_rx^r+\quad+C_{2n}x^{2n}$$

where $\quad C_r={}^{2n}C_r \qquad (3)$

Replacing x by $1/x$ in (3) we get

$$\left(1-\frac{1}{x}\right)^{2n}=C_0-C_1\frac{1}{x}+C_2\frac{1}{x^2}-\quad C_{2n}\frac{1}{x^{2n}} \qquad (4)$$

Multiplying (2) and (4) we get

$$2n\,(1+x)^{2n-1}\left(1-\frac{1}{x}\right)^{2n}=(C_1+2C_2x+3C_3x^2+\quad+2n\,C_{2n}x^{2n-1})$$
$$\times\left(C_0-C_1\,\frac{1}{x}+C_2\,\frac{1}{x^2}-\quad+C_{2n}\,\frac{1}{x^{2n}}\right)$$

The coefficient of $1/x$ in R H S is

$$-(C_1^2-2C_2^2+3C_3^2\quad-2nC_{2n}^2) \qquad (5)$$

Also the coefficient of $\frac{1}{x}$ in $2n\,(1+x)^{2n-1}\left(1-\frac{1}{x}\right)^{2n}$

$$=\text{Coefficient of }\frac{1}{x}\text{ in }2n\frac{(1+x)^{2n-1}\,(x-1)^{2n}}{x^{2n}}$$
$$=\text{Coeff of }\frac{1}{x}\text{ in }\frac{2n}{x^{2n}}\,(1-x^2)^{2n-1}\,(1-x)$$
$$=\text{Coeff of }x^{2n-1}\text{ in }2n\,(1-x^2)^{2n-1}\,(1-x)$$
$$=2n\,(-1)^{n-1}\,{}^{2n-1}C_{n-1}\,(-1)$$

angle β Prove that the length of the flag staff is

$$\frac{2a \sin\alpha \sin\beta}{\cos\alpha - \sin(\beta-\alpha)}$$

63 A flag staff is on the top of tower which stands on a horizontal plane A person observes the angles α and β, subtended at a point on the horizontal plane by the flag staff and the tower he then walks a known distance a towards the tower and finds that the flag staff subtends the same angle as before, prove that the height of the tower and the length of the flag staff are respectively

$$\frac{a \sin\beta \cos(\alpha+\beta)}{\cos(\alpha+2\beta)} \text{ and } \frac{a \sin\alpha}{\cos(\alpha+2\beta)}$$

64 A tower on a hill subtends the same angle at two points A and B on the level ground and the angles of elevation of the top of the tower from these points are respectively α and β If the tower and the two points of observation are in the same vertical plane, prove that the height of the tower is

$$AB \frac{|\cos(\alpha+\beta)|}{\sin(\alpha-\beta)}$$

65 A man standing on a plane observes a row of equal and equidistant pillars, the 10th and 17th of which subtend the same angle that they would do if they were in the position of the first and were respectively $\frac{1}{2}$ and $\frac{1}{3}$ of their height Prove that, neglecting the height of the man's eye the line of pillars is inclined to the line drawn from his eye to the first at an angle whose secant is nearly 2 6

66 A tower stands on the edge of a circular lake $ABCD$ The foot of the tower is at D and the angles of elevation of its top at A, B, C are respectively α β, and γ If the angles BAC, ACB are each θ, show that

$$2\cos\theta \cot\beta = \cot\alpha + \cot\gamma$$

67 (a) From a point on the ground, the angles of elevation of a bird flying with a constant speed in a horizontal direction at equal intervals of time are α, β, γ and δ Prove that

$$\cot^2\alpha - \cot^2\delta = 3(\cot^2\beta - \cot^2\gamma)$$

(b) A person on the summit of mountain observes that the angles of depression of a car moving on a straight road at three consecutive kilo meter stones are α, β and γ respectively Prove that the height of the mountain is

$$\left[\frac{2}{\cot^2\alpha-2\cot^2\beta+\cot^2\gamma}\right]^{1/2}$$

Solutions

1 Let $AB=h$ be the height of tower at a distance x from the post $PQ=a$. AB subtends angles α and β at P and Q respectively. We have to determine h and x in terms of known quantities α, β and a.

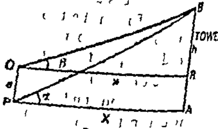


Fig 66

$$AB=x\tan\alpha,\quad BR=x\tan\beta$$

$$PQ=AR=AB-BR=x(\tan\alpha-\tan\beta)$$

$$x=\frac{PQ}{\tan\alpha-\tan\beta}=\frac{a}{\tan\alpha-\tan\beta}=\frac{a\cos\alpha\cos\beta}{\sin(\alpha-\beta)}$$

$$h=AB=x\tan\alpha=\frac{a\cos\alpha\cos\beta}{\sin(\alpha-\beta)}\cdot\frac{\sin\alpha}{\cos\alpha}=\frac{a\sin\alpha\cos\beta}{\sin(\alpha-\beta)}$$

2 M is the image of L in the water so that $BL=BM$. P is a point at height h above level of water at which the angle of elevation of L is α and the angle of depression of M is β. We have to find $BL=x$ say.

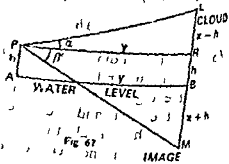


Fig 67

Let $AB=PR=y$

$LR=y\tan\alpha$, $MR=y\tan\beta$

or $x-h=y\tan\alpha$ and $x+h=y\tan\beta$

$$\frac{x+h}{x-h}=\frac{y\tan\beta}{y\tan\alpha}=\frac{\tan\beta}{\tan\alpha}.$$ Apply comp and divi

$$\frac{2x}{2h}=\frac{\tan\beta+\tan\alpha}{\tan\beta-\tan\alpha}\qquad x=h\,\frac{\tan\beta+\tan\alpha}{\tan\beta-\tan\alpha}\qquad (i)$$

Another form. If we divide above and below of (i) by $\tan\alpha\tan\beta$ then $x=h\,\dfrac{\cot\alpha+\cot\beta}{\cot\alpha-\cot\beta}$

If we change in terms of sin and cos, then from (i)

$$x=h\,\frac{\sin(\beta+\alpha)}{\sin(\beta-\alpha)}$$

3 Proceed as above

$$=(-1)^n 2n \frac{(2n-1)!}{(n-1)!\,n!}=(-1)^n \frac{(2n)!}{n(n-1)!\,n!}n$$

$$=(-1)^n \frac{(2n)!}{n!\,n!}n=-(-1)^{n-1} nC_n$$

From (5) and (6) we get

$$C_1^2-2C_2^2+3C_3^2- \quad -2nC^2_{2n}=(-1)^{n-1} nC_n$$

27 $(1+x)^{15}=C_0+C_1 x+C_2 x^2+ \quad C_{15} x^{15}$

$$\frac{(1+x)^{15}}{x}=\frac{C_0}{x}+C_1+C_2 x+C_3 x^2+ \quad +C_{15} x^{14}$$

Differentiate both sides w r t x

$$\frac{x\, 15\,(1+x)^{14}-1\,(1+x)^{15}}{x^2}=-\frac{C_0}{x^2}+C\ +2C_3 x+3C_4 x^2$$

$$+ \quad 14C_{15} x^{13}$$

Putting $x=1$ on both sides

$$15\ 2^{14}-2^{15}=-C_0+C_2+2C_3+3C_4+ \quad +14C_{15}$$

$$2^{14}(15-2)+1=C_2+2C_3+3C_4+ \quad +14C_{15}$$

The given series$=2^{14}\ 13+1=219923$

28 Putting the values of C_0 C_1 C, C_3 etc

$$C_1+C_1=1+n$$

$$C_1+C\ =n+\frac{n(n-1)}{2}=\frac{n}{2}(2+n-1)=(n+1)\frac{n}{2}=\frac{n+1}{2}C_1$$

$$C_2+C_3=\frac{n(n-1)}{1\ 2}+\frac{n(n-1)(n-2)}{1\ 2\ 3}=\frac{n(n-1)}{1\ 2}\left(1+\frac{n-2}{3}\right)$$

$$=\frac{n+1}{3}\ C$$

$$C_3+C_4= \qquad =\frac{n+1}{4}\ C_3$$

$$(C_0+C_1)(C_1+C\ )(C_2+C_3)(C_3+C_4)$$

$$=\frac{(n+1)^n}{1\ 2\ 3\ \ n}C_1 C_2\ \ C_n$$

$$=\frac{(n+1)^n}{n!}C_1 C_2\ \ C_n$$

29 $(1+x)^n=C_0+C_1x+C_2x+ \quad +C_nx^n$ (1)

We have to prove that

$$\frac{2^2 C_0}{1\ 2}+\frac{2^3C_1}{2\ 3}+ \quad +\frac{2^{n+}\ C_n}{(n+1)\ (n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}$$

Integrating both sides of (1) we get

10

Determinants

§ 1 **Definition** Consider the equations

$$a_1x+b_1y=0,\ a_2x+b_2y=0$$

These give $-\dfrac{a_1}{b_1}=\dfrac{y}{x}=-\dfrac{a_2}{b_2}$,

whence $\quad \dfrac{a_1}{b_1}=\dfrac{a_2}{b_2} \quad$ or $\quad a_1b_2-a_2b_1=0$

We shall express the above eliminant as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}=0 \tag{A}$$

We have suppressed the letters x and y to be eliminated and enclosed their coefficients as above in two parallel lines The left hand member of (A) is called a determinant of second order and its value as we have seen is $a_1b_2-a_2b_1$

Aid to memory

$$\begin{array}{ccc} a_1 & & b_1 \\ & \times & \\ -a_2 & & b_2 \end{array}$$

Similarly a determinant of 3rd order will consist of 3 rows and 3 columns enclosed in two vertical lines and is thus of the form

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \tag{B}$$

If can be seen that this determinant is the eliminant of x, y, z from the equations

$$\begin{aligned} a_1x+b_1y+c_1z&=0 \\ a_2x+b_2y+c_2z&=0 \\ a_3x+b_3y+c_3z&=0 \end{aligned}$$

The value of determinant (B) is

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

$$\frac{(1+x)^{n+1}}{(n+1)}=C_0x+C_1\frac{x^2}{2}+C_2\frac{x^3}{3}+\quad+C_n\frac{x^{n+1}}{n+1}+A \quad (2)$$

where A is arbitrary constant

Putting $x=0$, we get $\frac{1}{n+1}=A$

Integrating (2) again w r t x we get

$$\frac{(1+x)^{n+2}}{(n+1)(n+2)}=C_0\frac{x^2}{2}+C_1\frac{x^3}{2\,3}+C\ \frac{x^4}{3\,4}+$$

$$+\frac{C_nx^{n+2}}{(n+1)(n+2)}+\frac{1}{n+1}x+B$$

Putting $x=0$ we get $\frac{1}{(n+1)(n+2)}=B$

$$\frac{(1+x)^{n+2}-1}{(n+1)(n+2)}=C_0\frac{x^2}{2}+C_1\frac{x^3}{2\,3}+C_2\frac{x^3}{3\,4}+$$

$$+\frac{C_n\,x^{n+2}}{(n+2)(n+1)}+\frac{x}{n+1}$$

Putting $x=2$ in the above

$$\frac{3^{n+2}-1}{(n+1)\ n+2)}-\frac{2}{n+1}=C_0\frac{2^2}{2}+C_1\frac{2^3}{2\,3}+C_2\frac{2^4}{3\,4}$$

$$+\quad\frac{C_n\,2^{n+}}{(n+2)(n+1)}$$

R H S is $\frac{3^{n+}-1-2(n+2)}{(n+1)(n+2)}=\frac{3^{n+2}-2n-5}{(n+1)(n+2)}=$ L H S

30 Let the given expression be denoted by S_n then on putting the values of C_1 C C_3 etc

$$S_n=n-\frac{n(n-1)}{2\,2!}+\frac{n(n-1)(n-2)}{3\,3!}+\quad\text{to } n \text{ terms}$$

Replacing n by $(n+1)$ in the above we get

$$S_{n+1}=(n+1)-\frac{(n+1)\,n}{2\,2!}+\frac{(n+1)\,n\,(n-1)}{3\,3!}+\quad\text{to } (n+1)\ \text{terms}$$

$$S_{n+1}-S_n=1-\frac{n}{2\,2!}[n+1-n+1]+\frac{n(n-1)}{3\,3!}$$

$$[n+1-n+2]+$$

$$=1-\frac{n}{2!}+\frac{n(n-1)}{3!}\qquad (n+1)\ \text{term}$$

$$=\frac{1}{\ }\left[(n+1-\frac{(n+1)\,n}{\ }+\frac{(n+1)\,n\,(n-1)}{\ }+\quad\right]$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2) \qquad (1)$$

Rule a_1 (determinant obtained by removing the row and column intersecting at a_1)

$-b_1$ (determinant obtained by removing the row and column intersecting at b_1)

$+c_1$ (determinant obtained by removing the row and column intersecting at c_1)

Above is called expansion of the determinant w r t first row

Expansion with respect to first column

$$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1) \qquad (2)$$

If you compare (1) and (2) term by term you will observe that they are same

§ 2 **Properties**

1 The value of determinant is not altered by changing rows into columns and columns into rows

$$e\,g \quad \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

2 If any two adjacent rows or two adjacent columns of a determinant are interchanged the determinant retains its absolute value but changes its sign—

$$e\,g \quad \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix} = -\begin{vmatrix} x & y & z \\ 1 & 1 & 1 \\ x^2 & y^2 & z^2 \end{vmatrix}$$

Here we have interchanged 1st and 2nd rows and hence changed the sign

3 If any line of a determinant Δ be passed over p parallel lines the resultant determinant is $(-1)^p\,\Delta$

$$e\,g \quad \begin{vmatrix} 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^2 & y^2 & z^2 & u^2 \\ x^3 & y^3 & z^3 & u^3 \end{vmatrix} = (-1)^3 \begin{vmatrix} x^3 & y^3 & z^3 & u^3 \\ 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^2 & y^2 & z^2 & u^2 \end{vmatrix}$$

$$=\frac{1}{N}\left[N-\frac{N(N-1)}{2!}+\frac{N(N-1)(N-2)}{3!}+\ \ \right]$$

$$=\frac{1}{N}\left[{}^{N}C_1-{}^{N}C_2+{}^{N}C_3\ \ \right]$$

$$=\frac{1}{N}[1]=\frac{1}{n+1} \qquad N=n+1$$

[$(1+x)^n=1+{}^nC_1x+{}^nC_2x^2+{}^nC_3x^3+$ Put $x=-1$

$(1-1)^n=1-{}^nC_1+{}^nC_2-{}^nC_3+$

${}^nC_1-{}^nC_2+{}^nC_3\ \ =1$]

Now $S_{n+1}-S_n=\frac{1}{n+1}$ Put $n=1, 2, 3\ \ n-1$

$S_2-S_1=\frac{1}{2}$, $S_3-S_2=\frac{1}{3}$, $S_4-S_3=\frac{1}{4}$

$S_n-S_{n-1}=\frac{1}{n}$ Adding all these

$$S_n\ \ S_1=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ +\frac{1}{n}$$

$$S_n=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ \ +\frac{1}{n} \qquad S_1=1$$

Alternative $(1-x)^n=1-C_1x+C_2x^2-C_3x^3+\ \ +(-1)^n C_nx^n$

or $\frac{(1-x)^n}{x}-\frac{1}{x}=-C_1+C_2x-C_3x^2+\ \ +(-1)^nC_nx^{n-1}$

or $\frac{1-(1-x)^n}{1-(1-x)}=C_1-C_2x+C_3x^2\ \ +(-1)^{n-1}C_nx^{n-1}$

L H S =Sum of a G P

$$1+(1-x)+(1-x)^2+\ \ +(1-x)^{n-1}=C_1-C_2x+C_3x^2 +(-1)^{n-1}C_nx^{n-1}$$

Now integrating between the limits 0 to 1, we get

$$\left[x-\frac{(1-x)^2}{2}-\frac{(1-x)^3}{3}-\ \ -\frac{(1-x)^n}{n}\right]_0^1=\left[C_1x-\frac{C_2}{2}x^2+\frac{C_3}{3}x^3\ \ +(-1)^{n-1}C_n\frac{x^n}{n}\right]_0^1$$

or $1+\frac{1}{2}+\frac{1}{3}+\ \ +\frac{1}{n}$

$$=C_1-\frac{C_2}{2}+\frac{C_3}{3}-\ \ +(-1)^{n-1}\frac{C_n}{n}$$

$$=(-1)^2\begin{vmatrix} x^2 & y^2 & z^2 & u^2 \\ 1 & 1 & 1 & 1 \\ x & y & z & u \\ x^3 & y^3 & z^3 & u^3 \end{vmatrix}$$

$$=(-1)^1\begin{vmatrix} x & y & z & u \\ 1 & 1 & 1 & 1 \\ x^2 & y^2 & z^2 & u^2 \\ x^3 & y^3 & z^3 & u^3 \end{vmatrix}$$

In the first we have crossed fourth row over three parallel rows and hence $(-1)^3$ and in the second we have crossed 3rd row over two parallel rows and hence $(-1)^2$ and in the last we have crossed 2nd row over one parallel row and hence $(-1)^1$ Similar is the rule for crossing any column over other columns

4 **If any two rows or two columns of a determinant are identical then the determinant vanishes** Thus

$$\begin{vmatrix} a_1 & c_1 & c_1 \\ a_2 & c_2 & c_2 \\ a_3 & c_3 & c_3 \end{vmatrix}=0$$

5 **If each constituent in any row or in any column be multiplied by the same factor then the determinant is multiplied by that factor**

i e
$$\begin{vmatrix} pa_1 & b_1 & c_1 \\ pa_2 & b_2 & c_2 \\ pa_3 & b_3 & c_3 \end{vmatrix}=p\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have taken p common from 1st column

or
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ qa_2 & qb_2 & qc_2 \\ ra_3 & rb_3 & rc_3 \end{vmatrix}=qr\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

We have taken q and r from 2nd and 3rd rows respectively

31 We have

$$s_n=\frac{1-q^{n+1}}{1-q} \quad (1) \text{ and } S_n=\frac{1-\left(\frac{q+1}{2}\right)^{n+1}}{1-\frac{q+1}{2}}=\frac{2^{n+1}-(q+1)^{n+1}}{2^n\,(1-q)}$$

Now ${}^{n+1}C_1+{}^{n+1}C_2\, s_1+{}^{n+1}C_3\, s_2+\dots+{}^{n+1}C_{n+1}\, s_n$ (2)

$$=\frac{1}{1-q}\left[{}^{n+1}C_1\,(1-q)+{}^{n+1}C_2\,(1-q^2)+{}^{n+1}C_3\,(1-q^3)+\dots+{}^{n+1}C_{n+1}\,(1-q^{n+1})\right] \text{ by (1)}$$

$$=\frac{1}{1-q}\left[({}^{n+1}C_1+{}^{n+1}C_2+\dots+{}^{n+1}C_{n+1})-({}^{n+1}C_1\,q+{}^{n+1}C_2\,q^2+\dots+{}^{n+1}C_{n+1}\,q^{n+1})\right]$$

$$=\frac{1}{1-q}\left[(2^{n+1}-1)-\{(1+q)^{n+1}-1\}\right]=\frac{2^{n+1}-(1+q)^{n+1}}{(1-q)}$$

$$=2^n S_n \text{ by (2)}$$

32 Series $=\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left[\left(\frac{1}{2}\right)^r+\left(\frac{3}{4}\right)^r+\left(\frac{7}{8}\right)^r+\left(\frac{15}{16}\right)^r+\dots \text{ to } m \text{ terms}\right]$

Now $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{1}{2}\right)^r$

$$=1-{}^nC_1\,\frac{1}{2}+{}^nC_2\,\frac{1}{2^2}-{}^nC_3\,\frac{1}{2^3}+\dots$$

$$=\left(1-\frac{1}{2}\right)^n=\frac{1}{2^n}$$

Similarly $\sum_{r=0}^{n}(-1)^r\,{}^nC_r\left(\frac{3}{4}\right)^r=\left(1-\frac{3}{4}\right)^n=\frac{1}{4^n}$ etc

Hence the given series

$$=\frac{1}{2^n}+\frac{1}{4^n}+\frac{1}{8^n}+\frac{1}{16^n}+\dots \text{ to } m \text{ terms}$$

$$=\frac{\frac{1}{2^n}\left[1-\left(\frac{1}{2^n}\right)^m\right]}{1-\frac{1}{2^n}} \text{ [summing the G.P.]}$$

$$=\frac{2^{mn}-1}{2^{mn}\,(2^n-1)}$$

6 If each constituent in any row or column consists of r terms then the determinant can be expressed as the sum of r determinants Thus

$$\begin{vmatrix} a_1+\alpha_1 & b_1 & c_1 \\ a_2+\alpha_2 & b_2 & c_2 \\ a_3+\alpha_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1 & b_1 & c_1 \\ \alpha_2 & b_2 & c_2 \\ \alpha_3 & b_3 & c_3 \end{vmatrix}$$

and

$$\begin{vmatrix} a_1 & b_1 & c_1+\alpha_1+\beta_1 \\ a_2 & b_2 & c_2+\alpha_2+\beta_2 \\ a_3 & b_3 & c_3+\alpha_3+\beta_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \alpha_1 \\ a_2 & b_2 & \alpha_2 \\ a_3 & b_3 & \alpha_3 \end{vmatrix} + \begin{vmatrix} a_1 & b_1 & \beta_1 \\ a_2 & b_2 & \beta_2 \\ a_3 & b_3 & \beta_3 \end{vmatrix}$$

7 If to each constituent of a row (or column) of a determinant are added or subtracted the equi multiples of the corresponding constituent of any other row (or column) the determinant remains unaltered

e g consider $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Now suppose we add to Ist column, p times the corresponding elements of 2nd column and subtract q times the corresponding elements of 3rd column then the value of the determinant remains unaltered

Thus $\begin{vmatrix} a_1+pb_1-qc_1 & b_1 & c_1 \\ a_2+pb_2-qc_2 & b_2 & c_2 \\ a_3+pb_3-qc_3 & b_3 & c_3 \end{vmatrix} = \Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Notation The above change will be expressed as

$$C_1+(pC_2-qC_3)$$

Similarly $R_1+p\,R_2-q\,R_3$ would stand for the corresponding alteration with respect to rows

$C_1+(C_2+C_3)$ would mean add to the first column the corresponding elements of 2nd and 3rd columns

$R_2+(3R_1-6R_3)$ would mean add 3 times the elements of first row and subtract 6 times the elements of third row from the corresponding elements of second row

In particular if there is a column which contains all elements same (e g 1 or a) then applying

$$R_2-R_1,\ R_3-R_1,\ R_4-R_1$$

we will have three zeros in that column Similarly for rows

General rule It is always desired that we should try to bring in as many zeros as possible in any row or any column and then expand the determinant with respect to that column

$$\text{Thus}\quad \begin{vmatrix} a & x & y \\ 0 & p & q \\ 0 & r & s \end{vmatrix} = a\begin{vmatrix} p & q \\ r & s \end{vmatrix} - 0 + 0$$

$$= a\,(ps \quad qr)$$

$$\text{If } \Delta = \begin{vmatrix} a & x & y \\ b & 0 & q \\ c & 0 & s \end{vmatrix},$$

we would like to expand with respect to second column as it has two zeros

In order to make 2nd column first we have to make it cross over one column and hence we shall attach $(-1)^1$

$$\Delta = (-1)^1\left[x\begin{vmatrix} b & q \\ c & s \end{vmatrix} - 0 + 0\right]$$

$$= (-1)\,x\,(bs - cq)$$

In order to bring zeros in any row or column we have to use property 7 for alteration Mere expansion should be avoided as far as possible

8 **Multiplication of two determinants of same order**

$$\Delta\Delta = D \text{ say}$$

Rule Take the first row of Δ and multiply it successively with 1st, 2nd and 3rd rows of Δ The three expressions thus obtained will be elements of 1st row of D In a similar manner the elements of 2nd and third rows of D are obtained

$$\text{Thus}\quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \times \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

$$=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r+5\sum_{r=0}^{n}(-1)^r\,r\,{}^nC_r$$

But $r\,{}^nC_r=r\,\dfrac{n!}{r!(n-r)!}=r\,\dfrac{n(n-1)!}{r(r-1)!(n-r)!}$
$=n\,{}^{n-1}C_{r-1}$

$$S=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r+5n\sum_{r=1}^{n}(-1)^r\,{}^{n-1}C_{r-1}$$

$$=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r-5n\sum_{r=1}^{n}(-1)^{r-1}\,{}^{n-1}C_{r-1}$$

$=3[{}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\quad+(-1)^n\,{}^nC_r]$
$-5n\,[{}^{n-1}C_0-{}^{n-1}C_1+{}^{n-1}C_2\quad+(-1)^{n-1}\,{}^{n-1}C_{n-1}]$
$=3\times0-5n\times0=0$ by result (7) P 360

Problem Set (C)

(Objective Questions)

1 Sum of coefficients in the expansion of $(x+2y+z)^{10}$ is
(i) 2^{10}, (ii) 3^{10},
(iii) 1, (iv) None of these

2 The numb r of terms in th expansion of $(x+y+z)^{10}$ is
(A) 11 (B) 33,
(C) 66, (D) None of these

3 The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplifications is
(a) 202, (b) 51,
(c) 50, (d) None of these

4 The coefficient of middle term in the expansion of $(1+x)^{10}$ is
(a) $10!/5!\,6!$ (b) $10!/5!^2$,
(c) $10!/5!\,7!$, (d) None of these

5 The coefficient of $1/x$ in the expansion of $(1+x)^n(1+1/x)^n$ is
(a) $n!/[(n-1)!\,(n+1)!]$
(b) $2n!/[(n-1)!\,(n+1)!]$,
(c) $n!/[(2n-1)!\,(2n+1)!]$,

$$=\begin{vmatrix} R_1R_1' & R_1R_2' & R_1R_3' \\ R_2R_1' & R_2R_2' & R_2R_3' \\ R_3R_1' & R_3R_2' & R_3R_3' \end{vmatrix}$$

$$=\begin{vmatrix} a_1\alpha_1+b_1\beta_1+c_1\gamma_1 & a_1\alpha_2+b_1\beta_2+c_1\gamma_2 & a_1\alpha_3+b_1\beta_3+c_1\gamma_3 \\ a_2\alpha_1+b_2\beta_1+c_2\gamma_1 & a_2\alpha_2+b_2\beta_2+c_2\gamma_2 & a_2\alpha_3+b_2\beta_3+c_2\gamma_3 \\ a_3\alpha_1+b_3\beta_1+c_3\gamma_1 & a_3\alpha_2+b_3\beta_2+c_3\gamma_2 & a_3\alpha_3+b_3\beta_3+c_3\gamma_3 \end{vmatrix}$$

Similarly it could be written with respect to columns as

$$\begin{vmatrix} c_1c_1' & c_2c_1' & c_3c_1' \\ c_1c_2' & c_2c_2' & c_3c_2' \\ c_1c_3' & c_2c_3' & c_3c_3' \end{vmatrix}$$

9 Cofactors and minors

$$\text{Let } \Delta=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$\text{Then cofactor of } a_1=\begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix}=A_1, \text{ say,}$$

$$\text{cofactor of } a_2=-\begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix}=A_2, \text{ say etc}$$

Thus cofactor of an element of Δ is the determinant obtained by ommitting the row and column to which that element belongs with proper sign

Minor of $a_1=A_1$, minor of $a_2=-A_2$,

minor of $a_3=A_3$ etc

Remark It is easy to see that

$\Delta=a_1A_1+a_2A_2+a_3A_3=b_1B_1+b_2B_2+b_3B_3=a_1A_1+b_1B_1+c_1C_1$, that is, the value of a determinant is obtained by multiplying the elements of any row (or column) with the corresponding cofactors and adding the resulting products

Also it can be easily verified that if we multiply the elements of any row (or column) with the corresponding cofactors of any other row (or column) and add them, then the result is zero

33 Expanding by the binomial theorem, we get

$(1-x^3)^n = C_0 - C_1 x^3 + C_2 x^6 - C_3 x^9 + \dots + (-1)^n C_n x^{3n}$

Integrating between the limits 0 and 1 we get

$$I = \int_0^1 (1-x^3)^n \, dx = \Big[C_0 x - C_1 \frac{x^4}{4} + C_2 \frac{x^7}{7} - C_3 \frac{x^{10}}{10} + \dots + (-1)^n C_n \frac{x^{3n+1}}{3n+1} \Big]_0^1$$

$$= C_0 - \frac{C_1}{4} + \frac{C_2}{7} - \frac{C_3}{10} + \dots + (-1)^n \frac{C_n}{3n+1} \quad (1)$$

For the second part, we compute $I = \int_0^1 (1-x^3)^n \, dx$

Putting $x = \sin^{2/3}\theta$, $dx = \frac{2}{3}\sin^{-1/3}\theta\cos\theta\, d\theta$, we get

$$I = \int_0^{\pi/2} \cos^{2n}\theta \, \frac{2}{3} \sin^{-1/3}\theta \cos\theta \, d\theta = \frac{2}{3}\int_0^{\pi/2} \cos^{2n+1}\theta \sin^{-1/3} d\theta$$

$$= \frac{2}{3} \cdot \frac{\Gamma\left(\frac{2n+2}{2}\right)\Gamma\left(\frac{-\frac{1}{3}+1}{2}\right)}{2\Gamma\left(\frac{2n+2-\frac{1}{3}+1}{2}\right)} = \frac{1}{3} \cdot \frac{\Gamma(n+1)\Gamma(\frac{1}{3})}{\Gamma(n+1+\frac{1}{3})}$$

$$= \frac{1}{3} \cdot \frac{n!\,\Gamma\frac{1}{3}}{\left(n+\frac{1}{3}\right)\left(n-\frac{2}{3}\right)\left(n-\frac{5}{3}\right) \dots \frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{1}{3} \cdot \frac{3^{n+1}\, n!}{1\cdot 4\cdot 7 \dots (3n+1)}$$

$$= \frac{3^n\, n!}{1\cdot 4\cdot 7 \dots (3n+1)} \quad (2)$$

From (1) and (2), we obtain

$$\frac{C_0}{1} - \frac{C_1}{4} + \frac{C_2}{7} - \frac{C_3}{10} + \dots + \frac{(-1)^n C_n}{3n+1} = \frac{3^n\, n!}{1\cdot 4\cdot 7 \dots (3n+1)}$$

34 Let S denote the sum of the series. General term of the series is given by

$$T_r = (-1)^r (3+5r)\, {}^nC_r,$$

where $r = 0, 1, 2, \dots n$

$$S = \sum_{r=0}^{n} (-1)^r (3+5r)\, {}^nC_r$$

Theorem If Δ' is the determinant formed by replacing the elements of a determinant Δ by their corresponding co-factors then

$$\Delta'=\Delta^2$$

Proof If suffices to prove the theorem for a determinant of third order So let

$$\Delta=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \quad \text{Then } \Delta'=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$$

Then in view of the remark preceeding this theorem, we h

$$\Delta\Delta'=\begin{vmatrix} \Delta & 0 & 0 \\ 0 & \Delta & 0 \\ 0 & 0 & \Delta \end{vmatrix}=\Delta^3$$

or $$\Delta'=\Delta^2$$

Problem Set (A)

Evaluate the following determinants without expansion as f as possible

1 (i) $\begin{vmatrix} 43 & 1 & 6 \\ 35 & 7 & 4 \\ 17 & 3 & 2 \end{vmatrix}$ 2 $\begin{vmatrix} 1 & w & w^2 \\ w & w^2 & 1 \\ w^2 & 1 & w \end{vmatrix}$

where w is an imaginary cube root of unity

3 (a) $\begin{vmatrix} a-b & b-c & c-a \\ b-c & c-a & a-b \\ c-a & a-b & b-c \end{vmatrix}$ (b) $\begin{vmatrix} a-b & b-c & c-a \\ x-y & y-z & z-x \\ p-q & q-r & r-p \end{vmatrix}$

M N R 87

4 $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$

5 (a) $\begin{vmatrix} 0 & a-b & a-c \\ b-a & 0 & b-c \\ c-a & c-b & 0 \end{vmatrix}$ (b) $\begin{vmatrix} 0 & b & -c \\ -b & 0 & a \\ c & -a & 0 \end{vmatrix}$

$$=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r+5\sum_{r=0}^{n}(-1)^r\,r\,{}^nC_r$$

But $r\,{}^nC_r=r\,\frac{n!}{r!\,(n-r)!}=r\,\frac{n\,(n-1)!}{r\,(r-1)!\,(n-r)!}$
$=n\,{}^{n-1}C_{r-1}$

$$S=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r+5n\sum_{r=1}^{n}(-1)^r\,{}^{n-1}C_{r-1}$$

$$=3\sum_{r=0}^{n}(-1)^r\,{}^nC_r-5n\sum_{r=1}^{n}(-1)^{r-1}\,{}^{n-1}C_{r-1}$$

$=3\,[{}^nC_0-{}^nC_1+{}^nC_2-{}^nC_3+\quad+(-1)^n\,{}^nC_r]$
$-5n\,[{}^{n-1}C_0-{}^{n-1}C_1+{}^{n-1}C_2\quad+(-1)^{n-1}\,{}^{n-1}C_{n-1}]$
$=3\times0-5n\times0=0$ by result (7) P 360

Problem Set (C)

(Objective Questions)

1 Sum of coefficients in the expansion of $(x+2y+z)^{10}$ is

(i) 2^{10}, (ii) 3^{10},
(iii) 1, (iv) None of these

2. The numb r of terms in the expinsion of $(x+y+z)^{10}$ is

(A) 11 (B) 33
(C) 66, (D) None of these

3 The total number of terms in the expansion of $(x+a)^{100}+(x-a)^{100}$ after simplifications is

(a) 202, (b) 51,
(c) 50, (d) None of these

4 The coefficient of middle term in the expansion of $(1+x)^{10}$ is

(a) $10!/5!\,6!$ (b) $10!/5!^2$,
(c) $10!/5!\;7!$, (d) None of these

5 The coefficient of $1/x$ in the expansion of $(1+x)^n\,(1+1/x)^n$ is

(a) $n!/[(n-1)!\,(n+1)!]$
(b) $2n!/[(n-1)!\;(n+1)]$,
(c) $n!/[(2n-1)!\,(2n+1)!]$,

6 $\begin{vmatrix} 1 & a & a^2 & a^3+bcd \\ 1 & b & b^2 & b^3+cda \\ 1 & c & c^2 & c^3+abd \\ 1 & d & d^2 & d^3+abc \end{vmatrix}$

7 (a) The value of the determinant

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$$

(I I T 88)

(b) The determinants is

$$\begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix} \text{ and } \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

are not identically equal (i) True, (ii) False (I I T 1983)

8 $\begin{vmatrix} \frac{1}{a} & a_2 & bc \\ \frac{1}{b} & b^2 & ca \\ \frac{1}{c} & c^2 & ab \end{vmatrix}$

9 $\begin{vmatrix} 1 & bc & bc(b+c) \\ 1 & ca & ca(c+a) \\ 1 & ab & ab(a+b) \end{vmatrix}$

10 $\begin{vmatrix} 1^2 & 2^2 & 3^2 & 4^2 \\ 2^2 & 3^2 & 4^2 & 5^2 \\ 3^2 & 4^2 & 5^2 & 6^2 \\ 4^2 & 5^2 & 6^2 & 7^2 \end{vmatrix}$

11 $\begin{vmatrix} 265 & 240 & 219 \\ 240 & 225 & 198 \\ 219 & 198 & 181 \end{vmatrix}$

(d) $2n!/[(2n-1)!(2n+1)!]$

6 coefficient of x^4 in $\left(\frac{3}{2}-\frac{3}{x^2}\right)^{10}$ is

(a) $\frac{405}{256}$ (b) $\frac{504}{259}$,

(c) $\frac{450}{263}$, (d) None of these (I I T 83)

7 The coefficient of y in the expansion of $\left(y^2+\frac{c}{y}\right)^5$ is

(a) $20c$, (b) $10c$, (c) $10c^3$, (d) $20c^2$

(M N R 83)

8 The coefficients of x^p and x^q (p and q are positive integers) in the expansion of $(1+x)^{(p+q)}$ are

(a) equal, (b) equal to with opposite signs,

(c) reciprocals to each other (d) none of these

(M N R 83)

9 Given positive integers $r>1$, $n>2$ and that the coefficients of $(3r)^{th}$ and $(r+2)^{th}$ terms in the binomial expansion of $(1+x)^{2n}$ are equal, then

(a) $n=2r$ (b) $n=3r$, $n=2r+1$, (d) None of these

(I I T 83)

10 If C_r stands for ${}^nC_r$ then the sum of the series

$$\frac{2(n/2)!\,(n/2)!}{n!}[C_0^2-2C_1^2+3C_2^2+\ \ +(-1)^n(n+1)C_n^2]$$

where n is an even positive integer, is equal to

(A) 0, (B) $(-1)^{n/2}(n+1)$, $(-1)^n(n+2)$,

(D) $(-1)^n n$ (E) none of these, (I I T 86)

Answer and solutions

1 Ans (iv). [Hint, Sum of coeff $=7^{10}$]

2. Ans (a). 3 Ans (b) 4 Ans (b) 5 Ans (b)

6 Ans (a) 7 Ans (c) 8 Ans (a) 9 Ans (a)

10 Ans (B). We have

$C_0^2-2C_1^2+3C_2^2-\ \ +(-1)^n(n+1)C_n^2$

Theorem
the elements of
factors, then

Proof If
third order S

$$\Delta = \begin{vmatrix} a_1 \\ b_1 \\ c_1 \end{vmatrix}$$

Then in w

$$\Delta\Delta =$$

or

Evaluate
as product
1 (a)

3 (a)

4

12.

15	40	57
40	83	1?3
57	193	4?3

13.

21	17	7	10
24	22	6	10
6	8	2	3
5	7	1	2

14.

1	1	1	1	[illegible]
1	2	3	4	[illegible]
1	3	6	10	[illegible]
1	4	10	20	35
1	5	15	35	[illegible]

15. If ...

$$=(C_0^2-C_1^2+C_2^2- \quad +(-1)^n C_n^2)-(C_1^2-2C_2^2+3C_3^2 -(-1)^n nC_n^2)$$

$$=(-1)^{n/2} \frac{n!}{(n/2)!(n/2)!}-(-1)^{n/2-1} \tfrac{1}{2}n\, {}^nC_{n/2}$$

$$=(-1)^{n/2} \frac{n!}{(n/2)!(n/2)!} (1+\tfrac{1}{2}n)$$ [See Q 22 and Q 26 of Page 383]

$$\text{L H S} = \frac{2\,(n/2)!\,(n/2)!}{n!} \;(-1)^{n/2}\tfrac{1}{2}(n+2)\; \frac{n!}{(n/2)!\,(n/2)!}$$

$$=(-1)^{n/2}(n+2)$$

18 Prove that

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}=0$$

where a b, c are given to be in A P

19 Find the value of the determinant without expansion

$$\begin{vmatrix} b-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$$ (M N R 1988)

Solutions to problem Set (A)

1 $$\Delta=\begin{vmatrix} 7\,6+1 & 1 & 6 \\ 7\,4+7 & 7 & 4 \\ 7\,2+3 & 3 & 2 \end{vmatrix}$$

$$=7\begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

$=7\,0+0+0$

Both the determinants are zero because of identical columns

2 We know that if w is a cube root of unity then $1+w+w^2=0$ and $w^3=1$ and $w^4=w^3\ w=w$ etc
Hence applying $C_1+C_2+C_3$ we see that each element of the first column becomes $1+w+w^2$ i e zero $\Delta=0$

3 (a) Apply $C_1+C\ +C_3$ each element of first column is zero $\Delta=0$ (b) $\Delta=0$ (4) Apply C_3+C_2 etc $\Delta=0$

5 (a) Applying R_1-R and R_2-R_3 we get

$$\Delta=\begin{vmatrix} a-b & a-b & a-b \\ b-c & b-c & b-c \\ c-a & c-b & 0 \end{vmatrix}$$

$$=(a-b)\,(b-c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ c-a & c-b & 0 \end{vmatrix}=0$$ R_1 and R_2 are identical

(b) Take (-1) common from each row

$$\Delta=(-1)^3\begin{vmatrix} 0 & -b & c \\ b & 0 & -a \\ -c & a & 0 \end{vmatrix}=-\Delta$$

4 Proceed as above $h=2500$, $\beta=45^\circ$, $\alpha=15^\circ$

$$x=2500\times\frac{\sin(45^\circ+15^\circ)}{\sin(45^\circ-15^\circ)}=2500\times\frac{\sin 60^\circ}{\sin 30^\circ}$$

$$=2500\,(\sqrt{3})\text{ meters}$$

5 With the same figure as of Q 2, we have to find PL. Replacing α by θ and β by ϕ we have

$$x=h\frac{\tan\phi+\tan\theta}{\tan\phi-\tan\theta} \qquad (1)$$

Also from the figure, $\frac{x-h}{PL}=\sin\theta$

$$PL=\frac{1}{\sin\theta}(x-h)=\frac{1}{\sin\theta}\left[h\frac{\tan\phi+\tan\theta}{\tan\phi-\tan\theta}-h\right]$$

$$=\frac{h}{\sin\theta}\;\frac{2\tan\theta}{\tan\phi-\tan\theta}$$

Changing to sin and cos, we get

$$PL=\frac{2h}{\cos\theta}\;\frac{\cos\theta\cos\phi}{\sin(\phi-\theta)}=\frac{2h\cos\phi}{\sin(\phi-\theta)}$$

Note The horizontal distance of the cloud from the point of observation is PR where $PR=y$

Also $x-h=y\tan\theta$, $x+h=y\tan\phi$

Subtracting, $2h=y(\tan\phi-\tan\theta)=y\frac{\sin(\phi-\theta)}{\cos\phi\cos\theta}$

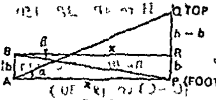
$$y=\frac{2h\cos\theta\cos\phi}{\sin(\phi-\theta)}$$

6 From the figure

$h=x\tan\alpha$ $b=x\tan\beta$

$$h=\frac{b}{\tan\beta}\tan\alpha$$

$$=b\tan\alpha\cot\beta$$



7 We have to find the value of $BC=x$ in terms of known quantities b, h and θ

Let us suppose that

$\angle BDE=\alpha$

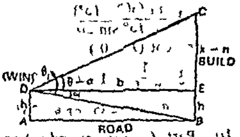
$h=b\tan\alpha$, (1)

$x-h=b\tan(\theta-\alpha)$

$$=b\frac{\tan\theta-\tan\alpha}{1+\tan\theta\tan\alpha}$$

Put for $\tan\alpha$ from (1)

$$x=h+b\frac{\tan\theta-h/b}{1+\tan\theta\,h/b}$$



$$= h + b\,\frac{(b \sin\theta - h\cos\theta)}{b\cos\theta + h\sin\theta}$$

or $$x = \frac{bh\cos\theta + h^2\sin\theta + b^2\sin\theta - bh\cos\theta}{b\cos\theta + h\sin\theta}$$

$$= \frac{(b^2+h^2)\sin\theta}{b\cos\theta + h\sin\theta}$$

8 (a) PQ subtends an angle of 15° at A and an angle of 30° at B where $AB = 100$ ft From the figure it is clear that

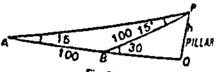


Fig 70

$AB = BP = 100$ and $\frac{h}{BP} = \sin 30°$

$$h = 100 \times \frac{1}{2} = 50 \text{ ft}$$

Also $\frac{h}{BQ} = \tan 30° = \frac{1}{\sqrt{3}}$ $\qquad BQ = h\sqrt{3} = 50\sqrt{3}$ ft

Hence $AQ = AB + BQ = 100 + 50\sqrt{3} = 50\,(2+\sqrt{3})$ ft

(b) Exactly as above $\frac{h}{40} = \sin 60°$ $\qquad h = 20\sqrt{3}$ ft

$\frac{h}{BQ} = \tan 60° = \sqrt{3}$, $BQ = \frac{h}{\sqrt{3}} = 20$ ft = breadth of the river

(c) Here $AB = BP = 150$ $\qquad$ Also $\frac{h}{BP} = \sin 60°$

$h = 150\sqrt{3}/2 = 75\sqrt{3}$

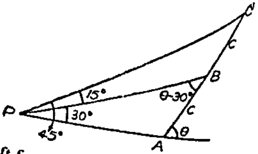
9 Applying $m-n$ theorem of trigonometry we get

$(C+C)\cot(\theta - 30°)$
$= C\cot 15° - C\cot 30°$

or $\cot(\theta-30°)$
$= \frac{1}{2}\,\frac{\sin(30° - 15°)}{\sin 15° \sin 30°}$

or $\cot(\theta - 30°)$
$= \frac{1}{2}\,\frac{1}{\sin 30°} = 1 = \cot 45°$

$\theta - 30° = 45°$ or $\theta = 75°$



10 Parts (a) (b) (c) and (d) are left for the reader

(e) In the figure OB represents the rod inclined at 10° to the vertical towards the sun and $OA = 2\,05$ m represents the shadow

12 $\begin{vmatrix} 18 & 40 & 89 \\ 40 & 89 & 198 \\ 89 & 198 & 440 \end{vmatrix}$ (b) $\begin{vmatrix} 38 & 7 & 63 \\ 16 & 3 & 29 \\ 27 & 5 & 46 \end{vmatrix}$

13 $\begin{vmatrix} 21 & 17 & 7 & 10 \\ 24 & 22 & 6 & 10 \\ 6 & 8 & 2 & 3 \\ 3 & 7 & 1 & 2 \end{vmatrix}$

14 $\begin{vmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 69 \end{vmatrix}$

15 If a, b, c (all +ive) are the pth, qth and rth terms respectively of a geometric progression, then prove that

$$\begin{vmatrix} \log a & p & 1 \\ \log b & q & 1 \\ \log c & r & 1 \end{vmatrix} = 0$$

16 If $D_r = \begin{vmatrix} r & x & \dfrac{n(n+1)}{2} \\ 2r-1 & y & n \\ 3r-2 & z & \dfrac{n(3n-1)}{2} \end{vmatrix}$

then prove that $\sum_{r=1}^{n} D_r = 0$

17 If $D_r = \begin{vmatrix} 2^{r-1} & 2(3^{r-1}) & 4(5^{r-1}) \\ x & y & z \\ 2^n-1 & 3^n-1 & 5^n-1 \end{vmatrix}$

then prove that $\sum_{r=1}^{n} D_r = 0$

$$=\begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

$$=1+a^2+b^2+c^2+d^2$$

16 $$\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ 1 & 1 & 1 \end{vmatrix}=\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}$$

17 If $p+q+r=0$, prove that

$$\begin{vmatrix} pa & qb & rc \\ qc & ra & pb \\ rb & pc & qa \end{vmatrix}=pqr\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

18 $$\begin{vmatrix} x^3 & 3x^2 & 3x & 1 \\ x^2 & x^2+2x & 2x+1 & 1 \\ x & 2x+1 & x+2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix}=(x-1)^6$$

19 $$\begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}=(x-1)^3$$

20 $$\begin{vmatrix} (x-2)^2 & (x-1)^2 & x^2 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}=-8$$

21 $$\begin{vmatrix} 0 & x & y & z \\ -x & 0 & c & b \\ -y & -c & 0 & a \\ -z & -b & -a & 0 \end{vmatrix}=(ax-by+cz)^2$$

18 Prove that

$$\begin{vmatrix} x+1 & x+2 & x+a \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}=0$$

where a, b, c are given to be in A P

19 Find the value of the determinant without expansion

$$\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$$ (M N R 1988)

Solutions to problem Set (A)

1 $$\Delta=\begin{vmatrix} 7\cdot 6+1 & 1 & 6 \\ 7\cdot 4+7 & 7 & 4 \\ 7\cdot 2+3 & 3 & 2 \end{vmatrix}$$

$$=7\begin{vmatrix} 6 & 1 & 6 \\ 4 & 7 & 4 \\ 2 & 3 & 2 \end{vmatrix}+\begin{vmatrix} 1 & 1 & 6 \\ 7 & 7 & 4 \\ 3 & 3 & 2 \end{vmatrix}$$

$=7\cdot 0+0+0$

Both the determinants are zero because of identical columns

2 We know that if w is a cube root of unity then $1+w+w^2=0$ and $w^3=1$ and $w^4=w^3 \cdot w=w$ etc

Hence applying $C_1+C_2+C_3$ we see that each element of the first column becomes $1+w+w^2$ i e zero $\Delta=0$

3 (a) Apply $C_1+C_2+C_3$ each element of first column is zero $\Delta=0$ (b) $\Delta=0$ (4) Apply C_3+C_2 etc $\Delta=0$

5 (a) Applying R_1-R_2 and R_2-R_3 we get

$$\Delta=\begin{vmatrix} a-b & a-b & a-b \\ b-c & b-c & b-c \\ c-a & c-b & 0 \end{vmatrix}$$

$$=(a-b)(b-c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ c-a & c-b & 0 \end{vmatrix}=0$$ R_1 and R_2 are identical

(b) Take (-1) common from each row

$$\Delta=(-1)^3\begin{vmatrix} 0 & -b & c \\ b & 0 & -a \\ -c & a & 0 \end{vmatrix}=-\Delta$$

22. $$\begin{vmatrix} a & b-c & c+b \\ a+c & c & c-a \\ a-b & b+a & c \end{vmatrix} = (a+b+c)\,(a^2+b^2+c^2)$$

23 $$\begin{vmatrix} (b+c)^2 & a^2 & a^2 \\ b^2 & (c+a)^2 & b^2 \\ c^2 & c^2 & (a+b)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

24 $$\begin{vmatrix} (a+b)^2 & ca & cb \\ ca & (b+c)^2 & ab \\ bc & ab & (c+a)^2 \end{vmatrix} = 2abc\,(a+b+c)^3$$

25 If $2s = a+b+c$, prove that

$$\begin{vmatrix} a^2 & (s-a)^2 & (s-a)^2 \\ (s-b)^2 & b^2 & (s-b)^2 \\ (s-c)^2 & (s-c)^2 & c^2 \end{vmatrix} = 2s^3\,(s-a)\,(s-b)\,(s-c)$$

26 $$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

27 $$\begin{vmatrix} a^2 & b^2 & c^2 \\ (a+1)^2 & (b+1)^2 & (c+1)^2 \\ (a-1)^2 & (b-1)^2 & (c-1)^2 \end{vmatrix} = 4\begin{vmatrix} a^2 & b^2 & c^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

28 Prove that

$$\begin{vmatrix} a^2+\lambda^2 & ab+c\lambda & ca-b\lambda \\ ab-c\lambda & b^2+\lambda^2 & bc+a\lambda \\ ca+b\lambda & bc-a\lambda & c^2+\lambda^2 \end{vmatrix} \times \begin{vmatrix} \lambda & c & -b \\ -c & \lambda & a \\ b & -a & \lambda \end{vmatrix} = \lambda^3\,(\lambda^2+a^2+b^2+c^2)^3$$

Now change rows into columns and columns into rows

$$\Delta=-\begin{vmatrix} 0 & b & -c \\ b & 0 & a \\ c & -a & 0 \end{vmatrix}=-\Delta$$

$$2\Delta=0 \quad \text{or} \quad \Delta=0$$

Note Part (a) can be done like part (b)

6 Split into two determinants

$$\Delta=\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}+\begin{vmatrix} 1 & a & a^2 & bcd \\ 1 & b & b^2 & cda \\ 1 & c & c^2 & dab \\ 1 & d & d^2 & abc \end{vmatrix}$$

$$=\Delta_1+\Delta_2$$

In Δ_2 multiply R_1, R_2, R_3, R_4 by a, b, c, d respectively and hence divide it by $abcd$

$$\Delta_2=\frac{1}{abcd}\begin{vmatrix} a & a^2 & a^3 & abcd \\ b & b^2 & b^3 & abcd \\ c & c^2 & c^3 & abcd \\ d & d^2 & d^3 & abcd \end{vmatrix}$$

$$=\frac{abcd}{abcd}\begin{vmatrix} a & a^2 & a^3 & 1 \\ b & b^2 & b^3 & 1 \\ c & c^2 & c^3 & 1 \\ d & d^2 & d^3 & 1 \end{vmatrix}$$

Now cross the 4th column over three columns and hence write $(-1)^3$

$$\Delta_1=(-1)^3\begin{vmatrix} 1 & a & a^2 & a^3 \\ 1 & b & b^2 & b^3 \\ 1 & c & c^2 & c^3 \\ 1 & d & d^2 & d^3 \end{vmatrix}$$

29 Without expanding at any stage show that

$$\begin{vmatrix} x^2+x & x+1 & x-2 \\ 2x^2+3x-1 & 3x & 3x-3 \\ x^2+2x+3 & 2x-1 & 2x-1 \end{vmatrix} = xA+B$$

where A and B are determinants of order 3 not involving x

(I I T 1982)

30 (a) Without expanding as far as passible, evaluate

$$\begin{vmatrix} \sin^2 A & \sin A \cos A & \cos^2 A \\ \sin^2 B & \sin B \cos B & \cos^2 B \\ \sin^2 C & \sin C \cos C & \cos^2 C \end{vmatrix} \text{ given that } A+B+C=\pi$$

(b) If $y=\sin px$, prove that

$$\Delta = \begin{vmatrix} y & y_1 & y_2 \\ y_3 & y_4 & y_5 \\ y_6 & y_7 & y_8 \end{vmatrix} = 0$$

31 Prove that

$$\begin{vmatrix} -bc & b^2+bc & c^2+bc \\ a^2+ac & -ac & c^2+ac \\ a^2+ab & b^2+ab & -ab \end{vmatrix} = (bc+ca+ab)^3$$

32
$$\begin{vmatrix} ax-by-cz & ay+bx & cx+az \\ ay+bx & by-cz-ax & bz+cy \\ cx+az & bz+cy & cz-ax-by \end{vmatrix}$$

$$=(x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

33 Show that

$$\begin{vmatrix} {}^xC_r & {}^xC_{r+1} & {}^xC_{r+2} \\ {}^yC_r & {}^yC_{r+1} & {}^yC_{r+2} \\ {}^zC_r & {}^zC_{r+1} & {}^zC_{r+2} \end{vmatrix}$$

$= -\Delta_1 \qquad \Delta = \Delta_2 + \Delta_1 = 0$

7 (a) Ans 0 Here $\Delta = \Delta_1 - \Delta_2$ and Δ_2 can be shown to be equal to Δ_1

(b) Ans (ii) Since by part (a), Δ_1 and Δ_2 are identical

8 Ans $= 0$

9 Ans $= 0$ Apply $R_2 - R_1$ and $R_3 - R_1$ first

10 Apply $C_4 - C_3$, $C_3 - C_2$, $C_2 - C_1$ (Note)

$$\Delta = \begin{vmatrix} 1 & 3 & 5 & 7 \\ 4 & 5 & 7 & 9 \\ 9 & 7 & 9 & 11 \\ 16 & 9 & 11 & 13 \end{vmatrix}$$

Again apply $C_4 - C_3$ and $C_3 - C_2$

$$\Delta = \begin{vmatrix} 1 & 3 & 2 & 2 \\ 4 & 5 & 2 & 2 \\ 9 & 7 & 2 & 2 \\ 16 & 9 & 2 & 2 \end{vmatrix} = 0 \qquad [\because C_4 \text{ and } C_3 \text{ are identical columns}]$$

11 *In order to make smaller numbers we apply* $C_1 - C_2$ and $C_2 - C_3$

$$\Delta = \begin{vmatrix} 25 & 21 & 219 \\ 15 & 27 & 198 \\ 21 & 17 & 181 \end{vmatrix}$$

Now applying $C_1 - C_2$ and $C_3 - 10C_2$, we get

$$\Delta = \begin{vmatrix} 4 & 21 & 219-210 \\ -12 & 27 & 198-270 \\ 4 & 17 & 181-170 \end{vmatrix} = 4 \begin{vmatrix} 1 & 21 & 9 \\ -3 & 27 & -72 \\ 1 & 17 & 11 \end{vmatrix}$$

Now make two zeros by $R_2 + 3R_1$, $R_3 - R_1$

$$\Delta = 4 \begin{vmatrix} 1 & 21 & 9 \\ 0 & 90 & -45 \\ 0 & -4 & 3 \end{vmatrix} = 4\,(180 - 180) = 0$$

12 (a) Applying $R_2 - 2R_1$, $R_3 - 5R_1$, we get

$$\Delta = \begin{vmatrix} 18 & 40 & 89 \\ 4 & 9 & 20 \\ -1 & -2 & -5 \end{vmatrix} \qquad [\text{Apply } C_3 - 2C_2]$$

$$=\begin{vmatrix} {}^{x}C_{r} & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_{r} & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_{r} & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$

(I I T 85)

Solutions to Problem Set B

1 Operating R_1+R_2 and R_1+R_3, we get

$$\Delta=\begin{vmatrix} x & y & z \\ 0 & 2y & 2z \\ 0 & 0 & 2z \end{vmatrix}=2z\,(2xy-0)=4xyz,$$

2 Operating $C_1+(C_2+C_3)$ and taking out 2 from newly formed C_1, we get

$$\Delta=2\begin{vmatrix} a+b+c & c+a & a+b \\ a+b+c & b+c & c+a \\ a+b+c & a+b & b+c \end{vmatrix}$$

$$=2\begin{vmatrix} a+b+c & -b & -c \\ a+b+c & -a & -b \\ a+b+c & -c & -a \end{vmatrix}, \text{ operating } C_2-C_1 \,\&\, C_3-C_1$$

$$=2\,(-1)^2\begin{vmatrix} a+b+c & b & c \\ a+b+c & a & b \\ a+b+c & c & a \end{vmatrix}$$

$$=2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}, \text{ operating } C_1-(C_2+C_3)$$

3 Apply $C_1+C_2+C_3$ and take out $(a+b)^2$ common from C_1

$$\Delta = \begin{vmatrix} 18 & 40 & 9 \\ 4 & 9 & 2 \\ -1 & -2 & -1 \end{vmatrix}$$

Now apply $C_1 - 2C_3$ to make two zeros in C_1

$$\Delta = \begin{vmatrix} 0 & 40 & 9 \\ 0 & 9 & 2 \\ 1 & 2 & -1 \end{vmatrix} = 1 \begin{vmatrix} 40 & 9 \\ 9 & 2 \end{vmatrix} = 80 - 81 = -1$$

(b) Apply $C_1 - 5C_2$, $C_3 - 9C_2$ to reduce the numbers

$$\Delta = \begin{vmatrix} 3 & 7 & 0 \\ 1 & 3 & 2 \\ 2 & 5 & 1 \end{vmatrix} \text{ Apply } R_2 - 2R_3$$

$$\Delta = \begin{vmatrix} 3 & 7 & 0 \\ -3 & -7 & 0 \\ 2 & 5 & 1 \end{vmatrix} = 0, \text{ as } R_1 \text{ and } P_2 \text{ are identical}$$

13 Here we shall try to make three zeros in any line and keep ing this in view we apply the following operrtions

$$R_1 - (R_2 + R_4),\ R \quad 3R_3,\ R_3 - 2R_4$$

and it will make three zeros in column no 3

$$\Delta = \begin{vmatrix} -8 & -12 & 0 & -2 \\ 6 & -2 & 0 & 1 \\ -4 & -6 & 0 & -1 \\ 5 & 7 & 1 & 2 \end{vmatrix}$$

$$= (-2)(-1) \begin{vmatrix} 4 & 6 & 0 & 1 \\ 6 & -2 & 0 & 1 \\ 4 & 6 & 0 & 1 \\ 5 & 7 & 1 & 2 \end{vmatrix} = 0$$

as R_1 and R_3 are identical

$$\Delta=(a+b)^2\begin{vmatrix}1 & a^2 & b^2\\ 1 & b^2 & 2ab\\ 1 & 2ab & a^2\end{vmatrix}$$

Apply R_2-R_1, R_3-R_1 to make two zeros

$$\Delta=(a+b)^2\begin{vmatrix}1 & a^2 & b^2\\ 0 & b^2-a^2 & b^2(2a-b)\\ 0 & a(2b-a) & a^2-b^2\end{vmatrix}$$

$$=(a+b)^2\begin{vmatrix}b^2-a^2 & b\,(2a-b)\\ a(2b-a) & a^2-b^2\end{vmatrix}$$

$$=(a+b)^2\,[-(a^2-b^2)^2-ab\,(2b-a)\,(2a-b)]$$
$$=-(a+b)^2\,[a^4+b^4-2a^2b^2+ab(5ab-2a^2-2b^2)]$$
$$=-(a+b)^2\,[a^4+b^4+3a^2b^2-2ab\,(a^2+b^2)]$$
$$=-(a+b)^2\,[(a^2+b^2)^2+a^2b^2-2ab\,(a^2+b^2)]$$
$$=-(a+b)^2\,[(a^2+b^2)-ab]^2$$
$$=-[(a+b)\,(a^2+b^2-ab)]^2$$
$$=-(a^3+b^3)^2$$

4 Apply $C_1-(C_2+C_3)$ to make one zero we get

$$\Delta=\begin{vmatrix}0 & z & y\\ -2x & z+x & x\\ -2x & x & x+y\end{vmatrix}$$

Take out $-2x$ from C_1 and then apply R_2-R_3

$$\Delta=-2x\begin{vmatrix}0 & z & y\\ 0 & z & -y\\ 1 & x & x+y\end{vmatrix}$$

$$=-2x\,[-zy-zy]=4xyz$$

(b) Apply $R_1-R_2-R_3$ etc

5 (a) Multiply R_1, R_2 and R_3 by c, a and b respectively an[d] hence divide Δ by abc

14 Here we apply R_5-R_4, R_4-R_3, R_3-R_2, R_2-R_1 Note that we have started changing from the last row so that the numbers are reduced and then we expand with 1st column and we are left with a determinant of 4th order as under

$$\Delta=\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 6 & 10 \\ 1 & 4 & 10 & 20 \\ 1 & 5 & 15 & 34 \end{vmatrix}$$

Again apply R_4-R_3, R_3-R_2, R_2-R_1 and expand

$$\Delta=\begin{vmatrix} 1 & 3 & 6 \\ 1 & 4 & 10 \\ 1 & 5 & 14 \end{vmatrix}$$

Again apply R_3-R_2, R_2-R_1 and expand

$$\Delta=\begin{vmatrix} 1 & 4 \\ 1 & 4 \end{vmatrix}=4-4=0,$$

15 Let A be the 1st term and R the common ratio of G P then

$$a=T_p=AR^{p-1}$$

$$\log a=\log A+(p-1)\log R$$

Similarly $b=T_q$, $c=T_r$ etc

$$\Delta=\begin{vmatrix} \log A+(p-1)\log R & p & 1 \\ \log A+(q-1)\log R & q & 1 \\ \log A+(r-1)\log R & r & 1 \end{vmatrix}$$

Split into two determinants and in the first take $\log A$ common and in the second take $\log R$ common

$$\Delta=\log A\begin{vmatrix} 1 & p & 1 \\ 1 & q & 1 \\ 1 & r & 1 \end{vmatrix}+\log R\begin{vmatrix} p-1 & p & 1 \\ q-1 & q & 1 \\ r-1 & r & 1 \end{vmatrix}$$

$$\Delta=\frac{1}{abc}\begin{vmatrix} a^2+b^2 & c^2 & c^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

Now apply $R_1-(R_2+R_3)$

$$\Delta=\frac{1}{abc}\begin{vmatrix} 0 & -2b^2 & -2a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}$$

$$=-\frac{2}{abc}\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & b^2+c^2 & a^2 \\ b^2 & b^2 & c^2+a^2 \end{vmatrix}=-\frac{2}{abc}\begin{vmatrix} 0 & b^2 & a^2 \\ a^2 & c^2 & 0 \\ b^2 & 0 & c^2 \end{vmatrix}$$

[Operating R_2-R_1 and R_3-R_1]

$$=-\frac{2}{abc}[0-b^2(a^2c^2-0)+a^2(0-b^2c^2)]$$

$$=-\frac{2}{abc}(-2a^2b^2c^2)$$

$$=4abc$$

(b) Do yourself

6 Take out a, b, c from C_1, C_2 and C_3 respectively

$$\Delta=abc\begin{vmatrix} a & c & a+c \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$$

Apply $C_1+(C_2-C_3)$ and take $2b$ common from C_1

$$\Delta=abc\,2b\begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$$

Now, appply R_2-R_3

$$\Delta=2ab^2c\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 1 & b+c & c \end{vmatrix}$$

Apply $C_1 - C_2 + C_3$ in the second

$$\Delta = 0 + \log R \begin{vmatrix} 0 & p & 1 \\ 0 & q & 1 \\ 0 & r & 1 \end{vmatrix} = 0$$

16 $\sum_{r=1}^{n} r = 1+2+3+\quad n = \frac{n(n+1)}{2}$

$$\sum_{r=1}^{n} (2r-1) = 1+3+5+\quad +(2n-1) = \frac{n}{2}[1+(2n-1)] = n^2$$

$$\sum_{r=1}^{n} (3r-2) = 1+4+7+\quad (3n-2) = \frac{n}{2}[1+3n-2] = \frac{n(3n-1)}{2}$$

We have used the formula for A P

$$S_n = \frac{n}{2}[a+l]$$

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \Sigma r & x & \frac{n(n+1)}{2} \\ \Sigma (2r-1) & y & n^2 \\ \Sigma (3r-2) & z & \frac{n(3n-1)}{2} \end{vmatrix}$$

$\sum_{r=1}^{n} D_r$ consists of n determinants in L H S and in R H S every constituent of first column consists of n elements and hence it can be splitted into sum of n determinants

$$\sum_{r=1}^{n} D_r = \begin{vmatrix} \frac{n(n+1)}{2} & x & \frac{n(n+1)}{2} \\ n^2 & y & n^2 \\ \frac{n(3n-1)}{2} & z & \frac{n(3n-1)}{2} \end{vmatrix} = 0$$

[C_1 and C_3 are identical]

$=2ab^2c\,\{c(a-c)+c(a+c)\}$
$=2ab^2c\,(2ca)=4a^2b^2c^2$

7 Multiply C_1 by a, C_2 by b and C_3 by c and hence divide by abc

$$\Delta=\frac{1}{abc}\begin{vmatrix} a(b^2+c^2) & ab^2 & ac^2 \\ a^2b & b(c^2+a^2) & bc^2 \\ a^2c & cb^2 & c(a^2+b^2) \end{vmatrix}$$

Take out a, b, c common from R_1, R_2 and R_3 respectively

$$\Delta=\frac{abc}{abc}\begin{vmatrix} b^2+c^2 & b^2 & c^2 \\ a^2 & c^2+a^2 & c^2 \\ b^2 & b^2 & a^2+b^2 \end{vmatrix}$$

Apply $C_1-C_2-C_3$

$$\Delta=\begin{vmatrix} 0 & b^2 & c^2 \\ -2c^2 & c^2+a^2 & c^2 \\ -2b^2 & b^2 & a^2+b^2 \end{vmatrix}=-2\begin{vmatrix} 0 & b^2 & c^2 \\ c^2 & a^2 & 0 \\ b^2 & 0 & a^2 \end{vmatrix}$$

[Operating C_2-C_1 and C_3-C_1 after taking -2 common from C_1]

$=-2\,[0-b^2\,(c^2a^2-0)+c^2\,(0-a^2b^2)]$
$=4a^2b^2c^2$

8 Apply $C_1+C_2+C_3$ and take out $a+b+c$ from column no 1

$$\Delta=(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 2 & c+a & a+b \end{vmatrix}$$

Apply R_3-2R_1

$$=(a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & b-c & c-a \\ 0 & c+a-2b & a+b-2c \end{vmatrix}$$

$=(a+b+c)\,\{(b-c)\,(a+b-2c)-(c-a)\,(c+a-2b)\}$
$=(a+b+c)\,\{(ab+b^2-2bc-ca-bc+2c^2)$
$\qquad -(c^2+ac-2bc-ac-a^2+2ab)\}$

17 $\sum_{r=1}^{n} 2^{r-1}=1+2+2^2+2^3+ \quad +2^{n-1}=1 \ \frac{(2^n-1)}{2-1}=2^n-1,$

$\Sigma\, 2\,(3^{r-1})=2\,(1+3+3^2+ \quad +3^{n-1})=2 \ 1 \ \frac{(3^n-1)}{3-1}=3^n-1,$

$\Sigma\, 4\,(5^{r-1})=4\,(1+5+5^2+ \quad +5^{n-1})=4 \ 1 \ \frac{(5^n-1)}{5-1}=5^n-1$

We have used the formula for G P

$$S_n=\frac{a\,(r^n-1)}{r-1}\,(r>1)$$

Rest is as in Q 16

18 Apply $R_1+R_3-2R_2$

$$\begin{vmatrix} 0 & 0 & a+c-2b \\ x+2 & x+3 & x+b \\ x+3 & x+4 & x+c \end{vmatrix}=0$$

Since a b, c are in A P

$$a+c=2b \quad \text{or} \quad a+c-2b=0$$

and as such R_1 is a row of zeros only

19 Taking out $(b-a)$ from C_1 and C_3, we get

$$\Delta=(b-a)^2\begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$$

Now operating C_1-C_3 we get C_1 and C identical in the newly formed determinant Hence $\Delta=0$

Problem Set (B)

Prove the following

1 $$\begin{vmatrix} x & y & z \\ -x & y & z \\ -x & -y & z \end{vmatrix}$$

$=4xyz$

$=(a+b+c)\{a^2+b^2+c^2-ab-bc-ca\}$

$=a^3+b^3+c^3-3\,abc$

9 Apply C_1-C_2 and C_2-C_3 and take $(a+b+c)$ common from each of C_1 and C_2

$$\Delta=(a+b+c)^2\begin{vmatrix} -1 & 0 & 2a \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Apply $R_1+R_2+R_3$

$$=(a+b+c)^2\begin{vmatrix} 0 & 0 & a+b+c \\ 1 & -1 & 2b \\ 0 & 1 & c-a-b \end{vmatrix}$$

Expand with Ist row

$$\Delta=(a+b+c)^3\begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix}=(a+b+c)^3$$

10 Apply $C_1+C_2+C_3$ and take out $2\,(a+b+c)$ from C_1

$$\Delta=2\,(a+b+c)\begin{vmatrix} 1 & a & b \\ 1 & b+c+2a & b \\ 1 & a & c+a+2b \end{vmatrix}$$

Apply R_2-R_1 and R_3-R_1

$$=2\,(a+b+c)\begin{vmatrix} 1 & a & b \\ 0 & b+c+a & 0 \\ 0 & 0 & c+a+b \end{vmatrix}=2\,(a+b+c)^3$$

Aliter Operate R_1-R_2 and R_2-R_3 etc

11 Apply $C_1+C_2+C_3+C_4$ and take out $1+a_1+a_2+a_3+a_4$

$$\Delta=(1+\Sigma a_1)\begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ 1 & 1+a_2 & a_3 & a_4 \\ 1 & a & 1+a_3 & a_4 \\ 1 & a_2 & a_3 & 1+a_4 \end{vmatrix}$$

2. $$\begin{vmatrix} b+c & c+a & a+b \\ a+b & b+c & c+a \\ c+a & a+b & b+c \end{vmatrix} = 2\begin{vmatrix} a & b & c \\ c & a & b \\ b & c & a \end{vmatrix}$$

3 $$\begin{vmatrix} 2ab & a^2 & b^2 \\ a^2 & b^2 & 2ab \\ b^2 & 2ab & a^2 \end{vmatrix} = -(a^3+b^3)^2$$

4 (a) $$\begin{vmatrix} y+z & z & y \\ z & z+x & x \\ y & x & x+y \end{vmatrix} = 4xyz$$ (b) $$\begin{vmatrix} y+z & x & x \\ y & z+x & y \\ z & x & x+y \end{vmatrix} = 4xyz$$ (Roorkee 80)

5 (a) $$\begin{vmatrix} \frac{a^2+b^2}{c} & c & c \\ a & \frac{b^2+c^2}{a} & a \\ b & b & \frac{c^2+a^2}{b} \end{vmatrix} = 4abc$$

(b) $$\begin{vmatrix} a & b & ax+by \\ b & c & bx+cy \\ ax+by & bx+cy & 0 \end{vmatrix} = (b^2-ac)(ax^2+2bxy+cy^2)$$

6 $$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

7 $$\begin{vmatrix} b^2+c^2 & ab & ac \\ ab & c^2+a^2 & bc \\ ca & cb & a^2+b^2 \end{vmatrix} = 4a^2b^2c^2$$

8 $$\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix} = a^3+b^3+c^3-3abc$$

Apply $R_2-R_1,\ R_3-R_1,\ R_4-R_1$

$$\Delta=(1+\Sigma a_1)\begin{vmatrix} 1 & a_2 & a_3 & a_4 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

Expand with first column

$$\Delta=(1+\Sigma a_1)\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}=1+\Sigma a_1$$

$$=1+a_1+a_2+a_3+a_4$$

12 Divide C_1, C_2, C_3, C_4 by a, b, c, d respectively and hence multiply Δ by $abcd$

$$\Delta=abcd\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 & \frac{1}{d} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c} & \frac{1}{d}+1 \end{vmatrix}$$

Now apply $C_1+C_2+C_3+C_4$ as in Q 11 etc

$$\Delta=abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

13 Apply $C_1+(C_2+C_3+C_4)$ and take out $x+1+2+3+4$ i e $x+10$ common from new C_1, rest as in Q 11

14 Apply $C_1+(C_2+C_3+C_4)$ and take out $x+3a$ common from new C_1 and rest as in Q 11

15 Multiply C_1, C_2, C_3 and C_4 by a, b, c and d respectively and hence divide by $abcd$

$$\Delta=\frac{1}{abcd}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 & ad^2 \\ a^2b & b(b^2+1) & bc^2 & bd^2 \\ a^2c & b^2c & c(c^2+1) & cd^2 \\ a^2d & b^2d & c^2d & d(d^2+1) \end{vmatrix}$$

9 $$\begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} = (a+b+c)^3,$$

10 $$\begin{vmatrix} a+b+2c & a & b \\ c & b+c+2a & b \\ c & a & c+a+2b \end{vmatrix} = 2\,(a+b+c)^3,$$

11 $$\begin{vmatrix} 1+a_1 & a_2 & a_3 & a_4 \\ a_1 & 1+a_2 & a_3 & a_4 \\ a_1 & a_2 & 1+a_3 & a_4 \\ a_1 & a_2 & a_3 & 1+a_4 \end{vmatrix} = 1+a_1+a_2+a_3+a_4,$$

12 $$\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

13 $$\begin{vmatrix} 1+x & 2 & 3 & 4 \\ 1 & 2+x & 3 & 4 \\ 1 & 2 & 3+x & 4 \\ 1 & 2 & 3 & 4+x \end{vmatrix} = x^3\,(x+10)$$

14 $$\begin{vmatrix} x & a & a & a \\ a & x & a & a \\ a & a & x & a \\ a & a & a & x \end{vmatrix} = (x+3a)\,(x-a)^3,$$

15 $$\begin{vmatrix} a^2+1 & ab & ac & ad \\ ab & b^2+1 & bc & bd \\ ac & bc & c^2+1 & cd \\ ad & bd & cd & d^2+1 \end{vmatrix}$$

Now take out a, b, c and d common from R_1, R_2, R_3 and R_4

$$\Delta=\frac{abcd}{abcd}\begin{vmatrix} a^2+1 & b^2 & c^2 & d^2 \\ a^2 & b^2+1 & c^2 & d^2 \\ a^2 & b^2 & c^2+1 & d^2 \\ a^2 & b^2 & c^2 & d^2+1 \end{vmatrix}$$

Now proceeding exactly as in Q 11 or 12 by $C_1+(C_2+C_3+C_4)$ we get

$$\Delta=1+a^2+b^2+c^2+d^2$$

16 Multiply R_3 by xyz and divide by xyz

$$\Delta=\frac{1}{xyz}\begin{vmatrix} ax & by & cz \\ x^2 & y^2 & z^2 \\ xyz & xyz & xyz \end{vmatrix}$$

Now take out x, y, z from C_1, C_2 and C_3

$$\Delta=\frac{xyz}{xyz}\begin{vmatrix} a & b & c \\ x & y & z \\ yz & zx & xy \end{vmatrix}=\begin{vmatrix} a & b & c \\ x & y & x \\ yz & zx & xy \end{vmatrix}$$

17 By expansion

L H S $=pa\,(qra^2-p^2bc)-qb\,(q^2ac-prb^2)+rc\,(pqc^2-r^2ab)$

$=pqr\,(a^3+b^3+c^3)-abc\,(p^3+q^3+r^3)$

$=pqr\,(a^3+b^3+c^3-3abc)-abc\,(p^3+q^3+r^3-3pqr)$

$=pqr\,(a^3+b^3+c^3-3abc)-0$

(When $p+q+r=0$, then $p^3+q^3+r^3-3pqr=0$)

Again R H S

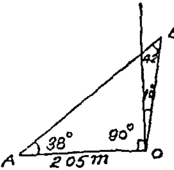
$=pqr\,[a\,(a^2-bc)-b\,(ca-b^2)+c\,(c^2-ab)]$

$=pqr\,[a^3+b^3+c^3-3abc]=$ L H S

18 Keeping in view that $x^3-3x^2+3x-1=(x-1)^3$, we apply $C_1-C_2+C_3-C_4$

$$\Delta=\begin{vmatrix} (x-1)^3 & 3x^2 & 3x & 1 \\ 0 & x^2+2x & 2x+1 & 1 \\ 0 & 2x+1 & x+2 & 1 \\ 0 & 3 & 3 & 1 \end{vmatrix}$$

where the elevation of the sun is $\angle OAB=38°$ and so we have $\angle OBA=180° -(38°+100°) =42°$



From $\triangle OAB$, we have

$$\frac{OB}{\sin 38°}=\frac{OA}{\sin 42°}=\frac{2\ 05}{\sin 42°}$$

$$OB=\frac{2\ 05 \sin 38°}{\sin 42°}$$

$$\log OB=\log 2\ 05+\log \sin 38°-\log \sin 42°$$

$$= 31175+\bar{1}78934-\bar{1}\ 82551 \quad \text{(from the tables)}$$

$$= 31175- 03617= 27458$$

Hence $OB=1\ 88$ meters (From anti log tabl s)

11 Proceed as in Q 8 Ans $10\sqrt{3}$ m

12 PQ is tower standing on the plane of the paper, and as such PQ is perpendicular to every line in the plane of the paper

PQ is perpendicular to all the lines AQ, BQ and CQ

Also angles at $A, B\ C$ are 60°, 45° and 30° respectively

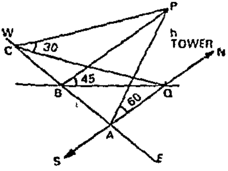


Fig 73

Also $\angle QAB=\angle QAC=90°$ and $PQ=h$

$$\frac{h}{AQ}=\tan 60°, \frac{h}{BQ}=\tan 45°, \frac{h}{CQ}=\tan 30°$$

$$AQ=h/\sqrt{3}, \qquad BQ=h, \qquad CQ=h/\sqrt{3}$$

From right angled $\triangle QAB$ we have

$$AB=\sqrt{\left(BQ^2-AQ^2\right)}=\sqrt{\left(h^2-\frac{h^2}{3}\right)}=\sqrt{\left(\frac{2}{3}\right)}\,h \quad (1)$$

Again from right angled triangle QAC

$$AC=\sqrt{(CQ^2-AQ^2)}=\sqrt{\left(3h^2-\frac{h^2}{3}\right)}=2\sqrt{\left(\frac{2}{3}\right)}h=2AB \text{ (by 1)}$$

or $AB+BC=2AB$ $\qquad AB=BC$ proved

13 $\dfrac{h}{AQ}=\tan 60°=\sqrt{3}$

$$AQ=\frac{h}{\sqrt{3}}$$

As the man walks $AB=400$ towards north the baloon sailing towards the north west comes to position above B as it is then vertically over his head From the figure $\angle AQB=\angle ABQ=45°$

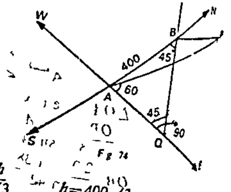


Fig 74

$$AB=AQ=\frac{h}{\sqrt{3}} \text{ or } 400=\frac{h}{\sqrt{3}} \qquad h=400\sqrt{3}$$

14 PQ is a tower of height h subtending an angle of 30° both at A and B where $AB=a$ and an angle of 60° at C where $BC=\dfrac{5a}{3}$ and BC is at right angles to AB

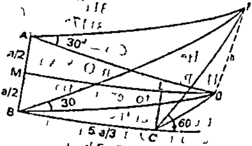


Fig 75

$\therefore h=AQ\tan 30°$ and also

$h=BQ\tan 30°$

$$AQ=BQ=h\sqrt{3} \qquad i\,e \quad \Delta ABQ \text{ is isoceles}$$

Also $h=CQ\tan 60°$ $\qquad CQ=\dfrac{h}{\sqrt{3}}$

Also $AM=MB=a/2=CL$ and $BC=LM=5a/3$

Now $QM^2=BQ^2-BM^2=3h^2-\dfrac{a^2}{4}$ from rt angled ΔMBQ

$QL^2=CQ^2-LC^2=\dfrac{h^2}{3}-\dfrac{a^2}{4}$ from rt angled ΔLCQ

$QM=QL+LM$

$$\sqrt{\left(3h^2-\frac{a^2}{4}\right)}=\sqrt{\left(\frac{h^2}{3}-\frac{a^2}{4}\right)}+\frac{5a}{3} \text{ square}$$

$$3h^2-\frac{a^2}{4}=\frac{h^2}{3}-\frac{a^2}{4}+\frac{25a^2}{9}+\frac{10a}{3}\sqrt{\left(\frac{h^2}{3}-\frac{a^2}{4}\right)}$$

$$=(x-1)^3 \begin{vmatrix} x^2+2x & 2x+1 & 1 \\ 2x+1 & x+2 & 1 \\ 3 & 3 & 1 \end{vmatrix}$$

Now make two zeros by R_1-R_2 and R_2-R_3

$$\Delta=(x-1)^3 \begin{vmatrix} x^2-1 & x-1 & 0 \\ 2(x-1) & x-1 & 0 \\ 3 & 3 & 1 \end{vmatrix}$$

$$=(x-1)^3 (x-1)^2 \begin{vmatrix} x+1 & 1 \\ 2 & 1 \end{vmatrix}$$

$$=(x-1)^5 (x+1-2)=(x-1)^6$$

19 Already done in Q 18

20 Apyly $R_1+R_3-2R_2$ and take 2 common from R_1

$$\Delta=2 \begin{vmatrix} 1 & 1 & 1 \\ (x-1)^2 & x^2 & (x+1)^2 \\ x^2 & (x+1)^2 & (x+2)^2 \end{vmatrix}$$

Make two zeros by C_2-C_1 and C_3-C_2

$$\Delta=2 \begin{vmatrix} 1 & 0 & 0 \\ (x-1) & 2x-1 & 4x \\ x^2 & 2x+1 & 4x+4 \end{vmatrix}$$

$$=2\times4 \begin{vmatrix} 2x-1 & x \\ 2x+1 & x+1 \end{vmatrix}$$

$$=8\,[(2x-1)(x+1)-x(2x+1)]$$

$$=8\,[2x^2+x-1-2x^2-x]=-8$$

21 Keeping in view the answer involving $ax-by+cz$ we multiply C_2, C_3, and C_4 by a, b, c respectively and divide Δ by abc

$$\Delta=\frac{1}{abc} \begin{vmatrix} 0 & ax & by & cz \\ -x & 0 & bc & bc \\ -y & -ac & 0 & ac \\ -z & -ab & -ac & 0 \end{vmatrix}$$

21 If $a \neq b \neq c$, one value of x which satisfies the equation

$$\begin{vmatrix} 0 & x-a & x-b \\ x+a & 0 & x-c \\ x+b & x+c & 0 \end{vmatrix} = 0 \text{ is given by}$$

(i) $x=a$, (ii) $x=b$, (iii) $x=c$, (iv) $x=0$ (M N R 80)

22 Let α be a repeated root of quadratic equation $f(x)=0$ and $A(x)$, $B(x)$, $C(x)$ be polynomials of degree 3, 4 and 5 respectively, then show that

$$\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

is divisible by $f(x)$, where dash denotes the derivative

(I I T. 84)

23 Prove that

$$\begin{vmatrix} a_1\alpha_1+b_1\beta_1 & a_1\alpha_2+b_1\beta_2 & a_1\alpha_3+b_1\beta_3 \\ a_2\alpha_1+b_2\beta_1 & a_2\alpha_2+b_2\beta_2 & a_2\alpha_3+b_2\beta_3 \\ a_3\alpha_1+b_3\beta_1 & a_3\alpha_2+b_3\beta_2 & a_3\alpha_3+b_3\beta_3 \end{vmatrix} = 0$$

24 If $ax^2+2hxy+by^2+2gx+2fy+c$

$$\equiv (lx+my+n)\,(l'x+m'y+n'),$$

then prove $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

25 If $\alpha_1\gamma_1+\alpha_2\gamma_2+\alpha_3\gamma_3=\beta_1\gamma_1+\beta_2\gamma_2+\beta_3\gamma_3=0$,
$\gamma_1^2+\gamma_2^2+\gamma_3^2=1$ and $\alpha_1\beta_1+\alpha_2\beta_2+\alpha_3\beta_3$

$$= \frac{\sqrt{3}}{2}\sqrt{\{(\alpha_1^2+\alpha_2^2+\alpha_3^2)\,(\beta_1^2+\beta_2^2+\beta_3^2)\}},$$

then show that $\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}^2 = \frac{1}{4}(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)$

Apply $C_2 - C_3 + C_4$

$$\Delta = \frac{1}{abc}\begin{vmatrix} 0 & ax-by+cz & by & cz \\ -x & 0 & bc & bc \\ -y & 0 & 0 & ac \\ -z & 0 & -ab & 0 \end{vmatrix}$$

Expand with 2nd column hence −ive sign

$$\Delta = \frac{-(ax-by+cz)}{abc}\begin{vmatrix} -x & bc & bc \\ -y & 0 & ac \\ -z & -ab & 0 \end{vmatrix}$$

Take −1, b and c common from C_1, C_2 and C_3,

$$\Delta = \frac{ax-by+cz}{a}\begin{vmatrix} x & c & b \\ y & 0 & a \\ z & -a & 0 \end{vmatrix}$$

$$= \frac{ax-by+cz}{a}\,[x\,(0+a^2) - y\,(0+ab) + z\,(ac+0)]$$

$$= (ax-by+cz)^2$$

22 Multiply C_1, C_2 and C_3 by a, b, c respectively and divide by abc

$$\Delta = \frac{1}{abc}\begin{vmatrix} a^2 & b^2-bc & c^2+cb \\ a^2+ac & b^2 & c^2-ac \\ a^2-ab & b^2+ab & c^2 \end{vmatrix}$$

Apply $C_1 + C_2 + C_3$ and take out $a^2+b^2+c^2$

$$\Delta = \frac{a^2+b^2+c^2}{abc}\begin{vmatrix} 1 & b^2-bc & c^2+cb \\ 1 & b^2 & c^2-ac \\ 1 & b^2+ab & c^2 \end{vmatrix}$$

Take b and c common from R_2 and R_3

$$= \frac{a^2+b^2+c^2}{abc}.bc\begin{vmatrix} 1 & b-c & c+b \\ 1 & b & c-a \\ 1 & b+a & c \end{vmatrix}$$

26 Show that

$$\begin{vmatrix} a+ib & c+id \\ -c+id & a-ib \end{vmatrix} \begin{vmatrix} \alpha-i\beta & \gamma-i\delta \\ -\gamma-i\delta & -\alpha+\beta i \end{vmatrix}$$

can be expressed in the form

$$\begin{vmatrix} A-iB & C-iD \\ -C-iD & A+iB \end{vmatrix}$$

Hence show that the product of two sums, each of four squares, can be expressed as the sum of four squares

27 By squaring the determinant

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \alpha & \beta & \gamma & \delta \\ \alpha^2 & \beta^2 & \gamma^2 & \delta^2 \\ \alpha^3 & \beta^3 & \gamma^3 & \delta^3 \end{vmatrix},$$

show that $\begin{vmatrix} s_0 & s_1 & s_2 & s_3 \\ s_1 & s_2 & s_3 & s_4 \\ s_2 & s_3 & s_4 & s_5 \\ s_3 & s_4 & s_5 & s_6 \end{vmatrix}$

$=[(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)(\beta-\gamma)(\beta-\delta)(\gamma-\delta)]^2$

where $s_r=\alpha^r+\beta^r+\gamma^r+\delta^r$

Let $f(x)=\begin{vmatrix} \sec x & \cos x & \sec^2 x+\cot x \operatorname{cosec} x \\ \cos^2 x & \cos^2 x & \operatorname{cosec}^2 x \\ 1 & \cos^2 x & \cos^2 x \end{vmatrix}$

then $\int_0^{\pi/2} f(x)\,dx=-\left(\frac{\pi}{4}+\frac{8}{15}\right)$ (I I T 87)

Solutions to Problem Set (C)

1 (a) Changing rows into columns and columns into rows we get the two forms

Apylying R_2-R_1 and R_3-R_1 thus making two zeros we get

$$\begin{vmatrix} 1 & a & a^2 \\ 0 & b-a & b^2-a^2 \\ 0 & c-a & c^2-a^2 \end{vmatrix}=(b-a)(c-a)\begin{vmatrix} 1 & b+a \\ 1 & c+a \end{vmatrix}$$

Apply R_2-R_1 and R_3-R_1

$$\Delta=\frac{a^2+b^2+c^2}{a}\begin{vmatrix} 1 & b-c & c+b \\ 0 & c & -a-b \\ 0 & a+c & -b \end{vmatrix}$$

$$=\frac{a^2+b^2+c^2}{a}\left[-bc+(a+c)(a+b)\right]$$

$$=\frac{a^2+b^2+c^2}{a}\left[-bc+a^2+ac+ab+bc\right]$$

$$=\frac{a^2+b^2+c^2}{a}\;a\,(a+b+c)$$

$$=(a+b+c)\,(a^2+b^2+c^2)$$

23 Apply C_2-C_1 and C_3-C_1

$$\Delta=\begin{vmatrix} (b+c)^2 & a^2-(b+c)^2 & a^2-(b+c)^2 \\ b^2 & c+a)^2-b^2 & 0 \\ c^2 & 0 & (a+b)^2-c^2 \end{vmatrix}$$

Take out $(a+b+c)$ common from each of C_2 and C_3

$$\Delta=(a+b+c)^2\begin{vmatrix} (b+c)^2 & a-b-c & a-b-c \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Apyly $R_1-(R_2+R_3)$ Then

$$\Delta=(a+b+c)^2\begin{vmatrix} 2bc & -2c & -2b \\ b^2 & c+a-b & 0 \\ c^2 & 0 & a+b-c \end{vmatrix}$$

Apply $C_2+\frac{1}{b}C_1$, $C_3+\frac{1}{c}C_1$ to make two zeros in R_1

$$\Delta=(a+b+c)^2\begin{vmatrix} 2bc & 0 & 0 \\ b^2 & c+a & \frac{b^2}{c} \\ c^2 & \frac{c^2}{b} & a+b \end{vmatrix}$$

$=(b-a)(c-a)(c+a-b-a)=(b-a)(c-a)(c-b)$
$=(a-b)(b-c)(c-a)$

(b) Hint Put $(m+n-l-p)^2=A$,

2 Applying C_2+C_1 and take $a+b+c$ common from C_2

3 Applying C_2-C_1, C_3-C_1 as in Q (1) thus making two zeros

$$\Delta=(b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b^2+a^2+ab & c^2+a^2+ca \end{vmatrix}$$

$$=(b-a)(c-a)[(c^2+a^2+ca)-(b^2+a^2+ab)]$$
$$=(b-a)(c-a)[(c^2-b^2)+a(c-b)]$$
$$=(b-a)(c-a)(c-b)(c+b+a)$$
$$=(a-b)(b-c)(c-a)(a+b+c)$$

(b) Do yourself

4 Multiplying C_1, C_2, and C_3 by x, y and z and hence dividing by xyz we get

$$\Delta=\frac{1}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ xyz & xyz & xyz \end{vmatrix}=\frac{xyz}{xyz}\begin{vmatrix} x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \\ 1 & 1 & 1 \end{vmatrix}$$

Now cross R_3 over two rows and hence $(-1)^2$ i e $+1$

$$\Delta=\begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

Apply C_2-C_1 and C_3-C_1 thus making two zeros

$$\Delta=\begin{vmatrix} 1 & 0 & 0 \\ x^2 & y^2-x^2 & z^2-x^2 \\ x^3 & y^3-x^3 & z^3-x^3 \end{vmatrix}$$

$$=(y-x)(z-x)\begin{vmatrix} y+x & z+x \\ y^2+xy+x^2 & z^2+xz+x^2 \end{vmatrix}$$

Apply C_2-C_1

$$\Delta=(y-x)(z-x)\begin{vmatrix} y+x & z-y \\ y^2+xy+x^2 & z^2-y^2+x(z-y) \end{vmatrix}$$

$$=(y-x)(z-x)(z-y)\begin{vmatrix} y+x & 1 \\ y^2+xy+x^2 & z+y+x \end{vmatrix}$$

$$=2bc\,(a+b+c)^2\,[(a+c)\,(a+b)-bc]$$
$$=2bc\,(a+b+c)^2\,(a^2+ab+ac+bc-bc)$$
$$=2abc\,(a+b+c)^3$$

24 Multiplying R_1, R_2 and R_3 by c, a and b respectively and hence dividing by abc we get

$$\Delta=\frac{1}{abc}\begin{vmatrix} c\,(a+b)^2 & c^2a & c^2b \\ a^2c & a(b+c)^2 & a^2b \\ b^2c & b^2a & b\,(c+a)^2 \end{vmatrix}$$

Now take c, a, b common from C_1, C_2 and C_3 respectively

$$\Delta=\frac{abc}{abc}\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$$

It is of the same form as Q 23

25 Let $s-a=A$, $s-b=B$, $s-c=C$

Then $\quad A+B+C=3s-(a+b+c)=3s-2s=s$

$$B+C=2s-b-c=a,\ C+A=b,\ A+B=c$$

$$\Delta=\begin{vmatrix} (B+C)^2 & A^2 & A^2 \\ B^2 & (C+A)^2 & B^2 \\ C^2 & C^2 & (A+B)^2 \end{vmatrix}$$

It is of the form of Q 23

$$\Delta=2ABC\,(A+B+C)^3=2\,(s-a)\,(s-b)\,(s-c)\,s^3$$

26 Apply C_1-bC_3, C_2+aC_3 and take out $1+a^2+b^2$ each common from both new C_1 and C_2

$$\Delta=(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b \end{vmatrix}$$

Again apply $R_3-b\,R_1$

$$\Delta=(1+a^2+b^2)^2\begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ 0 & -a & 1-a^2+b^2 \end{vmatrix}$$

$=(x-y)(y-z)(z-x)[(x+y)^2+z(x+y)-y^2-xy-x^2]$

$=(x-y)(y-z)(z-x)[xy+yz+zx]$ Proved

5 Splitting into two determinants, we get

$$0=\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}+xyz\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$=(1+xyz)[(x-y)(y-z)(z-x)]$ by Q 1

Since x, y, z are all different $x-y\neq 0, y-z\neq 0, z-x\neq 0$

$1+xyz=0$

6 Proceed as above and use the result of Q 2 and Q 4

7 Subtract R_1 from R_2 and R_3, etc

Alternative Put $bc+ad=x$ $b^2c^2+a^2d^2=x^2-2\,abcd$
Split into two and $\Delta=(x-y)(y-z)(z-x)$
where $x-y=(a-b)(d-c)$ etc

8 Split into two determinants first of which vanishes as C_1 and C_2 are identical

$$\Delta=0-\begin{vmatrix} x^2 & (y^2+z^2)-2yz & yz \\ y^2 & (z^2+x^2)-2xy & zx \\ z^2 & (x^2+y^2)-2xy & xy \end{vmatrix}$$

Again split into two determinants the second of which vanishes because C_2 and C_3 will be identical

$$\Delta=-\begin{vmatrix} x^2 & y^2+z^2 & yz \\ y^3 & z^2+x^2 & zx \\ z^2 & x+y^2 & xy \end{vmatrix}$$

Apply C_2+C_1 (Note) We have not applied C_1+C_2 and taking out $x^2+y^2+z^2$ we get

$$\Delta=-(x^2+y^2+z^2)\begin{vmatrix} x^2 & 1 & yz \\ y^2 & 1 & zx \\ z^2 & 1 & xy \end{vmatrix}$$

$$=\tfrac{1}{4}\, 4 \sin (A-B) \sin (C-A) \times \begin{vmatrix} -\cos (B+A) & \sin (B+A) \\ \cos (C+A) & -\sin (C+A) \end{vmatrix}$$

when $A+B+C=\pi$ then $\sin (A+B)=\sin C$, $\sin (C+A)$
but $\cos (B+A)=-\cos C$, $\cos (C+A)=-\cos B$

$$\Delta=\sin (A-B) \sin (C-A) \begin{vmatrix} \cos C & \sin C \\ -\cos B & -\sin B \end{vmatrix}$$

$$=\sin (A-B) \sin (C-A) [-\sin B \cos C+\sin C \cos$$

$$=\sin (A-B) \sin (C-A) \sin (C-B)$$

$$=-\sin (A-B) \sin (B-C) \sin (C-A)$$

(b) $$\Delta=\begin{vmatrix} \sin px & p \cos px & -p^2 \sin px \\ -p^3 \cos px & p^4 \sin px & p^5 \cos px \\ -p^6 \sin px & -p^7 \cos px & p^8 \sin px \end{vmatrix}$$

$$=-p^3 \begin{vmatrix} \sin px & \cos px & \sin px \\ -p^3 \cos px & p^2 \sin px & -p^3 \cos px \\ -p^6 \sin px & -p^6 \cos px & -p^6 \sin px \end{vmatrix}$$

$=0$, since C_1, C_3 are ident

31 Multiply R_1, R_2, R_3 by a, b, c respectively and then take a, b, c common from C_1, C_2 and C_3

$$\Delta=\frac{abc}{abc}\begin{vmatrix} -bc & ab+ac & ac+ab \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ca & -ab \end{vmatrix}$$

Now apply $R_1+R_2+R_3$ and take out $ab+bc+ca$ common

$$\Delta=(ab+bc+ca)\begin{vmatrix} 1 & 1 & 1 \\ ab+bc & -ac & bc+ab \\ ac+bc & bc+ca & -ab \end{vmatrix}$$

Again apply C_1-C_2 and C_2-C_3 and take $ab+bc+ca$ comm
from each new C_1 and C_2

Multiply R_1, R_2, R_3 by x, y and z respectively and hence divide by xyz

$$\Delta=-\frac{(x^2+y^2+z^2)}{xyz}\begin{vmatrix} x^3 & x & xyz \\ y^3 & y & xyz \\ z^3 & z & xyz \end{vmatrix}$$

$$=-\frac{(x^2+y^2+z^2)}{xyz}(xyz)\begin{vmatrix} x^3 & x & 1 \\ y^3 & y & 1 \\ z^3 & z & 1 \end{vmatrix}$$

Interchange C_1 and C_3 which means three interchanges of adjacent columns and hence chage the sign

$$\Delta=+(x^2+y^2+z^2)\begin{vmatrix} 1 & x & x^3 \\ 1 & y & y^3 \\ 1 & z & z^3 \end{vmatrix}$$

$=(x^2+y^2+z^2)(x-y)(y-z)(z-x)(x+y+z)$ by Q 3

9 (a) If we expand the first determinant, we get

$a(bc-a^2)-b(b^2-ca)+c(ab-c^2)$

$=3abc-a^3-b^3-c^3$

$\Delta=(3abc-a^3-b^3-c^3)^2=(a^3+b^3+c^3-3abc)^2$

2nd part $\Delta^2=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$

If we multiply them in the present form then the first element of first row will be $a^2+b^2+c^2$ whereas we want it to be $2bc-a^2$ Therefore we interchange C_2 and C_3 of first determinant and hence change of sign Again we want $-a^2$ in the result therefore we multiply the first column of first determinant by -1

$$\Delta=\begin{vmatrix} -a & b & c \\ - & & a \\ -c & & \end{vmatrix}$$

$$\Delta=(ab+bc+ca)^3\begin{vmatrix}0&0&1\\1&-1&bc+ab\\0&1&ab\end{vmatrix}$$

$$=(ab+bc+ca)^3\begin{vmatrix}1&-1\\0&1\end{vmatrix}=(ab+bc+ca)^3$$

32 Operating $xC_1+(yC_2+zC_3)$, we get

$$\Delta=\frac{1}{x}\begin{vmatrix}a(x^2+y^2+z^2)&ay+bx&cx+az\\b(x^2+y^2+z^2)&by-cz-ax&bz+cy\\c(x^2+y^2+z^2)&bz+cy&cz-ax-by\end{vmatrix}$$

and operating $aR_1+(bR_2+cR_3)$, we get

Taking out Σx^2,

$$\Delta=\frac{1}{ax}(x^2+y^2+z^2)\begin{vmatrix}a^2+b^2+c^2&y(a^2+b^2+c^2)&z(a^2+b^2+c^2)\\b&by-cz-ax&bz+cy\\c&bz+cy&cz-ax-by\end{vmatrix}$$

$$=\frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2)\begin{vmatrix}1&y&z\\b&by-cz-ax&bz+cy\\c&bz+cy&cz-ax-by\end{vmatrix}$$

$$=\frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2)\begin{vmatrix}1&0&0\\b&-cz-ax&cy\\c&bz&-ax-by\end{vmatrix}$$

operating C_2-yC_1 and C_3-zC_1

$$\frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2)\ [(-cz-ax)(-ax-by)-bzcy]$$

$$=\frac{1}{ax}(x^2+y^2+z^2)(a^2+b^2+c^2)(a^2x^2+abxy+aczx)$$

$$=(x^2+y^2+z^2)(a^2+b^2+c^2)(ax+by+cz)$$

33 Hint First observe that

${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$ and ${}^nC_{r+1}+{}^nC_{r+2}={}^{n+1}C_{r+2}$

Element of first row after multiplication are

$-aa+cb+bc, \quad -ab+cc+ba, \quad -ac+ca+bb$

or $\quad 2bc-a^2 \quad c^2 \quad b^2$

which is R_1 of 2nd determinant

Similarly we can find the elements of R and R_3

(b) $\Delta = -(a^3+b^3+c^3-3abc) = -(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

$= -\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$

Above is clearly negative because of given conditions

10 If we put $x=y$ then C_1 and C_2 are identical so that $\Delta=0$ and hence $x-y$ is a factor of Δ Similarly $y-z$ and $z-x$ are its factors

But $(x-y)(y-z)(z-x) = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ by Q 1

Similarly if we put $a=b$ then R_1 and R_2 are identical so that $\Delta=0$ and hence $(a-b)$ is a factor of Δ Similarly $b-c$ and $c-a$ are its factors

But

$(a-b)(b-c)(c-a) = \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$ by Q 1

$$\Delta = k \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \quad (2)$$

where k is constant

If we multiply in the present form then first element of first row will be $1+ax+a^2x^2$ where as we want it a^2+x^2-2ax

Therefore we interchange C_1 and C_3 of first determinant and hence $-$ive sign since it means three interchanges of adjacent columns

$$\Delta = -k \begin{vmatrix} x^2 & x & 1 \\ y^2 & y & 1 \\ z^2 & z & 1 \end{vmatrix} \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now applying C_3+C_2 and C_2+C_1, we get

$$\Delta=\begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+1}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+1}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+1}C_{r+2} \end{vmatrix}$$

Again applying C_3+C_2, we get

$$\Delta=\begin{vmatrix} {}^{x}C_r & {}^{x+1}C_{r+1} & {}^{x+2}C_{r+2} \\ {}^{y}C_r & {}^{y+1}C_{r+1} & {}^{y+2}C_{r+2} \\ {}^{z}C_r & {}^{z+1}C_{r+1} & {}^{z+2}C_{r+2} \end{vmatrix}$$ by the sam

Problem Set (C)

1 (a) Prove the following

$$\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}=\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=(a-b)(b-c)(c$$

(b) $$\begin{vmatrix} 1 & (m+n-l-p)^2 & (m+n-l-p)^4 \\ 1 & (n+l-m-p)^2 & (n+l-m-p)^4 \\ 1 & (l+m-n-p)^2 & (l+m-n-p)^4 \end{vmatrix}$$

$$=64\,(l-m)(l-n)(l-p)(m-n)(m-p)(n-p)$$

2 $$\begin{vmatrix} a & b+c & a^2 \\ b & c+a & b^2 \\ c & a+b & c^2 \end{vmatrix}=-(a+b+c)(a-b)(b-c)(c-a)$$

3 (a) $$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix}=(a-b)(b-c)(c-a)(a+b+c)$$

(b) $$\begin{vmatrix} -2a & a+b & c+a \\ b+a & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}=4\,(b+c)(c+a)(a+b)$$

Now the first element will be x^2+ax+a^2 whereas we want it $x^2-2ax+a^2$

Choose k=2 and then take -2 with 2nd column of first determinant

$$\Delta=\begin{vmatrix} x^2 & -2x & 1 \\ y^2 & -2y & 1 \\ z^2 & -2z & 1 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

Now after multiplication the elements of first row will be

$$x^2-2ax+a^2 \qquad x^2-2bx+b^2 \qquad x^2-2cx+c^2$$

or $\qquad (a-x)^2 \qquad (b-x)^2$ and $\qquad (c-x)^2$

Similarly other rows can be formed, as

$$(a-y)^2 \qquad (b-y)^2 \qquad (c-y)^2$$

$$(a-z)^2 \qquad (b-z)^2 \qquad (c-z)^2$$

Changing rows into columns and columns into rows we shall get the required form Hence from (1) on choosing $k=2$ the factors of Δ are

$$2\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$=2\,(x-y)\,(y-z)\,(z-x)\,(a-b)\,(b-c)\,(c-a)$ by Q 1

11 Put $x=a$ you find that R_1 and R_2 are identical after taking out a and m, and hence $\Delta=0$ $\quad (x-a)$ is a factor Similarly $x-b$ is a factor Also Δ on expansion gives a quadratic in x and hence $x=a$, b are the only roots

12 Apply C_3-C_2 and cancel -1, we get

$$\Delta=\begin{vmatrix} 15-2x & 11 & 1 \\ 11-3x & 17 & 1 \\ 7-x & 14 & 1 \end{vmatrix}=0$$

Apply R_2-R_1 and R_3-R_1

$$\Delta=\begin{vmatrix} 15-2x & 11 & 1 \\ -(x+4) & 6 & 0 \\ (x-8) & 3 & 0 \end{vmatrix}=0$$

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ yz & zx & xy \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)(xy+yz+zx)$$

If x, y, and z are all different and given that

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix}=0, \text{ prove } 1+xyz=0$$

If x, y and z are all different given that

$$\begin{vmatrix} x & x^3 & x^4-1 \\ y & y^3 & y^4-1 \\ z & z^3 & z^4-1 \end{vmatrix}=0, \text{ prove that}$$

$xyz\,(xy+yz+zx)=x+y+z$

$$\begin{vmatrix} 1 & bc+ad & b^2c^2+a^2d^2 \\ 1 & ca+bd & c^2a^2+b^2d^2 \\ 1 & ab+cd & a^2b^2+c^2d^2 \end{vmatrix}$$

$$=(a-b)(a-c)(a-d)(b-c)(b-d)(c-d)$$

$$\begin{vmatrix} x^2 & x^2-(y-z)^2 & yz \\ y^2 & y^2-(z-x)^2 & zx \\ z^2 & z^2-(x-y)^2 & xy \end{vmatrix}$$

$$=(x-y)(y-z)(z-x)(x+y+z)(x^2+y^2+z^2)$$

(a) Prove that

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}^2 = \begin{vmatrix} 2bc-a^2 & c^2 & b^2 \\ c^2 & 2ca-b^2 & a^2 \\ b^2 & a^2 & 2ab-c^2 \end{vmatrix}$$

$$= (a^3+b^3+c^3-3abc)^2$$

(b) Let a, b, c be positive and not all equal Show that the value of the determinant or $\begin{vmatrix} -x-4 & 2 \\ x-8 & 1 \end{vmatrix}=0$

or $-x-4-2x+16=0$

$-3x+12=0 \qquad x=4$

13 Apply R_2-R_1 and R_3-R_1, we get

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix}$$

or $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix}=0$

Expand with 1st row

or $(x-2)\,6-(2x-3)\,4+(3x-4)\,1=0$

or $x-4=0 \qquad x=4$

14 Apply $C_2-\frac{3}{2}C_1$ and C_3-2C_1 to remove x from C_2 and C_3

$\Delta=\begin{vmatrix} 4x & 2 & 1 \\ 6x+2 & 0 & -4 \\ 8x+1 & -\frac{3}{2} & 0 \end{vmatrix}=0$

Expand with 1st column

$4x\,(-6)-(6x+2)\,\frac{3}{2}+(8x+1)\,(-8)=0$

$24x+9x+3+64x+18=0$

or $97x+11=0 \qquad x=-\frac{11}{97}$

15 Apply $C_1+C_2+C_3$ and taking $3x-2$ out we get

$\Delta=(3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 1 & 3x-8 & 3 \\ 1 & 3 & 3x-8 \end{vmatrix}=0$

Apply R_2-R_1 and R_3-R_1 to make two zeros

$\Delta=(3x-2)\begin{vmatrix} 1 & 3 & 3 \\ 0 & 3x-11 & 0 \\ 0 & 0 & 3x-11 \end{vmatrix}$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \text{ is negative}$$

10 Express

$$\Delta = \begin{vmatrix} (a-x)^2 & (a-y)^2 & (a-z)^2 \\ (b-x)^2 & (b-y)^2 & (b-z)^2 \\ (c-x)^2 & (c-y)^2 & (c-z)^2 \end{vmatrix}$$

$$\text{or} \begin{vmatrix} (1+ax)^2 & (1+ay)^2 & (1+az)^2 \\ (1+bx)^2 & (1+by)^2 & (1+bz)^2 \\ (1+cx)^2 & (1+cy)^2 & (1+cz)^2 \end{vmatrix}$$

as the product of two determinants and evaluate it

11 Solve for x

$$\begin{vmatrix} a & a & x \\ x & m & m \\ b & x & b \end{vmatrix} = 0$$

12 $$\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$$

13 $$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

14 $$\begin{vmatrix} 4x & 6x+2 & 8x+1 \\ 6x+2 & 9x+3 & 12x \\ 8x+1 & 12x & 16x+2 \end{vmatrix} = 0$$

15 $$\begin{vmatrix} 3x-8 & 3 & 3 \\ 3 & 3x-8 & 3 \\ 3 & 3 & 3x-8 \end{vmatrix} = 0$$

Apply R_2-R_1 and R_3-R_1

$$\Delta=[3x+(a+b+c)]\begin{vmatrix}1 & b+x & c+x\\ 0 & c-b & a-c\\ 0 & a-b & b-c\end{vmatrix}=0$$

or $[3x+(a+b+c)][-(b-c)^2-(a-c)(a-b)]=0$

or $[3x+(a+b+c)][(b-c)^2-(c-a)(a-b)]=0$

It is given that $(b-c)^2\neq(a-b)(c-a)$ and hence from above

$$3x+a+b+c=0 \qquad x=-\tfrac{1}{3}(a+b+c)$$

21 On expansion

$\Delta=(x+a)(x-b)(x+c)+(x+b)(x-a)(x-c)=0$

$(x-b)\{x^2+ac+x(a+c)\}+(x+b)\{x^2+ac-x(a+c)\}=0$

$=x\{2x^2+2ac\}-b\{2x(a+c)\}=0$

or $2x\{x^2+ac-b(a+c)\}=0$ or $x=0$ (iv) is correct

22 Set $P(x)=\begin{vmatrix}A(x) & B(x) & C(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ (1)

Then $P'(x)=\begin{vmatrix}A'(x) & B'(x) & C'(x)\\ A(\alpha) & B(\alpha) & C(\alpha)\\ A'(\alpha) & B'(\alpha) & C'(\alpha)\end{vmatrix}$ (2)

Since $f(x)=0$ is a quadratic having a repeated root α, we can write $f(x)=a(x-\alpha)^2$ where a is a constant

Now $P(x)$ will be divisible by $f(x)$ if $P(x)$ and $P'(x)$ are divisible by $(x-\alpha)$ i.e. $P(\alpha)=0$ and $P'(\alpha)=0$ which is obvious by (1) and (2)

23 We write the given determinant Δ say as the product of two determinants as follows

$$\Delta=\begin{vmatrix}a_1 & b_1 & 0\\ a_2 & b_2 & 0\\ a_3 & b_3 & 0\end{vmatrix}\times\begin{vmatrix}\alpha_1 & \beta_1 & 0\\ \alpha_2 & \beta_2 & 0\\ \alpha_3 & \beta_3 & 0\end{vmatrix}$$

Since value of each det on R H S is zero we have $\Delta=0$

24 Equating coefficients of like terms, in the given identity, we get $a=ll'$, $b=mm'$, $c=nn'$

16 $$\begin{vmatrix} x+2 & 2x+3 & 3x+4 \\ 2x+3 & 3x+4 & 4x+5 \\ 3x+5 & 5x+8 & 10x+17 \end{vmatrix}=0$$

17 $$\begin{vmatrix} x+2 & x+6 & x-1 \\ x+6 & x-1 & x+2 \\ x-1 & x+2 & x+6 \end{vmatrix}=0$$

1 (a) Show that $x=2$ is a root of

$$\begin{vmatrix} x & -6 & -1 \\ 2 & -3x & x-3 \\ -3 & 2x & x+2 \end{vmatrix}=0$$ and solve it completely

(b) If $$\begin{vmatrix} 1 & 3 & 9 \\ 1 & x & x^2 \\ 4 & 6 & 9 \end{vmatrix}=0$$ then

(a) $x=3$ (b) $x=3$ or $x=6$

(c) $x=3$ or $3/2$ (d) None of these

(c) Show that $x=-9$ is a root of

$$\begin{vmatrix} x & 3 & 7 \\ 2 & x & 2 \\ 7 & 6 & x \end{vmatrix}=0,$$ other two roots are and (I I T 83)

19 Gigen $a+b+c=0$, solve

$$\begin{vmatrix} a-x & c & b \\ c & b-x & a \\ b & a & c-x \end{vmatrix}=0$$

20 If $(b-c)^2 \neq (a-b)(c-a)$, solve for x

$$\begin{vmatrix} a+x & b+x & c+x \\ b+x & c+x & a+x \\ c+x & a+x & b+x \end{vmatrix}=0$$

$2h=lm'+l'm,\ 2g=ln'+l'n,\ 2f=mn'+m'n$

Consider the product $\Delta_1\,\Delta_2$ where each of them is zero determinant

$$\begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix}\begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix}=0$$

$$\text{or}\quad \begin{vmatrix} 2ll' & lm'+l'm & ln+l'n \\ lm'+l\,m & 2n & mn+m\,n \\ nl'+n'l & mn'+m\,n & 2nn' \end{vmatrix}=0$$

$$\text{or}\quad \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix}=8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0$$

25 We have

$$\Delta^2=\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}\begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix}$$

$$=\begin{vmatrix} \Sigma\alpha_1^2 & \Sigma\alpha_1\beta_1 & \Sigma\alpha_1\gamma_1 \\ \Sigma\beta_1\alpha_1 & \Sigma\beta_1^2 & \Sigma\beta_1\gamma_1 \\ \Sigma\gamma_1\alpha_1 & \Sigma\gamma_1\beta_1 & \Sigma\gamma_1^2 \end{vmatrix}$$

$$=\begin{vmatrix} \alpha_1^2+\alpha_2^2+\alpha_3^2 & \frac{\sqrt{3}}{2}\sqrt{\{(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)\}} & 0 \\ \frac{\sqrt{3}}{2}\sqrt{\{(\quad)(\quad)\}} & \beta_1^2+\beta_2^2+\beta_3^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

(Using the given relations)

$$=(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)-\tfrac{3}{4}(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)$$

$$=\tfrac{1}{4}(\alpha_1^2+\alpha_2^2+\alpha_3^2)(\beta_1^2+\beta_2^2+\beta_3^2)$$

26 Do yourself 27 Do yourself

$$\frac{8h^2}{3}-\frac{25a^2}{9}=\frac{10a}{3}\sqrt{\left(\frac{h^2}{3}-\frac{a^2}{4}\right)}$$

or $$8h^2-\frac{25a^2}{3}=10a\sqrt{\left(\frac{h^2}{3}-\frac{a^2}{4}\right)}$$ Square again

$$64h^4-400\frac{a^2h^2}{3}+\frac{625}{9}a^4=\frac{100a^2h^2}{3}-25a^4$$

Multiply throughout by 9

$$576h^4-1500a^2h^2+850a^4=0 \quad \text{Cancel 2}$$

$$288h^4-750a^2h^2+425a^4=0$$

$$288h^4-510a^2h^2-240a^2h^2+425a^4=0$$

$$6h^2(48h^2-85a^2)-5a^2(48h^2-85a^2)=0$$

or $$(48h^2-85a^2)(6h^2-5a)=0$$

$$h=\sqrt{\left(\frac{85}{48}\right)}a \text{ or } \sqrt{\left(\frac{5}{6}\right)}a$$

15 $h=BD\tan\alpha=DC\tan\beta$

But $BD+DC=1$ mile

$$h\left(\frac{1}{\tan\alpha}+\frac{1}{\tan\beta}\right)=1$$

$$h=\frac{\tan\alpha\tan\beta}{\tan\alpha+\tan\beta}$$

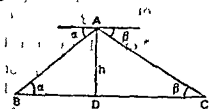


Fig 76

16 $PQ=MN=LN-LM$

$$=(1\cot 30-1\cot 60°)$$

$$=\left(\sqrt3-\frac{1}{\sqrt3}\right)=\frac{2}{\sqrt3}$$

PQ is covered in 10 seconds

$$=\frac{10}{3600}\text{ hours}=\frac{1}{360}\text{ hours}$$

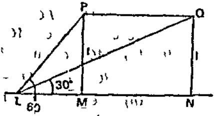


Fig 77

$$\text{speed}=\frac{s}{t}=\frac{2}{\sqrt3}\div\frac{1}{360}=\frac{2}{\sqrt3}(360)=240\sqrt3\text{ km/hour}$$

17 (*a*) Let $PD=h$ and θ be the angle subtended at A where θ is unknown Clearly $AB=BP=a$ from isoceles $\triangle ABP$

$h=a\sin 2\theta=2a\sin\theta\cos\theta$ from $\triangle PBD$ (1)

Again from $\triangle PBC$

$$\frac{PB}{\sin(\pi-3\theta)}=\frac{BC}{\sin\theta}$$

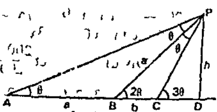


Fig 78

or $\frac{a}{b}=\frac{\sin 3\theta}{\sin \theta}=\frac{3\sin\theta-4\sin^3\theta}{\sin\theta}$

or $\frac{a}{b}=3-4\sin^2\theta$ $\quad \sin^2\theta=\frac{3b-a}{4b}$

$\cos^2\theta=1-\sin^2\theta=1-\frac{3b-a}{4b}=\frac{a+b}{4b}$

Now putting for $\sin\theta$ and $\cos\theta$ from above in (1), we get

$h=2a\sqrt{\left(\frac{3b-a}{4b}\right)}\sqrt{\left(\frac{a+b}{4b}\right)}=\frac{a}{2b}\sqrt{\{(a+b)(3b-a)\}}$

(b) Put $a=d$, $b=3d/4$ in part (a) and you get the result

18 From the figure of Q 17, we get

$\frac{AB}{BC}=\frac{AD-BD}{BD-CD}=\frac{h(\cot\theta-\cot 2\theta)}{h(\cot 2\theta-\cot 3\theta)}$

or $\frac{AB}{BC}=\frac{\cot\theta-\cot 2\theta}{\cot 2\theta-\cot 3\theta}$ (I)

Note In case C is the foot, then $3\theta=90°$

$\theta=30°$, and $\cot 30°=\sqrt{3}$, $\cot 90°=0$

$\frac{AB}{BC}=\frac{\sqrt{3}-\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{3}}-0}=\frac{2}{1}$

or $AB=2BC$

Hence B trisects AC

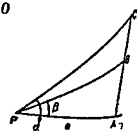


Fig. 79

19 (a) $AC=a\tan\alpha$ $AB=a\tan\beta$

$BC=AC-AB=a(\tan\alpha-\tan\beta)$

(b) Let $PQ=h$ be the height of tower where angles of depression of Q and P are 30° and 60° as seen from top B of cliff AB of height 200 ft Let $AP=x$

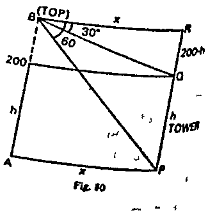


Fig. 80

$200-h=x\tan 30°=x/\sqrt{3}$ $\triangle BQR$

$200=x\tan 60°=\sqrt{3}x$, $\triangle BPR$

Dividing, we get $\frac{200-h}{200}=\frac{1}{3}$

or $3(200-h)=200$

or $400=3h$

$h=400/3$ ft

28 Applying $R_1 - R_3 \sec x$, $R_2 - R_3 \cos^2 x$, we will have two zero in the first column and expanding w r t this column

$$f(x) = -(\cos^2 x - \cos^4 x)\,(\sec^3 x + \cot x \operatorname{cosec} x - \cos$$

$$= -\cos^2 x \sin^2 x \left\{ \frac{1}{\cos^2 x} + \frac{\cos x}{\sin^2 x} - \cos x \right\}$$

$$= -\sin^2 x - \cos^3 x + \cos^3 x \sin^2 x$$

$$= -\sin^2 x - \cos^3 x\,(1 - \sin^2 x)$$

$$\int_0^{\pi/2} f(x)\, dx = \int_0^{\pi/2} (-\sin^2 x - \cos^5 x)\, dx$$

$$= -\left[\frac{1}{2} \cdot \frac{\pi}{2} + \frac{4 \cdot 2}{5 \cdot 3 \cdot 1} \right] = -\left[\frac{\pi}{4} + \frac{8}{15} \right]$$

§ 2 System of linear Equations

Definition 1 *A system of linear equations in n unknowns $x_1, x_2, x_3, \ldots, x_n$ is of the form*

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned}\right\} \quad (A)$$

If $b_1, b_2, \ldots, b_n$ are all zero the system is called **homogeneous** and non homogeneous if at least one b_i is non zero

Definition 2 *The* **solution set** *of the system (A) is an n tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of real numbers (or complex numbers if the coefficients are complex) which satisfy each of the equations of the system*

Definition 3 *A system of equations is called* **consistent** *if it has at least one solution* **inconsistent** *if it does not have any solution,* **determinate** *if it has a unique solution* **indeterminate** *if it has more than one solution*

(A) Non homogeneous Equations in two unknowns

Consider the system of equations

$$\left.\begin{aligned} a_1 x + b_1 y &= c_1 \\ a_2 x + b_2 y &= c_2 \end{aligned}\right\} \quad (1)$$

We consider the following cases

1 **a_i, b_i, c_i ($i = 1, 2$) are all zero** Then any pair of numbers (x, y) is a solution of the system (1) since in this case each equation reduces to an identity So in this case equations are always consistent and indeterminate

12 The three equations in two unknowns will be consistent if $D=0$

$$\text{or}\quad \begin{vmatrix} (a+1)^3 & (a+2)^3 & -(a+3)^3 \\ a+1 & (a+2) & -(a+3) \\ 1 & 1 & -1 \end{vmatrix}=0$$

Cancel minus from third column

Now put

$u=(a+1)$ $v=a+2$, $w=a+3$ Then $u-v=-1$, $v-w=-1$

$w-u=2$ and $u+v+w=3a+6$ (1)

Also $D=0$ reduces to

$$\begin{vmatrix} u^3 & v^3 & w^3 \\ u & v & w \\ 1 & 1 & 1 \end{vmatrix}=0$$

or $(u-v)(v-w)(w-u)(u+v+w)=0$ by Q 3 (a) P 436 of problem set C

or $(-1)(-1)(2)(3a+6)=0$ or $(a+2)=0$ or $a=-2$

13 Since

$$f(x)=ax^2+bx+c$$

where a b c are unknown

$$f(0)=a\,0+b\,0+c=6 \quad (1)$$

$$f(2)=a\,4+b\,2+c=11 \quad (2)$$

$$f(-3)=a\,9-b\,3+c=6 \quad (3)$$

Thus we have got three equations in a, b, c and on solving them we shall find the values of a b, c and hence of $f(x)$ and $f(1)$

$$\frac{a}{D_1}=\frac{b}{D_2}=\frac{c}{D_3}=\frac{1}{D}$$

Now $D=\Bigg|$ $=-$

6

2 a_i, b_i ($i=1, 2$) are all zero, but at least one of c_1 and c_2 is non-zero Then the system has no solution i e, the equations are inconsistent.

3 At least one of a_i, b_i, ($i=1, 2$) is non zero

Suppose $b_2 \neq 0$ Then system (1), is equivalent to the system

$$\left.\begin{aligned} a_1 x+b_1 y&=c_1, \\ \frac{a_2}{b_2} x+y&=\frac{c_2}{b_2} \end{aligned}\right\} \quad (2)$$

i e, if the pair (x_0, y_0) is a solution of system (1), then it is also a solution of system (2), and vice-versa

Multiplying the second equation of system (2) by b_1 and subtracting from first, we get

$$\left(a_1-\frac{a_2}{b_2} b_1\right) x=c_1-\frac{c_2}{b_2} b_1 \quad (3)$$

Now replacing the first equation of system (2) by equation (3), we obtain the system

$$\left.\begin{aligned} \left(a_1-\frac{a_2}{b_2} b_1\right) x&=c_1-\frac{c_2}{b_2} b_1 \\ \frac{a_2}{b_2} x+y&=\frac{c_2}{b_2} \end{aligned}\right\} \quad (4)$$

which is evidently equivalent to system (2)

(a) If $a_1-\frac{a_2}{b_2} b_1 \neq 0$ i e, if, $a_1 b_2-a_2 b_1 \neq 0$, then we find from the first equation of system (4) that

$$x=\frac{c_1 b_2-c_2 b_1}{a_1 b_2-a_2 b_1} \quad (5)$$

Substituting this value of x into the second equation of system (4), we obtain

$$y=\frac{a_1 c_2-a_2 c_1}{a_1 b_2-a_2 b_1}$$

For convenience we write

$$\Delta=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}, \Delta_x=\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix} \Delta_y=\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$$

[Note that Δ_x and Δ_y are obtained by replacing the first and second columns in Δ by the column c_1 and c_2 respectively]

Then (5) and (6) can be written as

$$x=\frac{\Delta_x}{\Delta}, y=\frac{\Delta_y}{\Delta} \quad (7)$$

Similarly $D_2=-45, \quad D_3=-180$

$$\frac{a}{-15}=\frac{b}{-45}=\frac{c}{-180}=\frac{1}{-30}$$

$a=\frac{1}{2},\ b=3/2,\ c=6$

Hence $f(x)=ax^2+bx+c=\frac{1}{2}x^2+\frac{3}{2}x+6$

$f(1)=\frac{1}{2}+3/2+6=2+6=8$

Alternative

From 1st equation $\qquad c=6$

Putting for c in (2) and (3) we get

$$4a+2b=5, \qquad 9a-3b=0$$

Solving $\quad a=\frac{1}{2},\ b=3/2$

Putting for a, b, c, we get

$f(x)=\frac{1}{2}x^2+\frac{3}{2}x+6$ and $f(1)=8$

14 Here

$$\Delta=\begin{vmatrix} 1 & 4 & -2 \\ 3 & 1 & 5 \\ 2 & 3 & 1 \end{vmatrix}$$

$=1\ (1-15)-3\ (4+6)+2\ (20+2)$

$=-14-30+44=0$

$$\text{And } \Delta_x=\begin{vmatrix} 3 & 4 & -2 \\ 7 & 1 & 5 \\ 5 & 3 & 1 \end{vmatrix}$$

$=3\ (1-15)-7\ (4+6)+5\ (20+2)=-2\neq 0$

Since $\Delta=0$ and $\Delta_x\neq 0$ the equations are not consistent and hence there is no solution of the given system of equations

15 Here

$$\Delta=\begin{vmatrix} 1 & 1 & -2 \\ 1 & -3 & 1 \\ 1 & -5 & 4 \end{vmatrix}$$

$=1\ (-12+5)-1\ (8-1)-2\ (-10+3)$

$=-7-7+14=0$

Thus $\Delta=0$ Hence if the equations are to be consistent, then we must have

$$\Delta_x=\Delta_y=\Delta_z=0$$

This is known as *Cramer's rule* Then *if* $a_1 b_2 - a_2 b_1 \neq 0$ *then the system* (4) *or system* (1) *has the unique solution given by* (7) *Hence in this case, the equations are* consistent and determinate

(b) Now let $\Delta = a_1 b_2 - a_2 b_1 = 0$

Then the system (4) has the form

$$\left.\begin{aligned} 0\,x &= c_1 b_2 - c_2 b_1 \\ \frac{a_2}{b_2}\,x + y &= \frac{c_2}{b_2} \end{aligned}\right\} \qquad (9)$$

Obviously this system has no solution if $c_1 b_2 - c_2 b_1 = \Delta_x \neq 0$

Thus in this case, the equations are inconsistent

But if $\Delta_x = 0$, then any pair of numbers (x, y), where

$y = \dfrac{c_2}{b_2} - \dfrac{a_2}{b_2}\,x,\ x \in R$, is a solution of system (9)

So in this case, the equations are consistent and indeterminate

We summarize the whole discussion given in (A) as follows

(i) *If* $\Delta \neq 0$, *then the system is* consistent *and* indeterminate and its solution is given by

$$x = \frac{\Delta_x}{\Delta},\ y = \frac{\Delta_y}{\Delta} \qquad (\textit{i e}\ \text{unique solution})$$

(ii) If $\Delta = 0$, but at least one of the numbers Δ_x, Δ_y is non zero, then the system in consistent *i e*, it has no solution

(iii) If $\Delta = 0$ and $\Delta_x = \Delta_y = 0$ but at least one of the numbers $a_1, b_1, a_2\ b_2$ is non zero, then the system has infinite number of solutions and hence it is consistent and indeterminate

(iv) If $a_i = b_i = c_i = 0\ (i = 1, 2)$ then the system has infinite number of solutions and so it is consistent and indeterminate

(B) Homogeneous linear equations in two unknowns

Consider the system of equations

$$\left.\begin{aligned} a_1 x + b_1 y &= 0, \\ a_2 x + b_2 y &= 0 \end{aligned}\right\} \qquad (10)$$

The system always has the solution $x = 0$, $y = 0$

It follows from the discussion in part (A) that if $\Delta \neq 0$, then the system (10) has the unique solution $x = 0$ $y = 0$

$$\Delta_{\bullet}=\begin{vmatrix} 0 & 1 & -2 \\ 0 & -3 & 1 \\ \lambda & -5 & 4 \end{vmatrix}=0$$

if $5k=0$ $k=0$

But when $k=0$, the equations become homogeneous

Since $\Delta=0$ for $k=0$ the system has non trivial solution

Also from 1st two

$$\frac{x}{1-6}=\frac{y}{-4-1}=\frac{z}{-3-2} \quad \text{or} \quad \frac{x}{1}=\frac{y}{1}=\frac{z}{1}=\lambda$$

$x=y=z=\lambda$ Thus there are infinite number of solutions

16 The given equations are

$$x-cy-bz=0$$
$$cx-y+az=0$$
$$bx+ay-z=0$$

Since x, y, z are not all zero, the system will have non trivial solution if

$$\Delta=\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix}=0$$

or $1(1-a^2)+c(-c-ab)-b(ac+b)=0$

or $1-a^2-c^2-abc-abc-b^2=0$

or $a^2+b^2+c^2+2abc=1$

17 For non trivial solution we must have

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

or $3abc-a^3-b^3-c^3=0$

or $(a+b+c)(a^2+b^2+c^2-bc-ca-ab)=0$

or $\frac{1}{2}(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2]=0$

Thus for non trivial solution, we have $a+b+c=0$

or $a=b=c$ When $a+b+c=0$,

then 1st two given equations can be written as

And if $\Delta=0$, and at least one of a_1, a_2, b_1, b_2 is non zero, then the system (10) reduces to the single equation so that any pair of numbers (x, y), is a solution

Thus system (10) is always consistent

(C) Non homogeneous linear equations

Consider the system of equations

$$\left.\begin{aligned} a_1x+b_1y+c_1z=d_1 \\ a_2x+b_2y+c_2z=d_2 \\ a_3x+b_3y+c_3z=d_3 \end{aligned}\right\} \quad (1)$$

Let us introduce the following notation

$$\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \quad \Delta_x=\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

$$\Delta_y=\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \quad \Delta_z=\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

without going into details, we give the following rule for testing the consistency of the system (1)

1 Let $a_i=b_i=c_i=d_i=0$, $i=1$ 2, 3
In this case any triplet (x, y, z) is a solution of the system Hence equations are consistent and indeterminate

2 If $a_i=b_i=c_i=0$, $i=1, 2, 3$ and at least one d_i $(i=1, 2, 3)$ is non zero, then the system has no solution i e, the equations in this case are inconsistent

3 Let $\Delta\neq0$ In this case the system (1) has the unique solution

$$x=\frac{\Delta_x}{\Delta},\ y=\frac{\Delta_y}{\Delta},\ z=\frac{\Delta_z}{\Delta} \quad (2)$$

This is known as Cramer s rule So equations in this case are consistent and determinate

4 If $\Delta=0$ and $\Delta_x\neq0$ (or $\Delta_y\neq0$ or $\Delta_z\neq0$), then the system has no solution so that the equations are inconsistent

$$ax+by-(a+b)z=0$$

and $\quad bx-(a+b)y+az=0$

Solving these equations we get

$$\frac{x}{ab-(a+b)^2}=\frac{y}{-b(a+b)-a^2}=\frac{z}{-a(a+b)-b^2}$$

or $\quad \frac{x}{1}=\frac{y}{1}=\frac{z}{1}=k$

When $a=b=c$, the three equations reduce to a single equation $x+y+z=0$

Since $1+\omega+\omega^2=0$, the solution set can be written as

$(k, k\omega, k\omega^2)$ or $(k, k\omega^2, k\omega)$

Thus in this case, we have $\frac{x}{1}=\frac{y}{w}=\frac{z}{w^2}=k$

or $\quad \frac{x}{1}=\frac{y}{w^2}=\frac{z}{w}=k$

18 Rewriting the given equations we have

$$1+bc+qr=0$$
$$1+ca+rp=0$$
$$1+ab+pq=0$$

Multiply 1st by ap, 2nd by bq and 3rd by cr we get

$$ap+(abc)p+(pqr)a=0$$
$$bq+(abc)q+(pqr)b=0$$
$$cr+(abc)r+(pqr)c=0$$

Put $\quad abc=x$ and $pqr=y$,

The above equations are

$$px+ay+ap=0$$
$$qx+by+bq=0$$
$$rx+cy+cr=0$$

These equations in x and y will be consistent if $D=0$

$$\text{or} \begin{vmatrix} p & a & ap \\ q & b & bq \\ r & c & cr \end{vmatrix}=0$$

Interchange C_1 and C_2

5 If $\Delta=\Delta_x=\Delta_y=\Delta_z=0$ and at least one of the cofactors of Δ is non zero, then the system will have an infinite number of solutions In this case, any one of the variables can be given arbitrary value and other variables can be expressed *in terms* of that variable

In such cases, the three equations reduce to two equations If all the cofactors Δ, Δ_x, Δ_y, Δ_z are zero but elements of Δ are not all zero then in this case the system will reduce to single equation and any two variables can be given arbitrary values So equations are consistent and indeterminate

(D) Homogeneous linear equations

If in (1) we take $d_i=0$ $(i=1, 2, 3)$ then the system is called the homogeneous system of equations

For such a system if $\Delta\neq0$, then it has the unique solution $x=0$ $y=0$ $z=0$

If $\Delta=0$, *then the system has an infinite number of solutions* So such system of equations is **always consistent**

§ 4 Three equations in two unknowns

Consider the equations

$$\left.\begin{aligned} a_1 x+b_1 y&=c_1 \\ a_2 x+b_2 y&=c_2 \\ a_3 x+b_3 y&=c_3 \end{aligned}\right\} \qquad (3)$$

The system (3) will be consistent if the solution set of any two satisfies the third equations *i e*, if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

Note The factors of the following two determinants be remembered

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}=\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b \\ 1 & c & c^2 \end{vmatrix}$$

$$=(a-b)\,(b-c)\,(c-a)$$

$$\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix}=0$$

19 Here

$$\Delta=\begin{vmatrix} 3 & m \\ 2 & -5 \end{vmatrix}=-15-2m,$$

$$\Delta_x=\begin{vmatrix} m & m \\ 20 & -5 \end{vmatrix}=-5m-20m=-25m$$

and

$$\Delta_y=\begin{vmatrix} 3 & m \\ 2 & 20 \end{vmatrix}=60-2m$$

If $\Delta=0$, then $m=-\frac{15}{2}$ and for this value of m, $\Delta_x\neq 0$ and $\Delta_y\neq 0$ Hence when $m=-\frac{15}{2}$, the given system of equations has no solution

If $\Delta\neq 0$, that is, if $m\neq -\frac{15}{2}$, the system has the unique solution given by

$$x=\frac{\Delta_x}{\Delta}=\frac{-15m}{-15-2m}=\frac{15m}{15+2m}$$

and $$y=\frac{\Delta_y}{\Delta}=\frac{60-2m}{-15-2m}=\frac{2m-60}{2m+15}$$

Now $\qquad x>0$ if $m>0$ or $m<-\frac{15}{2}$ (1)

and $\qquad y>0$ if $m>30$ or $m<-\frac{15}{2}$ (2)

Both the inequalities (1) and (2) are satisfied if $m<-\frac{15}{2}$

or $\qquad m>30$

Hence the given system of equations has a solution satisfying the conditions $x>0$, $y>0$ if $m<-\frac{15}{2}$ *or* $m>30$

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^3 & b^3 & c^3 \end{vmatrix} = \begin{vmatrix} 1 & a & a^3 \\ 1 & b & b^3 \\ 1 & c & c^3 \end{vmatrix}$$

$$=(a-b)(b-c)(c-a)(a+b+c)$$

Problem Set (D)

Solve the following by using Cramer's rule

1 $x+y-z=1$
$ax+by+cz=k$
$a^2x+b^2y+c^2z=k$

2 $ax+by+cz=k$
$a^2x+b^2y+c^2z=k^2$
$a^3x+b^3y+c^3z=k^3$

3 $x+2y+3z=6$
$2x+4y+z=7$
$3x+2y+9z=14$

4 $x+y+z=11$
$2x-6y-z=0$
$3x+4y+2z=0$

5 $x+y+z=6$
$x-y+z=2$
$2x+y-z=1$

6 $5x-6y+4z=15$
$7x+4y-3z=19$
$2x+y+6z=46$

7 Find the value of λ if the following equations are consistent

$x+y-3=0$
$(1+\lambda)x+(2+\lambda)y-8=0$
$x-(1+\lambda)y+(2+\lambda)=0$

8 If the equations

$ax+hy+g=0,\ hx+by+f=0,\ gx+fy+c=\lambda$

are consistent, show that $\lambda=\dfrac{abc+2fgh-af^2-bg^2-ch^2}{ab-h^2}$

9 If the equations

$(b+c)x+(c+a)y+(a+b)=0,\ cx+ay+b=0$

and $ax+by+c=0$ are cosistent then show that either $a+b+c=0$ or $a=b=c$

20 After simple calculations, it is easy to see that

$$\Delta=\begin{vmatrix} 3 & -1 & 4 \\ 1 & 2 & -3 \\ 6 & 5 & \lambda \end{vmatrix}=7\lambda+35,\ \Delta_x=\begin{vmatrix} 3 & -1 & 4 \\ -2 & 2 & -3 \\ -3 & 5 & \lambda \end{vmatrix}=4\lambda+20$$

$$\Delta_y=\begin{vmatrix} 3 & 3 & 4 \\ 1 & -2 & -3 \\ 6 & -3 & \lambda \end{vmatrix}=-9\lambda-45 \text{ and } \Delta_z=\begin{vmatrix} 3 & -1 & 3 \\ 1 & 2 & -2 \\ 6 & 5 & -3 \end{vmatrix}=0$$

If $\Delta\neq0$, that is, if $\lambda\neq-5$, the system of equations has the unique solution

$$x=\frac{4\lambda+20}{7\lambda+35},\ y=-\frac{9\lambda+45}{7\lambda+35} \text{ and } z=0$$

If $\Delta=0$, that is, if $\lambda=-5$, then we also have $\Delta_x=\Delta_y=\Delta_z=0$ So in this case, the system has infinite number of solutions

Again since one of the cofaotors, say

$$\begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix}$$

of Δ is not zero one variable is independent, that is, it can be given any arbitrary value, say we take, $z=c$ where c is real

Then from first two equations we have

$3x-y=3-4c$ and $x+2y=3c-2$

Solving these, we get

$$x=\frac{4-5c}{7},\ y=\frac{13c-9}{7}$$

Hence the solution set of the system is given by $x=\frac{1}{7}(4-5c)$,

$y=\frac{1}{7}(13c-9)$ and $z=c$, where c is any real number

The reader can verify that these values of x, y, z satisfy the given equations

21 Adding the given equations we get

$(a+b+c)(x+y+z)=0$ but $a+b+c\neq0$

$x+y+z=0$ or $y+z=-x$

Putting in 1st we get

$(b+c)(-x)-ax=b-c$

10 If a, b, c are all different and the equations

$$ax+a^2y+(a^3+1)=0$$
$$bx+b^2y+(b^3+1)=0$$
$$cx+c^2y+(c^3+1)=0$$

are consistent then prove that $abc+1=0$

11 For what value of k do the following system of equations possess a non trivial solution over the set of rationals

$$x+ky+3z=0$$
$$3x+ky-2z=0$$
$$2x+3y-4z=0$$

For that value of k, find all the solutions of the system

(I I T 1979)

12 Find the value of a if the three equations are consistent

$$(a+1)^3x+(a+2)^3y=(a+3)^3$$
$$(a+1)x+(a+2)y=a+3$$
$$x+y=1$$

13 Find a, b, c when $f(x)=ax+bx+c=0$ and

$f(0)=6, f(2)=11, f(-3)-6$

Determine $f(x)$ and find the value of $f(1)$

14 Show that there is no solution for the equations

$$x+4y-2z=3$$
$$3x+y+z=7$$
$$2x+3y+z=5$$

15 Find k for which the set of equations

$$x+y-2z=0$$
$$2x-3y+z=0$$
$$x-5y+4z=k$$

are consistent and find the solutions for all such values of k

16 Given

$$x=cy+bz,\ y=az+cx,\ z=bx+ay$$

where x, y, z are not all zero, prove that

$$a+b^2+c^2+2abc-1$$

(I I T 78)

17 If x, y, z are not all zero and if

$$ax+by+cz=0$$
$$bx+cy+az=0$$
$$cx+ay+bz=0$$

prove that $x\ y\ z=1\ 1\ 1$ or $1\ w\ w^2$ or $1\ w^2\ w$ where w is one of the complex roots of unity

$$x=-\frac{(b-c)}{a+b+c} \quad \text{Similarly}$$

$$y=-\frac{(c-a)}{a+b+c}, \quad z=-\frac{(a-b)}{a+b+c}$$

2nd Part

The given equations can be written as

$$x-ay+az=0$$
$$bx+y-bz=0$$
$$cx-cy-z=0$$

For a non trivial solution we have

$$\begin{vmatrix} 1 & -a & a \\ b & 1 & -b \\ c & -c & -1 \end{vmatrix}=0$$

or $1(-1-bc)+a(-b+bc)+a(-bc-c)=0$

or $-1-ab-bc-ca+abc-abc=0$

or $1+ab+bc+ca=0$

22 The system will have a non trivial solution if

$$\begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix}=0$$

or $(28-21)\sin 3\theta-(-7-7)\cos 2\theta+2(-3-4)=0$

or $7\sin 3\theta+14\cos 2\theta-14=0$

or $\sin 3\theta+2\cos 2\theta-2=0$

or $3\sin\theta-4\sin^3\theta+2(1-2\sin^2\theta)-2=0$

or $4\sin^3\theta+4\sin^2\theta-3\sin\theta=0$

or $\sin\theta(2\sin\theta-1)(2\sin\theta+3)=0$

$\sin\theta=\frac{1}{2}$ which gives $\theta=n\pi$,

or $\sin\theta=0$ which gives $\theta=n\pi+(-1)^n\pi/6$,

where n is an interger

[Note that $\sin\theta\neq-3/2$]

Hence values of θ are given by

$$\theta=n\pi \text{ or } \theta=n\pi+(-1)^n\pi/6,\ n\in I$$

23 we have the following relations

18 If $bc+qr=ca+rp=ab+pq=-1$ show that

$$\begin{vmatrix} ap & a & p \\ bq & b & q \\ cr & c & r \end{vmatrix}=0$$

19 For what values of m does the system of equations $3x+my=m$ and $2x-5y=20$ has a solution satisfying the conditions $x>0, y>0$ **(I I T 79)**

20 Show that the system of equations

$$3x-y+4z=3$$
$$x+2y-3z=-2$$
$$6x+5y+\lambda z=-3$$

has at least one solution for any real number λ Find the set of solutions if $\lambda=-5$ **(I I T 83)**

21 Solve

$$(b+c)(y+z)-ax=b-c$$
$$(c+a)(z+x)-by=c-a$$
$$(a+b)(x+y)\quad cz=a-b$$

where $a+b+c\neq 0$ *(I I T, 77)*

If $a=\dfrac{x}{y-z}$, $b=\dfrac{y}{z-x}$ and $c=\dfrac{z}{x-y}$

where $x, y\ z$ are not all zero, prove that

$$1+ab+bc+ca=0$$

22 Consider the system of linear equations in x, y, z

$$(\sin 3\theta)x-y+z=0$$
$$(\cos 2\theta)x+4y+3z=0$$
$$2x+7y+7z=0$$

Find the value of θ for which this system has non trivial solutions **(I I T 86)**

23 Let α_1, α_2 and β_1, β_2 be the roots of $ax^2+bx+c=0$ and $px^2+qx+r=0$ respectively If the system of equations $\alpha_1 y+\alpha_2 z=0$ and $\beta_1 y+\beta_2 z=0$ has a non trivial solution then, prove that

$$\frac{b^2}{q^2}=\frac{ac}{pr}$$ **(I I T 87)**

$$\alpha_1+\alpha_2=-\frac{b}{a} \quad \alpha_1\alpha_2=c/a \tag{1}$$

$$\beta_1+\beta_2=-\frac{q}{p} \quad \beta_1\beta_2=r/p \tag{2}$$

Since the given system has non trivial solution

$$\Delta=\begin{vmatrix} \alpha_1 & \alpha_2 \\ \beta_1 & \beta_2 \end{vmatrix}=0 \qquad \text{or} \quad \alpha_1\beta_2-\alpha_2\beta_1=0$$

or $\dfrac{\alpha_1}{\beta_1}=\dfrac{\alpha_2}{\beta_2}=k$ say $\qquad \alpha_1=k\beta_1 \quad \alpha_2=k\beta_2$

Hence from (I)

$k(\beta_1+\beta_2)=-b/a, \quad k^2\beta_1\beta_2=c/a$

or $\quad k\left(-\dfrac{q}{p}\right)=-b/a, \quad k^2\left(\dfrac{r}{p}\right)=c/a$ by (2)

$$k=\frac{pb}{qa} \quad \text{and } k^2=\frac{pc}{ra}$$

Eliminating k we get

$\dfrac{p^2b^2}{q^2a^2}=k^2=\dfrac{pc}{ra} \quad$ or $\quad \dfrac{b^2}{q^2}=\dfrac{ac}{pr}$ proved

Problem Set (E)

(Objective Questions)

1 Let a_{ij} denote the element of the i^{th} row and j^{th} column in a 3×3 determinant ($1\leqslant i\leqslant3$, $1\leqslant j\leqslant3$) and let $a_{ij}=-a_{ji}$ for every i and j. Then the determinant has all the principal diagonal elements as

(a) 1 (b) -1, (c) 0 (d) none of these

2 If each dement of a determinant of third order with value A is multiplied by 3, then the value of newly formed determinant is

(a) $3A$, (b) $9A$,
(c) $27A$, (d) None of these,

3 For $a>0$, $b>0$, $c>0$, the value of the determinant

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

is always positive,

or $3\lambda^2+2\lambda-5=0$

or $(\lambda-1)(3\lambda+5)=0$

$\lambda=1, -5/3$

8 As in Q (7) for consistency, we must have $D=0$ i e

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} a & h & g+0 \\ h & b & f+0 \\ g & f & c-\lambda \end{vmatrix}=0$$

$$\text{or} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}+\begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & -\lambda \end{vmatrix}=0 \quad \text{Expand}$$

$$a(bc-f^2)-h(ch-gf)+g(hf\ \ bg)-\lambda(ab-h^2)=0$$

$$\lambda=\frac{abc+2fgh-af^2\ \ bg^2-ch^2}{ab\ \ h^2}$$

9 For consistency of three equations in two unknowns as in Q 1 We must have $D=0$

$$\Delta=\begin{vmatrix} b+c & c+a & a+b \\ c & a & b \\ a & b & c \end{vmatrix}=0$$

$$\text{or} \quad \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}+\begin{vmatrix} c & a & b \\ c & a & b \\ a & b & c \end{vmatrix}=0$$

$$\text{or} \quad \begin{vmatrix} b & c & a \\ c & a & b \\ a & b & c \end{vmatrix}=0$$

or $3abc-a^3-b^3-c^3=0$ or $a^3+b^3+c^3-3abc=0$

or $(a+b+c)(a^2+b^2+c^2-ab-b\ \ -ca)=0$

or $\frac{1}{2}(a+b+c)[(b-c)^2+(c\ \ \ \ -b)^2]=0$

Hence either $c=0$

or $b-c=$ $-b$

Hence the ... $a=b=c$

(a) True, (b) False

4 The value of the determinant $\begin{vmatrix} a+b & a+2b & a+3b \\ a+2b & a+3b & a+4b \\ a+4b & a+5b & a+6b \end{vmatrix}$ is

5 The value of the determinant

$$\begin{vmatrix} x+1 & x+2 & x+4 \\ x+3 & x+5 & x+8 \\ x+7 & x+10 & x+14 \end{vmatrix}$$

is (a) -2, (b) x^2+2,
(c) 2, (d) none of these (M N R

6 If $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}$

then the two triangles with vertices (x_1, y_1) (x_2, y_2), (x_3, y_3) and $(a_1\ b_1)$, $(a_2\ b_2)$, $(a_3\ b_3)$ must be congruent
(a) True, (b) False (I I T 85)

7 If $f_r(x), g_r(x), h_r(x)$ $r=1, 2, 3$ are polynomials in x such that $f_r(a)=g_r(a)=h_r(a)$ $=1$ 2, 3

and $F(x)=\begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$

then $F'(x)$ at $x=a$ is (I I T 85)

8 If $\Delta_1=\begin{vmatrix} x & b & b \\ a & x & b \\ a & a & x \end{vmatrix}$ and $\Delta_2=\begin{vmatrix} x & b \\ a & x \end{vmatrix}$

are the given determinants then
(a) $\Delta_1=3(\Delta_2)^2$, (b) $(d/dx)\Delta_1=3\Delta_2$
(c) $(d/dx)\Delta_1=3(\Delta_2)^2$, (d) $\Delta_1=3\Delta_2^{3/2}$
(M N R 85)

10 For consistency $D=0$

$$\begin{vmatrix} a & a^2 & a^3+1 \\ b & b^2 & b^3+1 \\ c & c^2 & c^3+1 \end{vmatrix}=0$$

Hence as in Q 5, Page 437 of problem set C we have

$$1+abc=0$$

11 For non trivial solution $D=0$

$$\text{or} \quad \begin{vmatrix} 1 & k & 3 \\ 3 & k & -2 \\ 2 & 3 & -4 \end{vmatrix}=0$$

Apply R_2-3R_1 R_3-2R_1

$$\Delta=\begin{vmatrix} 1 & k & 3 \\ 0 & -2k & -11 \\ 0 & 3-2k & -10 \end{vmatrix}=0$$

or $\quad 20k+11\,(3-2k)=0 \quad$ or $\;33-2k=0 \qquad k=33/2$

Putting the value of λ the equations are

$x+\frac{33}{2}y+3z=0$ (1)

$3x+\frac{33}{2}y-2z=0$ (2)

$2x+3y-4z=0$ (3)

Multiply (1) by 3 and subtract from (2) and similarly multiply (1) by 2 and subtract from (3) Thus we get the equivalent system of equations as

$$x+\tfrac{33}{2}y+3z=0$$
$$-33y-11z=0$$
$$-30y-10z=0$$

From any of the last two we get $3y=-z$,

or $\quad \dfrac{y}{1}=\dfrac{z}{-3}=\lambda$ say $\qquad y=\lambda \;\; z=-3\lambda$

From 1st $x+\dfrac{33}{2}\lambda+3z=0$ we get

$$x+\frac{33}{2}\lambda-9\lambda=0 \qquad x=-\frac{15}{2}\lambda$$

$$x : y : z = \left(-\frac{15}{2} : 1 : -3\right)$$

9 The determinant

$$\begin{vmatrix} a & b & ax+b \\ b & c & bx+c \\ ax+b & bx+c & 0 \end{vmatrix}$$ is equal to zero, if

(a) a, b, c are in A P (b) a, b, c in G P
(c) a, b, c in H P
(d) α is a root of the equation
$ax^2+bx+c=0$
(e) $(x-\alpha)$ is a factor of $ax^2+2bx+c$ **(I I T 86)**

10 Let $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ and $\vec{c}=c_1\hat{i}+c_2\hat{j}+c_3\hat{k}$ be three non zero vectors such that $\vec{c}$ is a unit vector perpendicular to both the vectors $\vec{a}$ and $\vec{b}$. If the angle between a and $\vec{b}$ is $\frac{\pi}{6}$ then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$ is equal to

(a) 0, (b) 1
(c) $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$
(d) $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2)$,
(e) none of these **(I I T 86)**

11 The value of θ lying between $\theta=0$ and $\pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}=0$$

are

(a) $7\pi/24$ (b) $5\pi/24$
(c) $11\pi/24$ (d) $\pi/24$ **(I I T 88)**

(c) As in part (b)

$y-h=x\tan\theta$ from $\triangle BQR$

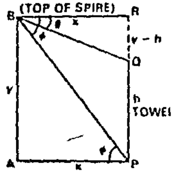
$y=x\tan\phi$ from $\triangle BPA$

$$\frac{y-h}{y}=\frac{\tan\theta}{\tan\phi}=\frac{\sin\theta\cos\phi}{\cos\theta\sin\phi}$$

$$1-\frac{h}{y}=\frac{\sin\theta\cos\phi}{\cos\theta\sin\phi}$$

or $$\frac{\sin\phi\cos\theta-\cos\phi\sin\theta}{\sin\phi\cos\theta}=\frac{h}{y}$$

or $$y=\frac{h\sin\phi\cos\theta}{\sin(\phi-\theta)} \quad (1)$$



Again $y=x\tan\phi$ or $\dfrac{h\sin\phi\cos\theta}{\sin(\phi-\theta)}=x\dfrac{\sin\phi}{\cos\phi}$ by (1)

$$x=\frac{h\cos\theta\cos\phi}{\sin(\phi-\theta)}$$

20 (a) As in (c), $\dfrac{y-h}{y}=\dfrac{\tan\theta}{\tan\phi}=\dfrac{4}{3}\ \dfrac{2}{5}=\dfrac{8}{15}$ or $1-\dfrac{h}{y}=\dfrac{8}{15}$

or $1-\dfrac{8}{15}=\dfrac{h}{150}$ $\quad y=150$

or $\dfrac{7}{15}=\dfrac{h}{150}$ $\quad h=70=PQ$ $\quad QR=150-70=80$

Also $PR=x\tan\phi$ or $150=x\frac{5}{2}$ $\quad x=60=BR$

$BQ^2=BR^2+RQ^2=60^2+80^2=20^2\,(9+16)=20^2\,5^2$

$BQ=20\times5=100$

(b) Let $AO=BM=a$, $AB=OM=h$

$PM=H-h$

$BM=AO=OP\cot\alpha=H\cot\alpha$

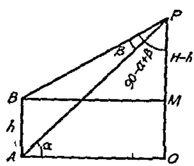
Again $BM=PM\tan(90-\alpha+\beta)$

$=(H-h)\cot(\alpha-\beta)$

$H\cot\alpha=(H-h)\cot(\alpha-\beta)$

or $h\cot(\alpha-\beta)=H\,[\cot(\alpha-\beta)-\cot\alpha]$

$$H=\frac{h\sin\alpha\cos(\alpha-\beta)}{\sin\beta}$$ On changing to sin and cos



21 Let $AB=2x$ so that $AC=CB=x$

By given condition

$AP=nAB=n\,2x,$

and let $\angle CPA=\alpha$

then $\tan\alpha=\dfrac{x}{2nx}=\dfrac{1}{2n}$ (1)

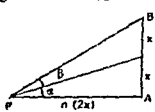


Fig 8

and

$$\tan(\alpha+\beta)=\frac{2x}{2nx}=\frac{1}{n}$$

$$\tan\beta=\tan\{(\alpha+\beta)-\alpha\}=\frac{\tan(\alpha+\beta)-\tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha}$$

$$=\frac{\frac{1}{n}-\frac{1}{2n}}{1+\frac{1}{n}\cdot\frac{1}{2n}}=\frac{n}{2n^2+1}$$

22 Let the baloon touch the horizontal plane through A in R. We have to determine AR

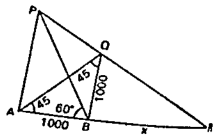


Fig 83

$AB=1000$ and let $BR=x$

$PA=AB\tan 60°=1000\sqrt{3}$ (1)

Also $AB=QB=1000$ (2)

as $\triangle ABQ$ is isosceles

Now from similar triangles APR and BQR we have

$$\frac{AP}{AR}=\frac{BQ}{BR}\quad\text{or}\quad\frac{1000\sqrt{3}}{1000+x}=\frac{1000}{x}$$

$$x\sqrt{3}=1000+x\quad\text{or}\quad x(\sqrt{3}-1)=1000$$

$$x=\frac{1000}{\sqrt{3}-1}=\frac{1000(\sqrt{3}+1)}{3-1}=500(\sqrt{3}+1)$$

$$AR=1000+x=1000+500(\sqrt{3}+1)$$

$$=500(2+\sqrt{3}+1)=500\sqrt{3}(\sqrt{3}+1)\text{ ft}$$

23 Proceed exactly as above

Here $\quad BQ=AB\tan 30°=\frac{1000}{\sqrt{3}}$

$$\frac{AP}{AR}=\frac{BQ}{BR}\quad\text{or}\quad\frac{1000\sqrt{3}}{1000+x}=\frac{1000}{\sqrt{3}x}$$

or $\quad 3x=1000+x,\quad x=500$

Hence $\quad AR=1000+x=1000+500=1500$

24 (a) $H=x\tan\beta$

$$H+h=x\tan\alpha$$

$$\frac{H+h}{H}=\frac{\tan\alpha}{\tan\beta}$$

or $\quad 1+\frac{h}{H}=\frac{\tan\alpha}{\tan\beta}$

$$\frac{h}{H}=\frac{\tan\alpha}{\tan\beta}-1$$

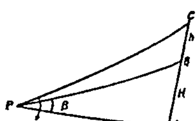


Fig 84

(d)

and as

in the ra

Solutions

1 Ans (*c*) 2 Ans (*c*)

3 Ans (*b*) 4 Ans 0

[Hint Operate R_3-R_2 and R_2-R_1]

5 Ans (a)

Operating C_3-C_2 and C_2-C_1, we get

$$\Delta=\begin{vmatrix} x+1 & 1 & 2 \\ x+3 & 2 & 3 \\ x+7 & 3 & 4 \end{vmatrix} \text{ Apply } R_3-R_2, -R_2-R_1$$

$$\Delta=\begin{vmatrix} 4 & 1 & ? \\ 2 & 1 & 1 \\ x+1 & 1 & 1 \end{vmatrix} \text{ Apply } C_3-C_2$$

$$\Delta=\begin{vmatrix} 4 & 1 & 1 \\ 2 & 1 & 0 \\ 4 & 1 & 0 \end{vmatrix}=1\,(2-4)=-2$$

6 Ans (b)

The two determinants denote twice the area of Δs whose vertices are (x_1, y_1), (x_2, y_2), $(x_3\ y_3)$ and $(a_1\ b_1)$, $(a_2\ b_2)$, $(a_3,\ b_3)$ Thus the equality of two determinants implies that their areas are equal But equality of the areas of two triangles does not imply that they are congruent

7 Ans $F(x)$ at $x=a$ is 0

$$F'(x)=\begin{vmatrix} f_1'(x) & f_2'(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1'(x) & h_2'(x) & h_3(x) \end{vmatrix}$$

Hence $F(a)=0$, since two rows in each determinant are identical by virtue of the relations

(ii) From case (i), we have

$$a^2+b^2 \geqslant 2ab \quad \text{or} \quad a^2-ab+b^2 \geqslant ab$$

$$(a+b)(a^2-ab+b^2) \geqslant ab(a+b)$$

or $\quad a^3+b^3 \geqslant ab(a+b)$

Similarly $\quad b^3+c^3 \geqslant bc(b+c)$

and $\quad c^3+a^3 \geqslant ca(c+a)$,

whence by addition we get

$$2(a^3+b^3+c^3) \geqslant ab(a+b)+bc(b+c)+ca(c+a)$$

7 Since a, b, c are the sides of a triangle, we have

$$a > 0, b > 0, c > 0$$

We may assume $a \geqslant b \geqslant c$

In a triangle difference of two sides is less than the third side

$$a-b < c \quad \text{or} \quad (a-b)^2 < c^2$$

or $\quad a^2+b^2-2ab < c^2$

or $\quad a^2+b^2-c^2 < 2ab$

Similarly $\quad b^2+c^2-a^2 < 2bc$

and $\quad c^2+a^2-b^2 < 2ca$

Adding all these inequalities we get

$$a^2+b^2+c^2 < 2(ab+bc+ca)$$

Adding $2(ab+bc+ca)$ to both sides of the above inequality

we get $\quad (a+b+c)^2 < 4(ab+bc+ca)$ (1)

Again $(a-b)^2 \geqslant 0$ or $a^2+b^2 \geqslant 2ab$

Similarly $b^2+c^2 \geqslant 2bc$ and $c^2+a^2 \geqslant 2ca$

Hence $\quad 2(a^2+b^2+c^2) \geqslant 2(ab+bc+ca)$

or $\quad a^2+b^2+c^2 \geqslant ab+bc+ca$

Adding $\quad 2(ab+bc+ca)$ to both sides

$$(a+b+c)^2 \geqslant 3(ab+bc+ca) \tag{2}$$

From (1) and (2), we get

$$3(ab+bc+ca) \leqslant (a+b+c)^2 < 4(ab+bc+ca)$$

8 Hint (i) Use $A.M. \geqslant G.M.$ in pairs a, b; b, c and c, a and multiply

(ii) Use $A.M. \geqslant G.M.$ in pairs

9 Since $x > 0, y > 0, z > 0$ and $x < y < z$, we have

$$3x^2 < x^2+y^2+z^2 < 3z^2 \tag{1}$$

and $\quad 3x < x+y+z < 3z$

or $\quad \dfrac{1}{3x} > \dfrac{1}{x+y+z} > \dfrac{1}{3z}$ i.e. $\dfrac{1}{3z} < \dfrac{1}{x+y+z} < \dfrac{1}{3x}$ (2)

Multiplying the inequalities (1) and (2) we get

$$\frac{3x^2}{3z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{3z^2}{3x} \text{ or } \frac{x^2}{z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{z^2}{x}$$

$f_r(a)=g_r(a)=h_r(a),\ r=1,2,3$

Ans (b)

$$\frac{d}{dx}\Delta_1=\begin{vmatrix}1&0&0\\a&x&b\\a&a&x\end{vmatrix}+\begin{vmatrix}x&b&b\\0&1&0\\a&a&x\end{vmatrix}+\begin{vmatrix}x&b&b\\a&x&b\\0&0&1\end{vmatrix}$$

$$=\begin{vmatrix}x&b\\a&x\end{vmatrix}+\begin{vmatrix}x&b\\a&x\end{vmatrix}+\begin{vmatrix}x&b\\a&x\end{vmatrix}$$

$$=3\begin{vmatrix}x&b\\a&x\end{vmatrix}$$

9 Ans (b) and (d)

Operating $R_3-(\alpha R_1+R_2)$ and expanding we shall easily get $\Delta=-(a\alpha^2+2b\alpha+c)(ac-b^2)$

Hence Δ is zero if $ac-b^2=0$ or $a\alpha^2+2b\alpha+c=0$, *i e* if a, b, c are in G P or $x-\alpha$ is a factor of $ax^2+2bx+c$

10 Ans (c)

$|\vec{c}|=1$, we have $|\vec{c}|^2-1$ or $c_1^2+c_2^2+c_3^2=1$ (1)

Again since $\vec{c}\perp\vec{a}$ and $\vec{c}\perp\vec{b}$, we have

$\vec{c}\,\vec{a}=0 \Rightarrow a_1c_1+a_2c_2+a_3c_3=0$ (2)

and $\vec{c}\,\vec{b}=0 \Rightarrow b_1c_1+b_2c_2+b_3c_3=0$ (3)

Also since angle between $\vec{a}$ and $\vec{b}$ is $\pi/6$ we have

$\mathbf{a}\,\mathbf{b}=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\ (b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})$

or $|\mathbf{a}|\ |\mathbf{b}|\cos\frac{\pi}{6}=(a_1b_1+a_2b_2+a_3b_3)$

or $(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)\ \frac{3}{4}=(a_1b_1+a_2b_2+a_3b_3)^2$ (4)

$$\text{Now}\ \begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}^2=\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}\begin{vmatrix}a_1&a_2&a_3\\b_1&b_2&b_3\\c_1&c_2&c_3\end{vmatrix}$$

$$=\begin{vmatrix}a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & a_1c_1+a_2c_2+a_3c_3\\ b_1a_1+b_2a_2+b_3a_3 & b_1^2+b_2^2+b_3^2 & b_1c_1+b_2c_2+b_3c_3\\ c_1a_1+c_2a_2+c_3a_3 & c_1b_1+c_2b_2+c_3b_3 & c_1^2+c_2^2+c_3^2\end{vmatrix}$$

10 Let $f(x)=x^8-x^5+x^2-x+1$

For $x=1$ or $0, f(x)=1>0$

Clearly for $x<0$, each term of $f(x)$ is positive and so $f(x)>0$ in this case For $0<x<1$, we write

$$f(x)=(1-x)+x^2(1-x^3)+x^8$$

which is positive [Note $1-x>0$ and

$1-x^3>0$ when $0<x<1$]

Finally for $x>1$, we write

$f(x)=x^5(x^3-1)+x(x-1)+1$ which is positive

Hence $f(x)$ is positive for all real x

11 First Solution We clearly have

$$(x-y)^2\geqslant 0 \text{ or } x^2+y^2\geqslant 2xy$$

whence $2(x^2+y^2)\geqslant x^2+2xy+y^2$ (1)

Since $x^2+y^2=1$, (1) gives

$$x^2+2xy+y^2\leqslant 2 \text{ or } (x+y)^2\leqslant 2$$

Whence $|x+y|\leqslant|\sqrt{2}|$ or $-\sqrt{2}\leqslant x+y\leqslant\sqrt{2}$

Second Solution If x and y satisfy the condition $x^2+y^2=1$, we can find an angle θ such that $\cos\theta=x$, $\sin\theta=y$ Then for any value of θ, we have to prove $-\sqrt{2}\leqslant\cos\theta+\sin\theta\leqslant\sqrt{2}$

$$\text{Now } \cos\theta+\sin\theta=\sqrt{2}\left[\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right]$$

$$=\sqrt{2}\left[\sin\frac{\pi}{4}\cos\theta+\cos\frac{\pi}{4}\sin\theta\right]$$

$$=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$$

Since $-1\leqslant\sin\left(\theta+\frac{\pi}{4}\right)\leqslant 1$, we have

$$-\sqrt{2}\leqslant\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)\leqslant\sqrt{2}$$

for all values of θ and the proof is complete

12 We have

$$\frac{1}{2n}=\frac{1}{2n},\ \frac{1}{2n-1}>\frac{1}{2n},\ \frac{1}{2n-2}>\frac{1}{2n},\ \ldots,$$

$$\frac{1}{n+2}>\frac{1}{2n} \text{ and } \frac{1}{n+1}>\frac{1}{2n}$$

[Above inequalities hold since

$n+1<2n, n+2<2n, \ldots 2n-1<2n$]

Adding these inequalities termwise, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}>\frac{1}{2n}+\frac{1}{2n}+\ldots+\frac{1}{2n}=\frac{n}{2n}=\frac{1}{2}$$

$$=\begin{vmatrix} a_1^2+a_2^2+a_3^2 & a_1b_1+a_2b_2+a_3b_3 & 0 \\ a_1b_1+a_2b_2+a_3b_3 & b_1^2+b_2^2+b_3^2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$ using the relations (1), (2), and (3)

$=(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)-(a_1b_1+a_2b_2+a_3b_3)^2$

$=(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

$\quad -(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)\,\frac{3}{4}$, by (4)

$=\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$

11 Ans (A) and (C)

Adding C_3 to C_1 we get

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 2 & 1+\cos^2\theta & 4\sin 4\theta \\ 1 & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix}=0$$

Now applying R_2-R_1 and R_3-R_1

$$\begin{vmatrix} 2 & \cos^2\theta & 4\sin 4\theta \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}=0$$

or $\quad 2+4\sin 4\theta=0$

or $\quad \sin 4\theta=-1/2=\sin(-\pi/6)=\sin(-\pi/6)$

$4\theta=n\pi+(-1)^n(-\pi/6)$

The value of θ lying between 0 and $\pi/2$ are $7\pi/24$ and $11\pi/24$ for $n=1$ and 2

13 $1^r, 2^r, 3^r, \ldots, n^r$ are unequal positive quantities, their
$A.M. > G.M.$

Hence $\dfrac{1^r+2^r+3^r+\ldots+n^r}{n} > \left(1^r\,2^r\,3^r\ldots n^r\right)^{1/n}$

or $(1^r+2^r+\ldots+n^r)^n > n^n\,(1^r\,2^r\ldots n^r)$

or $(1^r+2^r+\ldots n^r)^n > n^n\,(n\,!)^r$

14 Let $y+z-x=a,\ z+x-y=b,\ x+y-z=c$

Then $a+b=2z,\ b+c=2x,\ c+a=2y$

and $a+b+c=x+y+z$

Hence

(i) $\dfrac{a+b+c}{3} \geqslant (abc)^{1/3}$ [$\because$ $A.M. \geqslant G.M.$]

or $(a+b+c)^3 \geqslant 27\,abc$

or $(x+y+z)^3 \geqslant 27\,(y+z-x)\,(z+x-y)\,(x+y-z)$

(ii) $\dfrac{b+c}{2} \geqslant \sqrt{(bc)},\ \dfrac{c+a}{2} \geqslant \sqrt{(ca)},\ \dfrac{a+b}{2} \geqslant \sqrt{(ab)}$

whence multiplying we get

$\dfrac{(b+c)\,(c+a)\,(a+b)}{8} \geqslant \sqrt{(bc\;ca\;ab)} = abc$

or $\dfrac{2x\;2y\;2z}{8} \leqslant (y+z-x)\,(z+x-y)\,(x+y-z)$

or $xyz \leqslant (y+z-x)\,(z+x-y)\,(x+y-z)$

15 (i) $1+3+5+\ldots+(2n-1)$

$= \dfrac{n}{2}\,[1+(2n-1)] = n^2$ [Summing the A.P.]

$\dfrac{1+3+5+\ldots+(2n-1)}{n} > [1\;3\;5\ldots(2n-1)]^{1/n}$

[$\because$ A.M. > G.M.]

or $\dfrac{n^2}{n} > [1\;3\;5\ldots(2n-1)]^{1/n}$

or $n^n > 1\;3\;5\ldots(2n-1)$

(ii) We consider the series $1, 2, 2^2, 2^3, \ldots, 2^{n-1}$ and apply the inequality A.M. > G.M.

Thus $\dfrac{1+2+2^2+\ldots+2^{n-1}}{n} > (1\;2\;2^2\ldots2^{n-1})^{1/n}$

or $\dfrac{2^n-1}{n(2-1)} > [2^{1+2+3+\ldots+(n-1)}]^{1/n}$

[Summing the G.P. on left side]

11
Inequalities

1 Definition *Let a and b be real numbers If $a-b$ is negative we say that a is less than b and write $a < b$ If $a-b$ is positive then a is greater than b i e $a > b$*

2 Elementary properties of inequalities

(i) For any two real numbers a and b we have
$a > b$ or $a=b$ or $a < b$

(ii) If $a > b$ and $b > c$ then $a > c$

(iii) If $a > b$, then $a+m > b+m$, for any real number m

(iv) If $a > b$ then $am > bm$ for $m > 0$ and $am < bm$ for $m < 0$, that is when we multiply both sides of the inequality by a negative quantity the sign of inequality is reversed

(v) If $a \neq 0$, $b \neq 0$ and $a > b$, then
$\frac{1}{a} < \frac{1}{b}$

(vi) If $a_1 > b_1$, $a_2 > b_2$, $a_n > b_n$, then
$a_1+a_2+ \quad +a_n > b_1+b_2+ \quad +b_n$
and $a_1a_2 \quad a_n > b_1b_2 \quad b_n$
($a_i \geqslant 0$ and $b_i \geqslant 0$ $i=1, 2 \quad n$)

3 Some Important Properties

(i) If $x > 0$ and $a > b > 0$, then $a^x > b^x$,

(ii) If $a > 1$ and $x > y > 0$ then $a^x > a^y$,

(iii) If $0 < a < 1$ and $x > y > 0$, then $a^x < a^y$

(iv) If $a > 1$ and $x > y$ then $\log_a x > \log y$,

(v) If $0 < a < 1$ and $x > y$ then $\log_a x < \log_a y$

4 Some theorems on inequalities

We state some theorems on inequalities without proof

Theorem 1 *The arithmetic mean of two positive quantities is greater than or equal to their geometric mean*

Theorem 2 *If $a_i > 0$, $i=1, 2, \quad n$, then*

$$\frac{a_1+a_2+ \quad +a_n}{n} \geqslant (a_1a_2 \quad a_n)^{1/n},$$

or $2^n-1 > n\left[2^{n(n-1)/2}\right]^{1/n}=n\,2^{(n-1)/2}=n\sqrt{(2^{n-1})}$

or $2^n > 1+n\sqrt{(2^{n-1})}$

(iii) Hint Consider the numbers 2, 4, 6, , $2n$ and use A M $>$ G M

(iv) Consider the numbers $1^3, 2^3, 3^3,\ \ , n^3$ and use A M $>$ G M and $\Sigma n^3=\left[\dfrac{n(n+1)}{2}\right]^2$

16 We have to prove $\left(\dfrac{a}{b+c}+1\right)+\left(\dfrac{b}{c+a}+1\right)+\left(\dfrac{c}{a+b}+1\right)\geqslant 3+\dfrac{3}{2}$

or $(a+b+c)\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right)\geqslant\dfrac{9}{2}$ (1)

Now $\dfrac{(b+c)+(c+a)+(a+b)}{3}\geqslant[(b+c)(c+a)(a+b)]^{1/3}$

or $2(a+b+c)\geqslant 3[(b+c)(c+a)(a+b)]^{1/3}$ (1)

and $[(b+c)^{-1}+(c+a)^{-1}+(a+b)^{-1}]$

$\geqslant 3[(b+c)^{-1}(c+a)^{-1}(a+b)^{-1}]^{1/3}$ (2)

Multiplying (1) and (2), we get the result

17 Let m and M be the least and the greatest values of the given fractions Then

$$m\leqslant\frac{a_i}{b_i}\leqslant M \qquad (i=1, 2, 3,\ \ n)$$

$$mb_i\leqslant a_i\leqslant Mb_i$$

Summing all these inequalities

$$m\sum_{i=1}^{n} b_i\leqslant\sum_{i=1}^{n} a_i\leqslant M\sum_{i=1}^{n} b_i$$

or
$$m\leqslant\frac{\Sigma a_i}{\Sigma b_i}M\leqslant$$

that is,
$$m\leqslant\frac{a_1+a_2+\ \ +a_n}{b_1+b_2+\ \ +b_n}\leqslant M$$

18 Hint Use A M $\geqslant$ G M for the sets of numbers x_i and $\dfrac{1}{x_i}$ $(i=1, 2,\ \ , n)$ and multiply

that is, the geometric mean of n positive quantities cannot exceed their arithmetic mean

Theorem 3 *If the sum of two positive quantities is constant, then their product is greatest when they are equal, and if their product is constant, then their sum is least when they are equal*

Theorem 4 *If* $a_i > 0$ $(i=1, 2 \ldots n)$ *and* $a_1+a_2+\ldots a_n$ $=$*const then the product* $a_1a_2 \ldots a_n$ *is greatest when*

$$a_1=a_2=\ldots=a_n$$

Theorem 5 *If* $a_i \geqslant 0$, $i=1, 2, \ldots n$, *then*

$$(i)\quad \frac{a_1^m+a_2^m+\ldots+a_n^m}{n} \geqslant \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m$$

if $m \geqslant 1$ *or* m *is any negative quantity*

$$\text{and }(ii)\quad \frac{a_1^m+a_2^m+\ldots+a_n^m}{n} \leqslant \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^m,$$

if $0 < m < 1$

That is, arithmetic mean of the n^{th} *powers of n positive quantities is greater than the* n^{th} *power of their arithmetic mean in all cases except when* $0 < m < 1$

Theorem 6 *The product of the factorials of two numbers whose sum is constant is least when they are equal or consecutive according as their sum is even or odd*

Problem Set (A)

1 (i) For all $x > 0$, prove $x+\frac{1}{x} \geqslant 2$

(ii) For any real $x \neq 0$, prove

$\left|x+\frac{1}{x}\right| \geqslant 2$ or $\left|\frac{1+x^2}{2x}\right| \geqslant 1$

2 Prove that ${}^{n-1}C_3+{}^{n-1}C_4 > {}^nC_3$ if $n > 7$ [I I T 1975]

3 Find least integer n such that $7^n > 10^5$, given $\log 343=2.5353$ [I I T 1975]

4 If x may have any real value find which is greater, x^3+1 or x^2+x

5 Given $n^4 < 10^n$ for a fixed positive integer $n \geqslant 2$, prove that $(n+1)^4 < 10^{n+1}$ [I I T 1980]

13 $1^r, 2^r, 3^r, \ldots, n^r$ are unequal positive quantities, their
A M > G M

Hence $\dfrac{1^r+2^r+3^r+\ldots+n^r}{n} > \left(1^r\,2^r\,3^r \ldots n^r\right)^{1/n}$

or $(1^r+2^r+\ldots+n^r)^n > n^n\,(1^r\,2^r \ldots n^r)$

or $(1^r+2^r+\ldots n^r)^n > n^n\,(n\,!)^r$

14 Let $y+z-x=a,\ z+x-y=b,\ x+y-z=c$

Then $a+b=2z,\ b+c=2x,\ c+a=2y$

and $a+b+c=x+y+z$

Hence

(i) $\dfrac{a+b+c}{3} > (abc)^{1/3}$ [A M > G M]

or $(a+b+c)^3 \geqslant 27\,abc$

or $(x+y+z)^3 > 27\,(y+z-x)\,(z+x-y)\,(x+y-z)$

(ii) $\dfrac{b+c}{2} \geqslant \sqrt{(bc)},\ \dfrac{c+a}{2} \geqslant \sqrt{(ca)},\ \dfrac{a+b}{2} \geqslant \sqrt{(ab)}$

whence multiplying we get

$\dfrac{(b+c)\,(c+a)\,(a+b)}{8} \geqslant \sqrt{(bc\,ca\,ab)} = abc$

or $\dfrac{2x\,2y\,2z}{8} \leqslant (y+z-x)\,(z+x-y)\,(x+y-z)$

or $xyz \leqslant (y+z-x)\,(z+x-y)\,(x+y-z)$

15 (i) $1+3+5+\ldots+(2n-1)$

$= \dfrac{n}{2}\,[1+(2n-1)] = n^2$ [Summing the A P]

$\dfrac{1+3+5+\ldots+(2n-1)}{n} > [1\;3\;5 \ldots (2n-1)]^{1/n}$

[A M > G M]

or $\dfrac{n^2}{n} > [1\;3\;5 \ldots (2n-1)]^{1/n}$

or $n^n > 1\;3\;5 \ldots (2n-1)$

(ii) We consider the series $1, 2, 2^2, 2^3, \ldots, 2^{n-1}$ and apply the inequality A M > G M

Thus $\dfrac{1+2+2^2+\ldots+2^{n-1}}{n} > (1\;2\;2^2 \ldots 2^{n-1})^{1/n}$

or $\dfrac{2^n-1}{n(2-1)} > \left[2^{1+2+3+\ldots+(n-1)}\right]^{1/n}$

[Summing the G P on left side]

6 For $a > 0, b > 0, c > 0$, prove that

(i) $a^2+b^2+c^2 \geqslant bc+ca+ab$

(ii) $2(a^3+b^3+c^3) \geqslant bc(b+c)+ca(c+a)+ab(a+b)$

7 Show for any triangle with sides a, b and c

$3(ab+bc+ca) \leqslant (a+b+c)^2 < 4(bc+ca+ab)$

[I I T 1979]

8 For $a > 0$ $b > 0, c > 0$ prove that

(i) $(a+b)(b+c)(c+a) \geqslant 8abc$,

(ii) $\frac{bc}{a}+\frac{ca}{b}+\frac{ab}{c} \geqslant a+b+c$

9 If x, y, z are all positive and $x < y < z$, show that

$$\frac{x^2}{z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{z^2}{x}$$ [I I T 1971]

10 Prove that the polynomial $x^8-x^5+x^2-x+1$ is positive for all real values of x

11 If $x^2+y^2=1$, prove $-\sqrt{2} \leqslant x+y \leqslant \sqrt{2}$

12 Show that

$$\frac{1}{n+1}+\frac{1}{n+2}+ \quad +\frac{1}{2n} > \tfrac{1}{2} \text{ (} n \text{ a positive integer} > 1)$$

13 Show that

$$(1^r+2^r+3^r+ \quad +n^r)^n > n^n (n!)^r$$

where r is any real quantity

14 If $x > 0, y > 0, z > 0$ prove that

(i) $(x+y+z)^3 \geqslant 27(y+z-x)(z+x-y)(x+y-z)$

(ii) $xyz \geqslant (y+z-x)(z+x-y)(x+y-z)$

15 If n is a positive integer prove that

(i) $n^n > 1\ 3\ 5 \quad (2n-1)$

(ii) $2^n > 1+n\sqrt{2^{n-1}}$ $(n > 2)$

(iii) $2\ 4\ 6 \quad 2n < (n+1)^n$

(iv) $(n!)^3 < n^n\left(\frac{n+1}{2}\right)^{2n}$

16 Prove $\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geqslant \frac{3}{2}$ $(a\ b, c > 0)$

17 There are n fractions

$$\frac{a_1}{b_1}, \frac{a_2}{b_2}, \quad \frac{a_n}{b_n} \quad (b_i > 0, i=1\ 2, \quad n)$$

or $2^n-1 > n\left[2^{n(n-1)/2}\right]^{1/n} = n\,2^{(n-1)/2} = n\sqrt{(2^{n-1})}$

or $2^n > 1+n\sqrt{(2^{n-1})}$

(iii) Hint Consider the numbers 2, 4, 6, , $2n$ and use A M > G M

(iv) Consider the numbers $1^3, 2^3, 3^3,\ \ , n^3$ and use A M > G M. and $\Sigma n^3 = \left[\dfrac{n(n+1)}{2}\right]^2$

16 We have to prove $\left(\dfrac{a}{b+c}+1\right)+\left(\dfrac{b}{c+a}+1\right)$

$$+\left(\frac{c}{a+b}+1\right) > 3+\frac{3}{2}$$

or $(a+b+c)\left(\dfrac{1}{b+c}+\dfrac{1}{c+a}+\dfrac{1}{a+b}\right) > \dfrac{9}{2}$ (1)

Now $\dfrac{(b+c)+(c+a)+(a+b)}{3} \geqslant [(b+c)(c+a)(a+b)]^{1/3}$

or $2(a+b+c) \geqslant 3[(b+c)(c+a)(a+b)]^{1/3}$ (1)

and $[(b+c)^{-1}+(c+a)^{-1}+(a+b)^{-1}]$

$\geqslant 3[(b+c)^{-1}(c+a)^{-1}(a+b)^{-1}]^{1/3}$ (2)

Multiplying (1) and (2), we get the result

17 Let m and M be the least and the greatest values of the given fractions Then

$$m \leqslant \frac{a_i}{b_i} \leqslant M \qquad (i=1, 2, 3,\ \ n)$$

$$mb_i \leqslant a_i \leqslant Mb_i$$

Summing all these inequalities

$$m\sum_{i=1}^{n} b_i \leqslant \sum_{i=1}^{n} a_i \leqslant M\sum_{i=1}^{n} b_i$$

or $$m \leqslant \frac{\Sigma a_i}{\Sigma b_i} M \leqslant$$

that is $$m \leqslant \frac{a_1+a_2+\ \ +a_n}{b_1+b_2+\ \ +b_n} \leqslant M$$

18 Hint Use A M $\geqslant$ G M for the sets of numbers x_i and $\dfrac{1}{x_i}$ $(i=1, 2,\ \ , n)$ and multiply

Prove that the value of the fraction $\frac{a_1+a_2+\ \ \ \ +a_n}{b_1+b_2+\ \ \ \ +b_n}$ lies between the least and the greatest of these fractions

18 If $x_i > 0$, $(i=1, 2, \ \ \ n)$, prove

$$(x_1+x_2+\ \ \ \ +x_n)\left(\frac{1}{x_1}+\frac{1}{x_2}+\ \ \ \ +\frac{1}{x_n}\right) \geq n^2$$

19 If $a > 0, b > 0\ c > 0$ prove that

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) \geq 9$$ [I I T 1984]

20 Prove that

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\ \ \ \ +\frac{1}{\sqrt{n}} > 2\sqrt{(n+1)}-2$$

21 If n and p be positive integers and $n \geq 1\ p \geq 1$,

prove that $\frac{1}{n+1}-\frac{1}{n+p+1} < \frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ \ \ \ +$

$$\frac{1}{(n+p)^2} < \frac{1}{n}-\frac{1}{n+p}$$

22 If $a, b\ c$ are in $H\ P$ and $n > 1$ show that
$$a^n+c^n > 2\,b^n$$

23 If $a, b\ c$ are real numbers such that
$a^2+b^2+c^2=1$, show that $a\,b+b\,c+ca > -\frac{1}{2}$ [I I T 72]

24 If $x+y+z=a$ show that

(i) $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geq \frac{9}{a}$

that is, the least value of $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$ is $\frac{9}{a}$

(ii) $\frac{8}{27}a^3 > (a-x)(a-y)(a-z) \geq 8\,x\,y\,z$

25, If $a_i > 0$ $(i=1, 2, \ \ \ n)$ prove

$$\sqrt{(a_1a_2)}+\sqrt{(a_1a_3)}+\ \ \ \ +\sqrt{(a_{n-1}\,a_n)}$$
$$\leq \frac{n-1}{2}(a_1+a_2+\ \ \ \ +a_n)$$

26 Prove that for all x in the interval $0 < x < \frac{\pi}{2}$, the inequality $\tan x+\cot x > 2$ holds true

27 For real $a\ b$ and x, prove

19 Since A M $\geqslant$ G M, we have

$$\frac{a+b+c}{3} \geqslant (abc)^{1/3} \quad (1)$$

and $$\frac{a^{-1}+b^{-1}+c^{-1}}{3} \geqslant (a^{-1}\, b^{-1}\, c^{-1})^{1/3} \quad (2)$$

Multiplying (1) and (2), we get $(a+b+c)(a^{-1}+b^{-1}+c^{-1}) \geqslant 9$

20 We have

$$\frac{1}{\sqrt{r}}=\frac{2}{\sqrt{r}+\sqrt{r}} > \frac{2}{\sqrt{r}+\sqrt{(r+1)}}=\frac{2\,(\sqrt{(r+1)}-\sqrt{r})}{(r+1)-r}$$

Thus $$\frac{1}{\sqrt{r}} > 2\,(\sqrt{(r+1)}-\sqrt{r}) \quad (1)$$

Putting $r=1, 2, 3, \ldots, n$ in (1), we get

$$1 > 2\,(\sqrt{2}-1)$$

$$\frac{1}{\sqrt{2}} > 2\,(\sqrt{3}-\sqrt{2})$$

$$\frac{1}{\sqrt{3}} > 2\,(\sqrt{4}-\sqrt{3})$$

$$\frac{1}{\sqrt{n}} > 2\,(\sqrt{(n+1)}-\sqrt{n})$$

Adding these inequalities we obtain

$$1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}} > 2\sqrt{(n+1)}-2$$

21 We obviously have

$$\frac{1}{(n+r+1)\,(n+r)} < \frac{1}{(n+r)^2} < \frac{1}{(n+r-1)\,(n+r)}$$

But $$\frac{1}{(n+r+1)\,(n+r)}=\frac{1}{(n+r)}-\frac{1}{n+r+1}$$

and $$\frac{1}{(n+r-1)\,(n+r)}=\frac{1}{n+r-1}-\frac{1}{n+r}$$

Hence $$\frac{1}{n+r}-\frac{1}{n+r+1} < \frac{1}{(n+r)^2} < \frac{1}{n+r-1}-\frac{1}{n+r} \quad (1)$$

Putting $r=1, 2, 3, \ldots, p$ in (1) and adding all the resulting inequalities we shall obtain

$$\frac{1}{n+1}-\frac{1}{n+p-1} > \frac{1}{(n+1)^2}+\cdots+\frac{1}{(n+p)^2} < \frac{1}{n}-\frac{1}{n+p}$$

22 Since a, b, c are in H P b is the harmonic mean between a and c Also geometric mean between a and c is $\sqrt{(ac)}$.

$$-\sqrt{(a^2+b^2)} \leqslant a \sin x+b \cos x \leqslant \sqrt{(a^2+b^2)}$$

28 For all real θ, prove

$$\cos(\cos\theta) > 0$$

29 For all real θ prove

$$\cos(\sin\theta) > \sin(\cos\theta) \qquad \textbf{(IIT 1981)}$$

30 If $A+B+C=\pi$ ($A, B, C > 0$) and the angle C is obtuse, Prove that $\tan A \tan B < 1$

31 For all real x, prove

$$0 < \sin^8 x+\cos^{14} x \leqslant 1$$

32 If $a^2+b^2=1$, $m^2+n^2=1$ prove

$$|am+bn| \leqslant 1$$

33 If $A+B+C=\pi$, prove

$$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2} \geqslant 1$$

34 If $A+B+C=\pi$ and A, B C are acute angles prove

(i) $\cot A \cot B \cot C \leqslant \dfrac{1}{3\sqrt{(3)}}$

(ii) $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2} \geqslant 3\sqrt{(3)}$

35 Prove that a positive proper fraction with positive numerator and denominator increases with the increase both in the numerator and the denominator by one and the some positive number, and an improper fraction decreases

36 Prove that in any triangle the semi perimeter is greater than each of its sides

37 Prove that the sum of the cubes of the legs of a right angled triangle is less than the cube of the hypotenuse

38 Prove that the sum of the hypotenuse and the altitude of a right angled triangle dropped on the hypotenuse exceeds the half perimeter of the triangle

39 Prove that the area of an arbitrary triangle is less than one fourth the square of its semi perimeter

40 For $a \geqslant 0$, $b \geqslant 0$ and $\alpha > \beta > 0$ prove

$$(a^\alpha+b^\alpha)^{1/\alpha} \leqslant (a^\beta+b^\beta)^{1/\beta}$$

41 If $x_i > 0$, ($i=1, 2, \quad n$) and $x_1x_2x_3 \quad x_n=1$, prove $x_1+x_2+ \quad +x_n \geqslant n$

42 Prove that $\dfrac{1}{2} < \dfrac{1}{n}\left(\dfrac{1}{2}+\dfrac{2}{3}+ \quad +\dfrac{n}{n+1}\right) < 1$

Since geometric mean is greater than harmonic mean, we have

$$\sqrt{(ac)} > b \quad (1)$$

Now $\frac{a^n+c^n}{2} > \sqrt{(a^n c^n)}$ [A M > G M]

or $a^n+c^n > 2[\sqrt{(ac)}]^n$ (2)

From (1) and (2), we get

$$a^n+c^n > 2b^n$$

23 We have

$(a+b+c)^2 \geqslant 0$ or $a^2+b^2+c^2+2ab+2bc+2ca \geqslant 0$

or $ab+bc+ca \geqslant -\frac{1}{2}(a^2+b^2+c^2)$

$= -\frac{1}{2}$ [$a^2+b^2+c^2=1$]

24 (i) $\frac{x^{-1}+y^{-1}+z^{-1}}{3} \geqslant \left(\frac{x+y+z}{3}\right)^{-1} = \left(\frac{a}{3}\right)^{-1} = \frac{3}{a}$

[Theorem V of § 4]

or $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geqslant \frac{9}{a}$

(ii) $y+z=a-x$, $z+x=a-y$, $x+y=a-z$

Also $y+z \geqslant 2\sqrt{(yz)}$, $z+x \geqslant 2\sqrt{(zx)}$ $x+y \geqslant 2\sqrt{(xy)}$

Whence $(y+z)(z+x)(x+y) \geqslant 8xyz$

or $(a-x)(a-y)(a-z) \geqslant 8xyz$ (1)

Again $\frac{(a-x)+(a-y)+(a-z)}{3} \geqslant [(a-x)(a-y)(a-z)]^{1/3}$

or $8a^3=[3a-(x+y+z)]^3 \geqslant 27[(a-x)(a-y)(a-z)]$ (2)

[$x+y+z=a$]

From (1) and (2) we obtain,

or $\frac{8a^3}{27} \geqslant (a-x)(a-y)(a-z) \geqslant 8xyz$

25 Since G M $\leqslant$ A M, we have

$\sqrt{(a_1a_2)} \leqslant \frac{a_1+a_2}{2}$, $\sqrt{(a_1a_3)} \leqslant \frac{a_1+a_3}{2}$ $\sqrt{(a_{n-1}a_n)} \leqslant \frac{a_{n-1}+a_n}{2}$

Adding these inequalities termwise we get the required inequality

26 (i) Since $0 < x < \frac{\pi}{2}$, $\tan x > 0$ and $\cot x > 0$ Hence

$\frac{\tan x+\cot x}{2} \geqslant \sqrt{(\tan x \cot x)} = 1$ [A M $\geqslant$ G M]

or $\tan x+\cot x \geqslant 2$

27 We have

$a \sin x + b \cos x$

43 If n is a positive integer, prove

$$\tfrac{1}{2}n < 1+\tfrac{1}{2}+\tfrac{1}{3}+ \quad +\frac{1}{2^n-1} < n$$

44 If $ax^2+\dfrac{b}{x} > c$ for all positive x where $a > 0$ and $b > 0$, show that $27\,ab^2 > 4c^3$ (IIT 1982)

45 Divide an odd integer into two integral parts whose product is maximum

46 If a, b, c are distinct positive integers, show that

$$\left(\frac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} > a^a\, b^b\, c^c$$

47 If a, b, c are all positive and $0 < r < p$, show that $a^p+b^p \geqslant a^{p-r}\, b^r+b^{p-r}\, a^r$ and deduce that

$$3\,(a^p+b^p+c^p) \geqslant (a^r+b^r+c^r)\,(a^{p-r}+b^{p-r}+c^{p-r})$$

Solutions to Problem Set (A)

1 (i) Since $x > 0$ we have

$$\frac{1}{2}\left(x+\frac{1}{x}\right) > \sqrt{\left(x\ \frac{1}{x}\right)} \quad [\quad AM > GM]$$

or $x+\dfrac{1}{x} \geqslant 2$

(ii) Since both sides of the inequality are positive

$\left|x+\dfrac{1}{x}\right| > 2$ is equivalent to

$\left|x+\dfrac{1}{x}\right|^2 > 4$

or $\left(x+\dfrac{1}{x}\right)^2 > 4$ or $x^2+\dfrac{1}{x^2}+2 \geqslant 4$

or $x^2+\dfrac{1}{x^2}-2 \geqslant 0$ or $\left(x-\dfrac{1}{x}\right)^2 \geqslant 0,$

which is true

Alternative $\left|x+\dfrac{1}{x}\right| \geqslant 2 \Rightarrow \left|\dfrac{x^2+1}{x}\right| \geqslant 2 \Rightarrow |x^2+1| > 2|x|$
$\Rightarrow x^2+1 > 2\,|x| \Rightarrow |x|^2-2\,|x|+1 \geqslant 0 \Rightarrow (|x|-1)^2 \geqslant 0,$
which is true

2 Since ${}^nC_r+{}^nC_{r+1}={}^{n+1}C_{r+1}$, we have

$${}^{n-1}C_3+{}^{n-1}C_4={}^nC_4=\frac{n\,(n-1)\,(n-2)\,(n-3)}{1\ 2\ 3\ 4}$$

and $${}^nC_3=\frac{n\,(n-1)\,(n-2)}{1\ 2\ 3}$$

$$=\sqrt{(a^2+b^2)}\left[\frac{a}{\sqrt{(a^2+b^2)}}\sin x+\frac{b}{\sqrt{(a^2+b^2)}}\cos x\right]$$

Now put $\frac{a}{\sqrt{(a^2+b^2)}}=\cos\alpha$ Then $\frac{b}{\sqrt{(a^2+b^2)}}=\sin\alpha$

Hence $a\sin x+b\cos x=\sqrt{(a^2+b^2)}(\sin x\cos\alpha+\cos x\sin\alpha)$

$$=\sqrt{(a^2+b^2)}\sin(x+\alpha)$$

since $-1\leqslant\sin(x+\alpha)\leqslant 1$ and $\sqrt{(a^2+b^2)}>0$, we have

$$-\sqrt{(a^2+b^2)}\leqslant\sqrt{(a^2+b^2)}\sin(x+\alpha)\leqslant\sqrt{(a^2+b^2)}$$

that is, $-\sqrt{(a^2+b^2)}\leqslant a\sin x+b\cos x\leqslant\sqrt{(a^2+b^2)}$

28 For all real θ, we have $-1\leqslant\cos\theta\leqslant 1$

Let $\cos\theta=A$ so that $-1\leqslant A\leqslant 1$

Since $-\frac{\pi}{2}<-1$ and $1<\frac{\pi}{2}$, it follows that $-\frac{\pi}{2}<A<\frac{\pi}{2}$

Hence $\cos A>0$ that is $\cos(\cos\theta)>0$

29 We have to prove

$\cos(\sin\theta)-\sin(\cos\theta)>0$

or $\cos(\sin\theta)-\cos\left(\frac{\pi}{2}-\cos\theta\right)>0$

or $2\sin\left(\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\sin\theta+\cos\theta}{2}\right)>0$ (1)

We now show that both the factors on the right hand side of (1) are positive

Since $|\sin\theta-\cos\theta|=\left|\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)\right|$

$$\leqslant\sqrt{2}<\frac{\pi}{2}$$

We have $-\frac{\pi}{2}<\sin\theta-\cos\theta<\frac{\pi}{2}$

or $-\frac{\pi}{4}<\frac{\sin\theta-\cos\theta}{2}<\frac{\pi}{4}$

so that $0<\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}<\frac{\pi}{2}$

and therefore $\sin\left(\frac{\pi}{4}-\frac{\sin\theta-\cos\theta}{2}\right)>0$

Similarly we can prove $\sin\left(\frac{\pi}{4}-\frac{\sin\theta+\cos\theta}{2}\right)>0$

Hence (1) holds which is what we wanted to prove

30 Since $A+B=\pi-C$ we have

$\tan(A+B)=\tan(\pi-C)$

So we have to prove

$$\frac{n(n-1)(n-2)(n-3)}{1\,2\,3\,4} > \frac{n(n-1)(n-2)}{1\,2\,3} \qquad (1)$$

Since $\frac{n(n-1)(n-2)}{1\,2\,3} > 0$ the inequality (1) is equivalent to

$\frac{n-3}{4} > 1$ or $n > 7$

3 We are given $7^n > 10^5$ (1)

The inequality (1) will remain true if we take logarithms of both sides to base 10 which is greater than 1 (see (iv) § 3)

Hence $\log_{10} 7^n > \log_{10} 10^5 = 5$

or $n \log_{10} 7 > 5$

or $n > \frac{5}{\log_{10} 7} = \frac{3\times 5}{3\times \log_{10} 7} = \frac{15}{\log_{10} 7^3}$

or $n > \frac{15}{\log_{10} 343} = \frac{15}{2.5353}$

or $n > 5\,9$

Hence the least integer n satisfying (1) is 6

4 $x^3+1-(x^2+x) = x^3-x^2\ (x-1) = (x^2-1)\ (x-1)$
$= (x-1)^2\ (x+1)$

Now $(x-1)^2 > 0$ and so $x^3+1 >$ or $< x^2+x$ according as $x+1 > 0$ or < 0 i e accoording as $x >$ or < -1 For $x = -1$, the inequality becomes an equality

5 We are given $n^4 < 10^n$ (1)

Also for $n \geqslant 2$ we have

$$\left(1+\frac{1}{n}\right)^4 \leqslant \left(1\frac{3}{2}\right) < 10 \qquad (2)$$

From (1) and (2) we get

$$n^4\left(1+\frac{1}{n}\right)^4 < 10^n\,10$$

or $(n+1)^4 < 10^{n+1}$

6 (i) We have

$$\frac{a^2+b^2}{2} \geqslant \sqrt{(a^2b^2)} \quad [\quad A\,M \geqslant G\,M]$$

or $a^2+b^2 \geqslant 2ab$

Similarly $b^2+c^2 \geqslant 2bc$ and $c^2+a^2 \geqslant 2ca$

whence adding all these inequalities, we get

$$2\,(a^2+b^2+c^2) \geqslant 2\,(ab+bc+ca)$$

or $a^2+b^2+c^2 \geqslant ab+bc+ca$

or $\dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C>0,$ (1)

[$\because$ C is obtuse angle implies $\tan C<0$]

But since A and B are each less than $\pi/2$, it follows that

$$\tan A+\tan B>0$$

Hence (1) will hold if $1-\tan A\tan B>0$

or $\tan A\tan B<1$ as required

31 Clearly $\sin^8 x+\cos^{14} x\geqslant 0$ But the equality will hold if we have $\sin^8 x=0$ and $\cos^{14} x=0$ simultaneously But this is impossible Hence we have the strict inequality

$$\sin^8 x+\cos^{14} x<0 \quad (1)$$

Since $\sin^2 x\leqslant 1$ and $\cos^2 x\leqslant 1$, we have

$$\sin^8 x\leqslant\sin^2 x \text{ and } \cos^{14} x\leqslant\cos^2 x$$

Whence we get by addition

$$\sin^8 x+\cos^{14} x\leqslant\sin^2 x+\cos^2 x=1$$

i e $\sin^8 x+\cos^{14} x\leqslant 1$ (2)

From (1) and (2), we obtain $0<\sin^8 x+\cos^{14} x\leqslant 1$

[Note that equality holds when $x=0$ or $\pi/2$]

32 Since $a^2+b^2=1$ we can find an angle α such that $a=\cos\alpha$ and $b=\sin\alpha$

Similarly we can find an angle β such that

$$m=\cos\beta \text{ and } n=\sin\beta$$

We then have

$$|am+bn|=|\cos\alpha\cos\beta+\sin\alpha\sin\beta|$$
$$=|\cos(\alpha-\beta)|\leqslant 1$$

33 Since $A+B+C=\pi$ or $\dfrac{A}{2}+\dfrac{B}{2}=\dfrac{\pi}{2}-\dfrac{C}{2}$ we have

$$\tan\left(\frac{A}{2}+\frac{B}{2}\right)=\tan\left(\frac{\pi}{2}-\frac{C}{2}\right)$$

or $$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{1-\tan\frac{A}{2}\tan\frac{B}{2}}=\cot\frac{C}{2}=\frac{1}{\tan\frac{C}{2}}$$

or $\tan\dfrac{A}{2}\tan\dfrac{B}{2}+\tan\dfrac{B}{2}\tan\dfrac{C}{2}+\tan\dfrac{C}{2}\tan\dfrac{A}{2}=1$

Put $\tan\dfrac{A}{2}=x$, $\tan\dfrac{B}{2}=y$ and $\tan\dfrac{C}{2}=z$

Then we have to prove

$$x^2+y^2+z^2\geqslant 1$$

when $xy+yz+zx=1$

$$= \frac{\tan\alpha - \tan\beta}{\tan\beta}$$

$$H = \frac{h\tan\beta}{\tan\alpha - \tan\beta}$$

24 (b) $\tan\theta = 1/6$,

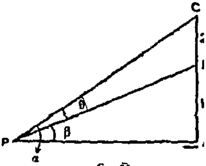


Fig. 85

$\tan\beta = \frac{H}{70}$, $\tan\alpha = \frac{H+20}{70}$

$\alpha = \theta + \beta$,

$$\tan\alpha = \frac{\tan\theta + \tan\beta}{1 - \tan\theta\tan\beta}$$

or $$\frac{H+20}{70} = \frac{\frac{1}{6} + \frac{H}{70}}{1 - \frac{1}{6}\frac{H}{70}} = \frac{70+6H}{420-H}$$

$$(20+H)(420-H) = 70(70+6H)$$
$$8400 + 400H - H^2 = 4900 + 420H$$

$H^2 - 20H - 3500 = 0$ or $(H+70)(H-50) = 0$ $\quad H = 50$

(c) $AD = 2$, $AB = 150$, $BC = 20$

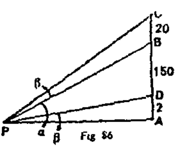


Fig 86

Let $PA = x$

$2 = x\tan\beta$, $\triangle PDA$

$150 = x\tan\alpha$ $\triangle PBA$

$170 = x\tan(\alpha+\beta)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$\frac{170}{x} = \frac{\frac{150}{x} + \frac{2}{x}}{1 - \frac{150}{x}\frac{2}{x}} = \frac{152x}{x^2 - 300}$$

$152x^2 = 170x^2 - 51000$ or $18x^2 = 51000$ $\quad x^2 = \frac{25500}{9}$

$x = \frac{1}{3}\sqrt{(25500)} = \frac{10}{3}\sqrt{(255)} = \frac{10}{3}\ 16 = \frac{160}{3}$

$= 53$ meters approx

(d) $\angle CQB = \angle AQB$ given

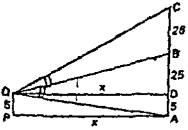


Fig 87

QB is bisector of angle AQC and as such it divides the base AC in the ratio of the arms of the angle

$$\frac{AB}{BC} = \frac{QA}{QC}$$

or $$\frac{25}{26} = \frac{\sqrt{(QD^2 + DA^2)}}{\sqrt{(QD^2 + DC^2)}}$$

or $\frac{25}{26}=\frac{\sqrt{(x^2+5^2)}}{\sqrt{(x^2+46^2)}}$ Square

$625\,(x^2+46^2)=676\,(x^2+25)$

$51x^2=(625\times 46^2-676\times 25)$

$=25\,[25\times 2116-676]=25\times 4\,[25\times 529-169]$

or $x^2=\frac{100\,[13225-169]}{51}$

or $x^2=100\times\frac{13056}{51}=100\times\frac{4352}{17}=100\times 256$

$x=10\times 16=160$ m

25 BP = ladder $=BC=20$ given

$\angle BCP=\angle CPB=30°$

$\angle BPA=60°-30°=30°$

$AB=20\sin 30°=20\cdot\frac{1}{2}=10$

Hence the height of flag staff

$=AB+BC=10+20=30$ ft

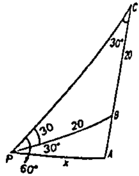


Fig 88

26 AB subtends an angle of 45° at P where $AP=120$

$AB=AP=120$ from isosceles triangle ABP

Again $\frac{120+h}{120}=\tan 60°=\sqrt{3}$

$h=120\,(\sqrt{3}-1)$

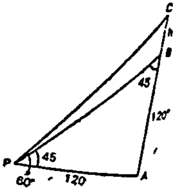


Fig 89

27 We are given that $\tan\theta=\frac{1}{2}$ where θ is the angle subtended by the upper portion

Also $\tan\alpha=\frac{3h}{40}$ $\tan\beta=\frac{h}{40}$

and $\theta=\alpha-\beta$, $\tan\theta=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

or $\frac{1}{2}=\frac{\frac{3h}{40}-\frac{h}{40}}{1+\frac{3h^2}{40^2}}=\frac{80h}{1600+3h^2}$

or $3h^2+1600=160h$

or $3h^2-160h+1600=0$

or $3h^2-120h-40h+1600=0$

or $(h-40)(3h-40)=0$

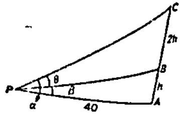


Fig 90

$$=\sqrt{(a^2+b^2)}\left[\frac{a}{\sqrt{(a^2+b^2)}}\sin x\right.$$

Now put $\frac{a}{\sqrt{(a^2+b^2)}}=\cos\alpha$ Then

Hence $a\sin x+b\cos x=\sqrt{(a^2+b^2)}$ (si

$=\sqrt{(a^2+b^2)}$ sir

since $-1\leqslant\sin(x+\alpha)\leqslant 1$ and $\sqrt{(a^2}$

$-\sqrt{(a^2+b^2)}\leqslant\sqrt{(a^2+b^2)}\sin(x$

that is, $-\sqrt{(a^2+b^2)}\leqslant a\sin x+b\cos x$

28 For all real θ, we have $-1\leqslant\cos\theta\leqslant$

Let $\cos\theta=A$ so that $-1\leqslant A\leqslant 1$

Since $-\frac{\pi}{2}<-1$ and $1<\frac{\pi}{2}$, it foll

Hence $\cos A>0$ that is, $\cos(\cos\theta)>$

29 We have to prove

$\cos(\sin\theta)-\sin(\cos\theta)>0$

or $\cos(\sin\theta)-\cos\left(\frac{\pi}{2}-\cos\theta\right.$

or $2\sin\left(\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}\right)\sin\left(\frac{\pi}{4}-\right.$

We now show that both the factors o

(1) are positive

Since $|\sin\theta-\cos\theta|=|\sqrt{2}$

$\leqslant\sqrt{2}<\frac{\pi}{2}$

We have $-\frac{\pi}{2}<\sin\theta$

or $-\frac{\pi}{4}<\frac{\sin\theta-}{2}$

so that $0<\frac{\pi}{4}+\frac{\sin\theta}{2}$

and therefore $\sin\left(\frac{\pi}{4}-\frac{\sin\theta-\cos\theta}{2}\right)$

Similarly we can prove $\sin\left(\frac{\pi}{4}\quad\frac{\sin\theta+}{}\right.$

Hence (1) holds which is what we wante

30 Since $A+B=\pi-C$ we have

$\tan(A+B)=\tan(\pi-C)$

From (1) and (2), we get

$$\tfrac{1}{2}n < S < n \text{ as required}$$

44 Since the inequality

$$ax^2+\frac{b}{x} \geqslant c \qquad (1)$$

holds for all positive values of x, it must hold for that positive value of x for which $ax^2+\frac{b}{x}$ is minimum

Let $u=ax^2+\frac{b}{x}$

For maximum or minimum of u, we have

$\frac{du}{dx}=0$, that is, $\frac{du}{dx}=2ax-\frac{b}{x^2}=0$ or $x=\left(\frac{b}{2a}\right)^{1/3}$

And $\frac{d^2u}{dx^2}=2a+\frac{2b}{x^3}>0$ for $x=\left(\frac{b}{2a}\right)^{1/3}$

Hence u is minimum for $x=\left(\frac{b}{2a}\right)^{1/3}$

Substituting in (1), we get

$$a\left(\frac{b}{2a}\right)^{2/3}+b\left(\frac{2a}{b}\right)^{1/3}\geqslant c$$

or $$a^{1/3}b^{2/3}\left[\frac{1}{2^{2/3}}+2^{1/3}\right]\geqslant c$$

or $$\frac{3}{2^{2/3}}\ a^{1/3}b^{2/3}\geqslant c$$

or $$\frac{27}{4}\ ab^2\geqslant c^3$$

or $$27\ ab^2\geqslant 4\ c^3$$

45 We denote the odd integer by $2n+1$

Let the two parts of $2n+1$ be x and $2n+1-x$. If y denotes the product of these parts then

$$y=x\ (2n+1-x)$$

or $$x^2-(2n+1)\ x+y=0$$

or $$2x=(2n+1)\pm\sqrt{[(2n+1)^2-4y]}$$

But the quantity under the radical must be positive, that is,

$$(2n+1)^2-4y>0$$

or $$y<\frac{1}{4}\ (2n+1)^2=n^2+n+\frac{1}{4}$$

Since y is a positive integer its greatest value is n^2+n and in this case we have

35 Let $\frac{p}{q}$ be any positive proper fraction with $p > 0, q > 0$

Since $\frac{p}{q}$ is a proper fraction we have $0 < \frac{p}{q} < 1$, i e $p<q$
Let x be any positive number Then we have to prove

$$\frac{p+x}{q+x} > \frac{p}{q}$$

or $$\frac{p+x}{q+x} - \frac{p}{q} > 0$$

or $$\frac{(q-p)x}{q(q+x)} > 0$$

which is true since $p > 0$ $q > 0$ $x > 0$ and $q - p > 0$
Similarly we can prove the reverse inequality in the case of improper fraction $\frac{p}{q}$ where $p > q$

36 We know that the sum of two sides of a triangle is greater than the third side Hence if a b c are the sides of a triangle, then $b+c > a$ or $a+b+c > 2a$

or $$\frac{a+b+c}{2} > a$$

Similarly $$\frac{a+b+c}{2} > b \text{ and } \frac{a+b+c}{2} > c$$

37 Let a and b be the lengths of the legs (i e the lengths of the sides containing the right angle) and c the length of its hypotenuse

Thus $$a^2+b^2=c^2 \qquad (1)$$

Then we have to prove

$$a^3+b^3 < c^3=(a^2+b^2)^{3/2} \text{ by } (1)$$

or $$(a^3+b^3)^2 < (a^2+b^2)^3$$

or $$a^6+b^6+2a^3b^3 < a^6+b^6+3a^2b^2\ (a^2+b^2)$$

or $$\tfrac{2}{3}\ ab < a^2+b^2 \quad \text{or} \quad a^2+b^2 > \tfrac{2}{3}\ ab$$

But $\frac{a^2+b^2}{2} \geqslant \sqrt{(a^2b^2)}=ab$ [$A\,M \geqslant G\,M$]

Hence $a^2+b^2 \geqslant 2ab > \frac{2}{3}\ ab$ as required

38 Let p be the length of perpendicular dropped from the right angle on the hypotenu e of lenght c of a right angled triangle of sides a and b Then

$$pc=ab = (\text{twice the area of } \Delta) \qquad (1)$$

$$2x=(2n+1) \pm \sqrt{[(2n+1)^2-4(n^2+n)]}$$
$$=2n+1\pm 1$$

Hence $x=n+1$ or n

The required two parts are n and $n+1$

16 Consider a quantities each equal to a, b quantities each equal to b and c quantities each equal to c

Their $A\,M=\dfrac{a\,a+b\,b+c\,c}{a+b+c}=\dfrac{a^2+b^2+c^2}{a+b+c}$

Their $G\,M=(a^a b^b c^c)^{1/(a+b+c)}$

$$A\,M > G\,M$$

$$\frac{a^2+b^2+c^2}{a+b+c} > (a^a b^b c^c)^{1/(a+b+c)}$$

or $\left(\dfrac{a^2+b^2+c^2}{a+b+c}\right)^{a+b+c} > a^a b^b c^c$

[Note that a, b, c are distinct positive integers, we have strict inequality]

17 Since a b c and $p-r$ are all positive a^r-b^r and $a^{p-r}-b^{p-r}$ are either both positive (when $a>b$), or both negative (when $a<b$) or both zero (when $a=b$) Hence we have the inequality

$$(a^r-b^r)(a^{p-r}-b^{p-r}) \geqslant 0$$

or $a^p+b^p - a^{p-r}b^r-a^r b^{p-r} \geqslant 0$

or $a^p+b^p \geqslant a^{p-r}b^r+b^{p-r}a^r$ (1)

Similarly $b^p+c^p \geqslant b^{p-r}c^r+c^{p-r}b^r$ (2)

and $c^p+a^p \geqslant c^{p-r}a^r+a^{p-r}c^r$ (3)

Adding (i), (ii) and (iii) we get

$$2(a^p+b^p+c^p) \geqslant a^r(b^{p-r}+c^{p-r})+b^r(c^{p-r}+a^{p-r})+c^r(a^{p-r}+b^{p-r})$$

Adding $a^p+b^p+c^p$ to both sides, we get

$$3(a^p+b^p+c^p) \geqslant a^r(a^{p-r}+b^{p-r}+c^{p-r})+b^r(a^{p-r}+b^{p-r}+c^{p-r})+c^r(a^{p-r}+b^{p-r}+c^{p-r})$$

or $3(a^p+b^p+c^p) \geqslant (a^r+b^r+c^r)(a^{p-r}+b^{p-r}+c^{p-r})$

Solving Inequalities

Problem Set (B)

1 Solve the following inequalities

(i) $|4x|<2$

(ii) $|x+3|>|2x-1|$,

But we have

$$2x^2+2y^2+2z^2-2xy-2yz-2zx=(x-y)^2+(y-z)^2+(z-x)^2 \geqslant 0$$

or $\quad 2(x^2+y^2+z^2) \geqslant 2(xy+yz+zx)=2$

or $\quad x^2+y^2+z^2 \geqslant 1$, which is what we wanted to prove

34 (i) Since $A+B+C=\pi$, we have

$$\tan(A+B)=\tan(\pi-C)$$

or $\quad \dfrac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$

or $\quad \tan A+\tan B+\tan C=\tan A\tan B\tan C \qquad (1)$

Since A, B, C are acute angles, $\tan A>0$, $\tan B>0$, $\tan C>0$ Hence

$$\frac{\tan A+\tan B+\tan C}{3} \geqslant (\tan A\tan B\tan C)^{1/3}$$

or $\quad (\tan A+\tan B+\tan C)^3 \geqslant 27\tan A\tan B\tan C \qquad (2)$

From (1) and (2), we get

$$(\tan A\tan B\tan C)^3 \geqslant 27\tan A\tan B\tan C$$

or $\quad \tan A\tan B\tan C \geqslant \sqrt{(27)}=3\sqrt{(3)}$

or $\quad \cot A\cot B\cot C \leqslant \dfrac{1}{3\sqrt{(3)}}$

(ii) Since $\dfrac{A}{2}+\dfrac{B}{2}+\dfrac{C}{2}=\dfrac{\pi}{2}$ we have

$$\cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)=\tan\frac{C}{2},$$

or $\quad \dfrac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\dfrac{1}{\cot\frac{C}{2}}$

or $\quad \cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}=\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2} \qquad (1)$

Now $\dfrac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{3} \geqslant \left(\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}\right)^{1/3} \qquad (2)$

Form (1) and (2) we get

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geqslant 3\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{1/3}$$

or $\left(\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}\right)^3 \geqslant 27\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

or $\quad \cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2} \geqslant 3\sqrt{(3)}$

(iii) $a(x+2) > x-1 \quad (a \neq 1)$

3' Solve the following system of inequalities

(i) $12x-6<0,\ 12-3x<0$

(ii) $\frac{x-3}{4}-x<\frac{x-1}{2}-\frac{x-2}{3},\ 2-x>2x-8$

3 Find the integral solutions of the following system of inequalities

(i) $5x-1<(x+1)^2<(7x-3)$ [I I T 1978]

(ii) $\frac{x}{2x+1}\geqslant\frac{1}{4},\ \frac{6x}{4x-1}<\frac{1}{2}$

4 Solve the inequality $(x-2)/(x+2)\geqslant(2x-3)/(4x-1)$

5 Solve the inequalities

(i) $|x^2+3x|+x^2-2\geqslant 0$

(ii) $x^2+x+|x|+1\leqslant 0$

6 Solve the inequalities

(i) $-1\leqslant(\frac{1}{2})^x<2$

(ii) $\log_{1/2} x\geqslant\log_{1/3}x$

7 Solve the inequalities

$$(x^2+x+1)^x<1$$

8 Solve the inequalities

(i) $\sin x>\frac{1}{2}$,

(ii) $\cos x>-\frac{1}{2}$,

(iii) $\sin x-\cos x>0$

9 Determine the values of θ such that $\pi/2\leqslant\theta\leqslant 3\pi/2$ which satisfy the inequality

$$2\cos^2\theta+\sin\theta\leqslant 2$$ [I I T 1976]

Solutions to Problem Set (B)

1 (i) We consider two cases (*a*) and (*b*) as follows

(*a*) Let $x\geqslant 0$ (1)

In this case, the given inequality reduces to $\frac{1}{2}x<2$ i e,

$x<4$ (2)

From (1) and (2) we conclude that the solution set of the given inequality consists of all x in the interval $0\leqslant x<4$ (3)

(*b*) Let $x<0$

In this case, the given inequality becomes $-\frac{1}{2}x<2$ i e,

$$x>-4 \quad (4)$$

From (3) and (4), we see that in this case the solution set consists of all x in the interval $-4<x<0$

35 Let $\frac{p}{q}$ be any positive proper fraction with $p > 0, q > 0$

Since $\frac{p}{q}$ is a proper fraction, we have $0 < \frac{p}{q} < 1$, i e $p<q$,

Let x be any positive number Then we have to prove

$$\frac{p+x}{q+x} > \frac{p}{q}$$

or $$\frac{p+x}{q+x} - \frac{p}{q} > 0$$

or $$\frac{(q-p)\,x}{q\,(q+x)} > 0$$

which is true since $p > 0$ $q > 0$, $x > 0$ and q $p > 0$

Similarly we can prove the reverse inequality in the case of improper fraction $\frac{p}{q}$ where $p > q$

36 We know that the sum of two sides of a triangle is greater than the third side Hence if a b c are the sides of a triangle, then $b+c > a$ or $a+b+c > 2a$

or $$\frac{a+b+c}{2} > a$$

Similarly $$\frac{a+b+c}{2} > b \text{ and } \frac{a+b+c}{2} > c$$

37 Let a and b be the lengths of the legs (i e the lengths of the sides containing the right angle) and c the length of its hypotenuse

Thus $$a^2+b^2=c^2 \quad (1)$$

Then we have to prove

$$a^3+b^3 < c^3=(a^2+b^2)^{3/2} \text{ by } (1)$$

or $$(a^3+b^3)^2 < (a^2+b^2)^3$$

or $$a^6+b^6+2a^3b^3 < a^6+b^6+3a^2b^2\,(a^2+b^2)$$

or $$\tfrac{2}{3}\,ab < a^2+b^2 \quad \text{or} \quad a^2+b^2 > \tfrac{2}{3}\,ab$$

But $\frac{a^2+b^2}{2} \geqslant \sqrt{(a^2b^2)}=ab$ [A M $\geqslant$ G M]

Hence $a^2+b^2 \geqslant 2ab > \frac{2}{3}\,ab$ as required

38 Let p be the length of perpendicular dropped from the right angle on the hypotenuse of lenght c of a right angled triangle of sides a and b Then

$$pc=ab=(\text{twice the area of } \Delta) \quad (1)$$

Combining the solutions in (*a*) and (b), we get the answer

$$-4 < x < 4$$

Alternative $\quad |\tfrac{1}{2}x| < 2$

$$\Rightarrow \quad -2 < \tfrac{1}{2}x < 2$$

$$\Rightarrow \quad -4 < x < 4$$

(ii) Clearly the inequality is not satisfied for $x = -3$ We now consider three cases (*a*), (*b*) and (*c*) as follows

(*a*) Let $x < -3$ (1)

Then the original inequality becomes $-(x+3) > -(2x-1)$, that is, $x > 4$ (2)

Since the inequalities (1) and (2) are inconsistent, there is no solution in this case,

(*b*) Let $-3 < x \leqslant \tfrac{1}{2}$

Then the original inequality reduces to $x+3 > -(2x-1)$, that is $\quad x > -\tfrac{2}{3}$

So in this case the solution consists of all x in the interval

$$-\tfrac{2}{3} < x \leqslant \tfrac{1}{2}$$

(*c*) Let $x > \tfrac{1}{2}$,

Then the original inequality reduces to

$x+3 > 2x-1$, that is, $x < 4$

So in this case the solution consists of all x in the interval

$$\tfrac{1}{2} < x < 4$$

Combining cases (*a*) (*b*) and (*c*), we get the answer

$$-\tfrac{2}{3} < x < 4$$

(iii) The inequality $a(x+2) > x-1$ is equivalent to

$(a-1)x > -(2a+1)$ (1)

If $a > 1$ so that $a-1 > 0$, then (1) gives,

$$x > -\frac{2a+1}{a-1} = \frac{2a+1}{1-a}$$

If $a < 1$ so that $a-1 < 0$, then dividing both members of (1) by $a-1$ will reverse the inequality sign So in this case we get

$$x < -\frac{2a+1}{a-1} = \frac{2a+1}{1-a}$$

So we get the answer

$$x > \frac{2a+1}{1-a} \text{ if } a > 1$$

and $x < \dfrac{2a+1}{1-a}$ if $a < 1$

But we have

$$2x^2+2y^2+2z^2-2xy-2yz-2zx=(x-y)^2+(y-z)^2+(z-x)^2 \geqslant 0$$

or $\quad 2(x^2+y^2+z^2) \geqslant 2(xy+yz+zx)=2$

or $\quad x^2+y^2+z^2 \geqslant 1$, which is what we wanted to prove

34 (i) Since $A+B+C=\pi$, we have

$$\tan(A+B)=\tan(\pi-C)$$

or $$\frac{\tan A+\tan B}{1-\tan A\tan B}=-\tan C$$

or $\quad \tan A+\tan B+\tan C=\tan A\tan B\tan C \qquad (1)$

Since A, B, C are acute angles, $\tan A>0$, $\tan B>0$, $\tan C>0$ hence

$$\frac{\tan A+\tan B+\tan C}{3} \geqslant (\tan A\tan B\tan C)^{1/3}$$

or $\quad (\tan A+\tan B+\tan C)^3 \geqslant 27\tan A\tan B\tan C \qquad (2)$

From (1) and (2), we get

$$(\tan A\tan B\tan C)^3 \geqslant 27\tan A\tan B\tan C$$

or $\quad \tan A\tan B\tan C \geqslant \sqrt{(27)}=3\sqrt{(3)}$

or $$\cot A\cot B\cot C \leqslant \frac{1}{3\sqrt{(3)}}$$

(ii) Since $\frac{A}{2}+\frac{B}{2}+\frac{C}{2}=\frac{\pi}{2}$ we have

$$\cot\left(\frac{A}{2}+\frac{B}{2}\right)=\cot\left(\frac{\pi}{2}-\frac{C}{2}\right)=\tan\frac{C}{2},$$

or $$\frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\frac{1}{\cot\frac{C}{2}}$$

or $$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \qquad (1)$$

Now $$\frac{\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}}{3} \geqslant \left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{1/3} \qquad (2)$$

Form (1) and (2) we get

$$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geqslant 3\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^{1/3}$$

or $$\left(\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}\right)^3 \geqslant 27\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$$

or $$\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2} \geqslant 3\sqrt{(3)}$$

2. (i) We have $12x-6<0 \Rightarrow 12x<6 \Rightarrow x<\frac{1}{2}$. (1)

And $12-3x<0 \Rightarrow 4<x \Rightarrow x>4$ (2)

Since there is no real number x satisfying both the inequalities (1) and (2), the given system of *inequalities has no solution*

(ii) We have

$$\frac{x-3}{4}-x<\frac{x-1}{2}-\frac{x-2}{3}$$

$\Rightarrow 3x-9-12x<6x-6-4x+8 \Rightarrow -11x<11$

$\Rightarrow -x<1 \Rightarrow x>-1$ (1)

And $2-x>2x-8 \Rightarrow -3x>-10$

$\Rightarrow x<\frac{10}{3}$ (2)

From (1) and (2) we see that the solution of the given system of inequalities is given by

$$-1<x<\frac{10}{3}$$

3 (i) The inequality

$5x-1<(x+1)^2 \Rightarrow 5x-1<x^2+x+1$

$\Rightarrow x^2-3x+2>0$

$\Rightarrow (x-1)(x-2)>0$ *i e*, +ive

$\Rightarrow x<1$ or $x>2$ (1)

x does not lie between 1 and 2

And $(x+1)^2<7x-3$

$\Rightarrow x^2+2x+1<7x-3$

$\Rightarrow x^2-5x+4<0$

$\Rightarrow (x-1)(x-4)<0$ *i e*, −ive

$\Rightarrow 1<x<4$

x lies between 1 and 4

From (1) and (2) we conclude that the given system of *inequalities is* satisfied for all values of x in the interval $2<x<4$

The only integral value of x in this interval is 3

Hence the integral solution of the given inequalities *is at* $x=3$

(ii) We have

$\frac{x}{2x+1}>\frac{1}{4} \Rightarrow \frac{x}{2x+1}-\frac{1}{4}>0$

$\Rightarrow \frac{2x-1}{4(2x+1)}>0$ (3)

35 Let $\frac{p}{q}$ be any positive proper fraction with $p > 0, q > 0$

Since $\frac{p}{q}$ is a proper fraction, we have $0 < \frac{p}{q} < 1$, i e $p<q$,
Let x be any positive number Then we have to prove

$$\frac{p+x}{q+x} > \frac{p}{q}$$

or $$\frac{p+x}{q+x} - \frac{p}{q} > 0$$

or $$\frac{(q-p)\,x}{q\,(q+x)} > 0$$

which is true since $p > 0$ $q > 0$ $x > 0$ and $q-p > 0$

Similarly we can prove the reverse inequality in the case of improper fraction $\frac{p}{q}$ where $p > q$

36 We know that the sum of two sides of a triangle is greater than the third side Hence if a b c are the sides of a triangle, then $b+c > a$ or $a+b+c > 2a$

or $$\frac{a+b+c}{2} > a$$

Similarly $$\frac{a+b+c}{2} > b \text{ and } \frac{a+b+c}{2} > c$$

37 Let a and b be the lengths of the legs (*i e* the lengths of the sides containing the right angle) and c the length of its hypotenuse

Thus $$a^2+b^2=c^2 \qquad (1)$$

Then we have to prove

$$a^3+b^3 < c^3=(a^2+b^2)^{3/2} \text{ by } (1)$$

or $$(a^3+b^3)^2 < (a^2+b^2)^3$$

or $$a^6+b^6+2a^3b^3 < a^6+b^6+3a^2b^2\,(a^2+b^2)$$

or $$\tfrac{2}{3}\,ab < a^2+b^2 \quad \text{or} \quad a^2+b^2 > \tfrac{2}{3}\,ab$$

But $\frac{a^2+b^2}{2} \geqslant \sqrt{(a^2b^2)}=ab$ [$\because$ $AM \geqslant GM$]

Hence $a^2+b^2 \geqslant 2ab > \frac{2}{3}\,ab$, as required

38 Let p be the length of perpendicular dropped from the right angle on the hypotenuse of lenght c of a right angled triangle of sides a and b Then

$$pc=ab = (\text{twice the area of } \Delta) \qquad (1)$$

$\Rightarrow \quad (2x-1)(2x+1)>0$ i e, +ive (4)

[*Multiplying both sides of* (3) *by* $4(2x+1)^2$ *which is positive*]

$x<-\frac{1}{2}, \quad \text{or} \quad x>\frac{1}{2}$ (5)

i e x does not lie between $-\frac{1}{2}$ and $\frac{1}{2}$

And $\frac{6x}{4x-1}<\frac{1}{2} \Rightarrow \frac{6x}{4x-1}-\frac{1}{2}<0$

$\Rightarrow \frac{8x+1}{2(4x-1)}<0$

$\Rightarrow (8x+1)(4x-1)<0$ *i e* $-$ive (6)

$-\frac{1}{8}<x<\frac{1}{4}$ (7)

i e x lies between $-\frac{1}{8}$ and $\frac{1}{4}$

Finally from (5) and (7), we conclude that the solution of the given inequalities consists of all x which satisfies the simultaneous inequalities

(a) $x<-\frac{1}{2}$ and $-\frac{1}{8}<x<\frac{1}{4}$

or (b) $x>\frac{1}{2}$ and $-\frac{1}{8}<x<\frac{1}{4}$

But the inequalities in (*b*) are inconsistent and inequalities in (*a*) are satisfied by all x in the interval

$$-\frac{1}{8}<x<-\frac{1}{2}$$

which is therefore the solution of the original system of inequalities, *clearly there is no integral value of* x *in the interval* $-\frac{1}{8}<x<-\frac{1}{2}$ Hence there exists no integral solution of the given inequalities

4 The domain of variable x in this inequality consists of all x except $x=\frac{1}{4}$ and $x=-2$

We shall therefore consider only those values of x which lie in the domain Now

$\frac{x-2}{x+2} \geqslant \frac{2x-3}{4x-1} \Rightarrow \frac{x-2}{x+2}-\frac{2x-3}{4x-1} \geqslant 0$

$\Rightarrow \frac{2(x^2-5x+4)}{(x+2)(4x-1)} \geqslant 0$

$\Rightarrow \frac{(x-1)(x-4)}{(x+2)(4x-1)} \geqslant 0$ (1)

Also $\qquad a^2+b^2=c^2 \qquad (2)$

Now we have to prove

$$p+c > \frac{a+b+c}{2}$$

or $\qquad 2p > a+b-c$

or $\qquad 2\frac{ab}{c} > a+b-c$ by (1)

or $\qquad 2ab > ac+bc-c^2=ac+bc-a^2-b^2$ by (2)

or $\qquad a^2+b^2+2ab > c(a+b)$

or $\qquad (a+b)^2 > c(a+b)$

or $\qquad a+b > c \qquad [\because a+b > 0]$

which is true since the sum of two sides of a triangle is greater than the third side

39 We have to prove

$$\sqrt{s(s-a)(s-b)(s-c)} < \tfrac{1}{4}s^2$$

or $\qquad s(s-a)(s-b)(s-c) < \frac{1}{16}s^4$

or $\qquad (s-a)(s-b)(s-c) < \frac{1}{16}s^3 \qquad (1)$

But $\dfrac{(s-a)+(s-b)+(s-c)}{3} \geqslant [(s-a)(s-b)(s-c)]^{1/3}$

$[\because$ A M $\geqslant$ G M$]$

or $\qquad \dfrac{s}{3} \geqslant [(s-a)(s-b)(s-c)]^{1/3} \quad [\because s-a+s-b+s-c$

$=3s-(a+b+c)=3s-2s=s]$

or $\qquad (s-a)(s-b)(s-c) \leqslant \dfrac{s^3}{27} < \dfrac{1}{16}s^3$

Hence (1) is proved.

40 In case $a=0$ or $b=0$ the equality will hold Now let $a > 0$ and $b > 0$

Since one of the numbers a and b cannot exceed the other, we may assume $0 < a \leqslant b$ Then $0 < \dfrac{a}{b} \leqslant 1$ and since $\alpha > \beta$ it follows that $0 < (a/b)^\alpha \leqslant (a/b)^\beta$

so that $\qquad 1+\left(\dfrac{a}{b}\right)^\alpha \leqslant 1+(a/b)^\beta \qquad (1)$

Since both sides of (1) are greater than 1 we have

$$[1+(a/b)^\alpha]^{1/\beta} \leqslant [1+(a/b)^\beta]^{1/\beta} \qquad (2)$$

Again since $\qquad 1+(a/b)^\alpha > 1$

and $\qquad 0 < 1/\alpha < 1/\beta$, we have

$$[1+(a/b)^\alpha]^{1/\alpha} < [1+(a/b)^\alpha]^{1/\beta} \qquad (3)$$

Clearly $x=1$ and $x=4$ are the solutions of (1)

We now assume $x\neq 4$ and $x\neq 1$, and solve the inequality

$$\frac{(x-1)(x-4)}{(x+2)(4x-1)}>0 \qquad (1)$$

Multiply both sides of (1) by $(x+2)^2$, $(4x-1)^2$, which is positive for the x under consideration. Then (1) is equivalent to the inequality

$$(x+2)(4x-1)(x-1)(x-4)>0 \qquad (2)$$

We now apply the method of intervals to solve the inequality (2). So we consider five intervals

$$x<-2,\ -2<x<\tfrac{1}{4},\ \tfrac{1}{4}<x<1,$$
$$1<x<4 \text{ and } x>4$$

Of these five intervals, the values of x in the three intervals

$$x<-2,\ \tfrac{1}{4}<x<1 \text{ and } x>4$$

satisfy the inequality (2). Since we have already found that $x=1$ and $x=4$ are solutions to the original inequality, we get the answer

$$x<-2,\ \tfrac{1}{4}<x\leqslant 1,\ x\geqslant 4$$

5 (i) Here we consider two cases

(a) $x^2+3x\geqslant 0$ i.e., $x(x+3)\geqslant 0$

(b) $x^2+3x<0$ i.e., $x(x+3)<0$

In case (*a*), the given inequality reduces to

$$2x^2+3x-2\geqslant 0 \quad\text{or}\quad (x+2)(2x-1)\geqslant 0$$

So in this case, we have to seek for the values of x which satisfy the inequalities

$$x(x+3)\geqslant 0 \qquad (1)$$

and $\quad (x+2)(2x-1)\geqslant 0 \qquad (2)$

simultaneously

Now the inequality (1) is satisfied by all x in the intervals

$$x\leqslant -3 \quad\text{and}\quad x\geqslant 0$$

And the inequality (2) is satisfied by all x in the intervals,

$$x\leqslant -2 \quad\text{and}\quad x\geqslant \tfrac{1}{2}$$

Hence we conclude that both the inequalities (1) and (2) are satisfied by all x in the intervals

$$x\leqslant -3 \quad\text{and}\quad x\geqslant \tfrac{1}{2}$$

Similarly in case (*b*), we will find that the solution consists of all x in the intervals

$$-3<x\leqslant -\tfrac{3}{2}$$

(This is left as an exercise)

Combining the solutions in (*a*) and (*b*), we get the answer

$$x\leqslant -\tfrac{3}{2} \quad\text{and}\quad x\geqslant \tfrac{1}{2}$$

From (2) and (3) we get

$$[1+(a/b)^\alpha]^{1/\alpha} < [1+(a/b)^\alpha]^{1/\beta} \leqslant [1+(a/b)^\beta]^{1/\beta}$$

or $\left(\frac{a^\alpha+b^\alpha}{b^\alpha}\right)^{1/\alpha} \leqslant \left(\frac{a^\beta+b^\beta}{b^\beta}\right)^{1/\beta}$

Since $b > 0$, it follows that

or $(a^\alpha+b^\alpha)^{1/\alpha} \leqslant (a^\beta+b^\beta)^{1/\beta}$

41 Hint Use $A.M. \geqslant G.M.$

42 We have $\frac{1}{2}=\frac{1}{2}$, $\frac{1}{2}<\frac{2}{3}$, $\frac{1}{2}<\frac{3}{4}$, ..., $\frac{1}{2}<\frac{n}{n+1}$

Adding these we get

$$n\cdot\tfrac{1}{2}<\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\ldots+\frac{n}{n+1} \tag{1}$$

Again $\frac{1}{2}<1$, $\frac{2}{3}<1$, $\frac{3}{4}<1$, ..., $\frac{n}{n+1}<1$

Adding these we get

$$\tfrac{1}{2}+\tfrac{2}{3}+\tfrac{3}{4}+\ldots+\frac{n}{n+1}<n \tag{2}$$

Therefore from (1) and (2) we obtain

$$\frac{n}{2}<\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1}<n$$

or $$\frac{1}{2}<\frac{1}{n}\left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\ldots+\frac{n}{n+1}\right)<1$$

43 Let $S=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n-1}$

Rewriting S in the following way, we get

$$S=1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+\ldots$$
$$+\left(\frac{1}{2^{n-2}+1}+\ldots+\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}$$
$$> \tfrac{1}{2}+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4})+(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8})+\ldots$$
$$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\ldots+\frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}$$
$$>\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots \text{ to } n \text{ terms}=\tfrac{1}{2}n$$

On the other hand S may be written as

$$S=1+(\tfrac{1}{2}+\tfrac{1}{3})+(\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7})+\ldots+$$
$$\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}+1}+\ldots+\frac{1}{2^n-1}\right)$$
$$<1+(\tfrac{1}{2}+\tfrac{1}{2})+(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4})+\ldots$$
$$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+\ldots+\frac{1}{2^{n-1}}\right)$$
$$=1+1+1+\ldots+1=n \tag{2}$$

(ii) The given inequality can be written as

$$|x| \leqslant -(x^2+x+1) \qquad (1)$$

Since $|x| \geqslant 0$, left hand member of (1) is non negative

Again since $x^2+x+1=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}>0$,

the right hand member of (1) is strictly negative Hence the inequality (1) has no solution

6 (i) To solve this inequality we have to find all those values of x which *simultaneously* satisfy the two inequalities

$$(1/3)^x \geqslant -1 \qquad (1) \quad \text{and} \quad (1/3)^x < 2 \qquad (2)$$

Since an exponential function is always positive, the inequality (1) is satisfied for all x

To solve the inequality (3), we rewrite it as

$$(1/3)^x < (1/3)^{\log_{1/3} 2}$$

Since the base of both members of the above inequality is 1/3 which is less than 1, this inequality is equivalent to the inequality

$$x > \log_{1/3} 2$$

Thus inequality (1) is satisfied for all x and inequality (2) is satisfied for all x such that $x > \log_{1/3} 2$

Hence both the inequalities (1) and (2) are satisfied by all x such that $x > \log_{1/3} 2$, which therefore is the solution of original inequality

(ii) Since $\log_{1/2} x = \log_{1/3} x \log_{1/2} 1/2$ the given inequality can be rewritten as

$$\log_{1/2} x - \log_{1/3} x \log_{1/3} (1/2) > 0$$

or $[1-\log_{1/3}(1/2)] \log_{1/3} x > 0$ (1)

Since $1/2 > 1/3$, it follows from the properties of logarithms to base less than 1 that $\log_{1/3} (1/2) < \log_{1/3} (1/3) = 1$ so that

$$1-\log_{1/3} (1/2) > 0$$

Hence dividing (1) by the positive quantity $1-\log_{1/3} (1/2)$ we obtain $\log_{1/3} x > 0$

Observing that $\log_{1/3} 1 = 0$, this inequality can be written as

$$\log_{1/3} x > \log_{1/3} 1$$

Since the base is $\frac{1}{3}$ which is less than 1, this last inequality is satisfied by all x in the interval $0 < x < 1$, which is therefore the solution of the original inequality

7 Since $x^2+x+1=(x+\frac{1}{2})^2+\frac{3}{4}$ which is positive for all real x, the domain of the variable consists of all real values of x

[Note that for exponential function to be meaningful, the base must be positive Here base is x^2+x+1]

Also $a^2+b^2=c^2$ (2)

Now we have to prove

$$p+c > \frac{a+b+c}{2}$$

or $2p > a+b-c$

or $2\frac{ab}{c} > a+b-c$ by (1)

or $2ab > ac+bc-c^2=ac+bc-a^2-b^2$ by (2)

or $a^2+b^2+2ab > c(a+b)$

or $(a+b)^2 > c(a+b)$

or $a+b > c$ [$a+b > 0$]

which is true since the sum of two sides of a triangle is greater than the third side

39 We have to prove

$$\sqrt{s(s-a)(s-b)(s-c)} < \tfrac{1}{4}s^2$$

or $s(s-a)(s-b)(s-c) < \frac{1}{16}s^4$

or $(s-a)(s-b)(s-c) < \frac{1}{16}s^3$ (1)

But $\frac{(s-a)+(s-b)+(s-c)}{3} \geqslant [(s-a)(s-b)(s-c)]^{1/3}$

[A M $\geqslant$ G M]

or $\frac{s}{3} \geqslant [(s-a)(s-b)(s-c)]^{1/3}$ [$s-a+s-b+s-c$

$=3s-(a+b+c)=3s-2s=s$]

or $(s-a)(s-b)(s-c) \leqslant \frac{s^3}{27} < \frac{1}{16}s^3$

Hence (1) is proved

40 In case $a=0$ or $b=0$ the equality will hold Now let $a > 0$ and $b > 0$

Since one of the numbers a and b cannot exceed the other, we may assume $0 < a \leqslant b$ Then $0 < \frac{a}{b} \leqslant 1$ and since $\alpha > \beta$ it follows that $0 < (a/b)^\alpha \leqslant (a/b)^\beta$

so that $1+\left(\frac{a}{b}\right)^\alpha \leqslant 1+(a/b)^\beta$ (1)

Since both sides of (1) are greater than 1, we have

$[1+(a/b)^\alpha]^{1/\beta} \leqslant [1+(a/b)^\beta]^{1/\beta}$ (2)

Again since $1+(a/b)^\alpha > 1$

and $0 < 1/\alpha < 1/\beta$, we have

$[1+(a/b)^\alpha]^{1/\alpha} < [1+(a/b)^\alpha]^{1/\beta}$ (3)

Now both sides of the inequality

$$(x^2+x+1)^x<1$$

are positive for all x. So we can take logarithms to base 10 to get to the equivalent inequality

$$x\log_{10}(x^2+x+1)<\log 1=0$$

This inequality holds true in two cases

(a) When x satisfies the system of inequalities

$$x>0,\ \log_{10}(x^2+x+1)<0$$

(b) When x satisfies the system of inequalities

$$x<0,\ \log_{10}(x^2+x+1)>0$$

From the properties of logarithms to base greater than 1, we find that system of inequalities in (*a*) is equivalent to the system

$$x>0,\ x^2+x+1<1$$

that is, $\quad x>0,\ x(x+1)<0$

But the solution of the inequality $x(x+1)<0$ consists of all x in the interval $-1<x<0$, which is inconsistent with the inequality $x>0$. Hence there is no solution in this case.

The system of inequalities in (b) is equivalent to

$$x<0,\ x^2+x+1>1$$

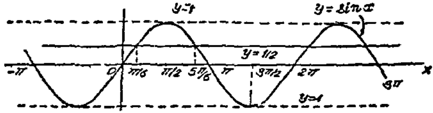
this is, $x<0,\ x(x+1)>0$

The solution of this system consists of all x in the interval $x<-1$ which is therefore the solution of the original inequality.

8 In such problems it is convenient to draw the graphs of the functions on each side of the inequality.

(i) We first draw the graphs of the functions $y=\sin x$ and $y=1/2$ (see the figure below).

The given inequality is satisfied for all those values of x for which the graph $y=\sin x$ lies above the graph $y=\frac{1}{2}$. Since the



period of the function $\sin x$, is 2π, it is sufficient first to solve the inequality on the interval $0\leqslant x\leqslant 2\pi$. From the graph it is clear that the solution is given by $\pi/6<x<5\pi/6$.

From (2) and (3) we get

$$[1+(a/b)^\alpha]^{1/\alpha} < [1+(a/b)^\alpha]^{1/\beta} \leqslant [1+(a/b)^\beta]^{1/\beta}$$

or $$\left(\frac{a^\alpha+b^\alpha}{b^\alpha}\right)^{1/\alpha} \leqslant \left(\frac{a^\beta+b^\beta}{b^\beta}\right)^{1/\beta}$$

Since $b > 0$, it follows that

or $(a^\alpha+b^\alpha)^{1/\alpha} \leqslant (a^\beta+b^\beta)^{1/\beta}$

41 Hint Use $A.M. \geqslant G.M.$

42 We have $\frac{1}{2}=\frac{1}{2}, \frac{1}{2}<\frac{2}{3}, \frac{1}{2}<\frac{3}{4}, \quad , \frac{1}{2}<\frac{n}{n+1}$

Adding these we get

$$n \cdot \frac{1}{2}<\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+ \quad +\frac{n}{n+1} \tag{1}$$

Again $\frac{1}{2}<1, \frac{2}{3}<1, \frac{3}{4}<1, \quad , \frac{n}{n+1}<1$

Adding these we get

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+ \quad + \frac{n}{n+1}<n \tag{2}$$

Therefore from (1) and (2) we obtain

$$\frac{n}{2} < \frac{1}{2}+\frac{2}{3}+\frac{3}{4}+ \quad + \frac{n}{n+1}<n$$

or $$\frac{1}{2} < \frac{1}{n}\left(\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+ \quad + \frac{n}{n+1}\right)<1$$

43 Let $S=1+\frac{1}{2}+\frac{1}{3}+ \quad +\frac{1}{2^n-1}$

Rewriting S in the following way, we get

$$S=1+\tfrac{1}{2}+(\tfrac{1}{3}+\tfrac{1}{4})+(\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7}+\tfrac{1}{8})+$$
$$+\left(\frac{1}{2^{n-2}+1}+ \quad + \frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}+1}+ \quad +\frac{1}{2^n-1}$$
$$> \tfrac{1}{2}+\tfrac{1}{2}+(\tfrac{1}{4}+\tfrac{1}{4})+(\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8}+\tfrac{1}{8})+$$
$$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+ \quad + \frac{1}{2^{n-1}}\right)+\frac{1}{2^{n-1}+1}+ \quad +\frac{1}{2^n-1}$$
$$>\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+ \quad \text{to } n \text{ terms}=\tfrac{1}{2}n$$

On the other hand, S may be written as

$$S=1+(\tfrac{1}{2}+\tfrac{1}{3})+(\tfrac{1}{4}+\tfrac{1}{5}+\tfrac{1}{6}+\tfrac{1}{7})+ \quad +$$
$$\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}+1}+ \quad + \frac{1}{2^n-1}\right)$$
$$< 1+(\tfrac{1}{2}+\tfrac{1}{2})+(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{4})+$$
$$+\left(\frac{1}{2^{n-1}}+\frac{1}{2^{n-1}}+ \quad + \frac{1}{2^{n-1}}\right)$$
$$=1+1+1+ \quad +1=n \tag{2}$$

Therefore the complete solution is

$$\pi/6+2n\pi<x<5\pi/6+2n\pi,\ n=0,\ \pm 1,\ \pm 2$$

(ii) Proceed as in (i)

Ans $\quad -\frac{2\pi}{3}+2n\pi\leqslant x\leqslant\frac{2\pi}{3}+2n\pi,\ n=0,\ \pm 1,\ \pm 2,$

(iii) We have

$$y=\sin x-\cos x=\sqrt{2}\sin\left(x-\frac{\pi}{4}\right)=\sqrt{2}\sin z$$

where $\quad x-\frac{\pi}{4}=z$

So we first solve the inequality

$\sqrt{2}\sin z>0$ that is, $\sin z>0$ (1)

From the graph of the function $y=\sin z$ in part (i), it is clear that the inequality (1) is satisfied by all z in the interval $0<z<\pi$

Replacing z by $x-\pi/4$, we see that the original inequality is satisfied by $\quad 0<x-\pi/4<\pi$

or $\quad \pi/4<x<5\pi/4$

Hence the complete solution is

$$\pi/4+2n\pi<x<5\pi/4+2n\pi,$$

where $n=0,\ \pm 1,\ \pm 2$

9 The given inequality can be written as

$2-2\sin^2\theta+\sin\theta\leqslant 2$ (1)

or $\quad \sin\theta\,(1-2\sin\theta)\leqslant 0$

The inequality (1) is satisfied if either

$\sin\theta\leqslant 0$ (2)

or $\quad \sin\theta\geqslant\frac{1}{2}$ (3)

Now we have to search for those values of θ in the interval $\pi/2\leqslant\theta\leqslant\ \pi/2$ which satisfy th inequalities (2) or (3) From the graph of the function $y=\sin\theta$ in part (i), we see that the inequality (2) is satisfied for all θ in the interval $\pi\leqslant\theta\leqslant\frac{3\pi}{2}$ and the inequality (3) is satisfied in the interval $\frac{\pi}{2}\leqslant\theta\leqslant\frac{5\pi}{6}$ Hence we get the answer

$$\frac{\pi}{2}\leqslant\theta\leqslant\frac{5}{6}\pi,\ \pi\leqslant\theta\leqslant 3\pi/2$$

Remarks Note that in problem 9, we have not given the complete solution since the solution was restricted by the

$h=40$ and hence $3h=120$ is the required length of the pole $3h \neq 40$ as the pole is given to be of more than 100 ft

28 BC (statue)$=30$,
AB (column)$=200$ AD (man)$=6$
$AP=x$ – breadth of river
$\angle CPB=\beta=\angle DPA$

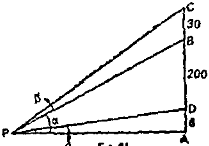


Fig 91

Let $\angle BPA=\alpha$ $\quad \tan\beta=\dfrac{6}{x}$

$$\tan\alpha=\frac{200}{x},\ \tan(\alpha+\beta)=\frac{230}{x}$$

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$\frac{230}{x}=\frac{\dfrac{200}{x}+\dfrac{6}{x}}{1-\dfrac{200}{x}\,\dfrac{6}{x}}=\frac{206x}{x^2-1200}$$

$230(x^2-1200)=206x^2$ $\quad$ or $\quad 24x^2=230\times1200$

$x^2=230\times50=11500$ $\quad x=10\sqrt{(115)}$ ft

29 $h=a\tan\alpha,\ h=b\tan(90^\circ-\alpha)$
$=b\cot\alpha$

Multiply $h^2=ab\tan\alpha\cot\alpha=ab$

$h=\sqrt{(ab)}$

$$\text{Also } \tan\alpha=\frac{h}{a}=\frac{ab}{a}=\sqrt{\left(\frac{b}{a}\right)} \qquad (1)$$

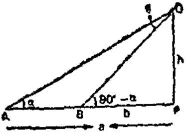


Fig 92

If AB subtends an angle θ at Q then

$90^\circ-\alpha=\theta+\alpha$ $\quad$ or $\quad \theta=90^\circ-2\alpha$

$$\sin\theta=\cos2\alpha=\frac{1-\tan^2\alpha}{1+\tan^2\alpha}$$

$$\text{or}\quad \sin\theta=\frac{1-b/a}{1+b/a}=\frac{a-b}{a+b}\ \text{by (1)}$$

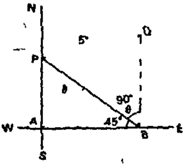
30 $h=AP\tan\theta,\ h=BP\tan(90^\circ-\theta)$

$h^2=AP\ BP \qquad (1)$

Clearly $AP=AB=100$

Also $\dfrac{AB}{BP}=\cos45^\circ=\dfrac{1}{\sqrt2}$

$BP=\sqrt2\,AB=100\sqrt2$



Putting the values of AP and BP in (1), we get

$$h^2=100\times 100\sqrt{2} \qquad h=100\,(2)^{1/4}$$

31 (a) The angle of elevation of summit A at C is α, and at D is γ where $CD=a$ and $\angle DCB=\beta$

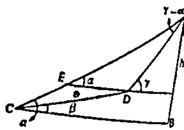


Fig 64

From $\triangle EAD$, $\gamma=\angle DAE+\alpha$

$\angle DAE=\gamma-\alpha$

Also from $\triangle CAD$, $\alpha-\beta+\gamma-\alpha+\angle CDA=180^\circ$

$\angle CDA=180^\circ-(\gamma-\beta)$

$\sin CDA=\sin(\gamma-\beta)$

Now $\dfrac{h}{AC}=\sin\alpha \qquad h=AC\sin\alpha \qquad (1)$

$\dfrac{AC}{\sin CDA}=\dfrac{CD}{\sin CAD}$ or $AC=\dfrac{a\sin(\gamma-\beta)}{\sin(\gamma-\alpha)} \qquad (2)$

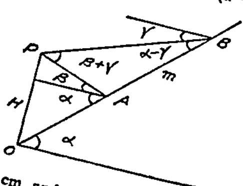
Hence from (1) by the help of (2)

$$h=\frac{a\sin\alpha\sin(\gamma-\beta)}{\sin(\gamma-\alpha)}$$

(b) In the figure, OP represents the tree and A and B the two points of observations in the hill inclined at an angle α to the horizon The elevation of P at A is β and the depression of P at B is γ We clearly have

$\angle POA=90^\circ-\alpha \qquad \angle OAP=\alpha+\beta,$

$\angle ABP=\alpha-\gamma$ and $\angle APB=(\alpha+\beta)-(\alpha-\gamma)=\beta+\gamma$



Also $OP=H$ cm and $AB=m$ cm

Now from $\triangle OAP$,

$$\frac{PA}{\sin(90^\circ-\alpha)}=\frac{OP}{\sin(\alpha+\beta)} \quad\text{or}\quad \frac{PA}{\cos\alpha}=\frac{H}{\sin(\alpha+\beta)} \qquad (1)$$

ahd from $\triangle PAB$

condition in the question to the interval $\frac{\pi}{2} \leqslant \theta \leqslant \frac{3\pi}{2}$ The complete solution in this case consists of all in the intervals

$$\pi/6+2n\pi \leqslant \theta \leqslant 5\pi/6+2n\pi$$

and $$\pi+2n\pi \leqslant \theta \leqslant 3\pi/2+2n\pi$$

where $n=0, \pm 1, \pm 2, \pm 3,$

Problem Set (C)
Objective Questions

1 If $|x|^2 < |y|^2$, then $x < y$
(a) True (b) False

2 $-4 < -5$
(a) True (b) False

3 If $x < 3$ and $x > -2$, then $-2 < x < 3$
(a) True (b) False

4 If $a > b$, then $-2a > -2b$
(a) True (b) False

5 If $a < b$, and $c < d$ then $a-c < b-d$
(a) True (b) False

6 If $a-b=2$, then $a < b$
(a) True (b) False

7 If $x+y=3$, then $x > -y$
(a) True (b) False

8 If $x^2-y^2=-z$, $z > 0$, then $|x| < |y|$
(a) True (b) False

9 If $\log_{1/2} x < \log_{1/2} y$ $(x > 0, y > 0)$, then $x < y$
(a) True (b) False

10 The inequalities $3x^2 < 6x$ and $x < 2$ are equivalent
(a) True (b) False

11 If $(x-9)^2-(x-5)^2 < 80$ then $x > -2$
(a) True (b) False

12 For very integer > 1, the in equality $(n!)^{1/n} < \frac{n+1}{2}$ holds
(a) True (b) False (I.I.T 1981)

13 If $0 < x \leqslant \pi/2$ then
$$\sin x+\operatorname{cosec} x \geqslant 2$$
(a) True (b) False

14 $\log_e 4+\log_4 e > 2$
(a) True (b) False

Thus when we toss a coin, either it must fall head or tail (the possibility of standing on the edge is ruled out)

§ 2 Classical Definition of probability

If there are n exhaustive, mutually exclusive and equally likely outcomes of an experiment and m of them are favourable to an event A, then the mathematical probability of A is defined as the ratio m/n

Odds in favour and odds against an event

If a of the outcomes are favourable to an event A and b of the outcomes are against it as a result of an experiment, then we say that *odds are a to b in favour of A, or odds are b to a against A*

§ 3 Axiomatic approach to Probability Theory

We shall be mainly concerned with discrete sample spaces, that is with those spaces which contain only a finite number of sample points or an infinite number of points which can be arranged as a sequence a_1, a_2, a_3

Axioms of Probability

Let the sample space S be the set

$$S=\{a_1, a_2, a_3, \}=A_1 \cup A_2 \cup A_3 \cup$$

where $A_i=\{a_i\}$ are the simple events in S

Then to each event A in S, we assign a **non negative real number** $P(A)$ called the probability of A satisfying the following axioms

$\mathbf{P}_1$ $P(A) \geqslant 0$ for every event A

$\mathbf{P}_2$ $P(S)=1$ for the certain event S

$\mathbf{P}_3$ Probability $P(A)$ of any event A is the sum of the probabilities of the simple events whose union is A

Remark 1 (i) If the sample space S is the union of the distinct simple events A_1 A_2 A_3, , then it follows from axioms $\mathbf{P}_2$ and $\mathbf{P}_3$ that

$$P(S)=P(A_1)+P(A_2)+P(A_3)+ \quad =1 \tag{1}$$

(ii) From axiom P_3 we easily conclude that if A and B are mutually exclusive events, so that $A \cap B=\phi$ then

$$P(A \cup B)=P(A)+P(B) \tag{2}$$

In general if A_1 A_2, A_3 are mutually exclusive events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \quad)=P(A_1)+P(A_2)+P(A_3)+ \tag{3}$$

15 $4^{\sin^2 x} + 4^{\cos^2 x} \geqslant 4$ for all real x
(a) True (b) False

16 If $y = \sin nx + \cos nx$ (x n real), then $-\sqrt{2} \leqslant y \leqslant \sqrt{2}$,
(a) True (b) False

17 If $x > 0$ and a is a known positive number then the least value of $ax + \frac{a}{x}$ is
(a) a^2, (b) a,
(c) $2a$ (b) None of these

18 If x, y and z are real and different and
$u = x^2 + 4y^2 + 9z^2 - 6yz - 3zx - 2xy$,
then u is always
(a) Non negative (b) Zero,
(c) Non positive, (d) None of these (I I T 1976)

19 Given $A = \sin^2\theta + \cos^4\theta$, then for real values of θ
(a) $1 \leqslant A \leqslant 2$ (b) $\frac{3}{4} \leqslant A \leqslant 1$,
(c) $\frac{13}{16} \leqslant A \leqslant 1$, (d) $\frac{3}{4} \leqslant A \leqslant \frac{13}{16}$ (I I T 1980)

20 If $y = 3^{x-1} + 3^{-x-1}$ (x real), then the least volue of y is
(a) 2, (b) 6
(c) 2/3, (d) None of these

21 The solution set of the inequality $\cos^2 x < \frac{1}{2}$ is
(i) $\{x : 2k\pi + (\pi/4) < x < 2k\pi + (3\pi/4)\}$
(ii) $\{x : 2k\pi - (3\pi/4) < x < 2k\pi - \pi/4\}$
(iii) $\{x : k\pi + (\pi/4) < x < k\pi + (3\pi/4)\}$,
(iv) None of these
where k is an integer

22 The solution set of the inequality $x + 1 > \sqrt{(x+3)}$ is
(i) $\{x : -1 < x \leqslant -3\}$, (ii) $\{x : x > -1\}$,
(iii) $\{x : -3 \leqslant x < -2\}$ (iv) $\{x : x > 1\}$

23 If $x_1, x_2, x_3, \ldots, x_n$ are any real numbers and n is any positive integer, then—
(i) $n \sum_{i=1}^{n} x_i^2 < (\sum_{i=1}^{n} x_i)^2$, (ii) $\sum_{i=1}^{n} x_i^2 \geqslant (\sum_{i=1}^{n} x_i)^2$
(iii) $\sum_{i=1}^{n} x_i^2 \geqslant n(\sum_{i=1}^{n} x_i)^2$, (iv) None of these (I I T 1982)

We now state some theorems without proof

Theorem 1 *Probability of an impossible event is zero i e ,* $P(\phi)=0$

Theorem 2 $P(A \cap \overline{B})=P(A)-P(A \cap B)$,

where $\overline{B}$ denotes the event complementary to the event B, that is, $\overline{B}$ contains the sample points of S not in B

Theorem 3 $P(\overline{A})=1-P(A)$

Theorem 4 (**Addition theorem**) *Probability of A or B*
$$P(A \cup B)=P(A)+P(B)-P(A \cap B)$$

Remark 2 By theorem 2

$$P(A)=P(A \cap B)+P(A \cap \overline{B})$$

and
$$P(B)=P(A \cap B)+P(\overline{A} \cap B)$$

Hence theorem 4 may be restated as

$$P(A \cup B)=P(A \cap B)+P(A \cap \overline{B})+P(\overline{A} \cap B)$$

i e, $P(A \cup B)$ denotes the probability that at least one of the events A and B will occur

Cor If A and B are mutually exclusive, that is, if $A \cap B=\phi$, then

$$P(A \cup B)=P(A)+P(B)$$

The extension of addition rule to three or more events is the following theorem

Theorem 5 (i) $P(A \cup B \cup C)=P(A)+P(B)+P(C)$
$$-P(A \cap B)-P(A \cap C)-P(B \cap C)+P(A \cap B \cap C)$$

(ii) $P(A_1 \cup A_2 \cup \ldots \cup A_n)=P(A_1)+P(A_2)+\ldots+P(A_n)$
$$-P(A_1 \cap A_2)-P(A_1 \cap A_3)-\ldots-P(A_{n-1} \cap A_n)$$
$$+P(A_1 \cap A_2 \cap A_3)+P(A_1 \cap A_2 \cap A_4)+\ldots$$
$$+P(A_{n-2} \cap A_{n-1} \cap A_n)$$
$$+\ldots+(-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)$$

Cor If $A_1, A_2, A_3, \ldots, A_n$ are mutually exclusive, then,
$$P(A_1 \cup A_2 \cup A_3 \cup \ldots \cup A_n)=P(A_1)+P(A_2)+\ldots+P(A_n)$$

Note For $A \cup B$ we often use the symbol $A+B$ and for $A \cap B$ we use AB

Remark 3 Since probability of an event is a non negative number it follows from theorem 4 that

$$P(A \cup B) \leqslant P(A)+P(B) \qquad (4)$$

The inequality (4) holds in general Thus for arbitrary events $A_1, A_2, A_3, \ldots$, we have

24 The largest interval for which $x^2 - x^3 + x^4 - x + 1 > 0$ is
(i) $-4 < x \leqslant 0$, (ii) $0 < x < 1$,
(iii) $-100 < x < 100$, (iv) $-\infty < x < \infty$
(*I I T* 1983)

25 If $a^2 + b^2 + c^2 = 1$ then $ab + bc + ca$ lies the interval
(i) $[\frac{1}{2}, 2]$ (ii) $[-1, 2]$ (iii) $[-\frac{1}{2}, 1]$ (iv) $[-1, \frac{1}{2}]$
(I I T 84)

26 If x satisfies $|x-1| + |x-2| + |x-3| \geqslant 6$
then (A) $0 \leqslant x \leqslant 4$, (B) $x \leqslant -2$ or $x \geqslant 4$, (C) $x \leqslant 0$ or $x \geqslant 4$, (D) None of these (I I T 83)

27 If a, b and c are distinct positive numbers, then the expression $(b+c-a)(c+a-b)(a+b-c) - abc$ is
(A) positive (B) negative (C) non positive (D) non negative
(*I I T* 86)

28 If p, q, r be three positive real numbers, then the value of $(p+q)(q+r)(r+p)$ is
(i) $> 8\, pqr$ (ii) $< 8\, pqr$ (iii) $8\, pqr$ (iv) none these

29 If p, q, r are any real numbers, then
(i) $\max(p, q) < \max(p, q, r)$
(ii) $\min(p, q) = \frac{1}{2}(p + q - |p - q|)$,
(iii) $\max(p, q) < \min(p, q, r)$
(iv) none of these (I I T 82)

30 If $\log_{0.3}(x-1) < \log_{0.09}(x-1)$ then x lies in the interval
(A) $(2, \infty)$, (B) $(1, 2)$, (C) $(2, -1)$ (D) None of these

31 The solution set of the inequality
$2^x + 3^x \geqslant 2$ is
(A) R (B) N (C) $[0, \infty]$, (D) none of these

Solutions

1 Ans (b)
Note that $|x|^2 < |y|^2 \Rightarrow |x| < |y|$

2 Ans (*b*)

3 Ans (*a*)

4 Ans (*b*) 5 Ans (*b*) 6 Ans (*b*) 7 Ans (*a*)

8 Ans (*a*) 9 Ans (*b*) 10 Ans (*b*) 11 Ans (*a*)

12 Ans (*a*) Consider A M > G M for first, *n* natural numbers

13 Ans (*a*) We have,

$$\sin x + \operatorname{cosec} x = \sin x + \frac{1}{\sin x} = \left(\sqrt{(\sin x)} - \frac{1}{\sqrt{(\sin x)}}\right)^2 + 2 \geqslant 2$$

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) \leqslant P(A_1)+P(A_2)+P(A_3)+\ldots \qquad (5)$$

The inequality (5) is known as Boole's inequality

§ 4 Conditional probability

A word of explanation is necessary to understand conditional probability Suppose a pair of dice is thrown Here the sample space S consists of 36 points (1, 1), (1, 2) , (1, 6), (2, 1), (2, 2), (2, 6) , (6, 1), (6, 2), (6, 6) Suppose that we now ask the question, If a pair of dice shows an even sum, what is the probability that this sum is less than 6 Here we are restricting the sample space to a subset of points corresponding to even sum only (18 such points) and asking "Which of these possible points (outcomes) represents a sum less than 6" There are four such points viz, (1, 1), (1, 3), (3, 1), (2, 2) Since all these outcomes are equally likely, the required probability

$$P=\frac{4}{18}=\frac{2}{9}$$

Observe that here we have imposed the condition that the sum, $x+y$, is even (event A), and then asked the probability for $x+y$ to be less than 6 (event B) We denote this conditional probability by the symbol $P(B/A)$, which is read ' the probability of the event B under the condition that the event A has already happened A little consideration will show that the probability $\frac{2}{9}$ was obtained when we divided the number $n(A \cap B)$ of points in the subset

$$A \cap B=\{(x, y) \mid x+y \text{ is even and } < 6\}$$

by the number, $n(A)$ of points in the subset

$$A=\{(x, y) \mid x+y \text{ is even}\}, \text{ so that}$$

$$P(B/A)=\frac{n(A \cap B)}{n(A)}=\frac{n(A \cap B)/n(S)}{n(A)/n(S)}=\frac{P(A \cap B)}{P(A)} \qquad (1)$$

where $n(S)$ is the number of points in the entire sample space Note that here it is assumed that

$$P(A) \neq 0$$

Although the equation (1) has been obtained for the special case where all points of S have the same probability $\frac{1}{36}$, it will be found to hold in those cases as well where the equality of probability does not hold *Thus equation (1) constitutes our formal definition of conditional probability of B on the hypothesis A (or for given A)*

Multiplication theorem $P(A \cap B)=P(A) \; P(B/A)$, *i e the probability of the simultaneous occurrence of the events A and B equals the probability of A multiplied by the conditional probability of B that A has already occured*

14 Ans (*a*)

Let $x = \log_e 4$ so that $x > 0$ Then $\log_4 e = 1/x$

Then $x + \frac{1}{x} = \left(\sqrt{x} - \frac{1}{\sqrt{x}}\right)^2 + 2 > 2$

[Note that we get the strict inequality since $x = \log_e 4 \neq 1$]

15 Ans (*a*) Using A M $\geq$ G M, we have

$$4^{\sin^2 x} + 4^{\cos^2 x} \geq 2\sqrt{(4^{\sin^2 x}\ 4^{\cos^2 x})} = 2\sqrt{4} = 4$$

16 Ans (*a*) We have

$$y = \sin nx + \cos nx = \sqrt{2}\left[\frac{1}{\sqrt{2}}\sin nx + \frac{1}{\sqrt{2}}\cos nx\right]$$

$$= \sqrt{2}\left[\cos\frac{\pi}{4}\sin nx + \sin\frac{\pi}{4}\cos nx\right]$$

$$= \sqrt{2}\sin\left(nx + \frac{\pi}{4}\right)$$

Since $-1 \leq \sin(nx + \pi/4) \leq 1$

we have $-\sqrt{2} \leq \sqrt{2}\sin(nx + \pi/4) \leq \sqrt{2}$

that is $-\sqrt{2} \leq y \leq \sqrt{2}$

17 Ans (iii)

18 Ans (*a*), We have

$$u = \tfrac{1}{2}[2x^2 + 8y^2 + 18z^2 - 12yz - 6zx - 4xy]$$
$$= \tfrac{1}{2}[(x-2y)^2 + (2y-3z)^2 + (3z-x)^2] \geq 0$$

19 Ans (*b*)

Since $0 \leq \cos^2\theta \leq 1$ we have

$$\cos^4\theta \leq \cos^2\theta$$

and so $\sin^2\theta + \cos^4\theta \leq \sin^2\theta + \cos^2\theta = 1$ (1)

Again $\sin^2\theta + \cos^4\theta = 1 - \cos^2\theta + \cos^4\theta$

$$= \left(\cos^2\theta - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4} \quad (2)$$

Hence from (1) and (2), we get

$3/4 \leq A \leq 1$ where $A = \sin^2\theta + \cos^4\theta$

20 Ans (*c*) We have

$3^{x-1} + 3^{-x-1} = \frac{1}{3}(3^x + 3^{-x}) \geq \frac{1}{3}\, 2\sqrt{(3^x\, 3^{-x})}$, [A M $\geq$ G M]

or $3^{x-1} + 3^{-x-1} \geq \frac{2}{3}$

Hence the least value of $3^{x-1} + 3^{-x-1}$ is $\frac{2}{3}$

20 Ans (iii) We have

$\cos^2 x < \frac{1}{2} \Rightarrow -1/\sqrt{2} < \cos x < 1/2$

$\Rightarrow \frac{\pi}{4} < x < \frac{3\pi}{4}$ or $-\frac{3\pi}{4} < x < -\frac{\pi}{4}$

Independent events. *If the happening of the event B is not influenced or conditioned by a second event A ($P(A) \neq 0$) so that*

$$P(B/A) = P(B) \qquad (2)$$

then B is said to be independent of A

Theorem 1 If $P(A) \neq 0$ and $P(B) \neq 0$ and B is independent of A, then A is independent of B In this case, we say that A and B are mutually independent

Theorem 2 Two events A and B are mutually independent if and only if

$$P(A \cap B) = P(A)\, P(B)$$

provided $\quad P(A) \neq 0$ and $P(B) \neq 0$

The generalization of this theorem is as follows The events $A_1, A_2, \ldots, A_n$ are mutually independent (or simply independent) if and only if the multiplication rule

$$P(A_1 \cap A_2 \cap \ldots \cap A_k) = P(A_1)\, P(A_2) \ldots P(A_k) \qquad (3)$$

holds for every k-tuples of events, $k = 2, 3, \ldots, n$

If (3) hold for $k=2$ and may or may not hold for $k = 3, 4, \ldots, n$, then events $A_1, A_2, \ldots, A_n$ are said to be pairwise independent Thus mutually independent events are pairwise independent but not conversely

Theorem 3 If the events A and B are mutually independent then the events A and $\overline{B}$ are also mutually independent

Theorem 4 If A and B are mutually independent events such that $P(A) \neq 0$ and $P(B) \neq 0$, then the events A and B have atleast one common sample point

Important Note The reader will do well to note *difference between mutually exclusive events and independent events* Mutually exclusive events in general are not independent For example, in the case of toss of two coins for which the sample space is

$$S = \{(H, H), (H\ T), (T, H), (T, T)\},$$

let A be the event that there are two heads and B the event that there are two tails Thus $A = \{(H, H)\}$ and $B = \{(T, T)\}$ Here $A \cap B = \phi$ so that A and B are mutually exclusive But A and B are not independent

since $\qquad P(A) = \frac{1}{4} = P(B)$ and $P(A \cap B) = 0$

so that $\qquad P(A \cap B) \neq P(A)\, P(B)$ in this case

Also we know from Theorem 4 above that independent events are not in general mutually exclusive

In fact, events are 'independent' if the probability of any of them is not affected by supplementary knowledge

Complete solution is

$$2n\pi+\frac{\pi}{4} < x < 2n\pi+\frac{3\pi}{4} \quad (1)$$

and $2n\pi-\frac{3\pi}{4} < x < 2n\pi-\frac{\pi}{4}$

or $(2n-1)\pi+\frac{\pi}{4} < x < (2n-1)\pi+\frac{3\pi}{4}$ (2)

Combining (1) and (2) we get the solution as

$k\pi+\frac{\pi}{4} < x < k\pi+\frac{3\pi}{4}$,

where k is any integer

22 Ans (iv) Do yourself

23 Ans (iv)

Hint By theorem 5 of § 4, the correct inequality must be

$$n\sum_{i=1}^{n} x_i^4 \geqslant \left(\sum_{i=1}^{n} x_i\right)^2$$

24 Ans (iv) We have

$f(x)=x^9(x^3-1)+x^3(x-1)+1>0$ if $x\geqslant 1$

and $f(x)=(1-x)+x^4(1-x^5)+x^{12}>0$ if $x<1$

25 Ans (iii) We have

$(a+b+c)^2\geqslant 0 \Rightarrow a^2+b^2+c^2+2ab+2bc+2ca\geqslant 0$

$\Rightarrow ab+bc+ca\geqslant -\frac{1}{2}$ [$a^2+b^2+c^2=1$]

Again $(b-c)^2+(c-a)^2+(a-b)^2\geqslant 0$

$\Rightarrow 2(a^2+b^2+c^2)-2(bc+ca+ab)\geqslant 0$

$\Rightarrow bc+ca+ab\leqslant a^2+b^2+c^2=1$

Hence $-\frac{1}{2}\leqslant bc+ca+ab\leqslant 1$

26 Ans (c)

Hint Consider four cases

(i) $x<1$, (ii) $1\leqslant x<2$,

(iii) $2\leqslant x<3$, (iv) $x\geqslant 3$

27 Ans (c)

Hint See Q 14 (ii) of Problem Set (A)

28 Ans (i) Hint Apply A M $>$ G M for each factor and multiply the resulting inequalities

29 Ans (ii)

30 Ans (a)

First note that $(x-1)$ must be greater than 0 that is $x>1$, the materialization of any number of the remaining events For example, if A and B represent white balls drawn from two different urns, the probability of A is the same whether the colour of the ball drawn from the other urn is known or not Similarly, for an unbiased coin, head at the first and head at the second throw are independent events but they are not mutually exclusive events As an example of dependent events consider an urn containing 2 white balls and 3 black ones Two balls are drawn in succession without replacement Let A be the event consisting in the white colour of the first ball and B the event consisting in the white colour of second ball Evidently, here A and B are dependent events

§ 5 **Partition of a sample space** *Let A_1, A_2, ..., A_n be subsets of a sample space S Then these subsets are said to form a partition of S if the following conditions hold*

(*i*) *Each A_i is a proper subset of S, that is*

$$A_i \subset S, \ i=1, 2, \ldots, n \text{ and } A_i \neq S$$

(*ii*) $A_1 \cup A_2 \cup \ldots \cup A_n = S$

(*iii*) *The subsets A_i are pairwise disjoint, that is,*

$$A_i \cap A_j = \phi, \ i, j = 1, 2, 3, \ldots, n, \ i \neq j$$

Theorem of Total Probability For Compound Events

Theorem 1 Let $\{A_1, A_2, \ldots, A_n\}$ be an partition of a sample space S and let A be any event Then

$$P(A) = \sum_{i=1}^{n} P(A_i) \, P(A \mid A_i) \tag{1}$$

provided $P(A_i) \neq 0$, $i = 1, 2, \ldots, n$

Proof If S is the sample space then

$$S = \bigcup_{i=1}^{n} A_i \text{ and therefore}$$

$$A = A \cap S = A \cap \left(\bigcup_{i=1}^{n} A_i\right) = \bigcup_{i=1}^{n} [A \cap A_i] \tag{2}$$

[By distributive law]

Since $A_i \cap A_j = \phi$ for $i \neq j$ it follows that

$(A \cap A_i) \cap (A \cap A_j) = \phi$ for $i \neq j$ Hence

$$P(A) = P\left[\bigcup_{i=1}^{n} (A \cap A_i)\right] = \sum_{i=1}^{n} P(A \cap A_i)$$

[By Cor of Theorem 5 of § 3]

Now $\log_{0.3}(x-1) < \log_{0.09}(x-1)$

$\Rightarrow \log_{0.09}(x-1) < \log_{(0.3)^2}(x-1)$

$\Rightarrow \log_{0.3}(x-1) < \frac{1}{2}\log_{0.3}(x-1)$

$\Rightarrow 2\log_{0.3}(x-1) < \log_{0.3}(x-1)$

$\Rightarrow \log_{0.3}(x-1)^2 < \log_{0.3}(x-1)$

$\Rightarrow (x-1)^2 > x-1$

[Note that the inequality is reversed since the base lies between 0 and 1]

$\Rightarrow (x-1)^2-(x-1) > 0$

$\Rightarrow (x-1)(x-2) > 0$ (1)

Since $x < 1$, the inequality (1) will hold if $x > 2$ that is if x lies in the interval $(2, \infty)$

31 Ans (*c*)

$$=\sum_{i=1}^{n} P(A_i)\, P(A/A_i)$$

Theorem 2 (Bayes's Rule) *Let the set of events* $\{A_1, A_2, \ldots, A_n\}$ *form a partition of the sample space S, where* $P(A_i)\neq 0$, $i=1, 2, \ldots n$. *Then for any event A for which* $P(A)\neq 0$ *and for* $1 \leqslant k \leqslant n$

$$P(A_k/A)=\frac{P(A_k)\, P(A/A_k)}{\sum_{i=1}^{n} P(A_i)\, P(A/A_i)}$$

Bayes's theorem can be put in the following useful form

Let $S=A_1 \cup A_2 \cup \ldots \cup A_n$ *where* A_i, *are* simple events *Then clearly* A_i *form a partition of S. If A is any non empty subset of S, then for each integer k* $(1 \leqslant k \leqslant n)$

$$P(A_k/A)=\frac{P(A/A_k)\, P(A_k)}{\sum_{i=1}^{n} P(A/A_i)\, P(A_i)} \tag{3}$$

Proof By Theorem 1

$$P(A)=\sum_{i=1}^{n} P(A_i)\, P(A/A_i) \tag{1}$$

Also $P(A \cap A_k)=P(A_k)\, P(A/A_k)$ (2)

$$P(A_k/A)=\frac{P(A \cap A_k)}{P(A)}=\frac{P(A_k)\, P(A/A_k)}{\sum_{i=1}^{n} P(A_i)\, P(A/A_i)},$$

by (1) and (2)

Note—Bayes's theorem serves to connect the conditional probability of A_k given A, with the conditional probability of A, given A_i and the probability of the A_i themselves. If the events A_i are called 'causes' then (2) or (3) can be regarded as a formula for the probability that the event A, which has occurred, is the result of the cause A_i

The probabilities $P(A_i)$, $i=1, 2, \ldots, n$ are called the *a priori probabilities* because they exist before we get any information from the experiment itself

The probabilities $P(A_i/A)$ $i=1, 2, \ldots n$ are called *posteriori probabilities*' because they are determined after the results of the experiment are known

The probabilities $P(A/A_i)$ $i=1, 2, \ldots, n$ are called '*likelihoods*' because they show how likely the event A under consideration is to occur, given a priori probabilities

For the application of Baye's theorem, it is essential that the events A_i $(i=1, 2, \ldots, n)$ *are mutually exclusive and sum of their probabilities equal to* 1

§ 6 **Binomial Distribution** When we know the probability of the happening of an event in one trial, the probability of the happening exactly once, twice, thrice etc in n trials can be found as shown below

If p is the probability of happening (success) of an event in one trial, then the probability of its non happening (failure) is $1-p=q$, say Hence, by the multiplication theorem, the probability of its happening r times and failing to happen $(n-r)$ times in a *prescribed order is* $p^r q^{n-r}$ But the number of orders is equal to the number of combinations of n things taken r at a time, *i e* ${}^nC_r$ and these orders are all mutually exclusive and equally likely Hence the probability of the event happening r times (*i e* probability of r successes) exactly in n trials *in any order* is ${}^nC_r\, p^r\, q^{n-r}$ which is the $(r+1)$th terms in the binomial expansion of $(q+p)^n$ Thus probabilities of 0, 1, 2, $\ldots$, n successes are respectively

$$q^n,\ {}^nC_1\, q^{n-1}\, p,\ {}^nC_2\, q^{n-2}\, p^2,\ \ldots,\ {}^nC_r\, q^{n-r}\, p^r,\ \ldots,\ p^n$$

Problem Set (A)

1 An urn contains 3 white and 5 black balls One ball is drawn What is the probability that it is black ?

2 From a pack of 52 cards, four cards are drawn Find the chance that they will be the four honours of the same suit

3 In a hand at whist what is the chance that the 4 kings are held by a specified player ?

4 Two dice are thrown simultaneously What is the probability of obtaining a total score of seven ? (*I I.T 1974*)

5 A coin is tossed n times, what is the chance that the head will present itself an odd number of times ? (*I I T 1970*)

6 Six boys and six girls sit in a row randomly Find the probability that

(i) The six girls sit together

(ii) The boys and girls sit alternately (*I.I T 1979*)

7 A die is loaded so that the probability of face i is proportional to i, $i=1\ 2, \ldots 6$ What is the probability of an even number occuring when the die is rolled ?

PROBABILITY

8 There are three events A, B, C, one of which must, and only one can happen the odds are 8 to 3 against A, 5 to 2 against B , find the odds against C

9 A has 3 shares in a lottery containing 3 prizes and 9 blanks B has two shares in a lottery containing 2 prizes and 6 blanks, compare their chances of success

10 (a) A student is given a true false exam with 10 questions If he gets 8 or more correct answers he passes the exam Given that he guesses at the answer to each question, compute the probability that he passes the exam

(b) In a multiple choice question there are four alternative answers, of which one or more are correct A candidate will get marks in the question only if he ticks all the correct answers The candidate decides to tick answers at random If he is allowed upto three chances to answer the question, find the probability that he will get marks in the question (*I* I T 85)

11 A coin is tossed twice Events E and F are defined as follows E=heads on first toss, F=heads on second toss
Find the probability of $E \cup F$

12 (a) One card is drawn from each of two ordinary sets of 52 cards Find the probability that at least one of them will be the ace of hearts

(b) If two dice are thrown, what is the probability that at least one of the dice shows a number greater than 3 ?

13 Two cards are drawn simultaneously from the same set Find the probability that at least one of them will be the ace of hearts

14 A speaks truth in 75 percent cases, and B in 80 percent of the cases In what percentage of cases are they likely to contradict each other in stating the same fact ?
(I I T 1975, Aligarh 1982)

15 In bridge game of playing cards 4 players are distributed one card each by turn so that each player gets 13 cards Find out the probability of a specified player getting a black ace and a king

16 (a) Given the probability that A can solve a problem is 2/3 and the probability that B can solve the problem is 3/5, find the probability that (i) at least one of A and B will

be able to solve the problem (ii) none of the two will be able to solve the problem

(b) A man and a women appear in an interview for, two vacancies in the same post The probability of man's selection is 1/4 and that of the woman's selection is 1/3, what is the probability that

(a) both of them will be selected

(b) only one of them will be selected

(c) none of them will be selected [M N R ' 88]

17 A problem in mathematics is given to three students whose chances of solving it are respectively $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ What is the probability that the problem will be solved ?

18 If three squares are chosen at random on a chess board, show that the chance that they should be in a diagonal line is $\frac{7}{744}$

19 If *four whole* numbers taken at random are multiplied together, show that chance that the last digit in the product is *1, 3, 7* or *9* is $\frac{16}{625}$ (*Roorkee 86*)

20 What is the chance that a leap year selected at random will contain 53 Sundays ?

21 The chance of an event happening is the square of the chance, of a second event but the odds against the first are the cubes of the odds against the second Find the chance of each

22 (a) An anti aircraft gun can take a maximum of *four shots* at an enemy plane moving away from it The probabilities of hitting the plane at the first, second third and fourth shot are 0.4, 0.3 0.2 and 0.1 respectively What is the probability that the gun hits the plane ? (I I T 1981)

(b) A worker attends three machines each of which operates independently of the other two The probabilities of the event that the machines will not require operator's intervention during a shift are equal to $p_1 = 0.4$ $p_2 = 0.3$, $p_3 = 0.2$ Find the probability of the event that at least one machine will require worker's intervention during a shift

23 A box contains 2 black 4 white and 3 red balls One ball is drawn at random from the box and kept aside From the remaining balls in the box another ball is drawn at random and kept besides the first This process is repeated till all the balls are drawn from the box Find the probability that the balls drawn are in the sequence of 2 black 4 white and 3 red (*I I T 1978*)

Probability (iii)

§ 1 Some definitions

1.1 Sample Space The set S of all possible outcomes of an experiment (or observation) is called a sample space provided no two or more of these outcomes can occur simultaneously and exactly one of the outcomes must occur whenever the experiment is performed.

It should be noted that with one experiment we may succeed in associating more than one sample space. To determine a sample space we must know precisely the aim of the experiment.

For example, consider the experiment of tossing two coins. If we are interested in whether each coin falls heads (H) or tails (T) then the possible outcomes are

$$(H, H), (H, T), (T, H), (T, T) \qquad (1)$$

On the other hand, we may be interested in whether the coins fall alike (A) or different (D). Then the possible outcomes are

$$(A), (D)$$

1.2 Event *An event is a subset of a sample space.* For example, for the sample space given by (1) the subset

$$A = \{(H, H), (H, T), (T, H)\}$$

is the event that at least one head occurs. An event consisting of a single point is called a simple event.

1.3 **Mutually Exclusive Events.** If two or more events have no point in common (*i.e.* if they cannot occur simultaneously), the events are said to be mutually exclusive.

Thus the events A and B are mutually exclusive if $A \cap B = \phi$.

1.4 Equally likely Events Two events are said to be equally likely if one of them cannot be expected to occur in preference to the other.

1.5 Exhaustive Events A set of events is said to be totally exhaustive (or simply exhaustive) if no event outside this set occurs and at least one of these events must happen as a result of an experiment.

24 There are two groups of subjects one of which consists of 5 science subjects and 3 engineering subjects and the other consists of 3 science and 5 engineering subjects An unbiased die is cast If number 3 or number 5 turns up, a subject is selected at random from the first group Otherwise the subject is selected at random from the second group Find the probability that an engineering subject is selected ultimately (I I T 1977)

25 There are two bags, one of which contains three black and four white balls while the other contains four black and three white balls A die is cast if the face 1 or 3 turns up, a ball is taken from the first bag , and if any other face turns up, a ball is chosen from the second bag Find the probability of choosing a black ball (I I T 1973)

26 M telegrams are distributed at random over communication channels $(N > M)$ Find the probability that not more than one telegram will be sent over each channel

27 K cells are destributed at random and independently of one another among N cells which lie in a straight line $(N > K)$ Find the probability that they will occupy K adjacent cells

28 If on an average 1 vessel in every 10 is wrecked find the chance that out of 5 vessels expected 4 at least will arrive safely

29 A lot of 100 bulbs from manufacturing process is known to contain 10 defective and 90 non defective bulbs If a sample of 8 bulbs is selected at random what is the probability that
(a) The sample has 3 defective and 5 non defective bulbs
(b) the sample has at least one defective bulb ?

30 In five throws with a single die what is the chance of throwing (1) three aces *exactly*, (2) three aces at least

31 In each of a set of games it is 2 to 1 in favour of the winner of the previous game What is the chance that the player who wins the first game shall win three at least of the next four ?

32 A, B, C in order cut a pack of cards replacing them after each cut on the condition that the first who cuts a spade shall win a prize find their respective chances

33 Three white balls and five black balls are placed in a bag, and three men draw a ball in succession (the balls drawn not

$$\frac{AB}{\sin(\beta+\gamma)}=\frac{PA}{\sin(\alpha-\gamma)} \quad \text{or} \quad \frac{m}{\sin(\beta+\gamma)}=\frac{PA}{\sin(\alpha-\gamma)} \qquad (2)$$

Multiplying (1) and (2), we get

$$\frac{m\,PA}{\cos\alpha\sin(\beta+\gamma)}=\frac{H\,PA}{\sin(\alpha+\beta)\sin(\alpha-\gamma)}$$

$$\text{or} \quad H=\frac{m\sin(\alpha+\beta)\sin(\alpha-\gamma)}{\cos\alpha\sin(\beta+\gamma)}$$

(c) We have marked the angles as given

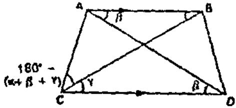


Fig 104

$$\angle ACD=180^\circ-(\alpha+\beta)-\gamma$$
$$=180^\circ-(\alpha+\beta+\gamma)$$
$$\sin ACD=\sin(\alpha+\beta+\gamma) \qquad (1)$$

Using sin formula on $\triangle ABC$

$$\frac{AB}{\sin ACB}=\frac{AC}{\sin\gamma}$$

$$AC=\frac{a\sin\gamma}{\sin(\alpha+\beta+\gamma)} \text{ by (1)} \qquad (2)$$

Again from $\triangle ACD$ by sine formula we have

$$\frac{CD}{\sin\alpha}\ \frac{AC}{\sin\beta}$$

$$CD=\frac{\sin\alpha}{\sin\beta}\ \frac{a\sin\gamma}{\sin(\alpha+\beta+\gamma)} \text{ by (2)}$$

$$=\frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$$

32 (a) From triangle ABC by sine formula $\frac{BC}{\sin 75^\circ}=\frac{AB}{\sin 30^\circ}$

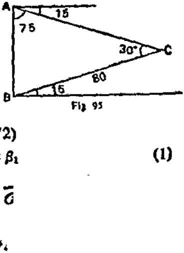


Fig 95

$$h=80\ \frac{\sin 30}{\sin(45+30)}=\frac{40\times 2\sqrt{2}}{(\sqrt{3}+1)}$$

$$=\frac{40\ 2\sqrt{2}\sqrt{(3-1)}}{3-1}$$

$$=40\sqrt{2}(\sqrt{3}-1)=40(\sqrt{6}-\sqrt{2})$$

(b) $h=PS_1\sin\beta_1$ or $PS_1=h\operatorname{cosec}\beta_1$ (1)

From $\triangle PS_1S_2$ by sine formula

$$\frac{PS_1}{\sin\gamma_1}=\frac{S_1S_2}{\sin\delta_1}$$

or $\dfrac{h}{\sin\beta_1 \sin\gamma_1} = \dfrac{S_1S_2}{\sin\delta_1}$ by (1)

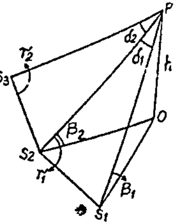
or $\dfrac{h}{S_1S_2} = \dfrac{\sin\beta_1\sin\gamma_1}{\sin\delta_1}$

Similarly $\dfrac{h}{S_2S_3} = \dfrac{\sin\beta_2\sin\gamma_2}{\sin\delta_2}$

But $S_2S_3 = S_1S_2$ given

$$\frac{\sin\beta_1\sin\gamma_1}{\sin\delta_1} = \frac{\sin\beta_2\sin\gamma_2}{\sin\delta_2}$$

$$= \frac{h}{S_1S_2}$$



33 Let the length of ladder be

$x = PB = QC$

$PA = x\cos\alpha,\ QA = x\cos\beta$

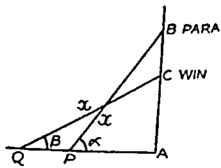
$AC = x\sin\beta,\ AB = x\sin\alpha$

$CB = AB - AC = x\,(\sin\alpha - \sin\beta)$

$QP = a = QA - PA = x(\cos\beta\ \ \cos\alpha)$

$$\frac{CB}{a} = \frac{\sin^7\alpha - \sin\beta}{\cos\beta - \cos\alpha}$$

$$= \frac{2\sin\frac{(\alpha-\beta)}{2}\cos\frac{\alpha+\beta}{2}}{2\sin\frac{\alpha+\beta}{2}\sin\left(\frac{\alpha-\beta}{2}\right)}, \qquad CB = a\cot\frac{\alpha+\beta}{2}$$



34 $\dfrac{b}{a} = \tan\theta,\ \dfrac{b+x}{a} = \tan 2\theta$

or $\dfrac{b+x}{a} = \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{2\,b/a}{1-b^2/a^2}$

$\dfrac{b+x}{a} = \dfrac{2ab}{a^2-b^2} \qquad x = \dfrac{2a^2b}{a^2-b^4} - b$

$\dfrac{a^2b+b^3}{a^2-b^2}$ or $x = \dfrac{b(a^2+b^2)}{a^2-b^2}$

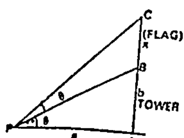


Fig 91

Alt Since BP is the bisector of $\angle APC$, we have

$\dfrac{BA}{BC} = \dfrac{PA}{PC}$ or $PC^2\,BA^2 = BC^2\,PA^2$

or $\{a^2 + (b+x)^2\}.b^2 = x^2.a^2$

or $a^2b^2 + b^2\,(b^2 + 2bx + x^2)$

or $(a^2\ \ b^2)x^2 - 2b^3x - b^2\,(a$

being replaced) until a white ball is drawn show that their respective chances are as 27 18 11

34 A is one of 6 horses entered for a race and is to be ridden by one of two jockeys B and C It is 2 to 1 that B rides A in which case all the horses are equally likely to win, if C rides A, his chance is trebled, what are the odds against his winning ?

35 The odds that a book will be reviewed favourably by three independent critics are 5 to 2, 4 to 3, and 3 to 4 respectively, what is the probability that of three reviews a majority will be favourable ?

36 A and B are two independent witnesses (i e there is no collusion between them) in a case The probability that A will speak the truth is x and the probability that B will speak the truth is y A and B agree in a certain statement Show that the probability that the statement is true is

$$\frac{xy}{1-x-y+2xy}$$

37 Three groups of children contain 3 girls and one boy 2 girls and 2 boys, one girl and 3 boys One child is selected at random from each group Show that the chance that the three selected consist of 1 girl and 2 boys is $\frac{13}{32}$ (Roorkee 1985)

38 Out of $3n$ consecutive integers, three are selected at random Find the chance that their sum is divisible by 3

39 (i) Out of $(2n+1)$ tickets consecutively numbered three are drawn at random Find the chance that the numbers on them are in A P

(ii) Out of 21 tickets marked with numbers from 1 to 21 three are drawn at random find the probability that the three numbers on them are in A P (Roorkee 88)

40 If $6n$ tickets numbered 0, 1 2 $6n-1$ are placed in a bag and three are drawn out, show that the chance that the sum of the numbers on them is equal to $6n$ is $\frac{3n}{(6n-1)(6n-2)}$

41 A coin is tossed $(m+n)$ times $(m > n)$, show that the probability of at least m consecutive heads is $\frac{n+2}{2^{m+1}}$

(x, y), $1 \leqslant x \leqslant 6$, $1 \leqslant y \leqslant 6$ and only 9 of these cases are favourable to the event $A \cap B$, so that

$$P(A \cap B)=\tfrac{9}{36}=\tfrac{1}{4}$$

Hence $\quad P(A \cup B)=P(A)+P(B)-P(A \cap B)=\tfrac{1}{2}+\tfrac{1}{2}-\tfrac{1}{4}=\tfrac{3}{4}$

13 Let A=event the first card is an ace of hearts,
and $\quad B$=event that second card is an ace of hearts
Since here both the cards are drawn from the same set, they both cannot be the ace of hearts so that here events A and B are mutually exclusive

$$P(A \cup B)=P(A)+P(B)$$

and $\quad P(A)=P(B)=\tfrac{1}{52}$ as in example 12

Hence $\quad P(A \cup B)=\tfrac{1}{52}+\tfrac{1}{52}=\tfrac{2}{52}=\tfrac{1}{26}$

14 If E_1 denotes the event that A speaks the truth,

then $\quad \overline{E_1}$ is the event that A does not speak the truth

Similarly, we define the events E_2 and E_2 for B
Let E be the event that A and B contradict each other
Then according to the question, we have

$$P(E_1)=\frac{75}{100}=\frac{3}{4} \text{ and } P(\overline{E_1})=1-\frac{3}{4}=\frac{1}{4},$$

$$P(E_2)=\frac{80}{100}=\frac{4}{5} \text{ and } P(\overline{E_2})=1-\frac{4}{5}=\frac{1}{5}$$

They will contradict each other if one speaks the truth and other does not

Hence $\quad E=E_1\overline{E_2}+\overline{E_1}E_2$

Now $\quad P(E_1\overline{E_2})$=the probability that A speaks the truth and B tells a lie

$$=P(E_1)\,P(\overline{E_2})=\frac{3}{4}\times\frac{1}{5}=\frac{3}{20}$$

Similarly $\quad P(\overline{E_1}E_2)=\frac{1}{4}\times\frac{4}{5}=\frac{1}{5}$

[Note the E_1 and $\overline{E_2}$ are independent events and so are $\overline{E_1}$ and E_2]

Since $\quad E_1\overline{E_2}$ and $\overline{E_1}E_2$ are mutually exclusive events, we have

$$P(E)=P(E_1\overline{E_2}+\overline{E_1}E_2)=P(E_1\overline{E_2})+P(\overline{E_1}E_2)$$

$$=\frac{3}{20}+\frac{1}{5}=\frac{7}{20}=\frac{35}{100} \text{ i e } 35\%$$

Hence in 35% cases, A and B will contradict each other

42 In a purse there are 10 coints, all five nP's except one which is a rupee, in another there are 10 coins all five nP's Nine coins are taken from the former and put into the latter and then nine coins are taken from the latter and put into the former, find the chance that the rupee is still in the first purse

43 An urn contains a white and b black balls and a second urn c white and d black balls One ball is transfered from the first urn into the second and one ball is then drawn from the second urn What is the probability that it is a white ball ?

44 (a) If m things are distributed among a men and b women, show that the chance that the number of things received by men is odd is $\frac{1}{2}\frac{(q+a)^m-(b-a)^m}{(b+a)^m}$

(b) If n biscuits are distributed among N different beggars, what is the probability that a particular beggar gets exactly r biscuits

45 A coin whose faces are marked 3 and 5 is tossed 4 times, what are the odds against the sum of the numbers thrown being less than 15 ?

46 A and B throw with 3 dice if A throws 8, what is As chance of throwing a higher number ?

47 A person throws two dice, one the common cube and the other a regular tetrahedron, the number on the lowest face being taken in the case of the tetrahedron What is the chance that the sum of the numbers thrown is not less than 5 ?

48 A and B throw with a pair of dice A wins if he throws 6 before B throws 7 and B if he throws 7 before A throws 6 If A begins, shows that his chance of winning is 30/61

49 The probability of at least one double six being thrown in n throws with two ordinary dice is greater than 99 percent Calculate the least numerical value of n

Given log 36=1 5563 and log 35=1 5441

50 A seed merchant finds that 90% of his cucumber seeds germinate under standard conditions He accordingly claims 90% germination when he sells them in packets of 10 Show that about one quarter of his customers will be entitled to complain that seeds in their packets do not reach the standard of germination

Note Here we have used the notation $E_1\overline{E_2}$ for $E_1 \cap \overline{E_2}$ and $E_1\overline{E_2}+\overline{E_1}E$ for $(E_1 \cap \overline{E}) \cup (\overline{E_1} \cap E_2)$ We shall make use of both the notations so that the students may be familiar with both of them

15 This problem is similar to problem 3

Here $n={}^{5}D_{13}$ and $m={}^{2}C_1 \times {}^{4}C_1 \times {}^{46}C_{11}$, since one black ace out of 2 black aces can be chosen in ${}^{2}C_1$ ways, one king out of 4 kings in ${}^{4}C_1$ ways and the remaining 11 cards from the remaining 46 cards in ${}^{46}C_{11}$ ways

The required probability

$$p=\frac{m}{n}=\frac{{}^{2}C_1 \times {}^{4}C_1 \times {}^{46}C_{11}}{{}^{52}C_{13}}=\frac{164502}{978775} \text{ [Calculate yourself]}$$

16 (a) Let E_1 be the event that A can solve a problem and E the event that B can solve the same problem

Then $P(E_1)=\frac{2}{3}$ and $P(\overline{E_1})=1-\frac{2}{3}=\frac{1}{3}$,

$P(E)=\frac{3}{5}$ and $P(\overline{E_2})=1-\frac{3}{5}=\frac{2}{5}$

(i) **First Method** Here we want the probability $P(E_1 \cup E)$

Now $P(E_1 \cup E)=P(E_1)+P(E)-P(E_1 \cap E_2)$

$=P(E_1)+P(E_2)-P(E_1)\,P(E)$

$[P(E_1 \cap E)=P(E_1)\,P(E)$ since the events E_1 and E_2 are independent]

$=\frac{2}{3}+\frac{3}{5}-\frac{2}{3}\times\frac{3}{5}=\frac{2}{3}+\frac{3}{5}-\frac{2}{5}=\frac{13}{15}$

Second Method $P(E_1 \cup E)=$Probability that A solves and B does not solve or B solves and A does not solve or both A and B solve the problem

$=P(E_1 \cap \overline{E_2})+P(E_2 \cap \overline{E})+P(E_1 \cap E_2)$

$=P(E_1)\,P(\overline{E_2})+P(E)\,P(\overline{E_1})+P(E_1)\,P(E_2)$

[Events are independent]

$=\frac{2}{3}\,\frac{2}{5}+\frac{3}{5}\,\frac{1}{3}+\frac{2}{3}\,\frac{3}{5}=\frac{13}{15}$

(ii) Here we have to find the probability of the event $\overline{E_1} \cap \overline{E_2}$ We have

$$P(\overline{E_1} \cap \overline{E_2})=P(\overline{E_1})\,P(\overline{E_2})=\frac{1}{3}\;\frac{2}{5}=\frac{2}{15}$$

Remark Probability of case (i) can be deduced from that of case (ii)

51 A bag contains a certain number of balls, some of which are white, a ball is drawn and replaced, another is then drawn and replaced, and so on If p be the chance of drawing a white ball in a single trial, find the number of white balls that is most likely to have been drawn in n trials For p $\frac{1}{2}$ and $n=12$, calculate the number of white balls

52 If a fair coin is tossed 15 times what is the probability of getting heads as many times in the first ten throws as in the last five

53 Two dice are thrown together first and secondly three dice are thrown together Find the probability that the total in the first throw is 4 or more and at the same time the total in the second throw is 6 or more

54 In a bag there are three tickets numbered 1, 2, 3 A ticket is drawn at random and put back and this is done four times Show that it is 41 to 40 that the sum of the numbers drawn is even

55 A die is thrown three times and sum of three numbers thrown is 15 Find the chance that the first throw was a four

56 From each of two equal lines of length l a portion is cut off at random, and removed What is the chance that the sum of the remainders is less than l ?

57 Three tangents are drawn at random to a given circle Show that the odds are 3 to 1 against the circle being inscribed in the triangle formed by them

58 (*The birthday problem*) What is the probability that in a group of N people at least two of them will have the same birthday ?

59 *The problem of points* Two players A and B want respectively m and n points of winning a set of games their chances of winning a single game are p and q respectively where $p+q=1$ The stake is to belong to the player who first makes up his set find the probabilities in favour of each player

60 For two events A, B prove the following relations

(i) $P(\overline{A}/B) = 1 - P(A/B)$

(ii) $P(\overline{A} \cap \overline{B}) = 1 - P(A) - P(B) + P(AB)$

61 Two dice are tossed If (m, n) denotes a typical sample point find whether the following two events A and B are independent

$A = \{(m, n) \quad m+n=11\}$ $B = \{(m, n) \quad n \neq 5\}$

$$P(\overline{E_1} \cap \overline{E_2}) = P\,\overline{(E_1 \cup E_2)} = 1 - P(E_1 \cup E_2)$$

$$P(E_1 \cup E_2) = 1 - P(\overline{E_1} \cap \overline{P_2}) = 1 - \frac{2}{15} = \frac{13}{15}$$

(b) Let E_1 be the event that man will be selected and E_1 the event that woman will be selected Then

$P(E_1) = \frac{1}{4}$, $P(\overline{E_1}) = 1 - \frac{1}{4} = \frac{3}{4}$

$P(E_2) = \frac{1}{3}$, $P(\overline{E_2}) = 1 - \frac{1}{3} = \frac{2}{3}$

clearly events, E_1, E_2 are independent Hence

(a) $P(E_1 \cap E_2) = P(E_1)\,P(E_2)$
$= \frac{1}{4} \times \frac{1}{3} = \frac{1}{12}$

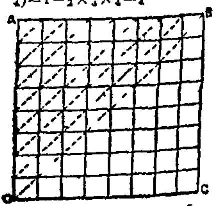
(b) $P(E_1 \cap \overline{E_2}) + P(\overline{E_1} \cap E_2)$
$= P(E_1)\,P(\overline{E_2}) + P(E_1)\,P(E_2)$
$= \frac{1}{4} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{5}{12}$

(c) $P(\overline{E_1} \cap \overline{E_2}) = P(\overline{E_1})\,P(\overline{E_2})$
$= \frac{3}{4} \times \frac{2}{3} = \frac{1}{2}$

17 Proceeding as in the remark of problem 16 we have
$P(E_1) = \frac{1}{2}$, $P(E_2) = \frac{1}{3}$ and $P(E_3) = \frac{1}{4}$

$P(E_1 \cup E_2 \cup E_3) = 1 - P(\overline{E_1})\,P(E_2)\,P(\overline{E_3})$
$= 1 - (1 - \frac{1}{2})(1 - \frac{1}{3})(1 - \frac{1}{4}) = 1 - \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} = \frac{3}{4}$

18 A chess board is a square divided into 64 equal squares parallel to the sides of the outer square as shown in the figure We can choose three squares in a diagonal line parallel to BD in the $\triangle ABD$ along the dotted lines It can be seen that in one of the dotted lines there are only three squares and hence the selection can be in ${}^3C_3$ ways In the next lower dotted line, the selection can be in ${}^4C_3$ ways and so on Similarly in the $\triangle BCD$ three squares can be chosen in an equal number of ways



Hence the number of ways in which 3 squares can be chosen parallel to BD and along BD itself is

62. A and B are two candidates seeking admission in I I T The probability that A is selected is 0 5 and the probability that both A and B are selected is at most 0 3 Is it possible that the probability of B getting selected is 9 (I I T 1982)

63 (a) Let A and B be two independent events such that the probability is $\frac{1}{8}$ that they will occur simultaneously and $\frac{3}{8}$ that neither of them will occur Find $P(A)$ and $P(B)$

(b) A and B are two independent events The probability that both A and B occur is $\frac{1}{6}$ and the probability that neither of them occurs is $\frac{1}{3}$ Find the probability of the occurrence of A (I I T 1984)

64 Two coins are tossed What is the conditional probability that two heads result given that there is at least one head ?

65 Assume that two coins are tossed one a rupee and the other an eight anna piece Let A be the event that rupee shows heads and B the event that the coins show different faces Are A and B independent ?

66 An urn contains five balls alike in every respect save colour If three of these balls are white and two are black and we draw two balls at random from this urn without replacing them If A is the event that the first ball drawn is white and B the event that the second ball drawn is black, are A and B independent ?

67 An urn contains four tickets with numbers 112, 121, 211, 222 and one ticket is drawn Let A_i ($i=1, 2, 3$) be the event that the ith digit of the number of tickets drawn is 1 Discuss the independence of the events A_1, A_2, A_3

68 Cards are dealt one by one from a well-shuffled pack until an ace appears Show that the probability that exactly n cards are dealt before the first ace appears is

$$\frac{4\,(51-n)\,(50-n)\,(49-n)}{52\;51\;50\;49}$$

(b) Cards are drawn one-by one at random from a well-shuffled full pack of 52 playing cards until 2 aces are obtained for the first time If N is the number of cards required to be drawn then show that

$$P_r\,(N=n)=\frac{(n-1)\,(52-n)\,(51-n)}{50\times 49\times 17\times 13} \text{ where } 2 \leqslant n \leqslant 50$$

(c) A person draws cards one by one from a pack Show that the probability that exactly n cards are drawn until all the aces appear is

$$2\,({}^3C_3+{}^4C_3+{}^5C_3+{}^6C_3+{}^7C_3)+{}^8C_3$$

[Note that we do not have 2 ${}^8C_3$ since the line BD is common to both the Δs ABD and ACD

The same argument applies to squares parallel to AC Hence the total no of favourable ways

$$m=4\,({}^3C_3+{}^4C_3+{}^5C_3+{}^6C_3+{}^7C_3)+2\,{}^8C_3=392$$

And n=total number of ways$={}^{64}C_3=\dfrac{64\ 63\ 62}{1\ 2\ 3}=32\ 21\ 62$

Hence the required probability$=\dfrac{m}{n}=\dfrac{392}{32\ 21\ 62}=\dfrac{7}{744}$

19 Total number of digits used in any number at the last (*i e* at unit's) place is 10 since 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9 can occur at that place It is clear that if the last digit in any of the four numbers is 0, 2, 4, 5, 6 or 8, then the last digit in the product of the four numbers cannot be 1, 3, 7 or 9 Hence in order that the last digit in the product may be 1, 3, 7 or 9, it is necessary that the last digit in each number must be 1, 3, 7 or 9 Hence for any one number, we have

$$n=10 \text{ and } m=4$$

p=The probability that the last digit in any number is 1, 3, 7 or 9

$$=\frac{m}{n}=\frac{4}{10}=\frac{2}{3}$$

Hence the required probability$=p^4=(\frac{2}{5})^4=\frac{16}{625}$

20 A leap year consists of 366 days and so it shall have 52 complete weeks and two extra days These two days can be

(i) Monday and Tuesday (ii) Tuesday and Wednesday
(iii) Wednesday and Thursday (iv) Thursday and Friday
(v) *Friday and Saturday* (vi) *Saturday and Sunday*
(vii) Sunday and Monday

Of these 7 cases, the last two are favourable and hence the required probability$=\frac{2}{7}$

21 Let the chance of the second event be p Then the chance of the first event is p^2

Odds against the *first* event are as $1-p^2$ p^2
and odds against the second event are $1-p$ p

Hence according to the condition given in the question, we have

$$\frac{4(n-1)(n-2)(n-3)}{52.51\ 50\ 49}$$

69 (a) A, B, C are events such that
$P_r(A)=0\ 3$, $P_r(B)=0\ 4$, $P(C)=0\ 8$
$P_r(AB)=0\ 08$, $P_r(AC)=0\ 28$
$P_r(ABC)=0\ 09$
If $P_r(A \cup B \cup C) \geqslant 0\ 75$, then show that $P_r(BC)$ lies in the interval $0\ 23 \leqslant x \leqslant 0\ 48$ *(I I T 1983)*

(b) There are ten pairs of gloves in a bag First A draws one glove from the bag, then B draws one glove, then A draws one glove and finally B draws one glove, drawn gloves being not replaced Show that the chance of A drawing a pair is the same as that of B drawing a pair Also find the probability that neither draws a pair

70 In a certain city only two newspapers A and B are published It is known that 25% of the city population reads A and 20% reads B while 8% reads both A and B It is also known that 30% of those who read A but not B look into advertisements and 40% of those who read B but not A look into advertisements while 50% of those who read both A and B look into advertisements What is the percentage of the population who reads an advertisement ? (I I T 1984)

71 (i) A bag A contains 2 white and 3 red balls and a bag B contains 4 white and 5 red balls One ball is drawn at random from one of the bags and is found to be red Find the probability that it was drawn from bag B (I I T 1976)

(ii) We have two boxes, B_1 and B_2 Box B_1 contains one red and one white marble Box B_2 contains three red marbles and one green marble A box is selected by the toss of a fair coin and one marble is drawn at random from the box selected Given that a red marble is obtained, what is the probability that the marble was drawn from B_1 ?

72 (a) Suppose there are three urns containing 2 white and 3 black balls 3 white and 2 black balls, and 4 white and one black ball respectively There is equal probability of each urn being chosen One ball is drawn from an urn chosen at random What is the probability that a white ball is drawn ?

$$\frac{1-p^4}{p^2}=\left(\frac{1-p}{p}\right)^3 \text{ or } \frac{(1-p)(1+p)}{p^4}=\frac{(1-p)^3}{p^3}$$

or $p\,(p^2+1)=(1-p)^2$ or $p^2+p=p^2-2p+1$ or $3p=1$

$p=\frac{1}{3}$ and $p^2=\frac{1}{9}$

Hence the probability of the first event $=\frac{1}{9}$

and the probability of the second event $=\frac{1}{3}$

22 (a) Let E_i = the event that ith shot from the guns hits the enemy $i=1, 2, 3, 4$. Then according to the question,

$P(E_1)=0.4$, $P(E_2)=0.3$, $P(E_3)=0.2$ and $P(E_4)=0.1$

We have to find, $P(E_1 \cup E_2 \cup E_3 \cup E_4)$

As in Problem 17, we have

$$P(E_1 \cup E_2 \cup E_3 \cup E_4)=1-P(\overline{E_1})\,P(\overline{E_2})\,P(\overline{E_3})\,P(\overline{E_4})$$
$$=1-(1-0.4)(1-0.3)(1-0.2)(1-0.1)$$
$$=1-0.6\times0.7\times0.8\times0.9$$
$$=1-0.3024=.6976$$

(b) Ans $p=1-p_1\,p_2\,p_3=1-.4\times.3\times.2$
$$=.976$$

23 Let $p_1, p_2, p_3, \ldots p_9$ denote the probabilities of drawing black, black, white, white, white, white, red, red and red respectively in this order without replacement. Then

p = the required probability $=p_1\,p_2\,p_3 \ldots p_9$

$$p_1=\frac{{}^2C_1}{{}^9C_1}=\frac{2}{9},$$

since one black ball can be drawn out of 2 in ${}^2C_1$ ways and total number of ways is ${}^9C_1$

$$p_2=\frac{{}^1C_1}{{}^8C_1}=\frac{1}{8},$$

since one black ball remains after the first draw

$$p_3=\frac{{}^4C_1}{{}^7C_1}=\frac{4}{7},$$

since in the remaining 7 balls 4 are white

Similarly $p_4=\frac{{}^3C_1}{{}^6C_1}=\frac{3}{6}=\frac{1}{2},\quad p_5=\frac{{}^2C_1}{{}^5C_1}=\frac{2}{5},\quad p_6=\frac{1}{4}$

Now the remaining three balls are all red so that

$$p_7=p_8=p_9=1$$

Hence $p=\frac{2}{9}\cdot\frac{1}{8}\cdot\frac{4}{7}\cdot\frac{1}{2}\cdot\frac{2}{5}\cdot\frac{1}{4}=\frac{1}{1260}$

(b) If in part (a), we are told that a white ball has been drawn, find the probability that it was drawn from the first urn

73 An urn contains five balls Two balls are drawn and are found to be white What is the probability that all the balls are white?

74 In a bolt factory, machines *A*, *B* and *C* manufacture 25%, 35% and 40% respectively Of the total of their output 5, 4 and 2% are defective A bolt is drawn and is found to be defective What are the probabilities that it was manufactured by the machines *A*, *B* and *C*?

75 A lot contains 20 articles The probability that the lot contains exactly 2 defective articles is 0 4 and the probability that the lot contains exactly 3 defective articles is 0 6 Articles are drawn from the lot at random one by one without replacement and tested till all the defective articles are found What is the probability that the testing procedure ends at the twelth testing? (I I T 86)

76 *n* letters to each of which corresponds an addressed envelope are placed in the envelopes at random What is the probability that no letter is placed in the right envelope?

Solutions to Problem Set (A)

1 Total no of ways $n={}^8C_1=8$
and favourable no of ways $m={}^5C_1=5$

Hence required probability

$$=\frac{m}{n}=\frac{5}{8}$$

2 Here $n=$Total no of ways $m={}^{52}C_4$

$$=\frac{52\times51\times50\times49}{1\times2\times3\times4}=13\times17\times25\times49$$
$$=270725$$

There are four honours (ace king, queen and knave) in each suit and so there are 4 sets of 4 honours each To obtain the favourable number of ways we have to select one suit of four honours from these four sets

$$m={}^4C_1=4$$

$$p=\text{required probability}=\frac{m}{n}=\frac{4}{270725}$$

24 Let E_1 be the event that a subject is selected from first group, E_2 the event that a subject is selected from the second group and E the event that an engineering subject is selected

It is given that subject is selected from first group if the die shows 3 or 5 otherwise it is selected from the second group

Now the probability that die shows 3 or $5 = \frac{2}{6} = \frac{1}{3}$

$$P(E_1) = \tfrac{1}{3} \text{ and } P(E_2) = P(\overline{E_1}) = 1 - P(E_1) = 1 - \tfrac{1}{3} = \tfrac{2}{3}$$

Now $P(E/E_1)$ = Prob of choosing an engineering subject from first group

$$= \frac{^3C_1}{^8C_1} = \frac{3}{8}$$

Similarly $P(E/E_2) = \frac{^5C_1}{^8C_1} = \frac{5}{8}$

Hence $P(E) = P(E_1)\ P(E/E_1) + P(E\ )\ P(E/E\ )$

$$= \frac{1}{3}\ \ \frac{3}{8} + \frac{2}{3}\ \ \frac{5}{8} = \frac{13}{24}$$

25 Let F_1 be the event that a ball is drawn from first bag, E_2 the event that a ball is drawn from the second bag and E the event a black ball is chosen Then as in Problem 24, we have

$$P(E) = P(E_1)\ P(E/E_1) + P(E_{\_})\ P(E/E\ )$$
$$= \tfrac{1}{3}\ \tfrac{3}{7} + \tfrac{2}{3}\ \tfrac{4}{7} = \tfrac{11}{21}$$

[Students are advised to write the detailed solution as in Problem 24]

26 n=the total number of ways of distributing M telegrams over Nchannels$=N^M$ The number of ways of choosing M channel's out of N in order to send one telegram over each channel is N_{C_M} and corresponding to each of these ways of choosing M channels, there are $M\,!$ ways of sending the M telegrams Hence

m=the favourable number of ways$=N_{C_M}\ M\,!$

$$\text{The required probability} = \frac{N_{C_M}\ M\,!}{N^M}$$

27 n=the total number of ways of distributing

K balls over N cells$=N^K$

Now we find the favourable no of ways K adjacent cells out of N can be chosen in $N-K+1$ ways For if we denote the N cells by $C_1, C_{\_}\ C_3\ \ , C_N$ then the following groupings of K consecutive cells are possible

3 We know that 13 cards are delivered to a hand at whist so that

$$n = \text{total no of ways} = {}^{52}C_{13} = \frac{52!}{13!\,39!}$$

Since 4 kings are held by a specified player, 9 more cards are to be delivered to him from the remaining 48 cards so that

$$m = \text{favourable no of ways} = {}^{48}C_9 = \frac{48!}{9!\,39!}$$

$$\text{Hence } p = \text{required probability} = \frac{m}{n} = \frac{48!}{9!\,39!} \times \frac{13!\,39!}{52!}$$

$$= \frac{13\cdot 12\cdot 11\cdot 10}{52\cdot 51\cdot 50\cdot 49} = \frac{11}{4165}$$

4 There are 6 numbers (1, 2, 3, 4, 5, 6) written on the six faces of each die Thus there are six possible ways as to the number of points on the first die, and to each of these ways, there correspond 6 possible numbers of points on the second die Hence the total no of ways

$$n = 6 \times 6 = 36$$

We now find out how many ways are favourable to the total of 7 points This may happen only in the following ways (1, 6), (6, 1), (2, 5), (5, 2), (3, 4), and (4, 3), that is, in 6 ways, where first member of each ordered pair denotes the number on the first die and second member denotes the number of the second die

$$m = 6$$

$$\text{Hence required probability} = \frac{m}{n} = \frac{6}{36} = \frac{1}{6}$$

5 In one throw of a coin, the number of possible ways is 2 since either head (H) or tail (T) may appear In two throws of a coin, the total no of ways is 2×2 i e 4 since corresponding to each way of the first coin there are 2 ways of second Similarly in three throws of a coin, the number of ways is 2^3 and thus in n throws, the number of ways is 2^n

Hence the total number of ways $N = 2^n$

And M = the favourable no of ways

= the number of ways in which head will occur once or thrice or 5 times, or 7 times and so on,

$$= {}^nC_1 + {}^nC_3 + {}^nC_5 + \dots = 2^{n-1}$$

$$\text{Hence required probability } p = \frac{M}{N} = \frac{2^{n-1}}{2^n} = \tfrac{1}{2}$$

$$C_1C_2 \quad C_K,$$
$$C_2C_3 \quad C_{K+1}$$
$$C_3C_4 \quad C_{K+2}$$

$$C_{N-K}, C_{N-K+1}, \quad C_{N-1}$$
$$C_{N-K+1}, C_{N-K+2}, \quad , C_N$$

Now K balls can be distributed over each of these groups of K consecutive cells in $K!$ ways Hence
m=the favourable no of ways$=(N-K+1)\,K!$

The required probability$=\dfrac{(N-K+1)\,K!}{N^K}$

28 Let the probability of a vessel wrecking be q and of safe arrival be p so that

$$q=\frac{1}{10} \text{ and } p=1-\frac{1}{10}=\frac{9}{10}$$

The probabilities of no vessel, one vessel, two vessels, *etc*, arriving safely are the first, second, third terms, etc., in the binomial expansion

$$(q+p)^5=q^5+{}^5C_1\, q^4p+{}^5C_2\, q^3p^2+{}^5C_3\, q^2p^3+{}^5C_4\, qp^4+p^5$$

The probability of at least 4 vessels arriving safely is the sum of last two terms

Hence the required probability$={}^5C_4\, qp^4+p^5$

$$=5\,\frac{1}{10}\left(\frac{9}{10}\right)^4+\left(\frac{9}{10}\right)^5$$
$$=\frac{45927}{50000}$$

29 As in Problem 28, we have

$$q=\text{prob of a bulb being defective}=\frac{10}{100}=\frac{1}{10}$$

and $\quad p=\text{prob of the bulb being defective}=\dfrac{9}{10}$

The probabilities of no defective one defective, two defective bulbs, etc, are the first, second, third terms etc, in the binomial expansion

$$(p+q)^8=p^8+{}^8C_1\, p^7q+{}^8C_2\, p^6q^2+ \quad +q^8$$

(a) The probability of 3 defective and 5 non defective bulbs

$$={}^8C_3p^5q^3=56\times\left(\frac{9}{10}\right)^5\left(\frac{1}{10}\right)^3=\frac{413343}{12500000}$$

Alternative Probability of tail (T) in one trial$=\frac{1}{2}=q$, say,
and Probability of head (H) in one trial$=\frac{1}{2}=p$, say
Then the required probability

$$=\text{sum of even terms in the expansion}$$
$$(q+p)^n=q^n+{}^nC_1q^{n-1}p+{}^nC_2q^{n-2}p^2+{}^nC_3q^{n-3}p^3+\quad+p^n$$
$$={}^nC_1\,q^{n-1}p+{}^nC_3\,q^{n-3}p^3+{}^nC_5\,q^{n-5}p^5+$$
$$={}^nC_1\,(\tfrac{1}{2})^{n-1}\,\tfrac{1}{2}+{}^nC_3\,(\tfrac{1}{2})^{n-3}\,(\tfrac{1}{2})^3+{}^nC\;(\tfrac{1}{2})^{n-}\;(\tfrac{1}{2})^5+$$
$$=(\tfrac{1}{2})^n\,[{}^nC_1+{}^nC_3+{}^nC_5+\quad]=\frac{2^{n-1}}{2^n}=\tfrac{1}{2}$$

6 (i) n=total no of ways$=12\,!$
and m=favourable no of ways$=7\,!\;6\,!$,
since 7 objects (considering 6 boys as 6 different objects and *all the six girls together as one object) can be arranged in 7 !* ways corresponding to each of these ways, the six girls can be arranged amongst themselves in 6 ! ways

Hence $p=\dfrac{m}{n}=\dfrac{7\,!\;6\,!}{12\,!}=\dfrac{1}{132}$

(ii) Here $n=12\,!$
and $m=2\times6\,!\;6\,!$,
since the boys and girls can sit alternately in 6 ! 6 ! ways if we begin with a boy and similarly they can sit alternately in 6 ! 6 ! ways if we begin with a girl

Hence $p=\dfrac{m}{n}=\dfrac{2\times6\,!\;6\,!}{12\,!}=\dfrac{1}{462}$

7 Since the probability of the faces are proportional to the numbers on them, we can take the probabilities of faces 1 2, 6 as k, $2k$, , $6k$ respectively
Since one of the faces must occur, we have

$$k+2k+3k+4k+5k+6k=1 \text{ or } k=\frac{1}{21}$$

The probability of an even number$=2k+4k+6k$

$$=12k=12\times\frac{1}{21}=\frac{4}{7}$$

8 Since odds against A are as 8 3, the probability of A's occurring,

$$P(A)=\frac{3}{8+3}=\frac{3}{11}$$

Similarly $P(B)=\dfrac{2}{5+2}=\dfrac{2}{7}$

Since the events A, B C are mutually exclusive and totally exhaustive, the sum of their probabilities must be unity, that is,

$$P(A)+P(B)+P(C)=1$$

or $\dfrac{3}{11}+\dfrac{2}{7}+P(C)=1$

(b) The prob of at least one defective bulb

$$=1-p^8=1-\left(\frac{9}{10}\right)^8=1-\frac{43046721}{100000000}$$

$$=\frac{56953279}{100000000}$$

30 Let p be the chance of throwing an ace (*i e* 1) and q the chance of not throwing an ace in a single throw with one die

Then $p=\frac{1}{6}$ and $q=\frac{5}{6}$ (why ?)

Now the chances of throwing no ace, one ace, two aces, etc , in five throws with a single die are the first, second, third terms, etc., in the binomial expansion

$$(q+p)^5=q^5+{}^5C_1\,q^4p+{}^5C_2\,q^3p^2+{}^5C_3\,q^2p^3+{}^5C_4\,qp^4+p^5$$

Hence (1) chance of three aces exactly

$$={}^5C_3\,q^2p^3=10\ \left(\tfrac{5}{6}\right)^2\left(\tfrac{1}{6}\right)^3$$

$$=\frac{125}{3888}$$

(2) Chance of throwing three aces at least

$$={}^5C_3\,q^2p^3+{}^5C_4\,qp^4+p^5=10\left(\tfrac{5}{6}\right)^2\left(\tfrac{1}{6}\right)^3+5\left(\tfrac{5}{6}\right)\left(\tfrac{1}{6}\right)^4+\left(\tfrac{1}{6}\right)^5$$

$$=\frac{1}{6^5}\ [250+25+1]=\frac{276}{7776}=\frac{23}{648}$$

31 Let w stand for the winning of a game and l for losing it Then there are 4 mutually exclusive possibilities

(i) $w, w, w,$ (ii) $w, w, l, w,$

(iii) $w, l, w, w,$ (iv) l, w, w, w

[Note that case (i) includes both the cases whether he loses or wins the fourth game]

According to the conditions of the question, the probabilities for (i), (ii), (iii) and (iv) are respectively

$\frac{2}{3}\ \frac{2}{3}\ \frac{2}{3}$, $\frac{2}{3}\ \frac{2}{3}\ \frac{1}{3}\ \frac{1}{3}$, $\frac{2}{3}\ \frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}$ and $\frac{1}{3}\ \frac{1}{3}\ \frac{2}{3}\ \frac{2}{3}$

Hence the required probability

$$=\tfrac{8}{27}+\tfrac{4}{81}+\tfrac{4}{81}+\tfrac{4}{81}=\tfrac{36}{81}=\tfrac{4}{9}$$

32. Let p be the chance of cutting a spade and q the chance of not cutting a spade from a pack of 52 cards

Then $p=\dfrac{{}^{13}C_1}{{}^{52}C_1}=\frac{1}{4}$ and $q=1-\frac{1}{4}=\frac{3}{4}$

$$P(C)=1-\frac{3}{11}-\frac{2}{7}=\frac{34}{77}$$

Hence the odds against C are as $77-34$: 34 i.e. 43 : 34

9 Since A has three shares in a lottery, his chance of success means that he gets at least one prize, that is, he gets either one prize or 2 prizes or 3 prizes and his chance of failure means that he gets no prize. It is certain that either he succeeds or fails. If p denotes his chance of success and q the chance of his failure, then

$p+q=1$ or $p=1-q$

We now find q

$$n=\text{total no. of ways}={}^{12}C_3=\frac{12\times 11\times 10}{1\times 2\times 3}=220$$

since out of 12 tickets in the lottery, he can draw any three tickets by virtue of his having three shares in the lottery and

$$m=\text{favourable number of ways}={}^{9}C_3=\frac{9\times 8\times 7}{1\times 2\times 3}=84,$$

Since he will fail to draw a prize if all the tickets drawn by him are blanks

$$q=\frac{m}{n}=\frac{84}{220}=\frac{21}{55}$$

$$p=A\text{'s chance of success}=1-\frac{21}{55}=\frac{34}{55}$$

Similarly B's chance of success

$$p=1-q=1-\frac{{}^{6}C_2}{{}^{8}C_2}=1-\frac{6\times 5}{8\times 7}=1-\frac{15}{28}=\frac{13}{28}$$

A's chance of success : B's chance of success

$$=p:p=\frac{34}{55}:\frac{13}{28}=\frac{952}{1540}:\frac{715}{1540}=952:715$$

10 (a) $n=\text{total no. of ways}=2^{10}=1024$

since each answer can be true or false

and $m=\text{favourable number of ways}$

$={}^{10}C_8+{}^{10}C_9+{}^{10}C_{10}=45+10+1=56$

since to pass the exam, he must give 8 *or* 9 or 10 true answers

Hence $p=\frac{m}{n}=\frac{56}{1024}=\frac{7}{128}$

(b) The total number of way of ticking the answers in any one attempt $=2^4-1=15$ and only one of these will be the correct answer. Hence the probability of getting

Now A will win a prize if he cuts spade at 1st, 4th, 7th, 10th turns, etc. Note that A will get a second chance if A, B, C all fail to cut a spade once and then A cuts a spade at the 4th turn.

Similarly he will cut a spade at the 7th turn when A, B, C fail to cut spade twice, etc.

Hence A's chance of winning the prize

$$=p+q^3p+q^6p+q^9p+\ldots$$

$$=\frac{p}{1-q^3}=\frac{\frac{1}{4}}{1-(\frac{3}{4})^3}=\frac{16}{37}$$

Similarly B's chance $=qp+q^4p+q^7p+\ldots$

$$=q(p+q^3p+q^6p+\ldots)$$

$$=\frac{3}{4}\cdot\frac{16}{37}=\frac{12}{37}$$

and C's chance $=\frac{3}{4}$ of B's chance $=\frac{3}{4}\cdot\frac{12}{37}=\frac{9}{37}$

33 Let A, B, C be the three men. This problem is different from problem 32 since here the balls are not replaced. To find A's chance, we must find the sum of probabilities of his drawing a white ball at 1st, 4th draw. (He will not need to draw at the 7th turn since the number of black ball is 5 which is less than 6)

Hence $p_1=A$'s chance $=\frac{3}{8}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{3}{5}=\frac{27}{56}$

[Note that A will get a second chance when A, B, C in this order draw black balls the probabilities for which are clearly $\frac{5}{8}$, $\frac{4}{7}$ and $\frac{3}{6}$ and at the fourth turn the chance of A's drawing white ball is $\frac{3}{5}$.]

Similarly,

$$p_2=B\text{'s chance}=\frac{5}{8}\cdot\frac{3}{7}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{3}{4}=\frac{18}{56}$$

and $p_3=C$'s chance $=\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}+\frac{5}{8}\cdot\frac{4}{7}\cdot\frac{3}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}\cdot 1$

$$=\frac{11}{56}$$

marks in any attempt is $\frac{1}{15}$ and of not getting marks is $1-\frac{1}{15}$ or $\frac{14}{15}$ Now the candidate will get marks if he ticks all the correct answers in at least one of the three chances.

Hence the required probability $=1-\left(\frac{14}{15}\right)^3=\frac{631}{3375}$

Remark Note that here it is given that the candidate decides to tick answers, that is, the possibility that he does not tick the answers at all is ruled out That is why we take 2^4-1 and not 2^4

11 We denote the appearance of head by H and of tail by T The sample space T consists of four points, that is,

$$S=\{(H, H), (H, T), (T, H), (T, T)\}$$

Then $E=\{(H, H), (H, T)\}$ and $F=\{(H, H), (T, H)\}$

so that $E \cup F=\{(H, H), (H, T), (T, H)\}$

Hence $n=n(S)=$ the number of points in $S=4$

and $m=n(E \cup F)=3$

Hence $P(E \cup F)=\frac{n(E \cup F)}{n(S)}=\frac{3}{4}$

12 (a) We denote by A the event that the first card is the ace of hearts and B the corresponding event in respect of the second card

We have to find $P(A \cup B)$ But by addition theorem

$$P(A \cup B)=P(A)+P(B)-P(A \cap B) \qquad (1)$$

Now $P(A)=P(B)=\frac{1}{52}$

since there is only one ace of hearts in each suit of 52 cards To find $P(A \cap B)$, we see that there are in all 52^2 cases which are represented by all possible pairs consisting of one card from first and one from the second Only one of these cases is favourable to the event $A \cap B$, so that

$$P(A \cap B)=\frac{1}{52^2}$$

Hence from (1), we obtain,

$$P(A \cup B)=\frac{1}{52}+\frac{1}{52}-\frac{1}{52^2}=\frac{103}{2704}$$

(b) Let A be the event that the first die shows a number greater than 3 and B the event that the second die shows a number greater than 3 Then $P(A)=\frac{3}{6}=\frac{1}{2}$ and $P(B)=\frac{3}{6}=\frac{1}{2}$ To find $P(A \cap B)$, we see that there are in all 36 cases which are represented by all possible pairs

$p_1 : p_2 : p_3 = \frac{27}{56} : \frac{18}{56} : \frac{11}{56} = 27 : 18 : 11$

34 Let E_1 be the event that B rides A, E_2 the event that C rides A and E the event that A wins. Then according to the question,

$P(E_1)=\frac{2}{3}$, $P(E_2)=1-\frac{2}{3}=\frac{1}{3}$

$P(E/E_1)=\frac{1}{6}$ (since all the 6 horses are equally likely to win when B rides A)

$P(E/E_2)=3\cdot\frac{1}{6}=\frac{1}{2}$ (since A's chance of winning is trebled when C rides A)

$P(E)=P(E_1)\,P(E/E_1)+P(E_2)\,P(E/E_2)$

$=\frac{2}{3}\cdot\frac{1}{6}+\frac{1}{3}\cdot\frac{1}{2}=\frac{5}{18}$

Hence probability of A's win is 5/18 so that odds against A's win are as $18-5 : 5$ that is, 13 : 5

35 Let p_1, p_2, p_3 denote the probabilities that first, second and third critics review the book favourably. Then

$$p_1=\frac{5}{5+2}=\frac{5}{7},\ p_2=\frac{4}{4+3}=\frac{4}{7},\ p_3=\frac{3}{3+4}=\frac{3}{7}$$

$\overline{p_1}=1-\frac{5}{7}=\frac{2}{7},\ \overline{p_2}=1-\frac{4}{7}=\frac{3}{7},\ \overline{p_3}=1-\frac{3}{7}=\frac{4}{7}$

Now the majority will be favourable if any of the two critics is favourable and third is unfavourable or all the three critics are favourable.

Hence the required probability

$$p=p_1p_2\overline{p_3}+p_1\overline{p_2}p_3+\overline{p_1}p_2p_3+p_1p_2p_3$$
$$=\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{4}{7}+\frac{5}{7}\cdot\frac{3}{7}\cdot\frac{3}{7}+\frac{2}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}+\frac{5}{7}\cdot\frac{4}{7}\cdot\frac{3}{7}=\frac{209}{343}$$

36 A and B both agree when either both of them speak truth or both make false statements. Hence the total number of cases of agreeing is proportional to $xy+(1-x)(1-y)$ and the number of cases of speaking truth is proportional to xy.

Hence the required probability

$$=\frac{xy}{xy+(1-x)(1-y)}=\frac{xy}{1-x-y+2xy}$$

37 Selection can be made in the following manner

(i) Boy, Boy, girl Probability $p_1=\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{1}{4}=\frac{1}{32}$

(ii) Boy, girl, boy $p_2=\frac{1}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}=\frac{3}{32}$

(iii) Girl, boy, boy $p_3=\frac{3}{4}\cdot\frac{2}{4}\cdot\frac{3}{4}=\frac{9}{32}$

Since these are mutually exclusive cases, therefore the required probability $p_1+p_2+p_3=\frac{13}{32}$

38 Let the sequence of numbers start with the integer m so that the $3n$ consecutive integers are

$m,\ m+1,\ m+2,\ m+3,\ \ldots,\ m+3n-1$

or $x=\frac{2b^3\pm\sqrt{\{4b^6+4b^2(a^4-b^4)\}}}{2(a^2-b^2)}=\frac{b^3\pm a^2b}{a^2-b^2}=\frac{b(a^2+b)}{a^2\ \ b^2}$

rejecting $x=-b$

35 Proceed as above $\tan 2\theta=40/x$, $\tan\theta=15/x$ $x=3$
where $AB=15$, $BC=25$, $AP=x$ etc

36 $AB=y$ tower, $BC=x$, flag staff $AP=9$, $AQ=11$ B
subtends equal angles α at P and Q s t $\tan\alpha=1/10$ Hence th

circle through C and B must also pass through P and Q as angles in the same segment are equal From the figure it is clear that segment BP subtends equal angles say β at C and Q

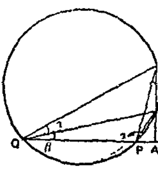


Fig 98

$y=11\tan\beta$ (1)

and $x+y=11\tan(\alpha+\beta)$ (2)

$AP=9=(x+y)\tan\beta$ (3)

$9=11\tan(\alpha+\beta)\tan\beta$ from (2), and (3)

or $9=11\tan\beta\left(\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\right)$ Put $\tan\alpha=1/10$

$9(1-1/10\tan\beta)=11\tan\beta(1/10+\tan\beta)$

$9(10-\tan\beta)=11\tan\beta(1+10\tan\beta)$

$110\tan^2\beta+20\tan\alpha-90=0$ or $11\tan^2\beta+2\tan\beta-9=0$

$(11\tan\beta-9)(\tan\beta+1)=0$ $\tan\beta=9/11$ (β acute)

Hence from (1), $y=11\tan\beta=11\ \frac{9}{11}=9$

From (3), $9=(x+y)\tan\beta=(x+9)\,9/11$

$x+9=11$ or $x=2$

37 (*a*) $OC=50$ is the tower standing on a sloping ground OB

$BA=2h$ is the flag pole whose mid point M is in level with O The angles of depression of A and B as seen from C, the top of the tower, are 15° and 45° respectively

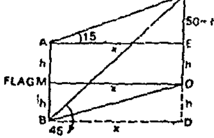


Fig 99

Let $BD=x=OM=AE$

$50-h=x\tan 15°$, $\triangle CAE$

$50+h=x\tan 45°$, $\triangle CBD$

Dividing in order to eliminate x we get

$$\frac{50+h}{50-h}=\frac{\tan 45^\circ}{\tan 15^\circ}$$

Applying componendo and dividendo, we get

$$\frac{2h}{100}=\frac{\sin(45^\circ-15^\circ)}{\sin(45^\circ+15^\circ)}=\frac{\sin 30^\circ}{\sin 60^\circ}=\frac{1}{\sqrt{3}}$$

$$2h=\frac{100}{\sqrt{3}}=\frac{100\sqrt{3}}{3}$$

38 $h=PA\tan\alpha$

and $h=PB\tan\beta$

and $PB=h\cot\beta$ $\quad PA=h\cot\alpha$,

Also from right angled triangle PAB, we get $PB^2=PA^2+AB^2$

$$AB^2=PB^2-PA^2$$

$$=h^2(\cot^2\beta-\cot^2\alpha),$$

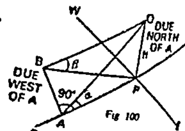


Fig 100

$$h=\frac{AB}{\sqrt{(\cot^2\beta-\cot^2\alpha)}}=\frac{AB\tan\alpha\tan\beta}{\sqrt{(\tan^2\alpha-\tan^2\beta)}}$$

On changing to tan

or $$h=\frac{AB\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha\cos^2\beta+\cos^2\alpha\sin^2\beta)}}$$ changing to sin and cos

$$=\frac{AB\sin\alpha\sin\beta}{\sqrt{\{\sin^2\alpha(1-\sin^2\beta)-\sin^2\beta(1-\sin^2\alpha)\}}}$$

$$=\frac{AB\sin\alpha\sin\beta}{\sqrt{(\sin^2\alpha-\sin^2\beta)}}=\frac{AB\sin\alpha\sin\beta}{\sqrt{\{\sin(\alpha+\beta)\sin(\alpha-\beta)\}}}$$

$$\sin^2\alpha-\sin^2\beta=\sin(\alpha+\beta)\sin(\alpha-\beta)$$

39 (a) PQ is $\perp$ to PA and PB

both $h=PA\tan\alpha$ and $h=PB\tan\beta$

$$PA=h\cot\alpha,\ PB=h\cot\beta$$

Also $AB^2=PA^2+PB^2$

or $d^2=h^2(\cot^2\alpha+\cot^2\beta)$

$$h=\frac{d}{\sqrt{(\cot^2\alpha+\cot^2\beta)}}$$

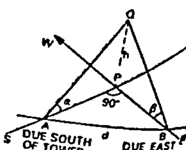


Fig 101

Now they can be classified as

$m, m+3, m+6, \ldots, m+3n-3,$

$m+1, m+4, m+7, \ldots, m+3n-2,$

$m+2, m+5, m+8, \ldots, m+3n-1$

The sum of the three numbers shall be divisible by 3 if either all the three numbers are from the same row or all the three numbers are from different rows. The number of ways that the three numbers are from the same row is $3\,{}^nC_3$ and the number of ways that the numbers are from different rows is $n\times n\times n=n^3$ since a number can be selected from each row in n ways.

Hence the favourable no. of ways $M=3\,{}^nC_3+n^3$

And the total number of ways $N={}^{3n}C_3$

$$\text{The required probability}=\frac{3\,{}^nC_3+n^3}{{}^{3n}C_3}=\frac{3n^2-3n+2}{(3n-1)(3n-2)}$$

39 (i) If the smallest number is 1, the groups of three numbers in A.P. are as 1, 2, 3,; 1, 3, 5; 1, 4, 7; ..., 1, $n+1$, $2n+1$, and they are n in number.

If the smallest number selected is 2, the possible groupings are 2, 3, 4; 2, 4, 6; 2, 5, 8; ... 2, $n+1$, $2n$, and their number is $n-1$.

If the lowest number is 3, the groupings are 3, 4, 5; 3, 5, 7; 3, 6, 9; ..., 3, $n+2$, $2n+1$, their number being $n-1$.

Similarly it can be seen that if the lowest numbers selected are 4, 5, 6, ..., $2n-2$, $2n-1$, the number of selections respectively are $n-2$, $n-2$, $n-3$, $n-3$, ..., 2, 2, 1, 1. Thus the favourable ways for 2, 3 are the same and similarly they are the same for 4, 5 and so on.

Hence number of favourable ways

$$M=2\,(1+2+3+\ldots n-1)+n$$

$$=2\,\frac{(n-1)\,n}{2}+n=n^2-n+n=n^2$$

Also the total number of ways

$$N={}^{2n+1}C_3=\frac{(2n+1)\,2n\,(2n-1)}{1\cdot2\cdot3}$$

$$=\frac{n\,(4n^2-1)}{3}$$

$$\text{Hence the required probability}=\frac{M}{N}=\frac{3n^2}{n\,(4n^2-1)}=\frac{3n}{4n^2-1}$$

The chance of this is ${}^{m+r-1}C_{m-1}\, p^{m-1} q^r p$, i e ${}^{m+r-1}C_{m-1}\, p^m q^r$ Now the set will definitely be decided in $m+n-1$ games, and A may win his m games in exactly m games, or $m+1$ games , or $m+n-1$ games Hence we shall obtain the chance of A's winning the set by giving to r the values 0, 1, 2, , $n-1$ in succession in the expression ${}^{m+r-1}C_{m-1}\, p^m q^r$ Hence A's probability to win the set is

$$p^m \left[1+mq+\frac{m\,(m+1)}{1\cdot 2}\, q^2+\cdots+\frac{(m+n-2)\,!}{(m-1)\,!\,(n-1)\,!}\, q^{n-1} \right],$$

Similarly B's probability to win the set is

$$q^n \left[1+np+\frac{n\,(n+1)}{1\cdot 2}\, p^2+\cdots+\frac{(m+n-2)\,!}{(m-1)\,!\,(n-1)\,!}\, p^{m-1} \right]$$

60 (i) $P(\bar{A}/B)=\dfrac{P(\bar{A}\cap B)}{P(B)}$, by definition of conditional probability

$$=\frac{P(B)-P(A\cap B)}{P(B)} \text{ by theorem 2 of § 3}$$

$$=1-\frac{P(A\cap B)}{P(B)}=1-P(A/B)$$

(ii) $P(\bar{A}\cup \bar{B})=P(\overline{A\cap B})$ by De-Morgan Laws

$=1-P(A\cap B)$ by Theorem 3 of § 3

$=1-P(A)\,P(B/A)$

61 Here A consists of two elements (6 5) and (5 6) only whereas B consists of 30 elements consisting of all the elements of sample space except the elements (1 5) (2 5), (3 5) (4 5) (5 5), (6, 5) Hence $A\cap B$ consists of a single element (5 6) only

Therefore $P(A)=\dfrac{2}{36}=\dfrac{1}{18}$

$$P(B)=\frac{30}{36}=\frac{5}{6} \text{ and } P(A\cap B)=\frac{1}{36}$$

Thus here $P(A)\,P(B)=\dfrac{1}{18}\times\dfrac{5}{6}=\dfrac{5}{108}$

giving $P(A\cap B)\neq P(A)\,P(B)$

Hence A and B are dependent events

62 Let A denote the event that the candidate A is selected and B that event that B is selected It is given that

$$P(A)=\cdot 5 \quad (1)$$

$$P(A\cap B)\leqslant \cdot 3 \quad (2)$$

Ans The required prob $=\frac{10}{133}$

[Hint Put $n=10$ in part (i) But students are advised to do it independently]

40 Total number of ways

$$N={}^{6n}C_3=\frac{6n\,(6n-1)\,(6n-2)}{1\ 2\ 3}=n\,(6n-1)\,(6n\ \ 2)$$

We now obtain the favourable number of ways

Let the lowest number drawn be 0 Then the remaining two tickets are to be drawn from the tickets numbered 1, 2, 3, , $3n-2$, $3n-1$, $3n$, $3n+1$, , $6n-2$, $6n-1$ such that their sum is $6n$ In this case, there are the following $(3n\ \ 1)$ triples of tickets having the sum $6n$

$(0\ 1, 6n-1)$, $(0, 2, n\ \ 2)$, $(0, 3\ \ 6n-3)$, $(0, 3n-1, 3n+1)$

If the lowest number drawn is 1, then the other two tickets will be chosen from the tickets numbered

$2, 3, 4,\ \ , 3n-1, 3n, 3n+1,\ \ , 6n-1$

such that the sum of all the three tickets is $6n$

There are the following $(3n-2)$ triples with their sum $6n$

$(1, 2, 6n-3)$, $(1, 3, 6n-4)$, $(1, 4, 6n-5)$, $(1, 3n-1, 3n)$

Similarly if the lowest number drawn is 2, then the other two tickets can be chosen from the tickets numbered 3, 4 5, , $3n-1$, $3n$, $3n+1$, , $(6n\ \ 1)$ in $(3n-4)$ ways and so on

There is only one way with the lowest number $2n-1$

It is the triple $(2n-1, 2n, 2n+1)$

Favourable number of ways

$$M=\{(3n-1)+(3n-2)\}+\{(3n-4)+(3n-5)\}+ \\ +(5+4)+(2+1)$$

$$=3+9+15+\ \ +(6n-9)+(6n-3)$$

$$=3\,[1+3+5+\ \ +(2n-1)]=3\,\frac{n}{2}[2\ 1+(n-1)\ 2]=3n^2$$

Hence the required probability

$$=\frac{M}{N}=\frac{3n^2}{n\,(6n-1)\,(6n-3)}=\frac{3n}{(6n-1)\,(6n-2)}$$

41 We denote by H the appearance of head and by T the appearance of tail and let X denote the appearance of head or tail Then

$$P(H)=P(T)=\tfrac{1}{2}\text{ and }P(X)=1$$

If the sequence of m consecutive heads starts from the first throw, we have

If A, B are independent we have

$$P(A \cap B) = P(A)\, P(B) \qquad (3)$$

From (1), (2) and (3), we have

$$5P(B) \leqslant 3 \text{ or } P(B) \leqslant \frac{3}{5} = 6$$

Hence $P(B)$ i.e. the probability of B getting selected cannot be 9

Remark The result will be obtained even if we regard the two events as dependent We then have

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

or $\quad P(A \cap B) = P(A) + P(B) - P(A \cup B)$

or $\quad P(B) = P(A \cap B) + P(A \cup B) - P(A)$

Now $\quad$ Max $P(A \cup B) = 1$ and

Max $P(A \cap B) = 3$ (given)

Hence Max $P(B) = 3 + 1 - 5 = 8$

Thus $P(B) \leqslant 8$ and so $P(B)$ cannot be 9

63 (a) We are given, $\quad P(A \cap B) = \frac{1}{8} \qquad (1)$

and $\quad P(\overline{A} \cap \overline{B}) = \frac{3}{8} \qquad (2)$

We have to find $P(A)$ and $P(B)$

Let $\quad P(A) = x$ and $P(B) = y$

Since the events are independent we have from (1),

$$P(A)\, P(B) = \frac{1}{8} \text{ i.e. } xy = \frac{1}{8} \qquad (3)$$

And from (2), we have

$$P(\overline{A} \cap \overline{B}) = P(\overline{A \cup B}) = \frac{3}{8}$$

or $\quad 1 - P(A \cup B) = \frac{3}{8}$

or $\quad 1 - P(A) - P(B) + P(A \cap B) = \frac{3}{8}$

or $\quad 1 - P(A) - P(B) + P(A)\, P(B) = \frac{3}{8}$

or $\quad 1 - x - y + xy = \frac{3}{8} \qquad (4)$

Subtracting (3) from (4), we get

$$1 - x - y = \frac{1}{4} \text{ or } x + y = \frac{3}{4} \qquad (5)$$

Now $\quad (x-y)^2 = (x+y)^2 - 4xy = \frac{9}{16} - 4 \cdot \frac{1}{8} = \frac{1}{16}$

or $\quad x - y = \pm \frac{1}{4} \qquad (6)$

Solving (5) and (6), we get

$$x = \frac{1}{2} \text{ and } y = \frac{1}{4}$$

or $\quad x = \frac{1}{4}$ and $y = \frac{1}{2}$

Hence $\quad P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{4}$

or $\quad P(A) = \frac{1}{4}$ and $P(B) = \frac{1}{2}$

(b) $P(A)\, P(B) = \frac{1}{8}$, $P(\overline{A})\, P(\overline{B}) = \frac{1}{2}$

$(HHH \ldots m \text{ times})(XXX \ldots n \text{ times})$

The chance of this event $= \frac{1}{2}\cdot\frac{1}{2}\cdot\frac{1}{2} \ldots m$ times $= \frac{1}{2^m}$

[Note that $(m+1)$ and subsequent throws may be head or tail since we are considering at least m consecutive heads]

If the sequence of m consecutive heads starts from the second throw the first must be a tail and we have

$T(HHH \ldots m \text{ times})(XXX \ldots \overline{n-m-1} \text{ times})$

The chance of this event $= \frac{1}{2}\cdot\frac{1}{2^m} = \frac{1}{2^{m+1}}$

If the sequence of heads starts with the $(r+1)^{th}$ throw, then the first $(r-1)$ throws may be head or tail but r^{th} throw must be tail and we have

$(XXX \ldots \overline{r-1} \text{ times})\, T(HHH \ldots m \text{ times})(XXX \ldots \overline{n-m-r} \text{ times})$

The probability of this event also $= \frac{1}{2}\cdot\frac{1}{2^m} = \frac{1}{2^{m+1}}$

Since all the above cases are mutually exclusive, the required

probability $= \frac{1}{2^m} + \left(\frac{1}{2^{m+1}} + \frac{1}{2^{m+1}} + \ldots \text{ to } n \text{ terms}\right)$

$$= \frac{1}{2^m} + \frac{n}{2^{m+1}} = \frac{2+n}{2^{m+1}}$$

42 **First Method** The rupee will be in the first purse if

(i) either it does not go from the first purse at all

or (ii) it goes to the second purse and comes back

The probability of case (i) $= \frac{{}^9C_9}{{}^{10}C_9}\times 1 = \frac{1}{10}$

and the probability of case (ii) $= \frac{{}^9C_8\times{}^1C_1}{{}^{10}C_9}\times\frac{{}^{18}C_8\times{}^1C_1}{{}^{19}C_9} = \frac{81}{190}$

Since the events in case (i) and (ii) are mutually exclusive,

the required probability $= \frac{1}{10} + \frac{81}{190} = \frac{100}{190} = \frac{10}{19}$

Second Method We find the probability of the event that the rupee goes to the second purse and does not return and subtract it from 1

Hence required probability $= 1 - \frac{{}^9C_8\times{}^1C_1}{{}^{10}C_9}\times\frac{{}^{18}C_9}{{}^{19}C_9}$

$$= 1 - \frac{9}{10}\times\frac{18!}{9!\,9!}\times\frac{9!\,10!}{19!} = 1 - \frac{9}{19} = \frac{10}{19}$$

or $[1-P(A)][1-P(B)]=\frac{1}{3}$

Put $P(A)=x$ and $P(B)=y$ Then

$xy=\frac{1}{6}$ and $1-x-y+xy=\frac{1}{3}$ whence eliminating y

$1-x-\frac{1}{6x}+\frac{1}{6}=\frac{1}{3}$ or $6x^2-5x+1=0$ $x=\frac{1}{2}, \frac{1}{3}$

Hence $P(A)=\frac{1}{3}$ or $\frac{1}{2}$

64 Let A be the event that two heads result and B the event that there is at least one head If S denote the sample space, then

$$S=\{(H, H), (H, T), (T, H), (T, T)\},$$
$$A=\{(H, H)\},$$
$$B=\{(H, H), (H, T), (T, H)\}$$

and so $A \cap B=\{(H, H)\}$,

$$P(B)=\frac{N(B)}{N(S)}=\frac{3}{4},$$

$$P(A \cap B)=\frac{N(A \cap B)}{N(S)}=\frac{1}{4}$$

Hence $P(A/B)=\frac{P(A \cap B)}{P(B)}=\frac{1/4}{3/4}=\frac{1}{3}$

65 Let S denote the sample space Then

$$S=\{(H, H)\ (H, T)\ (T, H), (T, T)\}$$

where the first letter of each pair refers to the outcome for the rupee, and the second to the eight anna piece We have

$A=\{(H, H), (H, T)\}$ so that $P(A)=\frac{1}{2}$

$B=\{(H, T), (T, H)\}$ so that $P(B)=\frac{1}{2}$

and $A \cap B=\{(H, T)\}$, so that $P(A \cap B)=\frac{1}{4}$

Since here $P(A \cap B)=P(A)\ P(B)$, the events A and B are independent

66 Here $P(A)=\frac{3}{5}$,

$$P(B)=P(B/A)+P(B/B)=\frac{2}{4}+\frac{1}{4}=\frac{3}{4}$$

Also $P(A \cap B)=\frac{3}{5}\cdot\frac{2}{4}=\frac{3}{10}$

and $P(A)\ P(B)=\frac{3}{5}\cdot\frac{3}{4}=\frac{9}{20}$

43 Let E_1 denote the event that first transferred ball is white and E_2 the event that it is black. Also let E denote the event consisting in the white colour of the ball drawn from the second urn. Clearly E_1, E_2 are mutually exclusive events. Hence

$$P(E)=P(E_1)\,P(E|E_1)+P(E_2)\,P(E|E_2) \quad (1)$$

Now $P(E_1)=\dfrac{a}{a+b}$ and $P(E_2)=\dfrac{b}{a+b}$

$P(E|E_1)=\dfrac{c+1}{c+d+1}$, because before the drawing there were $c+1$ white and d black balls in the second urn.

Similarly $P(E|E_2)=\dfrac{c}{c+d+1}$

Substituting in (1) we get

$$P(E)=\frac{a}{a+b}\cdot\frac{c+1}{c+d+1}+\frac{b}{a+b}\cdot\frac{c}{c+d+1}$$

$$=\frac{ac+bc+a}{(a+b)(c+d+1)}$$

44 (a) Let p denote the probability that a thing goes to a man and q the probability that it goes to a woman. Then

$$p=\frac{a}{a+b} \text{ and } q=\frac{b}{a+b}$$

Now the probabilities of 0, 1, 2, 3, ... things going to men are the first, second, third terms etc in the binomial expansion

$$(q+p)^n=q^n+{}^nC_1\,q^{n-1}p+{}^nC_2\,q^{n-2}p^2+\ldots+p^n \quad (1)$$

But men are to receive an odd number of things. Hence the required probability is the sum of even terms in (1).

To obtain the sum of even terms we write the expansion

$$(q-p)^n=q^n-{}^nC_1\,q^{n-1}p+{}^nC_2\,q^{n-2}p^2-\ldots+(-1)^n p^n \quad (2)$$

Subtracting (2) from (1), we get

$(q+p)^n-(q-p)^n=2\times$ sum of even terms in (1)

Hence required probability $=\frac{1}{2}[(q+p)^n-(q-p)^n]$

$$=\frac{1}{2}\left[1-\left(\frac{b-a}{b+a}\right)^n\right] \quad [\because q+p=1]$$

$$=\frac{1}{2}\,\frac{(b+a)^n-(b-a)^n}{(b+a)^n}$$

(b) Total no of ways $=N^r$

[Note That first biscuit can be given to any of the N beggars. Similarly second biscuit can be given to any of the N beggars and so on] Favourable no of ways is ${}^nC_r\,(N-1)^{n-r}$

Hence $P(A \cap B) \neq P(A)\, P(B)$

Therefore A and B are independent events

67 We have

$$P(A_1) = \tfrac{2}{4} = \tfrac{1}{2} = P(A_2) = P(A_3)$$

[Note that $P(A_1)$ is the probability of the event that the first digit is 1 and since there are two numbers having 1 at the first place out of four we have $P(A_1) = \tfrac{2}{4} = \tfrac{1}{2}$ Similarly for $P(A_2)$ and $P(A_3)$]

$A_1 \cap A_2$ is the event that the first two digits in the numbers drawn are each equal to 1 and so

$$P(A_1 \cap A_2) = \tfrac{1}{4} = \tfrac{1}{2}\ \tfrac{1}{2} = P(A_1)\ P(A_2)$$

Similarly

$P(A_2 \cap A_3) = P(A_2)\ P(A_3)$ and $P(A_3 \cap A_1) = P(A_3)\ P(A_1)$

Thus the events A_1, A_2 and A_3 are pairwise independent

Now $A_1 \cap A_2 \cap A_3$ is the event that all the three digits in the numbers are equal to 1 and since there is no such number, we have

$$P(A_1 \cap A_2 \cap A_3) = 0 \neq P(A_1)\ P(A_2)\ P(A_3)$$

Hence the events $A_1, A_2\ A_3$ are not mutually independent although they are pairwise independent

68 (a) Probability of not drawing an ace in the first n draws is ${}^{48}C_n / {}^{52}C_n$ and of drawing an ace in $(n+1)^{th}$ draw is ${}^4C_1 / {}^{(52-n)}C_1$

$$= \frac{4}{52-n}$$

Hence the required probability

$$= ({}^{48}C_n / {}^{52}C_n)\{4/(52-n)\}$$

$$= \frac{48\,!}{n\,!\ (48-n)\,!} \times \frac{n\,!\ (52-n)\,!}{52\,!} \times \frac{4}{52-n}$$

$$= \frac{4\ (51-n)\ (50-n)\ (49-n)}{52\ 51\ 50\ 49}$$

(b) We must have one ace in $(n-1)$ attempts and one ace in the n^{th} attempt The probability of one ace in first $(n-1)$ attempts is ${}^4C_1 \times {}^{48}C_{n-2} / {}^{52}C_{n-1}$ and of one ace in the nth attempt is ${}^3C_1 / \{52-(n-1)\} = \dfrac{3}{53-n}$ Hence the required prob-

Out of n biscuits, r biscuits can be given to a particular beggar in ${}^nC_r$ ways and the remaining $n-r$ biscuits can be distributed among the remaining $N-1$ beggars in $(N-1)^{n-r}$ ways. Hence, the required probability

$$=\frac{{}^nC_r\,(N-1)^{n-r}}{N^n}$$

45 Here total number of ways $n=2^4=16$ and favourable number of ways is m where

m=sum of coefficients of powers of x less than 15 in the expansion of $(x^3+x^5)^4$

Now $(x^3+x^5)^4=x^{12}(1+x^2)^4$

$$=x^{12}[1+4x^2+6x^4+4x^6+x^8]$$

m=sum of coefficients of x^{12} and $x^{14}=1+4=5$

Hence the required probability$=\dfrac{m}{n}=\dfrac{5}{16}$

The 'odds against' are as $16-5 : 5$ or $11 : 5$

46 Here total number of ways $n=6^3=216$. To find favourable number of ways, we have to find the sum of coefficients of powers of x less than 9 in the expansion $(x+x^2+x^3+x^4+x^5+x^6)^3$ and subtract this sum from 216

Now $(x+x^2+x^3+x^4+x^5+x^6)^3=x^3(1+x+x^2+x^3+x^4+x^5)^3$

$$=\frac{x^3(1+x+x^2+x^3+x^4+x^5)^3(1-x)^3}{(1-x)^3}$$

$$=x^3(1-x^6)^3(1-x)^{-3}$$

$$=x^3[1-3x^6+3x^{12}-x^{18}]$$

$$\times\left[1+3x+6x^2+\dots+\frac{(r+1)(r+2)}{2}x^r+\dots\right]$$

$$=[x^3-3x^9+3x^{15}-x^{21}]$$

$$\times\left[1+3x+6x^2+\dots+\frac{(r+1)(r+2)}{2}x^r+\dots\right] \quad (1)$$

It is evident from (1) that

Coeff^t of $x^8=1\times\dfrac{(5+1)(5+2)}{2}=21$

Coeff^t of $x^7=1\times\dfrac{(4+1)(4+2)}{2}=15$

Coeff^t of $x^6=1\times\dfrac{(3+1)(3+2)}{2}=10$

Coeff^t of $x^5=1\times\dfrac{(2+1)(2+2)}{2}=6$

$$\text{ability} = \frac{4 \cdot 48!}{(n-2)!\,(50-n)!} \times \frac{(n-1)!\,(53-n)!}{52!} \times \frac{3}{53-n}$$

$$= \frac{(n-1)(52-n)(51-n)}{50 \cdot 49 \cdot 17 \cdot 13}$$

(c) Proceed as in (b)

69 Let $P_r(B \cap C) = x$. Then we have

$$P_r(A \cup B \cup C) \geqslant 0.75$$

$$\Rightarrow P_r(A) + P_r(B) + P_r(C) - P_r(B \cap C) - P_r(C \cap A) - P_r(A \cap B) + P_r(A \cap B \cap C) \geqslant 0.75$$

$$\Rightarrow 0.3 + 0.4 + 0.8 - x - 0.28 - 0.08 + 0.09 \geqslant 0.75$$

$$\Rightarrow 0.48 - x \geqslant 0 \geqslant 0.48 \geqslant x \Rightarrow x \leqslant 0.48$$

Again similarly $P_r(A \cup B \cup C) \leqslant 1 \Rightarrow 0.23 - x \leqslant 0$

$$\Rightarrow 0.23 \leqslant x \qquad (2)$$

From (1) and (2) we get, $0.23 \leqslant x \leqslant 0.4$

Remark In our opinion some data in the question appears to be wrong since $P_r(A \cap B \cap C)$ cannot be greater than $P_r(A \cap B)$

(b) Let A denote the event that A draws a pair and B the event that B draws a pair. Then

$$P(A) = 1 \times \frac{^{18}C_1}{^{19}C_1} \times \frac{1}{^{18}C_1} \times 1 = \frac{1}{19}$$

Observe that at first draw A may draw any glove which is a certainty. Then B at the second draw, draws a glove out of the remaining 19 gloves, 18 of which are favourable since B must not draw the remaining one glove of the pair one of which was already drawn by A. Now 18 gloves are left. Since A has to draw a pair, at the third draw he must draw the remaining glove of the pair one of which was already drawn by him. At the fourth draw B may draw any glove. Note that we have to ensure only that A draws a pair, B may or may not draw a pair.

Arguing as before it is easy to see that

$$P(B) = 1 \times \frac{^{18}C_1}{^{19}C_1} \times \frac{^{1}C_1}{^{18}C_1} \times \frac{1}{^{1}C_1} = \frac{1}{19}$$

Thus $P(A) = P(B) = \frac{1}{19}$

This proves the first part

For the second part we have to find

Coeff' of $x^4=1\times\frac{(1+1)(1+2)}{2}=3$

and Coeff' of $x^3=1$

[Note that to obtain the coefficient of x^3, we put $r=5$ in the second bracket and multiply this coeff' of x^0 with the coefficient of x^3 (*i e* 1) in the first bracket etc]

Sum of these coefficients $=56$

Hence the favourable number of ways

$$m=216-56=160$$

Hence the required probability

$$=\frac{m}{n}=\frac{160}{216}=\frac{20}{27}$$

47 Total number of ways $n=6\times4=24$, because in a tetrahedron there are four faces and in a cube there are six faces To obtain the favourable number of ways we find the sum of coefficients of powers of x less than 5 in the expansion

$$(x+x^2+x^3+x^4)\ (x+x^2+x^3+x^4+x^5+x^6) \qquad (1)$$

and subtract this sum from 24

Now the coeff' of x^4 in (1) $=3$,

the coeff' of x^3 in (1)$=2$,

and coeff' of x^2 in (1)$=2$,

Sum of these coeff' $=6$

$m=$Favourable number of ways$=24-6=18$

The required probability$=\frac{m}{n}=\frac{18}{24}=\frac{3}{4}$

48 In a single throw with a pair of dice, the total number of ways is 36 The number of ways of throwing 6 is 5, and the number of ways of throwing 7 is 6 Hence if p_1 and p_2 denote the probabilities of throwing 6 and 7 respectively in a single throw with a pair of dice, then

$$p_1=\frac{5}{36} \text{ and } p_2=\frac{6}{36}=\frac{1}{6}$$

$$\overline{p_1}=1-\frac{5}{36}=\frac{31}{36} \text{ and } \overline{p_2}=1-\frac{1}{6}=\frac{5}{6}$$

Now A will win if he throws 6 at the first throw or at the third throw when A fails to throw 6 at the first and B fails to throw 7 at the second throw or at the 5^{th} throw when A fails twice to throw 6 and B fails twice to throw 8 and so on Hence A s chance of winning

$P\,(\overline{A} \cap B)$

But $P\,(\overline{A} \cap \overline{B}) = P\overline{[(A \cup B)]} = 1 - P\,(A \cup B)$

$= 1 - \{P\,(A) + P\,(B) - P\,(A \cap B)\}$ (1)

Now $P\,(A \cap B) =$ Prob that both A and B draw a pair

$$= 1 \times \frac{^{18}C_1}{^{19}C_1} \times \frac{1}{^{18}C_1} \times \frac{1}{^{17}C_1} = \frac{1}{19 \times 17} = \frac{1}{323}$$

Substituting in (1), we get

$$P(\overline{A} \cap \overline{B}) = 1 - \frac{1}{19} - \frac{1}{19} + \frac{1}{323} = \frac{290}{323}$$

70 Let $P\,(A)$ and $P\,(B)$ denote the percentage of city population who read newspapers A and B Then from given data, we have

$$P\,(A) = 25\% = \frac{1}{4},\ P(B) = 20\% = \frac{1}{5},$$

$$P\,(A \cap B) = 8\% = \frac{2}{25}$$

Percentage of those who read A but not B

$= P(A \cap \overline{B}) = P(A) - P(A \cap B)$ [see theorem 2 of § 3]

$$= \frac{1}{4} - \frac{2}{15} = \frac{17}{100} = 17\%$$

Similarly $P(\overline{A} \cap B) = P(B) - P\,(A \cap B) = \frac{1}{5} - \frac{2}{25} = \frac{3}{25} = 12\%$

If $P(C)$ denote the percentage of those who look into advertisements, then from the given data we obtain

$P(C) = 30\%$ of $P(A \cap \overline{B}) + 40\%$ of $P\,(\overline{A} \cap B) + 50\%$ of $P\,(A \cap B)$

$$= \frac{3}{10} \times \frac{17}{100} + \frac{2}{5} \times \frac{3}{25} + \frac{1}{2} \times \frac{2}{25} = \frac{51 + 48 + 40}{1000} = \frac{139}{1000}$$

$= 13.9\%$,

Thus the percentage of population who read an advertisement is 13.9%

71 (i) Let E_1 be the event that the ball is drawn from bag A, E_2 the event that it is drawn from bag B and E the event that the ball drawn is red

We have to find $P(E_2/E)$

Since both the bags are equally likely to be selected, we have

$$P(E_1) = P(E_2) = \tfrac{1}{2}$$

Also $P(E/E_1) = 3/5$ and $P(E/E_2) = ^5/9$

Hence by Baye's theorem, we have

$$P(E_2/E) = \frac{P(E_2)\,P(E/E_2)}{P(E_1)\,P(E/E_1) + P(E_2)\,P(E/E_2)}$$

$$= p_1 + \bar{p}_1 \bar{p}_2 p_1 + (\bar{p}_1)^2 (\bar{p}_2)^2 p_1 + \dots$$

$$= \frac{5}{36} + \frac{31}{36}\cdot\frac{5}{6}\cdot\frac{5}{36} + \left(\frac{31}{36}\right)^2\left(\frac{5}{6}\right)^2\frac{5}{36} + \dots$$

$$= \frac{\frac{5}{36}}{1 - \frac{31}{36}\cdot\frac{5}{6}} = \frac{5}{36}\times\frac{216}{61} = \frac{30}{61}$$

49. The probability of getting a double six in one throw with two dice $= \frac{1}{6}\times\frac{1}{6} = \frac{1}{36}$

The probability of not throwing a double six

$$= 1 - \frac{1}{36} = \frac{35}{36}$$

Let $p = \frac{1}{36}$ and $q = \frac{35}{36}$

The probability of not throwing a double six in any of the n throws $= q^n$

Hence probability of throwing a double six at least once in n throws $= 1 - q^n = 1 - \left(\frac{35}{36}\right)^n$

Now according to the question,

$$1 - \left(\frac{35}{36}\right)^n > 0.99$$

or $\left(\frac{35}{36}\right)^n < 0.01$ (1)

Since both sides of (1) are +ive the inequality will not be affected by taking logarithm to the base 10 (which is greater than 1)

or $n\,[\log_{10} 35 - \log_{10} 36] < \log_{10} 0.01$

or $n\,[1.5441 - 1.5563] < -2$

or $-0.0122\, n < -2$

or $0.0122\, n > 2$

or $n > \frac{2}{0.0122} = 163.9$

The least value of n is 164

50. Here $p = \frac{90}{100} = \frac{9}{10}$ and $q = 1 - \frac{9}{10} = \frac{1}{10}$ and $n = 10$

Let N be the number of customers to whom packets are sold. The frequency that success will be less than 90% is equal to

$$=\frac{\frac{1}{2}\cdot\frac{5}{9}}{\frac{1}{2}\cdot\frac{3}{5}+\frac{1}{2}\cdot\frac{5}{9}}=\frac{25}{52}$$

Note If in the problem, you read 'is found to be' then it is the problem based on Baye's Theorem

(ii) Same type as (i), Ans $\frac{2}{5}$

72 Let A_i $(i=1, 2, 3)$ be the event that ith urn is chosen and B the event that a white ball is drawn

Since all the urns are equally likely to be selected, we have

$$P(A_1)=P(A_2)=P(A_3)=\frac{1}{3} \text{ be}$$

and $P(B/A_1)=\frac{2}{5}$, $P(B/A_2)=\frac{3}{5}$, $P(B/A_3)=\frac{4}{5}$

Hence $P(B)=P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)$
$+P(A_3)\,P(B/A_3)$

$$=\frac{1}{3}\cdot\frac{2}{5}+\frac{1}{3}\cdot\frac{3}{5}+\frac{1}{3}\cdot\frac{4}{5}=\frac{9}{15}=\frac{3}{5}$$

(b) Here we have to find $P(A_1/B)$

By Baye's theorem

$$P(A_1/B)=\frac{P(A_1)\,P(B/A_1)}{P(A_1)\,P(B/A_1)+P(A_2)\,P(B/A_2)+P(A_3)\,P(B/A_3)}$$

$$=\frac{\frac{1}{3}\cdot\frac{2}{5}}{\frac{3}{5}} \text{ from part } (a)$$

$$=\frac{2}{9}$$

73 Let A_i $(i=1, 2, 3, 4)$ be the event that the urn contains 2, 3, 4 or 5 white balls and B the event that two white balls are drawn

We have to find $P(A_1/B)$

Since the four events A_1, A_2, A_3, A_4 are equally likely, we have

$$P(A_1)=P(A_2)=P(A_3)=P(A_4)=\frac{1}{4}$$

$P(B/A_1)$ is the probability of event that the urn contains 2 white balls and both have been drawn

Hence $P(B/A_1)=\frac{{}^2C_2}{{}^5C_2}=\frac{1}{10}$

Similarly $P(B/A_2)=\frac{{}^3C_2}{{}^5C_2}=\frac{3}{10}$

$$P(B/A_3)=\frac{{}^4C_2}{{}^5C_2}=\frac{6}{10}=\frac{3}{5}$$

$N({}^{10}C\ p^8q^2+\ \ +{}^{10}C_9pq^9+q^{10})$
$=N(p+q)^{10}-N(p^{10}+{}^{10}C_1p^9q)$
$=N[1-p^{10}-10p^9q]=N[1-(9)^{10}-10(9)^9(1)]$
$=N[1\quad 3483\quad 3871]$
$=2646\ N=\frac{1}{4}N$ nearly

51 The probability of drawing exactly r white balls in n trials is ${}^nC_r\,p^r q^{n-r}$ and it is required to find for what value of r this expression is greatest

Now ${}^nC_r\,p^r q^{n-r} > {}^nC_{r-1}\,p^{r-1}q^{n-(r-1)}$

so long as $\frac{n!}{r!(n-r)!}p > \frac{n!}{(r-1)!(n-r+1)!}q$

or $(n-r+1)p > rq$

or $(n+1)p > (p+q)r$

or $np+p > r\quad [\ \ p+q=1]$

or $r < np+p$

Hence the required value of r is the greatest integer in $np+p$

If n is such that np is an integer, the most likely case is that of np successes and nq failures

For the numerical part, we have $p=\frac{1}{3}$ and $n=12$

In this case, $r < 12\times\frac{1}{3}+\frac{1}{3}$ or $r < 4+\frac{1}{3}$

Hence 4 white balls are most likely to be drawn

52 In the last five throws there can be 0, 1 2 3, 4 or 5 heads and the same should be the case in the first ten throws

Hence the favourable number of cases

$m=C_0\,{}^{10}C_0+C_1\,{}^{10}C_1+C_2\,{}^{10}C_2+C_3\,{}^{10}C_3+C_4\,{}^{10}C_4+C_5\,{}^{10}C_5$
$=1+50+450+1200+1050+252$
$=3003$

And the total number of ways $n=2^{15}=32768$

Hence the required probability $=\frac{m}{n}=\frac{3003}{32768}$

53 In the first throw a number less than 4 can come as (1 1), (1, 2), (2, 1) that is, in 3 ways and so 4 or more can come in 36 3 i e in 33 ways since two dice can come up in 36 ways

Hence probability of this case $=\frac{33}{36}=\frac{11}{12}$

In the second case three dice are thrown In this case, the total of 3, 4 or 5 can come as

(1, 1, 1), (2, 1 1) (1, 2 1) (1 1 2) (1, 1, 3), (3, 1, 1) (1 3, 1) (1 2, 2) (2 1 2) and (2, 2, 1) i e in 10 ways

Hence the number 6 or more can come in 216 -10=206 ways and $\qquad P(B/A_4) = \frac{C}{C} = 1$

By Baye's theorem

$$P(A_1/B) = \frac{P(A_1)\,P(B/A_1)}{\sum_{i=1}^{4} P(A_i)\,P(B/A_i)} = \frac{\frac{1}{4}\cdot 1}{\frac{1}{4}\left(\frac{1}{10}+\frac{3}{10}+\frac{3}{5}+1\right)} = \frac{1}{2}$$

74 Do yourself

Ans $\frac{25}{69}, \frac{28}{69}, \frac{16}{69}$

75 Let A_1 be the event that the lot contains 2 defective articles and A_2 the event the lot contains 3 defective articles. Also let A be the event that the testing procedure ends at the twelfth testing. Then according to the question

$$P(A_1) = 0.4 \text{ and } P(A_2) = 0.6$$

Since $0 < P(A_1) < 1$, $0 < P(A_2) < 1$ and $P(A_1) + P(A_2) = 1$, the events A_1, A_2 form a partition of the sample space. Hence by the theorem of total probability for compound events we have

$$P(A) = P(A_1)\,P(A/A_1) + P(A_2)\,P(A/A_2) \qquad (1)$$

But $P(A/A_1) = \left(\frac{{}^{18}C_{10} \times {}^{2}C_1}{{}^{20}C_{11}}\right) \times \left(\frac{1}{9}\right) \qquad (2)$

$$= \frac{2}{9} \times \frac{18!}{10!\,8!} \times \frac{11!\times 9!}{20!} = \frac{11}{190}$$

[Note that out of 20 articles, first 11 draws must contain 10 non defective and 1 defective article and the 12^{th} draw must give a defective article. *Probabilities of these two events are shown in separate parantheses in (2)*]

Similarly $P(A/A_2) = \frac{{}^{17}C_9 \times {}^{3}C_2}{{}^{20}C_{11}} \times \frac{1}{9}$

$$= \frac{1}{3} \times \frac{17!}{9!\,8!} \times \frac{11!\,9!}{20!} = \frac{11}{228}$$

Now substituting the values of $P(A_1)$, $P(A_2)$, $P(A/A_1)$ and $P(A/A_2)$ in (1), we get

$$P(A) = 0.4 \times \frac{11}{190} + 0.6 \times \frac{11}{228}$$

$$= \frac{11}{475} + \frac{11}{380} = \frac{99}{1900}$$

76 Let A_i denote the event that the i^{th} letter is placed in the right envelope. Then the required probability is

Hence the probability of this case $=\frac{206}{216}=\frac{103}{108}$

The required probability $=\frac{11}{12}\times\frac{103}{108}=\frac{1133}{1296}$

54 Total number of ways $=3^4=81$

Favourable number of ways is the sum of the coefficients of $x^2, x^4, x^6, x^8, x^{10}$ and x^{12} in the expansion of $(x+x^2+x^3)^4$

Now $(x+x^2+x^3)^4=x^4(1+x+x^2)^4$

$=x^4[1+4(x+x^2)+6(x+x^2)^2+4(x+x^2)^3+(x+x^2)^4]$

Coeff^t of $x^2=0$, coeff^t of $x^4=1$, coeff^t of $x^6=10$, coeff^t of $x^8=6+12+1=19$ coeff^t of $x^{10}=4+6=10$ and coeff of $x^{12}=1$ Hence the favourable number of ways $=0+1+10+19+10+1=41$

The required probability $=\frac{41}{81}$ and so the odds in favour are as 41 : 81−41 or 41 : 40

55 We have to find the conditional probability of obtaining a sum of 15 when the first throw was a four

Let A be the event that the sum of the three numbers thrown is 15 and B the event that the first throw was a four So we have to find $P(A/B)$

But $P(A/B)=\frac{n(A\cap B)}{n(B)}$ [See (1) of § 4]

where $n(A\cap B)$ and $n(B)$ denote the number of points in $A\cap B$ and B respectively

Now $n(B)=36$, since first throw being a four, the other two throws can occur in 6×6 *i e* 36 ways There are *only two throws of three dice beginning with a four and giving a total of 15 namely*, (4, 5 6), (4 6 5)

Hence $n(A\cap B)=2$

$P(A/B)=\frac{2}{36}=\frac{1}{18}$

56 We place the two lines parallel to one another Suppose that after cutting the lines the right hand portions are removed Then the problem is equivalent to asking ' what is the chance that the sum of the left hand portions is less than the sum of right hand portions" It is equally likely that the first sum is equally likely to be less or greater than the second

Hence the required probability is $\frac{1}{2}$

$$P(\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A_n}) = P\,\overline{(A_1 \cup A_2 \cup \ldots A_n)},$$

[By De Morgan law]

$$= 1 - P(A_1 \cup A_2 \cup \ldots \cup A_n)$$

$$= 1 - [\Sigma P(A) - \sum_{i \neq j} P(A_i \cap A_j) + \sum_{i \neq j \neq k} P(A_i \cap A_j \cap A_k) - \ldots + (-1)^{n-1} P(A_1 \cap A_2 \cap \ldots \cap A_n)]$$

Now $P(A_i) = \frac{(n-1)!}{n!}$ as having placed i^{th} letter in the right envelope, the remaining letters can be placed in $(n-1)!$ ways. Similarly $P(A_1 \cap A_2 \cap \ldots \cap A_r)$

=Prob. of r particular letters in right envelopes

$$= \frac{(n-r)!}{n!}$$

$$\Sigma P(A_1 \cap A_2 \cap \ldots \cap A_r) = {}^nC_r \frac{(n-r)!}{n!} = \frac{1}{r!}$$

where $r = 1, 2, 3, \ldots, n$

$$\Sigma (\overline{A_1} \cap \overline{A_2} \cap \ldots \cap \overline{A^n}) = 1 - \left\{\frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \ldots + (-1)^{n-1} \frac{1}{n!}\right\}$$

$= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \ldots + (-1)^n \frac{1}{n!}$ which is equal to first $n-2$ terms in the expansion of e^{-1}

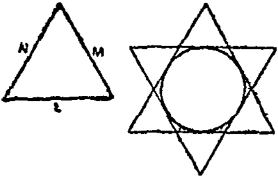
Objective Questions on Probability

Problem Set (B)

1. The probability of getting heads in both trials, when a balanced coin is tossed twice, will be
   (i) $\frac{1}{4}$ (ii) $\frac{1}{2}$ (iii) 1 (iv) $\frac{3}{4}$
2. Two cards are drawn at random from a pack of 52 cards. The probability of these two being aces is
   (a) $\frac{1}{26}$ (b) $\frac{1}{221}$ (c) $\frac{1}{2}$ (d) none of these
3. A and B throw with 2 dice, if A throws 9, then B's chance of throwing a higher number is $\frac{1}{6}$
   (a) True (b) False
4. Two cards are drawn successively with replacement from a well shuffled deck of 52 cards. The probability of drawing two aces is

57 We draw three lines L, M, N at random in the same plane as the circle and then draw the six tangents parallel to these lines (see the figure)

Then the total number of triangles so formed is 8 Out of these eight triangles, the circle is described to 6 and inscribed in 2 and this is true whatever be the original direction of L, M, N Hence the odds against the circle being inscribed in the triangle formed by the lines are as 6 2 or 3 1 as required



58 We first find the probability that no two persons have the same birthday and then subtract the result from 1 Excluding leap years, there are 365 different birthdays possible Any person might have any one of the 365 days of the year as a birthday A second person may likewise have any one of 365 birthdays, and so on Hence the total number of ways is given by

$$n=365^N$$

And the number of possible ways for none of the N birthdays to coincide is

$$m=365\ 364 \quad (365-\overline{N-1})$$
$$=\frac{365\,!}{(365-N)\,!}$$

The probability that no two birthdays coincide is

$$\frac{m}{n}=\frac{365\,!/(365-N)\,!}{365^N}$$

Hence the required probability

$$=1-\frac{m}{n}=1-\frac{365\,!/(365-N)\,!}{365^N}$$

59 Suppose A wins in exactly $m+r$ games To do so he must win the last game and $m-1$ out of the preceeding $m+r-1$ games

(a) $\frac{1}{13}\times\frac{1}{13}$ (b)

(c) $\frac{1}{52}\times\frac{1}{51}$ (d) $\overline{13}\wedge\overline{51}$ (M N R 88)

5 A person draws a card from a pack of playing cards, replaces it and shuffles the pack He continues doing this until he shows a spade The chance that he will fail the first two times is

(a) $\frac{9}{64}$ (b) $\frac{1}{64}$ (c) $\frac{1}{16}$ (d) $\frac{9}{16}$

6 If A and B are independent events, then $P(A \cap B)$ equals
(i) $P(A)+P(B)$ (ii) $P(A)\,P(B)$ (iii) $P(A/B)$ (iv) $P(B/A)$

7 If A and B are mutually exclusive events, then $P(A \cap B)$ equals
(i) 0 (ii) $\frac{1}{2}$ (iii) 1 (iv) $\frac{1}{4}$

8 Two mutually exclusive events are always independent
(a) True (b) false

9 Two independent events are always mutually exclusive
(a) True (b) false

10 Three letters are written to different persons, and addresses on the three envelopes are also written Without looking at the addresses, the probability that the letters go into right envelopes is

(a) $\frac{1}{27}$ (b) $\frac{1}{6}$ (c) $\frac{1}{9}$ (d) none of these

11 The probability that a students is not a swimmer is $\frac{1}{5}$ Then the probability that out of five students, four are swimmers is

(i) ${}^5c_4 \left(\frac{4}{5}\right)^4 \frac{1}{5}$ (ii) $\left(\frac{4}{5}\right)^4 \frac{1}{5}$

(iii) ${}^5c_1 \frac{1}{5}\left(\frac{4}{5}\right)^4$ (iv) none of these

12 The chance of throwing an ace in the first only of two successive throws with an ordinary die is

(i) $\frac{1}{36}$ (ii) $\frac{5}{36}$ (iii) $\frac{25}{36}$ (iv) $\frac{1}{6}$

13 The probability that a certain beginner at golf gets a good shot if he uses the correct club is $\frac{1}{3}$, and the probability of a good shot with an incorrect club is $\frac{1}{4}$ In his bag are 5 diffe

(*b*) PQ is $\perp$ to AP and DP where $h = AP \tan 60°$, $h = DP \tan 30°$

$$AP = h/\sqrt{3},\ DP = h\sqrt{3}$$

Now from rt angled triangle ADP we have

$$DP^2 = AD^2 + AP^2$$

$$(h\sqrt{3})^2 = 4^2 + (h/\sqrt{3})^2$$

or $\quad h^2 (3 - \tfrac{1}{3}) = 16$

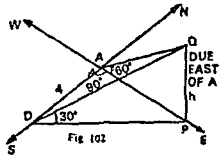


Fig 102

$$h^2 = 6 \quad \text{or} \quad h = \sqrt{6}\ \text{km}$$

(*c*) $PQ = h$ is the lamp post, AL the man due south of it and of height 6 ft and $AC = 24$ ft is the lenght of the shadow After walking 300 ft eastwards he comes to B and now the length of shadow is $BD = 30$ ft

Now let $AP = x$ and $BP = y$, $AC = 24$, $BD = 30$,

From $\triangle$s QPC and LAC

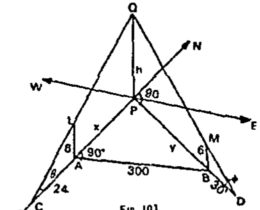


Fig 103

we have $\quad \dfrac{h}{6} = \dfrac{24+x}{24}$

$$x = 4\,(h-6) \qquad (1)$$

Similarly from $\triangle$s QPD and MBD

$$\frac{h}{6} = \frac{y+30}{30}$$

$$y = 5\,(h-6) \qquad (2)$$

From (1) and (2), we get $\dfrac{24+x}{4} = \dfrac{y+30}{5} \qquad y = \dfrac{5}{4}x \qquad (3)$

Again from rt angled triangle PAB in which $\angle PAB = 90°$

$PB^2 = PA^2 + AB^2$ or $y^2 - x^2 = (300)^2 \qquad (3)$

Putting for y and x from (1) and (2) in (3), we get

$(h-6)^2\,[25 - 16] = 300^2 = 9\ 100^2 \qquad h - 6 = 100$

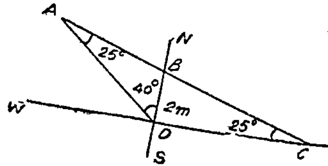
or $\quad h = 6 + 100 = 106$ ft

40 The figure is self explanatory In $\triangle ADC$, we have

$\angle ADC = 40° + 90° = 130°$

and $\angle DCA = 25°$

$\therefore \angle DAC = 180° - (130° + 25°) = 25°$



Hence $AD=DC=2\cot 25=2\dfrac{\sqrt{(1-\sin^2 25^\circ)}}{\sin 25^\circ}$

$$=2\frac{\sqrt{(1-0\ 423^2)}}{0\ 423}=\frac{2\sqrt{(\ 82107)}}{0\ 423}$$

$$=\frac{2\times\ 906}{0\ 423}=\frac{1\ 812}{0\ 423}=4\ 283 \text{ km}$$

41 Do yourself

42 It is the same question as Q 38

where $AB=a$, $\alpha=30^\circ$ and $\beta=18^\circ$

$$h=\frac{AB\sin\alpha\sin\beta}{\sqrt{\{(\sin^2\alpha-\sin^2\beta)\}}}$$

$\sin\alpha=\sin 30^\circ=\dfrac{1}{2}$ $\quad\sin\beta=\sin 18^\circ=\dfrac{\sqrt{5}-1}{4}$,

$$\sin^2\alpha-\sin^2\beta=\frac{1}{4}-\frac{6-2\sqrt{5}}{16}=\frac{\sqrt{5}-1}{8},$$

and $\quad\sin\alpha\sin\beta=\dfrac{\sqrt{5}-1}{8}$

$$h=a\frac{\sqrt{5}-1}{8}\sqrt{\left(\frac{8}{\sqrt{5}-1}\right)}=a\sqrt{\left(\frac{\sqrt{5}-1}{8}\right)}$$

$$=a\sqrt{\left(\frac{\sqrt{5}-1}{8}\ \frac{\sqrt{5}+1}{\sqrt{5}+1}\right)}=a\sqrt{\left(\frac{4}{8(\sqrt{5}+1)}\right)}$$

$$=a\sqrt{\left(\frac{1}{(2\sqrt{5}+2)}\right)}=\frac{a}{\sqrt{\{(2+2\sqrt{5})\}}}$$

43 Proceed as in Q 38

$$h=\frac{AB}{\sqrt{(\cot^2\beta-\cot^2\alpha)}}=\frac{32}{\sqrt{\left(\frac{9}{25}-\frac{4}{25}\right)}}=32\sqrt{5}$$

44 Proceed as in Q 38

45 (a) PQ is perpendicular to each of the lines QA, QC and QB

rent clubs only one of which is correct for the shot in question If he chooses a club at random and takes a stroke, the probability that he gets a good shot is

(i) $\frac{1}{3}$ (ii) $\frac{1}{12}$ (iii) $\frac{4}{15}$ (iv) $\frac{7}{12}$

14 The probability that in the toss of two dice we obtain the sum 7 or 11 is

(i) $\frac{1}{6}$ (ii) $\frac{1}{18}$ (iii) $\frac{2}{9}$ (iv) $\frac{23}{108}$

15 The probability that in the toss of two dice we obtain an even sum or a sum less than 5 is

(i) $\frac{1}{2}$ (ii) $\frac{1}{6}$ (iii) $\frac{2}{3}$ (iv) $\frac{5}{9}$

16 One of the two events must occur If the chance of one is $\frac{2}{3}$ of the other then odds in favour of the other are

(a) 1 : 3 (b) 3 : 1 (c) 2 : 3 (d) none of these

17 An ordinary cube has four blank faces, one face marked 2 another marked 3 Then the probability of obtaining a total of exactly 12 in 5 throws is

(a) $\frac{5}{1296}$ (b) $\frac{5}{1944}$ (c) $\frac{5}{2592}$ (d) none of these

18 The probability that a marksman will hit a target is given as 1/5 Then his probability of at least one of hit in 10 shots is

(a) $1-\left(\frac{4}{5}\right)^{10}$ (b) $\frac{1}{5^{10}}$ (c) $1-\frac{1}{5^{10}}$ (d) none of these

19. The odds in favour of A winning a game of chess against B are 5 : 2 If three games are to be played, then the odds in favour of A's winning at least one game are 355 : 8

(a) True (b) False

20 A cricket club has 15 members of whom only 5 can bowl If the names of the 15 members are put into a hat and 11 drawn random, then the chance of obtaining an eleven containing at least 3 bowlers is

(a) $\frac{7}{13}$, (b) $\frac{11}{15}$ (c) $\frac{12}{13}$ (d) none of these

21 On a toss of two dice A throws a total of 5 Then the probability that he will throw another 5 before he throws 7 is

that is, $1-P$ is the probability that A will throw more tails than B

[Reason $\lambda \leqslant \mu \Rightarrow n+1-\lambda' \leqslant n-\mu'$

$\Rightarrow 1-\lambda' \leqslant -\mu' \Rightarrow \lambda'-1 \geqslant \mu'$

$\Rightarrow \lambda' \geqslant \mu'+1 > \mu'$]

By reason of symmetry $1-P=P$ or $P=\frac{1}{2}$

34 Ans $\frac{3}{16}$

n=total no of ways$=2^4=16$, since each of the four places in a determinant of order 2 can be filled in two ways by 0 or 1

m=favourable no of ways$=3$ since the value of the determinant is positive when it is

$$\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \text{ or } \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \text{ or } \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix},$$

Hence $p=\frac{m}{n}=\frac{3}{16}$

35 Ans $\frac{5}{21}$

Let A be the event that face 1 turns up and B the event that face 2 turns up

Then $P(A)=0\ 10$ and $P(B)=0\ 32$

Since A, B are mutually exclusive, we have

$P(A \cup B)=P(A)+P(B)=0\ 10+0\ 32=0\ 42$

We are to find $P(A/A \cup B)$

But $P(A/A \cup B)=\frac{P[A \cap (A \cup B)]}{P(A \cup B)}=\frac{P(A)}{P(A \cup B)}$

Hence $P(A/A \cup B)=\frac{0\ 10}{0\ 42}=\frac{5}{21}$

36 Ans (B)

Hint Here $n=6^3=216$ and $m=6$

37 Ans (d)

Required prob $=\frac{1}{2}\ \frac{4}{7}+\frac{1}{2}\ \frac{5}{8}=\frac{27}{56}$

38 Ans (C)

Hint On trial, $n=15$ since any of the 15 numbers can be on the selected coin and $m=9$ since the largest, number is 9 and so it can be 1 or 2 or 3, or 9

Since seven trials are performed with replacement, we have the required probability$=(\frac{9}{15})=(\frac{3}{5})^7$

(a) $\frac{1}{9}$, (b) $\frac{1}{6}$, (c) $\frac{2}{5}$, (d) $\frac{5}{36}$

22 (a) If A and B are such events that $P(A) > 0$ and $P(B) \neq 1$, then $P(\overline{A}/\overline{B})$ is equal to

(i) $1-P(A/B)$, (ii) $1-P(\overline{A}/B)$, (iii) $\frac{1-P(A \cup B)}{P(\overline{B})}$

(iv) $\frac{P(\overline{A})}{P(\overline{B})}$ **(I I T 1982)**

(b) $P(A \cup B) = P(A \cap B)$ if and only if the relation between $P(A)$ and $P(B)$ is **(I I T 1985)**

23 A number is chosen at random among the first 120 natural numbers The probability of the number chosen being a multiple of 5 or 15 is

(a) $\frac{1}{5}$, (b) $\frac{1}{8}$, (c) $\frac{1}{6}$ (d) none of these

24 The probability of an event A occuring is 0 5 and of B occuring is 0 3 If A and B are mutually exclusive events, then the probability of neither A nor B occuring is

(a) 0 6, (b) 0 5, (c) 0 7, (d) none of these

25 Two events A and B have probabilities 0 25 and 0 50 respectively The probability that both A and B occur simultaneously is 0 14 Then the probability that neither A nor B occurs is

(a) 0 39, (b) 0 25 (c) 0 11, (d) none of these

(I I T 1980)

26 The probability that an event A happens in one trial of an experiment is 0 4 Three independent trials of the experiment are formed The probability that the event A happens at least once is

(a) 0 936, (b) 0 784, (c) 0 904 (d) none of these

(I I T 1980)

27 If A and B are two events such that $P(A \cup B) = 5/6$, $P(A \cap B) = \frac{1}{3}$ $P(\overline{B}) = \frac{1}{2}$, then then the events A and B are

(i) dependent, (ii) independent, (iii) mutually exclusive (iv) none of these

28 A and B each throw a die Then it is 7 5 that A's throw is not greater than B's

39 Ans (A), (C), (D)

The required probability

$=$ prob that M occurs and N does not occur or N occurs and M does not occur

$=P(M \cap N^c)+P(M^c \cap N)$ [This is (D)]

$=P(M)-P(M \cap N)+P(N)-P(M \cap N)$ [Theorem 2, § 3]

$=P(M)+P(N)-2P(M \cap N)$

[This is (A)]

$=1-P(M^c)+1-P(N^c)-2\{1-P(M \cap N)^c\}$

[By Theorem 3, § 3]

$=2P(M^c \cup N^c)-P(M^c)-P(N^c)$

[By De Morgan law]

$=2[P(M^c)+P(N^c)-P(M^c \cap N^c)]-P(M^c)-P(N^c)$

$=P(M^c)+P(N^c)-2P(M^c \cap N^c)$

[This is (C)]

40 Let A be the event that the maximum number on the two chosen tickets is not more than 10, that is, the no on them $\leqslant 10$ and B the event that the minimum no on them is 5 that is the no on them is $\geqslant 5$. We have to find $P(B/A)$

Now $P(B/A)=\dfrac{P(A \cap B)}{P(A)}=\dfrac{n(A \cap B)}{n(A)}$

Now the number of ways of getting a number r on the two tickets is the coeff^t of x^r in the expansion of

$(x^1+x^2+\ldots+x^{100})^2=x^2(1+x+\ldots+x^{99})^2$

$=x^2\left(\dfrac{1-x^{100}}{1-x}\right)^2=x^2(1-2x^{100}+x^{200})(1-x)^{-2}$

$=x^2(1-2x^{100}+x^{200})(1+2x+3x^2+\ldots+(r+1)x^r+\ldots)$

Thus coeff^t of $x^2=1$, of $x^3=2$, of $x^4=3$, ..., of x^{10} is 9

Hence $n(A)=1+2+3+4+5+6+7+8+9=45$

and $n(A \cap B)=4+5+6+7+8+9=39$

[Note that in finding $n(A)$ we have to add the coeff^s of $x^2, x^3, \ldots, x^{10}$ and in $n(A \cap B)$ we add the coefficients of $x^5, x^6, \ldots, x^{10}$]

Hence required prob $P(B/A)=\frac{39}{45}=\frac{13}{15}$

41 Ans (a) 42 Ans (c)

42 Ans (c)

Let A, B, C denote the events of passing the tests I, II and III respectively. Evidently A, B, C are independent events. Now according to the questions

(a) True (b) False

29 Two persons each make a single throw with a die The probability they get equal value is P_1, Four persons each make a single throw and probability of three being equal is P_2 Then

(a) $P_1=P_2$, (b) $P_1 < P_2$, (c) $P_1 > P_2$

30 A bag has 13 red, 14 green and 15 black balls The probability of getting exactly 2 blacks on pulling out 4 balls is P_1 Now the number of each colour ball is doubled and 8 balls are pulled out The probability of getting exactly blacks is P_2 Then

(a) $P_1=P_2$, (b) $P_1 > P_2$, (c) $P_1 < P_2$

31 If A and B are arbitrary events, then

(i) $P(A \cap B) \geqslant P(A)+P(B)-1$

(ii) $P(A \cap B) \leqslant P(A)+P(B)-1$

(iii) $P(A \cap B)=P(A)+P(B)-1$

32 (a) If A and B are arbitrary events, then

(i) $P(A \cap B) \geqslant P(A)+P(B)$,

(ii) $P(A \cap B) \leqslant P(A)+P(B)$,

(iii) $P(A \cap B)=P(A)+P(B)$

(b) For two events A and B, $P(A \cap B)$ is

(i) not less than $P(A)+P(B)-1$

(ii) not greater than $P(A)+P(B)$

(iii) equal to $P(A)+P(B)-P(A \cup B)$

(iv) equal to $P(A)+P(B)+P(A \cup B)$ (I I T 1988)

33 Two persons, A and B, have respectively $n+1$ and n coins, which they toss simultaneously Then the probability that A will have more heads than B is

(i) $\frac{1}{2}$, (ii) $> \frac{1}{2}$ (iii) $< \frac{1}{2}$

Fill in the blanks in the following

34 A determinant is chosen at random from the set of all determinants of order 2 with elements 0 or 1 only The probability that value of the determinant chosen is positive is (I I T 1982)

35 For a biased die the probabilities for different faces to turn up are given below

Face	1	2	3	4	5	6
Probability	0 1	0 32	0 21	0 15	0 05	0 17

$\frac{1}{2}=P[(A \cap B) \cup (A \cap C)]$

$=P(A \cap B)+P(A \cap C)-P(A \cap B \cap C)$

$=P(A)\,P(B)+P(A)\,P(C)\quad P(A)\,P(B)\,P(C)$

$=pq+p\,\frac{1}{2}-pq\,\frac{1}{2}$

or $1=2pq+p-pq=p\,(q+1)$ (1)

The equality (1) is satisfied for infinite no of values of p and q For, if we take any value of q such that $0 \leqslant q \leqslant 1$, then p takes the value $\frac{1}{q+1}$ It is evident that $0<\frac{1}{q+1} \leqslant 1$

i e $0<p \leqslant 1$

44 Ans $\frac{1}{3} \leqslant p \leqslant \frac{1}{2}$

Let A, B, C denote the mutually exclusive events so that $P(A)=\frac{1+3p}{3}$, $P(B)=\frac{1-p}{4}$, $P(C)=\frac{1-2p}{2}$ Since A B, C are mutually exclusive, we have

$0 \leqslant P(A)+P(B)+P(C)=P(A \cup B \cup C) \leqslant 1$

or $0 \leqslant \frac{1+3p}{3}+\frac{1-p}{4}+\frac{1-2p}{2} \leqslant 1$

or $0 \leqslant 13-3p \leqslant 12$ or $-13 \leqslant -3p \leqslant -1$

or $13 \geqslant 3p \geqslant 1$ or $\frac{1}{3} \leqslant p \leqslant \frac{13}{3}$ (1)

Also $0 \leqslant P(A) \leqslant 1 \Rightarrow 0 \leqslant \frac{1+3p}{3} \leqslant 1$

$\Rightarrow 0 \leqslant 1+3p \leqslant 3 \Rightarrow -1 \leqslant 3p \leqslant 2$

$\Rightarrow -\frac{1}{3} \leqslant p \leqslant \frac{2}{3}$

But $p \geqslant 0$, hence $0 \leqslant p \leqslant \frac{2}{3}$ (2)

Again $0 \leqslant P(B) \leqslant 1 \Rightarrow 0 \leqslant \frac{1-p}{4} \leqslant 1$

$\Rightarrow 0 \leqslant 1-p \leqslant 4 \Rightarrow -1 \leqslant -p \leqslant 3$

$\Rightarrow 1 \geqslant p \geqslant -3$

$0 \leqslant p \leqslant 1 \quad (\quad p \geqslant 0)$ (3)

Similarly from $0 \leqslant P(C) \leqslant 1$, we shall easily get

$0 \leqslant p \leqslant \frac{1}{2}$ (4)

Now the set of values of p which satisfy all the inqualities from (1) to (4) is given by $\frac{1}{3} \leqslant p \leqslant \frac{1}{2}$

45 Ans (D) we have

$^{100}C_{50}\,p^{50}\,(1-p)^{50}={}^{100}C_{51}\,p^{51}\,(1-p)^{49}$

The die is tossed and you are told that either face 1 or 2 has turned up Then the probability that it is face 1 is

(I I T 1981)

36 Three identical dice are rolled The probability that the same number will appear on each of them is

(a) $\frac{1}{6}$, (b) $\frac{1}{36}$, (c) $\frac{1}{18}$ (d) $\frac{3}{28}$ (I I T 1984)

37 A purse contains 4 copper coins 3 silver coins, the second purse contains 6 copper coins and 2 silver coins A coin is taken out of any purse, the probability that it is a copper coin is

(a) $\frac{4}{7}$, (b) $\frac{3}{4}$, (c) $\frac{3}{7}$, (d) $\frac{37}{56}$

(M N R. 1984)

38 Fifteen coupons are numbered 1, 2, , 15 respectively Seven coupons are selected at random one at a time with replacement *The probability that the largest number appearing on* a selected coin is 9, is

(a) $\left(\frac{9}{16}\right)^6$, (b) $\left(\frac{8}{15}\right)^7$, (c) $\left(\frac{3}{5}\right)^7$, (d) none of these (I I T 1983)

39 If M and N are any two events, the probability that exactly one of them occurs is

(a) $P(M)+P(N)-2P(M \cap N)$
(b) $P(M)+P(N)-P(M \cap N)$
(c) $P(M^c)+P(N^c)-2P(M^c \cap N^c)$
(d) $P(M \cap N^c)+P(M^c \cap N)$ (I I T 1984)

40 A box contains 100 tickets, numbered 1 2, , 100 Two tickets are chosen at random It is given that the maximum number on the two chosen tickets is not more than 10 The minimum number of them is 5 with probability

(I I T 1985)

41 A single letter is selected at random from the word ' PROBABILITY

The probability that it is a vowel is

(a) $\frac{3}{11}$, (b) $\frac{4}{11}$, (c) $\frac{2}{11}$, (d) 0

42 In a box containing 100 bulbs 10 are defective What is the probability that out of a sample of 5 bulbs, none is defective?

(a) 10^{-5} (b) $(\frac{1}{2})^5$ (c) $(9/10)^5$ (d) 9/10 or $\quad \dfrac{1-p}{p}=\dfrac{100\,!}{51\,!\,49\,!}\times\dfrac{50\,!\,50\,!}{100\,!}=\dfrac{50}{51}$

or $\quad 51-51p=50p$ giving $p=\dfrac{51}{101}$

46 Let R stand for drawing a red ball and B for drawing a black ball Then

Required probability

$$=RRR+RBR+BRR+BBR$$

$$=\frac{5}{10}\times\frac{5}{11}\times\frac{6}{10}+\frac{6}{10}\times\frac{6}{11}\times\frac{5}{10}+\frac{4}{10}\times\frac{4}{11}\times\frac{7}{10}$$

$$+\frac{4}{10}\times\frac{7}{11}\times\frac{6}{10}=\frac{640}{1100}=\frac{32}{55}$$

43 A student appears for tests I, II and III The student is successful if he passes either in tests I and II or tests I and III The probabilities of the student passing in tests I, II and III are p, q and $\frac{1}{2}$ respectively If the probability that the student is successful is $\frac{1}{2}$, then

(A) $p=q=1$ (B) $p=q=\frac{1}{2}$ (C) $p=1, q=0$

(D) $p=1, q=\frac{1}{2}$ (E) none of these **(I I T 1986)**

44 If $\frac{1+3p}{3}$, $\frac{1-p}{4}$ and $\frac{1-2p}{2}$ are the probabilities of three mutually exclusive events, then the set of all values of p is **(I I T 1986)**

45 One hundred identical coins, each with probability, p, of showing up heads are tossed If $0 < p < 1$ and the probability of heads showing on 50 coins is equal to that of the heads showing on 51 coins, then the value of p is

(A) $\frac{1}{2}$ (B) $\frac{49}{101}$ (C) $\frac{50}{101}$ (D) $\frac{51}{101}$ **(I I T 88)**

46 Urn A contains 6 red and 4 black balls and urn B contains 4 red and 6 black balls, one ball is drawn at random given urn A and placed in urn B Then one ball is drawn at random from urn B and placed in urn A If one ball is now drawn from urn A, the probability that it is found to be red is **(I I T 88)**

Solutions Problem Set (B)

1 Ans (i)

Probability of getting head in one trial$=\frac{1}{2}$

Probability of getting heads in both the trials$=\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$

2 Ans (b)

Required probability$=\frac{{}^4c_2}{{}^{52}c_2}=\frac{4}{52}\times\frac{3}{51}=\frac{1}{221}$

3 Ans (a)

n=Total no of ways$=6\times6=36$

The numbers higher than 9 are 10, 11, 12 in the case of two dice

m=favourable no of ways$=3+2+1=6$ (why ?)

Hence $p=\frac{m}{n}=\frac{6}{36}=\frac{1}{6}$

4 Ans (a)

Required probability$=\frac{{}^{13}c_1}{{}^{52}c_1}\cdot\frac{{}^4c_1}{{}^{52}c_1}=\frac{13\times4}{52\times52}=\frac{1}{52}$

Additional Problems

TRIGONOMETRY

1 If $\sin\theta+\cos\theta=a$, then find the values of $|\sin\theta-\cos\theta|$ and $\cos^4\theta+\sin^4\theta$

2 (a) Evaluate $4\cos 20^\circ-\sqrt{3}\cot 20^\circ$

(b) Prove that

$$4\sin 27^\circ=(5+\sqrt{5})^{1/2}-(3-\sqrt{5})^{1/2}$$

3 Prove that $8\left(1+\sin^8\frac{1}{2}\alpha+\cos^8\frac{1}{2}\alpha\right)=1+6\cos^2\alpha+\cos^4\alpha$

4 Prove that

$$\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)=\frac{1}{16}$$

5 Evaluate without using table values

(i) $1+\cos\frac{\pi}{7}+\cos\frac{2\pi}{7}+\quad+\cos\frac{6\pi}{7}$

(ii) $\dfrac{\cos^2 33^\circ-\cos^2 57^\circ}{\sin 21^\circ-\cos 21^\circ}$

6 Evaluate without using table values

$\cot^2 36^\circ\cot^2 72^\circ$

7 Without using tables, evaluate

(i) $(2\cos 40^\circ-\cos 20^\circ)/\sin 20^\circ$

(ii) $\cos 292^\circ 30'$,

(iii) $6\cos 40^\circ-8\cos^3 40^\circ$

(iv) $\cos^2 73^\circ+\cos^2 47^\circ+\cos 73^\circ\cos 47^\circ$,

(v) $(\cos^2 33^\circ-\cos^2 57^\circ)/(\sin 21^\circ-\cos 21^\circ)$

8 If $A=580^\circ$ prove that

$2\sin\frac{1}{2}A=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$

9 If $\sin A=4/5$ $(0<A<\pi)$, then find the value of

$3\sin(\pi/4+A)\quad 4\cos(\pi/4+A)$

10 If $\dfrac{\sin(x-\alpha)}{\sin(x-\beta)}=\dfrac{a}{b}$, $\dfrac{\cos(x-\alpha)}{\cos(x-\beta)}=\dfrac{A}{B}$ and $aB+bA\neq 0$, then prove that $\cos(\alpha-\beta)=\dfrac{aA+bB}{aB+bA}$

11 If $\tan\{(x+\alpha)/2\}\tan\{(x-\alpha)/2\}=\tan^2(\beta/2)$, prove that $\cos x=\cos\alpha\cos\beta$ when $(x\pm\alpha)\neq(2n+1)\pi$, $\beta\neq 2(m+1)\pi$, $m, n=0, \pm 1, \pm 2,$

5 Ans (a)

Required probability $= \frac{^{3}C_1}{C_1} \times \frac{^{3}C_1}{C_1} \times \frac{^{1}C_1}{C_1} = \frac{3}{4} \times \frac{3}{4} \times \frac{1}{4} = \frac{9}{64}$

6 Ans (ii) (see definition)

7 Ans (i)

Here $A \cap B = \phi$ and so $P(\phi) = 0$

8 Ans (*b*) (See the 'Important Note' of § 4)

9 Ans (*b*) (See the 'Important Note' of § 4)

10 Ans (*b*)

Total no of ways $= 3! = 6$ and favourable no of ways $= 1$

11 Ans (i)

12 Ans (ii)

Probability of throwing an ace $= \frac{1}{6}$, and of not throwing ace $= \frac{5}{6}$

Hence required probability $= \frac{1}{6} \times \frac{5}{6} = \frac{5}{36}$

13 Ans (iii)

Required probability = Prob of right club and good shot or Prob of wrong club and good shot

$= \frac{1}{5} \times \frac{1}{3} + \frac{4}{5} \times \frac{1}{4} = \frac{4}{15}$

14 Ans (iii)

Here $n = 36$ and $m = 8$ since 7 can be thrown in 6 ways and 11 in 2 ways

$p = \frac{m}{n} = \frac{8}{36} = \frac{2}{9}$

15 Ans (iv)

Let A be the event of obtaining an even sum and B the event of obtaining a sum less than five Then we have to find $P(A \cup B)$ Since A, B are not mutually exclusive, we have

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{18}{36} + \frac{6}{36} - \frac{4}{36} = \frac{5}{9},$$

since there are 18 ways to get an even sum and 6 ways to get a sum < 5, viz (1, 3) (3, 1) (2, 2), (1, 2) (2, 1) (1, 1) and 4 ways to get an even sum less than 5 namely, (1, 3), (3, 1), (2, 2) (1, 1)

16 Ans (d)

Let p be the probability of the other event Then the proba

12. If $\tan B = 2 \sin A \sin C \operatorname{cosec} (A+C)$, then prove that $\cot A$, $\cot B$, $\cot C$ are in A P

13 (a) If $2 \tan A = 3 \tan B$, prove that

$$\tan (A - B) = \frac{\sin 2B}{5 - \cos 2B}$$

(b) If $\dfrac{1}{\cos \alpha \cos \beta} + \tan \alpha \tan \beta = \tan \gamma$

show that $\cos 2\gamma \leqslant 0$

14 (a) If A and B be acute positive angles satisfying the equalities $3 \sin^2 A + 2 \sin^2 B = 1$ and $3 \sin 2A - 2 \sin 2B = 0$, prove that $A + 2B = \pi/2$

(b) If $\tan^2 \theta = \tan \phi$ and $\tan 2\theta = 2 \tan \alpha$, prove that

$$\theta + \phi = n\pi + \alpha$$

15 If $\sin \theta = \dfrac{\sin A - \sin B}{1 + \sin A \sin B}$ prove that

$$\cos \theta = \frac{\cos A \cos B}{1 + \sin A \sin B}$$

16 Prove that

$\cot A \cot 3A - \cot 2A \cot 3A - \cot 2A \cot A = 1$

17 If $\tan \alpha$ equals the integral solution of the inequality $4x^2 - 16x + 15 < 0$ and $\cos \beta$ equals the slope of the bisectors of the first quadrant, find the value of $\sin (\alpha+\beta) \sin (\alpha-\beta)$

18 Show that $\tan 18°$ is a root of the equation $5x^4 - 10x^2 + 1 = 0$, hence evaluate $\tan^2 18°$

19 If $\tan^{-1} y = 5 \tan^{-1} x$, find y, as an algebraic function of x, hence show that $\tan 18°$ is a root of the equation

$$5x^4 - 10x^2 + 1 = 0$$

20 (a) If $[\sin (\alpha-\beta) + \cos (\alpha+2\beta) \sin \beta]^2 = 4 \cos \alpha \sin \beta \sin (\alpha+\beta)$,

prove that $\tan \alpha = \tan \beta \left(\dfrac{1}{(\sqrt{2} \cos \beta - 1)} - 1\right)$

(b) If $\sin^4 A/a + \cos^4 A/b = 1/(a+b)$ prove that

$$\sin^8 A/a^3 + \cos^8 A/b^3 = 1/(a+b)^3$$

21 Show that for varying θ and fixed α (θ and α real), the minimum and maximum values of the expression $\cos \theta (\sin \theta + \sqrt{(\sin^2 \theta + \sin^2 \alpha)})$ are respectively $-\sqrt{(1+\sin^2 \alpha)}$ and $\sqrt{(1+\sin^2 \alpha)}$

22 If $0 < \alpha, \beta, \gamma < \frac{1}{2}\pi$, prove that $\sin \alpha + \sin \beta + \sin \gamma > \sin (\alpha+\beta+\gamma)$

bility of the first event is $\frac{2}{3} p$ Since the two events are tota exhaustive, we have

$$p+\frac{2}{3} p=1 \quad \text{or} \quad p=\frac{3}{5}$$

Hence odds in favour of the other are 3 5−3 *i e* 3 2

17 Ans (c)

n=Total no of ways$=6^5$

To find the favourable no of ways, a total of 12 in 5 throw can be obtained in the following two ways only

(i) One blank and four 3's

or (ii) Three 2 s and two 3's

The no of ways in case (i)$={}^5c_1=5$

and the no of ways in case (ii)$={}^5c_2=10$

m=the favourable no of ways$=5+10=15$

Hence the required probability$=\frac{15}{6^5}=\frac{5}{2592}$

18 Ans (*a*)

19 Ans (*a*)

We have $p=1-\left(\frac{2}{7}\right)^3=\frac{335}{343}$ Hence odds in favour are as 335 343−335 or 335 8

20 Ans (c)

$$\text{Required probability}=\frac{{}^5c_3\times{}^{10}c_8}{{}^{15}c_{11}}+\frac{{}^5c_4\times{}^{10}c_7}{{}^{15}c_{11}}+\frac{{}^5c_5\times{}^{10}c_6}{{}^{15}c_{11}}$$

$$=\frac{1}{{}^{15}c_{11}}[10\times45+5\times120+1\times210]$$

$$=\frac{1260\times1\times2\times3\times4}{15\times14\times13\times12}=\frac{12}{15}$$

21 Ans (c)

5 can be thrown in 3 ways and 7 can be thrown in 6 *ways in* a single throw with a pair of dice Hence *no* of ways of throwing neither 5 nor 7 is 36−(4+6)=26

Probability of throwing a five in a single throw with a pair of dice is $\frac{4}{36}=\frac{1}{9}$

And probability of throwing neither 5 nor 7 is $\frac{26}{36}=\frac{13}{18}$

Hence the required probability

23 Show that for varying θ and fixed α (θ and α real), the expression $\frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$ cannot lie between $\tan^2(\frac{1}{4}\pi-\alpha)$ and $\tan^2(\frac{1}{4}\pi+\alpha)$

24 Prove that (i) $\cot 16^\circ \cot 44^\circ + \cot 34^\circ \cot 76^\circ - \cot 76^\circ \cot 16^\circ = 3$

(ii) $\tan 70^\circ = \tan 20^\circ + 2\tan 40^\circ + 4\tan 10^\circ$

25 If x is real, prove that $\frac{x^2-2x\cos\alpha+1}{x^2-2x\cos\beta+1}$ lies between $(\sin^2\alpha/2)/(\sin^2\beta/2)$ and $(\cos^2\alpha/2)/(\cos^2\beta/2)$

26 Show that $\sin^2 12^\circ + \sin^2 21^\circ + \sin^2 39^\circ + \sin^2 48^\circ = 1 + \sin^2 9^\circ \sin^2 18^\circ$

27 If $k^2\sin^2(\theta+\phi) = \sin^2\theta + \sin^2\phi - 2\sin\theta\sin\phi\cos(\theta-\phi)$ show that $\tan\alpha = \frac{1\pm n}{1\mp n}\tan\beta$

28 Show that $-\frac{1}{4} \leqslant \sin\theta\sin(60^\circ-\theta)\sin(60^\circ+\theta) \leqslant \frac{1}{4}$

29 (a) If $\tan\theta = \frac{1}{2}x(y+1)$, $\tan\phi = \frac{1}{2}xy$ and $\cot\psi = \frac{2}{x} + \frac{xy(y+1)}{2}$, where $x, y, z > 0$ and θ, ϕ, ψ are acute angles, prove that $\theta - \phi = \psi$

(b) The positive acute angles θ, ϕ, ψ satisfy the relations $\tan\frac{1}{2}\phi = \frac{1}{3}\cot\frac{1}{2}\theta$ $\cot\frac{1}{2}\psi = \frac{1}{2}(3\tan\frac{1}{2}\theta + \cot\frac{1}{2}\theta)$
Find the sum $\theta + \phi = \psi$

30 Eliminate θ from the equations
$\sin(\theta+\alpha) = a$ and $\cos^2(\theta+\beta) = b$

31 Eliminate x from the equations
$m\tan x + n\cot 2x = p$
and $m\cot x - n\tan 2x = p$

32 If $\cos\alpha + \cos\beta + \cos\gamma = \sin\alpha + \sin\beta + \sin\gamma = 0$, prove that $\sin^2\alpha + \sin^2\beta + \sin^2\gamma = \frac{3}{2}$

33 If $p\cos\theta + q\sin\theta = r$ and $p\cos\phi + q\sin\phi = r$, show that
$$\frac{\cos\{(\theta+\phi)/2\}}{p} = \frac{\sin\{(\theta+\phi)/2\}}{q} = \frac{\cos\{(\theta-\phi)/2\}}{r}$$

34 Prove that
$\cos 2\theta = 2\sin^2\phi + 4\cos(\theta+\phi)\sin\theta\sin\phi + \cos 2(\theta+\phi)$

35 Eliminate θ and ϕ from the relations
$x\cot^2\theta + y\cot^2\phi = 1$, $x\cos^2\theta + y\cos^2\phi = 1$
and $x\sin\theta = y\sin\phi$

$$=\frac{1}{9}+\frac{13}{18}\cdot\frac{1}{9}+\left(\frac{13}{18}\right)^2\frac{1}{9}+\left(\frac{13}{18}\right)^3\frac{1}{9}+$$

$$=\frac{\frac{1}{9}}{1-\frac{13}{18}}=\frac{2}{5}$$

Explanation Since he has already thrown a five (*i e* a number different from 7), he may throw 5 at the next attempt the probability for which is $\frac{1}{9}$ or he may throw 5 at the second attempt when he fails to throw either 5 or 7 at the first attempt, the probability for which is $\frac{13}{18}\times\frac{1}{9}$ or at the third attempt the probability for which is $\left(\frac{13}{18}\right)^2\times\frac{1}{9}$ and so on

22 (a) Ans (iii)

$$P(\overline{A}/\overline{B})=\frac{P(\overline{A}\cap\overline{B})}{P(\overline{B})}=\frac{P(\overline{A\cup B})}{P(\overline{B})}=\frac{1-P(A\cup B)}{P(\overline{B})}$$

(b) $P(A\cup B)=P(A\cap B)$ iff $P(A)+P(B)-P(A\cap B)=P(A\cap B)$

iff $P(A)+P(B)=2P(A\cap B)$ iff $P(A)+P(B)=2P(A)\,P(B/A)$ which is the required relation

Note that we may write $P(B)\,P(A/B)$ for $P(A\cap B)$ instead of $P(A)\,P(B/A)$

23 Ans (a)

n=Total no of ways$={}^{120}c_1=120$

m=Favourable no of ways is the number of terms in the arithmetical series 5, 10, 15, 20 25 30, , 120

$120=5+(m-1)\,5$ or $m=24$

Hence $p=\frac{m}{n}=\frac{24}{120}=\frac{1}{5}$

24 Ans (d)

$P(\overline{A}\cap\overline{B})=A(\overline{A\cup B})=1-P(A\cup B)=1-[P(A)+P(B)]$

[Events A, B are mutually exclusive, we have $P(A\cup B)=P(A)+P(B)$]

$=1-0\,5-0\,3=0\,2$

36. Express $S=\sin 3A+\sin 3B+\sin 3C$ where A, B, C are the angles of a triangle as a product of three trigonometric ratios If $S=0$, show that, at least one of the angles is 60°

(IIT 1962)

37 If $A+B+C=2\pi$, prove that
$$\sin^3 A+\sin^3 B+\sin^3 C=3\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C-\sin\frac{3}{2}A\sin\frac{3}{2}B\sin\frac{3}{2}C$$

38 If $\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos(\alpha-\beta)+1=0$, prove that $\beta-\gamma$ or $\gamma-\alpha$ or $\alpha-\beta$ is a multiple of π

39 If A, B, C are the angles of a triangle and $\sin^3\theta=\sin(A-\theta)\sin(B-\theta)\sin(C-\theta)$, prove that
$$\cot\theta=\cot A+\cot B+\cot C$$

40 (a) If $A+B+C=\pi$ prove that
$$\cos^2 A+\cos^2 B+\cos^2 C=1-2\cos A\cos B\cos C$$
Hence show that if $\cot A=\cot y\cot z$, $\cot B=\cot z\cot x$, $\cot C=\cot x\cot y$, then $\cos^2 x+\cos^2 y+\cos^2 z=1$

(b) In a $\triangle ABC$ prove that
$$\tan A+\tan B+\tan C=1+\sec A\sec B\sec C$$

41 For real α, β, γ prove that
$$1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma+2\cos\alpha\cos\beta\cos\gamma$$
$$=4\sin\tfrac{1}{2}(\alpha+\beta+\gamma)\sin\tfrac{1}{2}(-\alpha+\beta+\gamma)\times\sin\tfrac{1}{2}(\alpha-\beta+\gamma)\sin\tfrac{1}{2}(\alpha+\beta-\gamma)$$

42 For real α, prove that
$$(1+\sec 2\alpha)(1+\sec 4\alpha)(1+\sec 8\alpha)\ \ (1+\sec 2^n\alpha)=\tan 2^n\alpha\cot\alpha$$

43 If $\cos(\alpha+\beta)\sin(\gamma+\delta)=\cos(\alpha-\beta)\sin(\gamma-\delta)$, prove that
$$\cot\alpha\cot\beta\cot\gamma=\cot\delta$$

44 If $\dfrac{\cos\theta}{\cos\alpha}+\dfrac{\sin\theta}{\sin\alpha}=\dfrac{\cos\phi}{\cos\alpha}+\dfrac{\sin\phi}{\sin\alpha}=1$ prove that
$$\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}+1=0$$

45 If $\alpha+\beta+\gamma+\delta=2\theta$, prove that
$$\cos(\theta-\alpha)\cos(\theta-\beta)\cos(\theta-\gamma)\cos(\theta-\delta)$$
$$+\sin(\theta-\alpha)\sin(\theta-\beta)\sin(\theta-\gamma)\sin(\theta-\delta)$$
$$=\cos\alpha\cos\beta\cos\gamma\cos\delta+\sin\alpha\sin\beta\sin\gamma\sin\delta$$

If $A+B+C+D=2\pi$, prove that

46 $\cos A+\cos B+\cos C+\cos D$
$$=-4\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A+C)\cos\tfrac{1}{2}(A+D)$$

25 Ans (a)

Here $P(A)=0.25$, $P(B)=0.50$, $P(A \cap B)=0.14$

Hence $P(\overline{A} \cap \overline{B})=P(\overline{A \cup B})=1-P(A \cup B)$

$=1-[P(A)+P(B)-P(A \cap B)]$

$=1-0.25-0.50+0.14=0.39$

26 Ans (a)

Here $P(A)=0.4$ and $P(\overline{A})=0.6$

Required Probability $=1-[P(\overline{A})]^3=1-(0.6)^3$

$=1-0.216=0.784$

27 Ans (ii)

$P(B)=1-P(\overline{B})=1-\frac{1}{2}=\frac{1}{2}$

and $P(A \cup B)=P(A)+P(B)-P(A \cap B)$

or $\frac{5}{6}=P(A)+\frac{1}{2}-\frac{1}{3}$ or $P(A)=\frac{2}{3}$

$P(A)\,P(B)=\frac{2}{3}\times\frac{1}{2}=\frac{1}{3}=P(A \cap B)$

Hence A and B are independent

28 Ans (a)

Let (p, q) denote a typical throw in which first number is thrown by A and second by B. Then A can throw a higher number than B in the following 15 ways

(2, 1), (3, 2), (3, 1), (4, 3), (4, 2), (4, 1), (5, 4), (5, 3), (5, 2), (5, 1), (6, 5), (6, 4), (6, 3), (6, 2), (6, 1)

Hence the number of ways in which A's throw is greater than B's $=36-15=21$

Hence odds in favour of A not throwing a number greater than B are as 21 : 15 i e 7 : 5

29 Ans (c)

$P_1=\frac{6}{36}=\frac{1}{6}$ [Out of total of 36 ways both the persons can throw equal values in 6 ways]

To find P the total no of ways $n=6^4$ and the favourable no of ways $m=15\times 8=120$

Since any two numbers out of 6 can be selected in ${}^6C_2$ i e, 15 ways and corresponding to each of these ways there are 8 ways e g, corresponding to the numbers 1 and 2 the eight

47 $\sin A - \sin B + \sin C - \sin D$
$$= -4 \cos \tfrac{1}{2}(A+B) \sin \tfrac{1}{2}(A+C) \cos \tfrac{1}{2}(A+D)$$

48 $\tan A + \tan B + \tan C + \tan D$
$$= \tan A \tan B \tan C \tan D (\cot A + \cot B + \cot C + \cot D)$$

49 (i) For any angles $\alpha, \beta, \gamma, \delta$, prove that
$$\sin(\alpha+\beta+\gamma+\delta) + \sin(\alpha+\beta-\gamma-\delta) + \sin(\alpha+\beta-\gamma+\delta) + \sin(\alpha+\beta+\gamma-\delta) = 4 \sin(\alpha+\beta) \cos\gamma \cos\delta$$

(ii) $\sin(\alpha-\beta)\cos(\alpha+\beta) + \sin(\beta-\gamma)\cos(\beta+\gamma) + \sin(\gamma-\delta)\cos(\gamma+\delta) + \sin(\delta-\alpha)\cos(\delta+\alpha) = 0$

(iii) $\sin(\alpha+\beta-2\gamma)\cos\beta - \sin(\alpha+\gamma-2\beta)\cos\gamma = \sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha) + \cos(\gamma+\alpha-\beta) + \cos(\alpha+\beta-\gamma)\}$

50 In a triangle ABC prove that
$$\sin 2mA + \sin 2mB + \sin 2mC = (-1)^{m+1}\, 4 \sin mA \sin mB \sin mC$$
where m is any integer

51 In any triangle ABC prove that
$$\cos mA + \cos mB + \cos mC - 1 = \pm 4 \sin \frac{mA}{2} \sin \frac{mB}{2} \sin \frac{mC}{2},$$
according as m is of the form $4n+1$ or $4n+3$

Solve the following equations

52 $2 \sin x \cos x + \sqrt{3} - \sqrt{2} \cos x - \sqrt{3} \sin x = 0$

53 $\sin 3x = (\cos x - \sin x)^2$

54 $2 \cos x (\cos x + \sqrt{8} \tan x) = 5$

55 $6 \sin^2 x + 2 \sin^2 2x = 5$

56 $\cos(10x + 4) + 4\sqrt{2} \sin(5x+2) = 4$

57 $\cos^3 \theta + \cos^2 \theta - 4 \cos \tfrac{1}{2}\theta = 0$

58 $3(\cos\theta - \sin\theta) = 1 + \cos 2\theta - \sin 2\theta$

59 $1 + \sin 2x = (\sin 3x - \cos 3x)^2$

60 $\cos 4x = \cos 3x$

61 $2^{\cos 2x} + 3 \cdot 2^{\cos^2 x} = 4$

62 (i) $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$

(ii) $\sin x + \cos x = 1 - \sin 2x$

63 $\cos\theta + \sqrt{3} \sin\theta = 2 \cos 2\theta$

64 $(1 - \tan x)(1 + \sin 2x) = 1 + \tan x$

65 $6 \tan^2 \theta - 2 \cos^2 \theta = \cos 2\theta$

66 $\sin^2 \theta - 3 \sin\theta \cos\theta + 2 \cos^2 \theta = 0$

67 $\tan x - \sin x + \tan x \sin x - 1 = 0$

68 $\cot\theta - \tan\theta = \sec\theta$

69 $\sin 8x - \cos 6x = \sqrt{3}(\sin 6x + \cos 8x)$

ways are (1, 1 1, 2), (1, 1, 2, 1), (1, 2, 1, 1), (2, 1, 1, 1), (2, 2, 2, 1), (2, 2 1, 2), (2, 1, 2 2), (1, 2, 2, 2)

Hence $p_- = \frac{120}{6^4} = \frac{5}{54}$

Since $\frac{1}{6} > \frac{5}{54}$, we have $P_1 > P$

30 Ans (b)

$$P_1 = \frac{{}^{15}C_2 \times {}^{27}C_2}{{}^{42}C_4} = \frac{15 \times 14 \times 27 \times 26 \times 1 \times 2 \times 3 \times 4}{1 \times 2 \times 1 \times 2 \times 42 \times 41 \times 40 \times 39} = \frac{27}{82}$$

$$P_2 = \frac{{}^{30}C_4 \times {}^{54}C_4}{{}^{84}C_8} = \frac{30\ 29\ 28\ 27\ 54\ 53\ 52\ 51\ \ 8!}{4!\ 4!\ 84\ 83\ 82\ 81\ 80\ 79\ 78\ 77}$$

$$= \frac{17\ 29\ 45\ 53}{11\ 79\ 82\ 83}, \text{ after simplification}$$

$$\frac{P_1}{P_2} = \frac{27}{82}\ \frac{11\ 79\ 82\ 83}{17\ 29\ 45\ 53} = \frac{33\ 79\ 83}{29\ 53\ 85} = \frac{216381}{130645} > 1$$

Hence $P_1 > P$

31 Ans (i)

For arbitrary events A, B, we have (1)

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Since probability of an event is less than or equal to 1, we have $P(A \cup B) \leqslant 1$ and so (1) gives

$$P(A) + P(B) - P(A \cap B) \leqslant 1$$

or $P(A) + P(B) - 1 \leqslant P(A \cap B)$

or $P(A \cap B) \geqslant P(A) + P(B) - 1$

32 (a) Ans (ii)

Since probability of an event is a non negative quantity equations (1) of Q 31 give

$$P(A) + P(B) - P(A \cap B) \geqslant 0$$

or $P(A) + P(B) \geqslant P(A \cap B)$

or $P(A \cap B) \leqslant P(A) + P(B)$

(b) Ans (A), (B), (C)

Hint See part (i) of Q 31, part (ii) of Q 32 (a) and Th 4

33 Ans (i)

Let λ, μ and λ', μ', be the numbers of heads and tails thrown by A and B respectively, so that $\lambda + \lambda' = n + 1$ and $\mu + \mu' = n$ The required probability P is the probability of the inequality $\lambda > \mu$ The probability $1 - P$ of the opposite event $\lambda \leqslant \mu$ is at the same time the probability of the inequality $\lambda' > \mu'$, 70 $\sin^4 x+\cos^4 x=\cos 4x$

71 $2\sin^3 x=\cos x$

72 $2\sin^3 x+\cos^3 2x=\sin x$

73 $\tan^2 2x+\cot 2x+2\tan 2x+2\cot 2x=6$

74 $2^{\sin^2 x}+5\, 2^{\cos^2 x}=7$

75 (a) Prove that the equation $\sin^{\circ} x+\cos^2 x=0$ has no real solution Solve the following equations

(b) Prove that the equation $\sin x+2\sin 2x=3+\sin 3x$ has no relation in the interval $0 < x < -$

76 $\sin 3x+\sin x+2\cos x=\sin 2x+2\cos^2 x$

77 $\cos x\cos y=3/4$ and $\sin x\sin y=-1/4$

78 $\cos^3\left(\frac{7}{2}--\frac{1}{4}-\right)+\cos x=0$

79 $3^{\sin 2x+2\cos x}+3^{1-\sin 2x+2\sin x}=28$

80 $\sin x\sqrt{(8\cos^{-} x)}=1$

81 $6|\sin x|\cos x-1=\cos 4x$

82 How many roots has the equation
$\sin x+2\sin 2x=3+\sin 3x$
in the interval $0\leqslant x\leqslant -$?

83 How many roots has the equation

$$x+2\tan x=\frac{-}{2}$$

in the interval $0\leqslant x\leqslant 2-$?

Solve the following equations

84 $\sin^2 x+\frac{1}{4}\sin 3x=\sin x\sin^2 3x$

85 $$\sin^4 x+\cos^4 x=\frac{5}{8}$$

86 $$\sin^6 2x+\cos^6 2x=\frac{7}{16}$$

87 $$\sin^8 x+\cos^8=\frac{17}{16}\cos 2x$$

88 $$\sin^4 x+\sin^4\left(x+\frac{1}{4}-\right)=\frac{1}{4}$$

89 $$\sin^4 x+\sin^4\left(x+\frac{1}{4}-\right)+\sin^4\left(x-\frac{1}{4}-\right)=\frac{9}{8}$$

90 $|\cos x-2\sin 2x\cos 3x|=1-2\sin x-\cos 2x$

Solve the following system of equations

91 $\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)-\frac{1}{2}\cos A\cos B=\frac{1}{4}$

$h = QB \tan\theta = QA \tan\theta$

$QB = QA = h\cot\theta \qquad (1)$

Also $h = QC \tan\phi$

$QC = h\cot\phi \qquad (2)$

$\triangle QAB$ is isosceles triangle in which C is mid point of AB. Therefore QC is perpendicular to AB

$BQ^2 = CB^2 + CQ^2$

or $h^2(\cot^2\theta - \cot^2\phi) = a^2$ by (1) and (2)

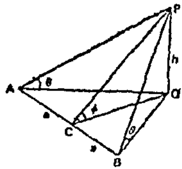


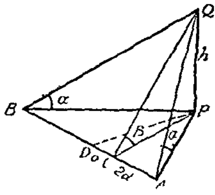
Fig 105

$$h = \frac{a}{\sqrt{(\cot^2\theta - \cot^2\phi)}} = \frac{a\sin\theta\sin\phi}{\sqrt{\{\sin(\phi+\theta)\sin(\phi-\theta)\}}}$$

as in Q 38

(b) Let PQ represent the tower on one bank and A, C, B be the points on the other bank such that $AB = 6d$ and $AC = 2d$. If h be the height of the tower then by the given condition $PA = PB = h\cot\alpha$

$PC = h\cot\beta \qquad (1)$



Since triangle PAB is isosceles therefore if D be mid point of AB then median PD will be perpendicular to AB and hence it will represent the width of the river

Also $CD = AD - AC = 3d - 2d = d$

$PA^2 = PD^2 + DA^2$ from rt angled $\triangle PDA \qquad (2)$

$PC^2 = PD^2 + CD^2$ from rt angled $\triangle PDC \qquad (3)$

Subtracting we get $PA^2 - PC^2 = DA^2 - CD^2$

or $h^2(\cot^2\alpha - \cot^2\beta) = 9d^2 - d^2 = 8d$ by (1)

$$h = \frac{2\sqrt{2}d}{\sqrt{(\cot^2\alpha - \cot^2\beta)}} = \text{height of the tower}$$

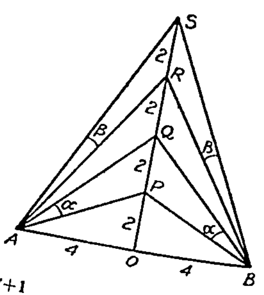
Again from (2) the width PD of the canal is given by

$$PD^2 = PA^2 - DA^2 = h^2\cot^2\alpha - 9d^2 = \frac{8d^2\cot^2\alpha}{\cot^2\alpha - \cot^2\beta} - 9d^2$$

$$PD = d\sqrt{\left(\frac{\cot^2\beta - \cot^2\alpha}{\cot^2\alpha - \cot^2\beta}\right)}$$

(c) Let O be the mid point of $AB = 8 \qquad OA = OB = 4$

Also $OP = 2$ is the initial position of the object which is two meters long



$\frac{ds}{dt}=2t+1$

$s=t^2+t+k$ when $t=0$ $s=2$ $k=2$

$s=t^2+t+2$

For $t=0$, $s=2=OP$

For $t=1$, $s=4=OQ$ $PQ=2$ where PQ is the position of the object after 1 second. For $t=2$ $s=8=OS$ but $RS=2$ where RS the position of the object after 2 seconds

$OR=OS-RS=6$ Also $OQ=4$

$QR=OR-OQ=6-4=2$

As per the condition of the question $OP=PQ=QR=RS=2$. PQ and RS the positions of the object at $t=1$ and $t=2$ subtend angles α and β at A and B respectively. Applying $a^2+b^2=c^2$ in various right angled triangles we have

$AP=2\sqrt{5}$ $AQ=4\sqrt{2}$, $AR=2\sqrt{13}$, $AS=4\sqrt{5}$

Again applying cosine formula $\cos A=\frac{b^2+c^2-a^2}{2bc}$

$$\cos\alpha=\frac{AP^2+AQ^2-PQ^2}{2AP\,AQ}=\frac{20+32-4}{2\cdot 2\sqrt{5}\cdot 4\sqrt{2}}=\frac{3}{\sqrt{10}}$$

$$\cos\beta=\frac{AR^2+AS^2-RS^2}{2AR\,AS}=\frac{52+80-4}{2\cdot 2\sqrt{13}\cdot 4\sqrt{5}}=\frac{8}{\sqrt{65}}$$

$\sin\alpha=\frac{1}{\sqrt{10}}$ and $\sin\beta=\frac{1}{\sqrt{65}}$

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta=\frac{24+1}{\sqrt{10}\sqrt{65}}=\frac{5}{\sqrt{(26)}}$$

92 (i) $\cos x \cos y=\frac{3}{4}$, $\sin x \sin y=-\frac{1}{4}$

(ii) $\sin x+\sin y=1$, $\cos x-\cos y=\sqrt{3}$

93 $\sqrt{3} \sin 2x=\sin 2y$, $\sqrt{3} \sin^2 x+\sin^2 y=\frac{1}{2}(\sqrt{3}-1)$

94 (a) Evaluate (i) $\sin [\frac{1}{2} \cot^{-1} (-\frac{3}{4})]$, (ii) $\cot \cos^{-1} (-\frac{1}{3})$

(b) Prove that

(i) $2 \tan^{-1} (-3)=-\cos^{-1} (-4/5)=-\pi+\cos^{-1} (4/5)$

$=(-\pi/2)+\tan^{-1} (-4/3)$

(ii) $\frac{1}{2} \cos^{-1} \frac{3}{5}=\tan^{-1} \frac{1}{2}=\pi/4-\frac{1}{2} \cos^{-1} (4/5)$

95 Evaluate arc sin (sin 10)

96 Solve $\tan \cos^{-1} x=\sin \cot^{-1} \frac{1}{2}$

97 Solve $2 \sin^{-1} x=\sin^{-1} \frac{10x}{13}$

98 Solve $\cos^{-1} \frac{x^2-1}{x^2+1}+\tan^{-1} \frac{2x}{x^2-1}=\frac{2\pi}{3}$

99 Solve $\sin^{-1} \frac{5}{x}+\sin^{-1} \frac{12}{x}=\frac{\pi}{2}$

100 Solve $\tan^{-1} x+2 \cot^{-1} x=\frac{2}{3}\pi$

101 Prove $\cos^{-1} \sqrt{\left(\frac{a-x}{a-b}\right)}=\sin^{-1} \sqrt{\left(\frac{x-b}{a-b}\right)}=\cot^{-1} \sqrt{\left(\frac{a-x}{x-b}\right)}$

$=\frac{1}{2} \sin^{-1} \left[\frac{2\sqrt{\{(a-x)(x-b)\}}}{(a-b)}\right]$

102 Prove $\sin^{-1} \left(\frac{1}{3}\right)+\sin^{-1} \left(\frac{3}{4}\right)=\cos^{-1} \left(\frac{2\sqrt{14}-3}{12}\right)$

103 Prove that

$\tan^{-1} a+\tan^{-1} \frac{2a}{1-a}=\tan^{-1} \frac{3a-a^3}{1-3a}$, if $0 < a < \frac{1}{\sqrt{3}}$ or $a > 1$

and $=-\pi+\tan^{-1} \frac{3a-a^3}{1-3a}$ if $\frac{1}{\sqrt{3}} < a < 1$

104 The medians of a triangle ABC make angles α, β, γ with each other Prove that

$\cot \alpha+\cot \beta+\cot \gamma+\cot A+\cot B+\cot C=0$

105 Show that the radius of the circle inscribed in the triangle formed by joining the centres of the escribed circles of a triangle ABC is

$(4R \cos A/2 \cos B/2 \cos C/2)/(\cos A/2+\cos B/2+\cos C/2)$

106 If p, q, r are the perpendiculars from the vertices of a triangle ABC upon any straight line meeting the sides externally in D, E, F prove that

(iii) $\int \frac{(x^2-1)\,dx}{x\sqrt{(x^4+3x^2+1)}}$ (iv) $\int \frac{dx}{\sqrt{(\sin^3 x \cos^5 x)}}$

(v) $\int \frac{x \log x}{\sqrt{\{(x^2-1)^3\}}}\,dx$ (vi) $\int \sqrt{\left\{\left(\frac{\cos x - \cos^3 x}{1-\cos^3 x}\right)\right\}}\,dx$

(vii) $\int e^{\sin x}\left(\frac{x\cos^3 x - \sin x}{\cos^2 x}\right)dx$

201 Prove that

$$\int \frac{dx}{(1+x^2)^n} = \frac{x}{2(n-1)(1+x^2)^{n-1}} + \frac{2n-3}{2(n-1)}\int \frac{dx}{(1+x^2)^{n-1}}$$

(n a positive integer) and with the help of it compute the integral $\int \frac{dx}{(1+x^2)^4}$

202 If $I_n = \int_1^e \ln^n x\,dx$, prove that

$I_n = e - n\,I_{n-1}$ (n a positive integer)

203 Evaluate $\int \frac{d\theta}{(1+e\cos\theta)^2}$

204 Prove that, if n is a positive integer,

$$\int_0^a e^{-x} x^n\,dx = n!\left\{1 - e^{-a}\left(1 + a + \frac{a^2}{2!} + \ldots + \frac{a^n}{n!}\right)\right\}$$

Deduce the value of $\int_0^\infty e^{-x} x^n\,dx$

205 Evaluate $\int \frac{dx}{(a+b\cos\theta)^2}$, $(a > b)$

206 Evaluate $\int_{-1}^{1} \log\left(\frac{1+x}{1-x}\right)\frac{x^3\,dx}{\sqrt{(1-x^2)}}$

207 Prove that $\int_0^\infty \frac{dx}{1+x^4} = \int_0^\infty \frac{x^2\,dx}{1+x^4} = \frac{\pi}{2\sqrt{2}}$

208 Prove that $\int_1^\infty \frac{(x-2)\,dx}{x^3\sqrt{(x^2-1)}} = 0$

209 Evaluate $\int_0^\infty \frac{x\log x}{(1+x^2)^2}\,dx$

210 Find the limit, when $n \to \infty$, of the series

(i) $\frac{1}{n}\left\{\sin^{2k}\frac{\pi}{2n} + \sin^{2k}\frac{2\pi}{2n} + \sin^{2k}\frac{3\pi}{2n} + \ldots + \sin^{2k}\frac{\pi}{2}\right\}$

(ii) $\frac{n}{(n+1)\sqrt{(2n+1)}} + \frac{n}{(n+2)\sqrt{\{2(2n+2)\}}}$

$$+ \frac{n}{(n+3)\sqrt{\{3(2n+3)\}}} + \ldots + \frac{n}{2n\sqrt{(n\cdot 3n)}}$$

$a^2(p-q)(p-r)+b^2(q-r)(r-p)+c^2(r-p)(r-q)=4\Delta^2$,
where Δ is the area of the triangle

107 The base of a triangle is divided into three equal parts, if t_1, t_2, t_3 be the tangents of the angles subtended by these parts at the opposite vertex prove that

$$\left(\frac{1}{t_1}+\frac{1}{t_2}\right)\left(\frac{1}{t}+\frac{1}{t_3}\right)=4\left(1+\frac{1}{t_2^2}\right)$$

108 A hill on a level plane has the form of portion of a sphere At the bottom the surface slopes at an angle α and from a point on the plane distant a from the foot of the hill the elevation of the highest visible point is β Prove that the height of the hill above the plane is $\dfrac{a \sin \beta \sin^2(\alpha/2)}{\sin^2\{(\alpha \quad \beta)/2\}}$

109 A tower and a flagstaff on its top subtend equal angles at the observer's eye If the heights of flagstaff, tower and the eye of the observer are respectively a b and h prove that the distance of the observer's eye from the base of the tower is

$$b\left(\frac{a+b-2h}{a-b}\right)^{1/}$$

110 A tower stands on the base of an inclined plane making an angle of 9° with the horizontal After climbing a distance of 100 metres up the plane a man finds that the tower subtends an angle of 54° at his eye Find the height of the tower, given $\log 2=0\ 30103$, $\log 114\ 4123=2\ 0584726$ and $L \sin 54°=9\ 9079576$

111 An observer on a hill observes that the three pillars on the horizontal plane through the foot of the hill make equal angles at his eye If the depressions of the bases of the pillars are α, β, γ respectively and their heights a, b, c, prove that

$$\frac{\sin(\beta-\gamma)}{a \sin \alpha}+\frac{\sin(\gamma-\alpha)}{b \sin \beta}+\frac{\sin(\alpha-\beta)}{c \sin \gamma}=0$$

112 If the angular elevations of the tops of two spires which appear in a straight line is α, and the angular depressions of their reflections in a lake, h metres below the point of observation are β and γ Show that the distance between the spires is

$2h \cos^2 \alpha \sin(\quad -\gamma) \operatorname{cosec}(\beta-\alpha) \operatorname{cosec}(\gamma-\alpha)$ metres $(\beta > \gamma)$

113 A hill, standing on a horizontal plane, has a circular base and forms a part of the sphere At two points on a diagonal

211 Prove that $\int_0^{b-c} f(x+c)\, dx = \int_c^b f(x)\, dx$

212 Prove that $\int_0^{\pi/2} \frac{x \sin x \cos x\, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi}{4ab^2 (a+b)^2}$

$(a, b > 0)$

213 Prove that $\int_0^{\pi/2} \varphi (\sin 2x) \sin x dx = \int_0^{\pi/2} \varphi (\sin 2x) \cos x\, dx$

$= \sqrt{2} \int_0^{\pi/4} \varphi (\cos 2x) \cos x\, dx$

214 Prove that $\int_0^1 \frac{\log (1+x)}{(1+x^2)} dx = \frac{1}{8}\pi \log_e 2$

215 Find a, b c such that the function $f(x) = ae^{2x} + be^x + cx$ satisfies the conditions $f(0) = -1$, $f'(ln\, 2) = 31$

and $\int_0^{ln 4} [f(x) - cx]\, dx = 19\frac{1}{2}$

216 Compute the area of the figure contained between the curve $y = 1/(1+x^2)$ and its asymptote

217 Find the area of the figure bounded by the curves $y = x$, $y = 1/x$, $y = 0$ and $x = e$

218 Compute the area cut off the parabola $y = x^2 - 4x + 5$ by the straight line $y = x + 1$

219 Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a$ is divided by the parabola $y^2 = 12ax$ is $(16/3)\, a^2 (8\pi - \sqrt{3})$

220 Trace the curve $a^4 y^2 = x^5 (2a + x)$ and prove that its area is to that of the circle whose radius is a, as 5 to 4

221 Find the whole area included between the curve $x^2 y^2 = a (y - x)$ and its asymptote

222 Find the area bounded by the curves $y = -x^2 + 6x - 5$, $y = -x^2 + 4x - 3$ and the straight line $y = 3x - 15$

223 Find the area enclosed between the circle $x^2 + y^2 - 2x + 4y - 11 = 0$ and the parabola $y = -x^2 + 2x + 1 - 2\sqrt{3} = 0$

Algebra

224 Factorize the expression $(a/b - b/a)$ into two factors whose sum is $(a/b + b/a)$

225 Simplify (i) $\frac{1}{\sqrt[4]{a} + \sqrt[8]{a} + 1} + \frac{1}{\sqrt[4]{a} - \sqrt[8]{a} + 1} - \frac{2\sqrt[4]{a} - 2}{\sqrt{a} - \sqrt[4]{a} + 1}$

(ii) $\frac{a+1}{a^{2/3} - a^{1/3} + 1} - \frac{a-1}{a - \sqrt{a}}$

of the base produced in the plane distant a and b from the base, the angular elevations of the highest visible points on the hill are θ and ϕ. Prove that the height of the hill is

$$2\left[\frac{\sqrt{\{b\cot(\phi/2)\}}-\sqrt{\{a\cot(\theta/2)\}}}{\cot(\phi/2)-\tan(\theta/2)}\right]$$

Coordinate Geometry*

114 (a_1, b_1), (a_2, b_2), (a_3, b_3) are the vertices A, B, C of a triangle ABC. The side AB is is divided by the point D in the ratio $\lambda : \mu$ and then the line segment DC is divided by the point E in the ratio $\nu : (\lambda+\mu)$. Find the coordinates of E.

115 The point $(1, -2)$ is reflected in the x axis and then translated parallel to the positive direction of x-axis through a distance of 3 units, find the coordinates of the point in the new position.

116 The line segment joining $A(2, 0)$ and $B(3, 1)$ is rotated about A in the anti clockwise direction through an angle of $15°$. Find the equation of the line in the new position. If B goes to B' in the new position, what will be coordinates of B'?

117 The line segment joining $A(0, 0)$ and $B(5, 2)$ is rotated about A in the anti clockwise direction through an angle of $45°$ so that B goes to C. If D is the reflection of C in x-axis, find the coordinates of D.

118 The diagonal of a square is the portion of the line $x/a+y/b=1$ intercepted between the coordinate axes OX and OY. Find the coordinates of its vertices.

119 Prove that the three lines whose equations are $15x-18y+1=0$, $12x+10y-3=0$ and $6x+66y-11=0$ are concurrent.

120 Prove that the four lines

$$\frac{x}{a}+\frac{y}{b}=1,\ \frac{x}{b}+\frac{y}{a}=1,\ \frac{x}{a}+\frac{y}{b}=2, \text{ and } \frac{x}{b}+\frac{y}{a}=2$$

enclose a rhombus whose area is $\dfrac{a^2b^2}{|a^2-b^2|}$.

121 One side of a square is inclined at an angle α with x axis and one of its vertices is the origin. Prove that the equations of its diagonals are $y(\cos\alpha-\sin\alpha)=x(\sin\alpha-\cos\alpha)$,
$y(\sin\alpha+\cos\alpha)+x(\cos\alpha-\sin\alpha)=a$,
where a is the length of a side of the square.

226 In what interval must the number m vary so that both roots of the equation $x^2-2mx+m^2-1=0$ lie between -2 and 4 ?

227 If $\sqrt{(a-x)}+\sqrt{(b-x)}+\sqrt{(c-x)}=0$, prove $(a+b+c+3x)(a+b+c-x)=4(ab+bc+ca)$

228 Prove that

$$\frac{ac(b+d)+bd(a+c)+(ac+bd+ad+bc)x}{(a+b)(b+d)+(a+b+c+d)}$$

is independent of x

229 If $|z|<\frac{1}{2}$, show that $|(1+i)z^3+iz|<\frac{3}{4}$

230 Show that $-3-4i=5e^{i(\pi+\tan^{-1}4/3)}$

231 Prove that $|1-\bar{z}_1z_2|^2-|z_1-z_2|=(1-|z_1|^2)(1-|z_2|^2)$

232 Show that $e^{2mi\cot^{-1}p}\left[\frac{pi+1}{pi-1}\right]^m=1$

233 Solve the equation

$$z^2+2z|z|+|z^2|=0$$

234 If z_1 and z_2 are complex numbers and $u=\sqrt{(z_1z_2)}$, prove that

$$|z_1|+|z_2|=\left|\frac{z_1+z_2}{2}+u\right|+\left|\frac{z_1+z_2}{2}-u\right|$$

235 (i) Show that if the points z_1, z_2, z_3, z_4 taken in order are concyclic, then the expression $\{(z_3-z_1)(z_4-z_2)\}/\{(z_3-z_2)(z_4-z_1)\}$ is purely real

(ii) Let z_1, z, z_3, z_4 are the vertices of a quadrilateral in order Prove that the quadrilateral is concyclic if $z_1z_2+z_3z_4=0$ and $z_1+z_2=0$

236 Prove that the reciprocal of any one of the twelth roots of unity is again a twelth root of unity

237 Find the roots of the equation $z^{10}-z^5-992=0$, whose real part is negative

238 For what values of m is one of the roots of the equation $(2m+1)x^2-mx+m-2=0$ greater than unity and the other less than unity ?

239 Show that the equations

$$\frac{A^2}{x-a}+\frac{B^2}{x-b}+\frac{C^2}{x-c}+\dots+\frac{H^2}{x-h}=k,$$

has no imaginary roots

240 If each pair of the three equations $x^2-p_1x+q_1=0$, $x^2-p_2x+q_2=0$, $x^2-p_3x+q_3=0$ have a common root, prove that

$$p_1^2+p_2^2+p_3^2+4(q_1+q_2+q_3)=2(p_2p_3+p_3p_1+p_1p_2)$$

122 The equal sides AB and AC of a right angled $\triangle ABC$ are produced to E and F such that $2BE \cdot CF = AB^2$, prove that the line EF always passes through a fixed point

123 If the sum of the lengths of perpendiculars from the points $(3, 4)$ and $(7, 2)$ on a variable line is thrice the length of perpendicular from $(1, 3)$ on that line Show that the line always passes through a fixed point Find the coordinates of that point

124 Prove that the area of the triangle enclosed by the lines $a_1x+b_1y+c_1=0$, $a_2x+b_2y+c_2=0$, $a_3x+b_3y+c_3=0$ is

$$\frac{1}{2}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & b_3 \end{vmatrix} \quad (a_1b_2-a_2b_1)(a_2b_3-a_3b_2)(a_3b_1-a_1b_3)$$

125 prove that the area of the triangle enclosed by the lines $x\cos\alpha+y\sin\alpha=p_1$, $x\cos\beta+y\sin\beta=p_2$, $x\cos\gamma+y\sin\gamma=p_3$

is $\frac{1}{2}\dfrac{\{p_1\sin(\gamma-\beta)+p_2\sin(\alpha-\gamma)+p_3\sin(\beta-\alpha)\}}{\sin(\gamma-\beta)\sin(\alpha-\gamma)\sin(\beta-\alpha)}$

126 Prove analytically that the internal bisectors of a triangle are concurrent

127 Prove analytically that in a triangle the external bisectors of two angles and the internal bisector of the third are concurrent

128 Prove analytically that the bisectors of angles between two straight lines are perpendicular

129 Find the coordinates of the in centre of the triangle formed by the y axis and the lines $3x+4y-10=0$ and $3y-4x-15=0$

130 Prove that the ortho centre of the triangle enclosed by the lines

$y=m_1x+\dfrac{a}{m_1}$, $y=m_2x+\dfrac{a}{m_2}$ and $y=m_3x+\dfrac{a}{m_3}$ is

$$\left(a,\ a\left\{\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_1m_2m_3}\right\}\right)$$

131 A variable line cuts two fixed straight lines OX and OY in the points A and B respectively and P and Q are the foot of the perpendiculars from A and B to the lines OBY and OAX Prove that if the line AB passes through a fixed point, then PQ also passes through some fixed point

241 If a, b, c, x, y, z are real numbers, and
$(a+b+c)^2=3(bc+ca+ab-x^2-y^2-z^2)$, show that
$a=b=c$ and $x=0, y=0, z=0$

242 If the roots of the equation
$(1-q+p^2/2)x^2+p(1+q)x+q(q-1)+p^2/2=0$
are equal, prove that $p^2=4q$

243 If $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=\frac{1}{a+b+c}$ show that

(i) $\frac{1}{a^n}+\frac{1}{b^n}+\frac{1}{c^n}=\frac{1}{a^n+b^n+c^n}$ where n is an odd integer

(ii) $a^{-2n+1}+b^{-2n+1}+c^{-2n+1}=(a+b+c)^{-2n+1}$ [I I T 1977]

244 Show that the expression
$(x^2-yz)^3+(y^2-zx)^3+(z^2-xy)^3-3(x^2-yz)(y^2-zx)(z^2-xy)$
is a perfect square, and find its square root

245 (i) Find the range of values of a for which two of the roots of the equation $(a-1)(1+x+x^2)^2=(a+1)(1+x^2+x^4)$ are real and distinct

(ii) If the roots of $ax^2+2bx+c$ be real and distinct, prove that the roots of $(a+c)(ax^2+2bx+c)=2(ac-b^2)(x^2+1)$ will be imaginary and vice versa

246 If the equations $ax^3+3bx^2+3cx+d=0$ and $ax^2+2bx+c=0$ have a common root prove that $(bc-ad)^2=4(ac-b^2)(bd-c^2)$

247 Find a and b so that $\frac{x}{3x}-\frac{ax^4+b}{3x-1}$ may be an integral function of x

248 (i) If p be greater than unity, prove that for all real values of x, the expression $\frac{x^2-2x+p}{x^2+2x+p}$ lies between $\frac{p-1}{p+1}$ and $\frac{p+1}{p-1}$

(ii) Prove that the expression
$(ax+b)(dx+c)/(bx+a)(cx+d)$
will take all real values when x is real provided that a^2-b^2 and c^2-d^2 have the same sign

249 For what values of m is the inequality (i) $\left|\frac{x^2+mx+1}{x^2+x+1}\right|<3$

(ii) $-3<\frac{x^2+mx-2}{x^2-x+1}<2$ satisfied for all the values

132 If (α, β), $(\bar{x}, \bar{y})$ and (p, q) are the coordinates of the circum centre, the centroid and the ortho centre of a triangle, prove that $3\bar{x}=2\alpha+p$ and $3\bar{y}=2\beta+q$

133 Find the coordinates of the in centre of the triangle the equation of whose sides are
$3x+4y-7=0$, $12x-5y-17=0$ and $5x+12y-34=0$
Also find the radius of the in circle

134 Prove that the coordinates of the in centre of the triangle whose vertices are (1, 2) (2, 3) and (3, 1) are $\frac{1}{6}\{8+\sqrt{(10)}\}$ and $\frac{1}{6}\{16-\sqrt{(10)}\}$

135 Find the coordinates of the centroid and circum centre of the triangle whose vertices are (2, 3), (3, 4) and (6, 8) and hence find the coordinates of its ortho-centre

136 How many circles can be drawn each touching all the lines $x+y-6=0$, $x-7y+42=0$, $7x+y-42=0$? Find the centres and radii of these circles

137 P and Q are two fixed points whose coordinates are respectively (3, 2) and (5, 1) On PQ an equilateral $\triangle PQR$ is described away from the origin Find the coordinates of R and of the ortho centre of $\triangle PQR$

138 Find the equation of the lines bisecting the angles between the line $Ax+By+C=0$ and the line passing through the point (h, k) and perpendicular to this line

139 A line is such that the algebraic sum of the perpendiculars on it from a number of points is zero Prove that the line always passes through a fixed point

140 A variable line cuts an intercept of constant length C between the coordinate axes and meets them in the points A and B respectively The rectangle $OAPB$ is completed Show that the locus of P is $x^2+y^2=c^2$

141 A variable line at a constant distance p from the origin meets x axis at A and y axis at B The rectangle $OAPB$ is completed Prove that the locus of P is $1/x^2+1/y^2=1/p^2$

142 If in Q 141, Q is the foot of perpendicular from P on the line, prove that the locus of Q is

$$xy\,(x^{2/3}+y^{2/3})^{3/2}=p^3$$

143 The line $Ax+By+C=0$ cuts the x-axis at P and y-axis at Q

250 If $a \neq 1$ and $a \neq -2$, show that the roots of the equation $(a^2+a-2)\ x^2+(2a^2+a+3)\ x+a^2-1=0$ are rational Hence solve the equation

251 *If the roots of* $x^2-ax+b=0$ *are real and differ by a quantity which is less than* $c\ (c>0)$, *prove that* b lies between $\frac{1}{4}(a^2-c^2)$ and $\frac{1}{4}a^2$

Solve the following equations

252 $|6x^2-5x+1|=5x-6x^2-1$

253 $\log_2 x/\log_4 2x=\log_8 4x/\log_8 8x$

254 $\left(\frac{2x+3}{2x-3}\right)^{1/3}+\left(\frac{2x+3}{2x+3}\right)^{1/3}=\frac{8}{13}\ \frac{4x^2+9}{4x^2-9}$

255 $|x-3|^{(x^2-8x+15)/(x-2)}=1$

256 $\log_{3x} x=\log_{9x} x$

257 $xy+6=2x-x^2,\ xy-9=2y-y^2$

258 (a) $\log_2 x=\log_4 y+\log_4 (4-x),\ \log_3 (x+y)=\log_3 x-\log_3 y$

(b) $11^{ax}-2\times 5^y=71$

$11^x+2\times 5^{y/2}=21$

$11^{(x-1)x}+5^{y/2}=16$

259 (i) $x^4+2x^3y+x^2y^2+2xy^3+y^4=41,\ \frac{x}{y}+\frac{y}{x}=\frac{5}{2}$

(ii) $x^2+xy+xz=45,\ y^2+yz+xy=75,\ z^2+zx+zy=105$

260 $x^2+xy+y^2=13,\ y^2+yz+z^2=49,\ z^2+zx+x^2=31$

261 Show that the number of ways in which three numbers in A P can be selected from 1, 2, 3, , n is $\frac{1}{4}(n-1)^2$ or $\frac{1}{4}n(n-2)$ according as n is odd or even

262 (i) n different objects are arranged round a circle In how many ways can 3 objects be selected when no *two* of the selected objects are consecutive ?

(ii) If n things are arranged in a row, find the no of such sets of three objects

263 How many triangles can be formed by joining the vertices of a polygon of n sides when no side of the polygon is used in forming the triangle

264 If n points in a plane be joined in all possible ways by straight lines, and if no two of the straight lines be coincident or parallel, and no three pass through the same point (with the

Find the coordinates of the otho centre, centroid and circum centre of the $\triangle\ OPQ$

144 Prove that the area of the parallelogram enclosed by the lines $ax+by+c=0$, $Ax+By+C=0$, $ax+by+d=0$ and

$Ax+By+D=0$ is

$$\left|\frac{(c-d)\,(C-D)}{aB-bA}\right|$$

145 A ray of light is sent along the line $x-y+1=0$ Upon reaching the line $3x-5y+3=0$, the ray is reflected from it Find the equation of the line containing the reflected ray

146 The equations of the lines PQ QR RP are $3y+x=2$, $y+3x=10$, $y=3x+2$ respectively Show that the triangle PQR has a right angle and find the equation of the altitude to the hypotenuse

147 $OABC$ is a square O is the origin, the coordinates of A are (5, 3) and the y coordinates of B and C are positive Find the equations of AB and BC

148 If the line $y+3x=4$, $ay=x+10$ and $2y+bx+9=0$ form three sides of a rectangle, find a and b given that they are integers If the fourth side passes through the point (1, -2), find its equation

149 The vertices A and B of a parallelogram $OABC$ lie on the x axis and y axis respectively, O is the origin AB produced passes through the point (2, 3) If the slope of AB is m, find the coordinates of C Hence find the locus of C for varying values of m If AC is produced its own length to P, find the locus of P

150 Find the projection of the line segment joining the points (2 3) and (5, 7) on the line $x+y=1$

151 Show that the five points (12, 43) (18, 39), (42, 3), (-54, -69), (-81, -38) are concyclic

152 On the line segment joining the points (1, 0) and (2, 0) an equilateral triangle is constructed whose vertex lies in the positive quadrant Find the equation of circles which are drawn on the sides of the triangle as diameter

153 Two circles are drawn through the points $(a, 5a)$ and $(4a, a)$ touching the axis of y Prove that they intersect at an angle

exception of n original points), then prove that the number of points of intersection exclusive of the n points is $\frac{1}{8}n(n-1)(n-2)(n-3)$

265 If each of the m distinct points on a straight line is joined to each of n distinct points on another line by straight lines terminated by the points Show that the number of points of intersection besides these points is $\frac{1}{4}mn(m-1)(m-2)$

266 In how many ways can r flags be displayed on n poles in a row, disregarding the limitation on the number of flags on a pole ?

267 The sides AB, BC, CA of a triangle ABC have 4, 5 and 6 interior points, respectively on them Find the number of triangles that can be constructed using these interior points as vertices

268 In how many ways can 16 apples be distributed among four persons each receiving not less than 3 apples ?

269 Show that a selection of 10 balls can be made from an unlimited number of red, white, blue and green balls in 286 different ways and that 84 of these contain balls of all four colours

270 Find the rank of the word THEORY when its letters are arranged as in a dictionary

271 Show that the no of ways in which m identical balls can be distributed among $2m$ boxes so that no box contains more than one ball cannot exceed $4^m/\sqrt{(2m+1)}$ Also show that the number of ways cannot be less than $4^m/2\sqrt{m}$

272 Prove by induction that the product of any number of factors, each of which is the sum of two squares can be expressed as the sum of two squares

273 Prove by induction that n^5-n is divisible by 30 for all natural numbers $n > 1$

274 Prove by induction that for every natural number n the equality $\sin\frac{\pi}{3}+\sin\frac{2\pi}{3}+\sin\frac{3\pi}{3}+ \quad +\sin\frac{n\pi}{3}=2\sin\frac{n\pi}{6}\sin\frac{n+1}{6}$ – holds true

275 If $a > 0$, $b > 0$, prove that $(a+b)^n > 2^{n-1}(a^n+b^n)$ for all n except when $0 < n < 1$ and then the inequality is reversed

276 If the common ratio of an infinite G P be less than $\frac{1}{2}$, show

$\tan^{-1}\left(\frac{40}{9}\right)$

154 Find the equation of the circle circumscribing the triangle whose sides are $x=0$, $y=0$ and $lx+my=1$. If, l, m can vary so that $l^2+m^2=4l^2m^2$, find the locus of the centre of the circle.

155 A tangent is drawn to the circle $(x-a)^2+y^2=b^2$ and a perpendicular tangent to the circle $(x+a)^2+y^2=c^2$. Find the locus of their point of intersection and prove that the bisector of the angle between them always touches one or other of two fixed circles.

156 Two rods of lengths a and b slide along the axes, which are rectangular, in such a way that their ends are always concyclic, prove that the locus of the centre of circle passing through these ends is the curve $4(x^2-y^2)=a^2-b^2$

157 A variable circle passes through the point of intersection O of any two straight lines and cuts off from them portions OP and OQ such that $mOP+nOQ$ is equal to unity, prove that this circle always passes through a fixed point (other than O).

158 A circle passes through $(-1, 1)$, $(0, 6)$ and $(5, 5)$. Find the points on this circle at which the tangents are parallel to the line joining the origin to its centre.

159 Find the equation of tangents from the point $(4, 5)$ to the circle $2x^2+2y^2-8x+12y+21=0$

160 A point moves in such a manner that the sum of the squares of its distances from the vertices of a triangle is constant. Prove that its locus is a circle.

161 A point moves such that the sum of the squares of its distances from n fixed points is constant. Prove that its locus is a circle.

162 If the length of the tangent from a point P to the circle $x^2+y^2-a^2=0$ is four times the length of the tangent from that point to the circle $(x-a)^2+y^2=a^2$, prove that the locus of P is $15x^2+15y^2-32ax+a^2=0$

163 From a point on a given line tangents are drawn to a given circle. Prove that the locus of the mid point of the chord of contact is another circle.

that each term will be greater than the sum of all the terms that follow it

277 First two terms of an A P as wall as an H P as well are a and b. If x be any term of the A P and y the corresponding term of H P, then will $\frac{x-a}{y-a}=\frac{b}{y}$

278 (a) Terms equidistant from a given term of an A P are multiplied together. Show that the differences of the successive terms of the series so formed are in A P

(b) Find a three digit number divisible by 45 such that its digits are terms of an A P

279 If a, b, c are all distinct-and positive and either in A P or in G P or H P and n be any positive integer, prove that $a^n+c^n > 2b^n$

280 (a) Show that if $a_1, a_2, a_3 \ldots a_n$ be all real and if $(a_1^2+a_2^2+a_3^2+\ldots+a^2_{n-1})(a_2^2+a_3^2+\ldots+a_n^2)$ $=(a_1a_2+a_2a_3+\ldots+a_{n-1}a_n)^2$ prove that $a_1, a_2, a_3, \ldots, a_n$ are in G P

(b) Two rays are drawn through a point O at an angle of 30°. A point P is taken on one of them at a distance a from O. A perpendicular is drawn from the point P to the other ray another perpendicular is drawn from its foot to OP, and so on. Find the length of the resulting infinite polygonal line

281 Show that any even square $(2n)^2$ is equal to the sum of n terms of one series of integers in A P and that any odd square $(2n+1)^2$ is equal to the sum of n terms of another A P increased by unity

282 Prove that any positive integral power (except the first) of any positive integer, is the sum of P consecutive terms of the series 1, 3, 5, 7, ... and find the first of the P terms when the sum is p^r

283 Show that if $a(b-c)x^2+b(c-a)xy+c(a-b)y^2$ is a perfect square, the quantities a, b, c are in H P

284 Show that the sum of the product of every pair of the squares of the first n natural numbers is $\frac{1}{360}n(n-1)(4n-1)(5n+6)$

285 The sixth term in the expansion of the binomial $\left(\frac{1}{x^{8/3}}+x^2\log_{10}x\right)^8$ is 5600. Prove that $x=10$

164 The polars of a point P with respect to two circles meet in the point Q Prove that the circle on PQ as diameter passes through two fixed points and cuts both the given circles at right angles

165 If two circles cut orthogonally, prove that the polar of any point P on the first circle with respect to the second passes through the other end of the diameter of the first circle which goes through P

166 Two straight lines OAB and OCD through a fixed point O are drawn to meet a circle in A B and C, D respectively Prove that the locus of the point of intersection of AD and BC as well as the locus of the point of intersection of AC and BD is the polar of O with respect to the circle

167 Find the locus of a point whose shortest distance from the circle $x^2+y^2-2x+6y-6=0$ is equal to its distance from the line $x-3=0$

Calculus

168 Find $\frac{dy}{dx}$ when $y=(1+\sqrt{x})(1+\sqrt{[2x]})(1+\sqrt{[3x]})$

169 If $f(x)=(x^2+x+1)(x^2-x+1)$, find $f'(0)$ and $f'(1)$

170 If $f(x)=(x-1)(x-2)(x-3)$, find $f'(0)$, $f'(1)$ and $f'(2)$

171 Differentiate the following functions

(i) $\left(x^{x}\right)^{x}$, (ii) $x^{x^{x}}$ (iii) $\left(\frac{x}{1+x}\right)^{x}$

(iv) $\sqrt{[\{x+\sqrt{(x+\sqrt{(x)})}\}]}$ (v) $\sqrt[3]{\{\arctan \sqrt[5]{[(\cos \ln^3 x)]}\}}$

172 Prove that the function $y=\frac{1}{2}x^2+\frac{1}{2}x\sqrt{(x^2+1)}+\ln[\sqrt{\{x+\sqrt{(x^2+1)}\}}]$ satisfies the relation $2y=xy'+\ln(y')$

173 Find $\frac{dy}{dx}$ when

(i) $2^x+2^y=2^{x+y}$, (ii) $\sin(xy)+\cos(xy)=\tan(x+y)$

174 If $xy-\ln(y)=1$, prove that $y^2+(xy-1)\frac{dy}{dx}=0$

175 If $y=(\tan^{-1}x)$, prove that $(x^2+1)^2\frac{d^2y}{dx^2}+2x(x^2+1)\frac{dy}{dx}-2=0$

176 If $y=x^n[a\cos(\ln x)+b\sin(\ln x)]$ show that $x^2y''+(1-2n)xy'+(1+n^2)y=0$

286 Find the number of terms which are free of radicals in the expansion of $(x^{1/5}+y^{1/10})^{55}$

287 If n is a positive integer, prove that

$$1-n\left(\frac{1+x}{1+nx}\right)+\frac{n(n-1)}{1.2}\frac{1+2x}{(1+nx)^2}-\frac{n(n-1)(n-2)}{1.2.3}\frac{1+3x}{(1+nx)^3}-\ldots=0$$

288 If a, b, c be three consecutive coefficients in the expansion of a power of $1+x$, prove that the index of the power is $\frac{2ac+b(a+c)}{b^2-ac}$ and that the number of the term of which a is the coefficient is $\frac{a(b+c)}{b^2-ac}$

289 If $C_0, C_1, C_2, \ldots$ be the coefficients in the expansion of $(1+x)^n$, prove that

(i) $C_1^2+2C_2^2+3C_3^2+\ldots+nC_n^2=\frac{(2n-1)!}{(n-1)!(n-1)!}$

(ii) $C_0^2+2C_1^2+3C_2^2+\ldots+(n+1)C_n^2=\frac{(n+2)(2n-1)!}{(n-1)!n!}$

290 Prove $^3C_1+{}^7C_2+{}^{11}C_3+\ldots+{}^{4n-1}C_n=1+(2n-1)2^n$

291 If $s=a+(a+d)+(a+2d)+\ldots+(a+nd)$ and $S=a+(a+d)C_1+(a+2d)C_2+(a+3d)C_3+\ldots+(a+nd)C_n$, prove that $nS=2^n s$, where $C_1, C_2, C_3, \ldots$ have their usual meanings

292 Prove that $\begin{vmatrix} 1 & 1 & 1 \\ b+c & c+a & a+b \\ b^2+c^2 & c^2+a^2 & a^2+b^2 \end{vmatrix}=(b-c)(c-a)(a-b)$

293 Prove that $\begin{vmatrix} bc & bc'+b'c & b'c' \\ ca & ca'+c'a & c'a' \\ ab & ab'+a'b & a'b' \end{vmatrix}=(bc'-b'c)(ca'-c'a)(ab'-ba')$

294 Prove that $\begin{vmatrix} bc-a^2 & ca-b^2 & ab-c^2 \\ -bc+ca+ab & bc-ca+ab & bc+ca-ab \\ (a+b)(a+c) & (b+c)(b+a) & (c+a)(c+b) \end{vmatrix}$
$=3(a-b)(b-c)(c-a)(a+b+c)(bc+ca+ab)$

177 Differentiating the following from first principles

(i) $\cos^2(\log x)$ (ii) $\sin^{-1}\{2x/(1+x^2)\}$

(iii) $(x^3+7)/(3x^2+5x+1)$ (iv) $e^{\sin^{-1} x}$

(v) $\sqrt[3]{(\sin^n ax)}$ (vi) $e^x \cos^n x$

178 Show that the function $y=f(x)$, defined by the parametric equations $x=e^t \sin t$ and $y=e^t \cos t$ satisfies the relation $y''(x+y)^2=2(xy'-y)$

179 Find the value of y' at the point $x=1$ if $x^3-2x^2y^2+5x+y-5=0$ and $y=1$ at $x=1$

180 Given the function $y=a e^{2x}+bx e^{2x}+e^x$ Show that this function satisfies the equation $y''-4y'+4y=e^x$

181 If $\frac{d}{dx}[x^m-A_1 x^{m-1}+A_2 x^{m-2}-A_3 x^{m-3}+\ldots+(-1)^m A_m]\, e^x \equiv x^m e^x$, find the value of A_r, $0<r\leqslant m$

182 Evaluate the following limits

(i) $\lim\limits_{x\to 0} \frac{\log \tan 2x}{\log \tan 3x}$ (ii) $\lim\limits_{x\to\pi/2} (\sin x)^{\tan x}$

(iii) $\lim\limits_{x\to 0} \frac{5\sin x-7\sin 2x+3\sin 3x}{\tan x-x}$

183 Find the values of a, b, c so that

$$\lim_{x\to 0}\frac{a e^x-b\cos x+c e^{-x}}{x\sin x}=2$$

184 If $F(x)\,F(y)=F(x)+F(y)+F(xy)-2$ for all real x and y and $F(2)=5$ determine $F(5)$

185 Does the limit of $\frac{\sqrt{(1-\cos 2x)}}{x}$ at $x=0$ exist ?

186 If $f(x)=\cos^{-1}\left\{\frac{b+a\cos x}{a+b\cos x}\right\}$ prove that

$$\lim_{x\to 0} f'(x)=\sqrt{\left(\frac{a-b}{a+b}\right)}$$

187 A function φ is defined as follows

$$\varphi(x)=\begin{cases}-2\sin x & \text{if } x\leqslant-\pi/2\\ a\sin x+b & \text{if } -\pi/2<x<\pi/2\\ \cos x & \text{if } x\geqslant\pi/2\end{cases}$$

For what values of a and b the function φ is continuous

188 Using "$\epsilon-\delta$" definition of continuity show that the function f defined by $f(x)=x^2$ when x is rational and $f(x)=-x^2$ when x is irrational is continuous at $x=0$ and discontinuous for $x\neq 0$

295 Show that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2-bc & b^2-ca & c^2-ab \end{vmatrix}=0$

296 Prove $\Delta=\begin{vmatrix} b^2+c^2 & ab & ca \\ ab & c^2+a^2 & bc \\ ca & bc & a^2+b \end{vmatrix}=\begin{vmatrix} -a^2 & ab & ac \\ ab & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix}$

by expressing each determinant as the square of a determinant, hence show that $\Delta=4a^2b^2c^2$

297 Let $\Delta=\begin{vmatrix} 2a_1b_1 & a_1b_2+a_2b_1 & a_1b_3+a_3b_1 \\ a_1b_2+a_2b_1 & 2a_2b_2 & a_2b_3+a_3b_2 \\ a_1b_3+a_3b_1 & a_3b_2+b_3a_2 & 2a_3b_3 \end{vmatrix}$

Express Δ as a product of two determinants Hence or otherwise show that $\Delta=0$

298 Prove that the sequence $\{a_n\}$ where $a_n=\left(1+\frac{1}{n}\right)^n$ increases monotonically, that is, prove the validity of the inequality $a_{n+1}=\left(1+\frac{1}{n+1}\right)^{n+1}>\left(1+\frac{1}{n}\right)^n=a_n$

299 Solve the inequality $|x-1|+|x+1|<4$

300 For what values of parameter λ the inequality $\frac{2-\lambda x-x^2}{1-x+x^2}\leqslant 3$ holds true for all values of x

301 Given two events A and B If odds against A are as 2 1 and those in favour of $A\cup B$ are as 3 1 Show that

$$\frac{5}{12}\leqslant P(B)\leqslant\frac{3}{4}$$

302 Out of m persons who are sitting in a circle three are selected at random Prove that the chance that no two of these selected are sitting next to each other is

$$(m-4)(m-5)/(m-1)(m-2)$$

303 (The Encounter Problem) Two persons P and Q have agreed to meet at a definite spot between 12 noon and one o clock The first one to come waits for 20 minutes and then leaves What is the probability of a meeting between P and Q if the arrival of each during indicated hour can occur at random

189 Show that the function f defined by

$$f(x)=\lim_{t\to\infty}\frac{(1+\sin \pi x)^t-1}{(1+\sin \pi x)^t+1}$$

is discontinuous at the points $x=0, 1, 2, 3, \ldots, n, \ldots$

190 Show that the function f defined by $f(x)=e^{-|x|}$ is continuous but non differentiable at $x=0$

191 Construct the graph of the function $y=-\sqrt{(x^2-4x+4)}$

192 If $\varphi(x)=f(x)=f(x)+f(1-x)$ and $f''(x)<0 \leqslant x \leqslant 1$, show that $\varphi(x)$ increases in $0 \leqslant x \leqslant \frac{1}{2}$ and decreases in $\frac{1}{2} \leqslant x \leqslant 1$

193 Prove that for the caternary $y=a \cosh (x/a)$ the perpendicular dropped from the foot of the ordinate upon any tangent is of constant length

194 Prove that in an ellipse $x^2/a^2+y^2/b^2=1$, the length of the normal at any point varies as the length of the perpendicular from the origin on the tangent at the point

195 A curve $y=px^2+qx+r$ touches the lines $y=x$ at the point $x=1$ and passes through the point $(-1, 1)$ Determine the values of p, q, r

196 A piece of wire of length l is cut into two parts, one of which is bent in the shape of a circle, and the other into the shape of a square How should the wire be cut so that the sum of the areas of the circle and the square is minimum

197 The sum of the surfaces of a sphere and a cube is given Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube

198 A fence h metres high runs parallel to and a metres from a vertical wall Find the length of the shortest ladder which will reach from the ground across the top of the fence to the wall, given that $h^{2/3}+a^{2/3}=16$

199 The tangent to the graph of the function $y=f(x)$ at the point with absissa $x=a$ forms the x axis an angle of $\pi/3$ and at the point with abscissa $x=b$ an angle of $\pi/4$ Compute $\int_a^b f''(x)\,dx$ and $\int_a^b f'(x) f''(x)\,dx$, $f''(x)$ is supposed to be continuous

200 Find

(i) $\displaystyle\int \frac{dx}{\cos^3 x\sqrt{(\sin 2x)}}$ (ii) $\displaystyle\int \frac{dx}{\sqrt{(1+e^x+e^{2x})}}$

and the times of arrival are independent (i e the time of arrival of one person does not affect the arrival time of the other)

304 A sportsman's chance of shooting an animal at a distance r is a^2/r^2 He fires when $r=2a$ He reloads if he misses and fires when $r=3a, 4a, 5a,$ If he misses at a distance na, the animal escapes Show that the odds against the sports man are $(n+1)$ $(n-1)$

305 A bag contains n counters marked 1, 2, 3, n If two coun ters are drawn, show that the chances that the difference of the counters exceeds m $(m \leqslant n-1)$ is

$$\frac{(n-m)(n-m-1)}{n(n-1)}$$

306 *A* man has coins *A B C* *A* is unbiased the probability that a head will result when *B* is tossed is 2/3, and the probability that a head will result when *C* tossed is $\frac{1}{3}$ If one of the counters, chosen at random, is tossed three times giving two heads and one tail Find the probability that the chosen coin was *A*

Vectors

307 If the vectors **a**, **b**, **c** satisfy the condition $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$, calculate the value of $\mathbf{a}\ \mathbf{b}+\mathbf{b}\ \mathbf{c}+\mathbf{c}\ \mathbf{a}$ if $|\mathbf{a}|=1, |\mathbf{b}|=3$ and $|\mathbf{c}|=4$

308 If $\mathbf{e}_1$ and $\mathbf{e}_2$ are non-collinear unit vectors compute $(2\mathbf{e}_1-5\mathbf{e}_2)$ $(3\mathbf{e}_1+\mathbf{e}_2)$ if $|\mathbf{e}_1+\mathbf{e}_2|=\sqrt{3}$

309 The vector collinear with the vector $\mathbf{b}=(12, -16, 15)$ makes an acute angle with the basis vector **k** Knowing that $|\mathbf{a}|=100$, find the coordinates of the vector **a**

310 Find λ and μ if the vectors $\mathbf{a}=3\mathbf{i}+\lambda\mathbf{j}-\mathbf{k}$ is perpendicular to the vector $\mathbf{b}=2\mathbf{i}+\mathbf{j}+\mu\mathbf{k}$ and $|\mathbf{a}|=|\mathbf{b}|$

311 If **a**, **b**, **c** are the pos tion vectors of three non collinear points *A*, *B*, *C* respectively, show that the shortest distance from *A* to *BC* is

$$\frac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{|\mathbf{b}-\mathbf{c}|}$$

312 Show that the volume of a tetrahedron, whose three coterminous edges in the right handed orientation are **a** **b**, **c** is $\frac{1}{6}$ [**abc**]

(*d*) $h=OA\tan\alpha,\quad h=OB\tan\beta$

$OA=h\cot\alpha,\quad OB=h\cot\beta$

Now from $\triangle OAB$ by cosine formula

$d^2=OA^2+OB^2-2OA\ OB\cos\gamma$

$=h^2[\cot^2\alpha+\cot^2\beta$

$-2\cot\alpha\cot\beta\cos\gamma]$

$$h=\frac{d}{[\cot^2\alpha+\cot^2\beta-2\cot\alpha\cot\beta\cos\gamma]^{1/2}}$$

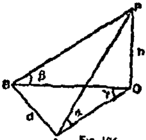


Fig 106

(*e*) Do yourself

46 Let PQ represent the flag staff on the house OP and let

$\angle PAQ=\angle PBQ=\beta$

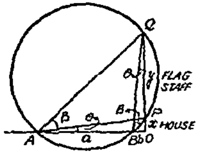
where $OA=a$ and $OB=b$

Let $OP=x$ and $PQ=y$

Clearly points A, B, P, Q are concyclic so that

$\angle PQB=\angle PAB=\theta$, say Now

$$\tan\theta=\frac{OP}{OA}=\frac{BO}{OQ}$$



or $\tan\theta=\dfrac{x}{a}=\dfrac{b}{x+y}$, $\quad x^2+xy=ab$ (1)

Again $\tan\beta=\tan(\beta+\theta-\theta)=\dfrac{\tan(\beta+\theta)-\tan\theta}{1+\tan(\beta+\theta)\tan\theta}$

$$\text{or}\quad \tan\beta=\frac{\dfrac{x+y}{a}-\dfrac{x}{a}}{1+\dfrac{x(x+y)}{a^2}}=\frac{ya}{a^2+x^2+xy}$$

or $x^2+xy+a^2=ya\cot\beta$ (2)

Subtracting (1) from (2), $\quad a^2=ya\cot\beta-ab$

or $y=(a+b)\tan\beta$

47 $AQ=x$, $BQ=y$ and

PQ is a tower leaning towards north at an angle θ From P draw $PM=h$ perpendicular and let $QM=a$ Both h and a are unknown and θ is to be determined

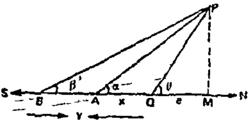


Fig 107

$h=a\tan\theta,$ (1) $\quad h=(a+x)\tan\alpha$ (2)

$=(a+y)\tan\beta$ (3)

$a = h\cot\theta,\quad a+x = h\cot\alpha,\quad a+y = h\cot\beta$

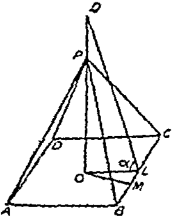
Multiply 2nd by y and 3rd by x and subtract

$$a(y-x) = h(y\cot\alpha - x\cot\beta)$$

or $\quad h\cot\theta\,(y-x) = h(y\cot\alpha - x\cot\beta)$ (by Ist)

$$\theta = \cot^{-1}\frac{(y\cot\alpha - x\cot\beta)}{y-x}$$

48 Let $PQ=h$ represent the flag staff on the top of a pyramid of height $OP=p$, say Elevation of Sun is the angle $QLO=\alpha$ The base of the pyramid is the square $ABCD$ It is given that $CL=x$ and $BL=y$ so that side of the square is $x+y$



Now from $\triangle OLQ$

$$OL = OQ\cot\alpha$$

or $\quad OL=(OP+PQ)\cot\alpha=(p+h)\cot\alpha$ (1)

If OM is perpendicular on BC then $\angle OML=90°$

and $OM=MC=\frac{x+y}{2}$ so that $LM=MC-CL=\frac{x+y}{2}-x=\frac{y-x}{2}$

Finally from $\triangle OML$

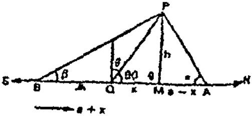
$$OL^2=OM^2+ML^2 \text{ or } OL^2=\left(\frac{x+y}{2}\right)^2+\left(\frac{y-x}{2}\right)^2=\frac{x^2+y^2}{2}$$

$$(p+h)^2\cot^2\alpha=\frac{x^2+y^2}{2}$$

or $\quad p+h=\sqrt{\left[\left(\frac{x^2+y^2}{2}\right)\right]}\tan\alpha$ or $p=\sqrt{\left(\frac{x^2+y^2}{2}\right)}\tan\alpha-h$

49 Proceed as in Q 47

50 Applying $(m-n)$ theorem of trigonometry on $\triangle APB$ in which angle $\angle PQA=90°-\theta$



$(a+a)\cot(90°-\theta)$

$=a\cot\beta - a\cot\alpha$ (1)

$$2\tan\theta=\frac{\sin(\alpha-\beta)}{\sin\alpha\sin\beta}$$

$$\theta=\tan^{-1}\left(\frac{\sin(\alpha-\beta)}{2\sin\alpha\sin\beta}\right)$$

313 A particle acted on by constant forces $4\mathbf{i}+3\mathbf{j}$ and $3\mathbf{i}+2\mathbf{j}$ is displaced from the point $\mathbf{i}+2\mathbf{j}$ to $5\mathbf{i}+4\mathbf{j}$ Find the total work done by the forces

314 Find the torque about the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ of a force represented by $3\mathbf{i}+\mathbf{k}$ acting through the point $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

315 The velocity of a boat relative to water is represented by $3\mathbf{i}+4\mathbf{j}$ and that of the water relative to the earth by $\mathbf{i}-3\mathbf{j}$ What is the velocity of the boat relative to the earth if $\mathbf{i}$ and $\mathbf{j}$ represent velocities of one kilometre an hour east and north respectively ?

Now the given equation is

$$\frac{\cos\theta}{\sin\theta}-\frac{\sin\theta}{\cos\theta}=\frac{1}{\cos\theta} \text{ or } \cos^2\theta-\sin^2\theta-\sin\theta=0$$

or $\cos 2\theta-\cos(\frac{1}{2}\pi-\theta)=0$

or $2\sin\left(\frac{1}{2}\theta+\frac{1}{4}\pi\right)\sin\left(\frac{3}{2}\theta-\frac{1}{4}\pi\right)=0$

$$\frac{1}{2}\theta+\frac{1}{4}\pi=n\pi \quad \text{or} \quad \theta=2n\pi-\frac{1}{2}\pi \tag{1}$$

and $$\frac{3}{2}\theta-\frac{\pi}{4}=n\pi \quad \text{or} \quad \theta=\frac{2}{3}n\pi+\frac{1}{6}\pi \tag{2}$$

But the values of θ given in (1) do not satisfy the given equation and the value of θ given in (2) do not satisfy the the given equation for $n=3m-1, m\in I$ Nence the solution of the equation is

$$\theta=\frac{2}{3}n\pi+\frac{1}{6}\pi, n\in I, n\neq 3m-1, m\in I$$

69 Ans $\frac{1}{7}n\pi+\frac{1}{12}\pi$ or $n\pi+\frac{1}{4}\pi, n\in I$

Hint $\sin 8x-\sqrt{3}\cos 8x=\sqrt{3}\sin 6x+\cos 6x$

Now divide both sides by 2 This gives

$$\sin\frac{\pi}{6}\sin 8x-\cos\frac{\pi}{6}\cos 8x=\sin\frac{\pi}{3}\sin 6x+\cos\frac{\pi}{3}\cos 6x$$

or $$-\cos\left(8x+\frac{1}{6}\pi\right)=\cos\left(6x-\frac{1}{3}\pi\right)$$

$$\cos\left\{\pi-\left(8x+\frac{1}{6}\pi\right)\right\}=\cos\left(6x-\frac{1}{3}\pi\right)$$

$$\pi-\left(8x+\frac{1}{6}\pi\right)=2n\pi\pm\left(6x-\frac{1}{3}\pi\right)$$

Taking +ve sign, we get

$$\pi-8x-\frac{1}{6}\pi=2n\pi+6x-\frac{1}{3}\pi$$

or $x=-\frac{1}{7}n\pi+\frac{1}{12}\pi$

or $x=\frac{1}{7}n\pi+\frac{1}{12}\pi$ [n is an integer, it maks no difference if we write $-n$ in place of n]

Answers and Hints

Trigonometry

1 Ans $\sqrt{(2-a^2)}$, $1-(a^2-1)^2/2$

[Hint $\sin\theta+\cos\theta=a \Rightarrow 1+\sin 2\theta=a^2 \Rightarrow \sin 2\theta=a^2-1$

$$|\sin\theta-\cos\theta|=\sqrt{(\sin\theta-\cos\theta)^2}$$
$$=\sqrt{(1-\sin 2\theta)}=\sqrt{(2-a^2)}$$

and $\cos^4\theta+\sin^4\theta=(\cos^2\theta+\sin^2\theta)^2-\frac{1}{2}(\sin 2\theta)^2$ etc]

2 (a) Ans -1

[Hint Expression $=4\cos 20°-2\sin 60°\cot 20°$

$$=(4\sin 20°\cos 20°-2\sin 60°\cos 20°)/\sin 20°$$
$$=\{2\sin 40°-(\sin 80°+\sin 40°)\}/\sin 20°$$
$$=(\sin 40°-\sin 80°)/\sin 20° \text{ etc]}$$

(b) $4\sin 27°=\sqrt{(16\sin^2 27°)}=\sqrt{\{8(1-\cos 54°)\}}$

$$=\sqrt{[8(1-\sin 36°)]}=\sqrt{\left[8\left(1-\frac{\sqrt{(10-2\sqrt{5})}}{4}\right)\right]}$$
$$=\sqrt{[8-2\sqrt{(10-2\sqrt{5})}]}=\sqrt{(5+\sqrt{5})}-\sqrt{(3-\sqrt{5})},$$

by trial

5 (i) Ans, $\frac{1}{8}$

(ii) Expression

$$=\frac{\sin^2 57°-\sin^2 33°}{\sin 21°-\sin 69°}=\frac{\sin(57°+33°)\sin(57°-33°)}{2\cos 45°\sin(-24°)}=-\frac{1}{\sqrt{2}}$$

6 $\cot^2 36°\cot^2 72°=(\cot^2 36°\cot^2 72°-1)+1$

$$=\frac{\cos^2 36°\cos^2 72°-\sin^2 36°\sin^2 72°}{\sin^2 36°\sin^2 72°}+1$$
$$=1+\frac{\cos(72°+36°)\cos(72°-36°)}{\sin^2 36°\sin^2 72°}=1+\frac{\cos 108°\cos 36°}{\sin^2 36°\sin^2 72°}$$
$$=1+\frac{\cos 72°\cos 36°}{\sin^2 72°\sin^2 36°}=1-\cot 72°\cot 36°\frac{1}{\cos 72°\cos 36°}$$
$$=1-\cot^2 72°\cot^2 36°\frac{4\sin 36°}{4\sin 36°\cos 36°\cos 72°}$$
$$=1-\cot^2 72°\cot 36°\frac{4\sin 36°}{\sin 144°}$$
$$=1-4\cot 72°\cot^2 36°\frac{4\sin 36°}{\sin 36°}=1-4\cot^2 72°\cot^2 36°$$
$$5\cot^2 72°\cot^2 36°=1 \text{ or } \cot 72°\cot 36°=\frac{1}{\sqrt{5}}$$

Similarly taking $-$ve sign, we shall get

$x = n\pi - \frac{1}{4}\pi$

70 Ans $\frac{1}{2} n\pi$, $n \in I$

Hint Adding $2\sin^2 x\cos^2 x$ to both sides we get
$(\sin^2 x+\cos^2 x)^2=\cos 4x+2\sin^2 x\cos^2 x$

or $1=\cos 4x+\frac{1}{2}\sin^2 2x=\cos 4x+\frac{1}{4}(1-\cos 4x)$

or $\cos 4x=1$, which gives $4x=2n\pi$ or $x=\frac{1}{2}n\pi$

71 Ans $n\pi+\frac{1}{4}\pi$, $n \in I$

Hint Since $\sin^2 x+\cos^2 x=1$, the given equation can be written as

$$2\sin^3 x=\cos x\,(\sin^2 x+\cos^2 x)$$

Dividing by $\cos^3 x$, we get
$2\tan^3 x=\tan^2 x+1$ or $2\tan^3 x-\tan^2 x-1=0$
which on factorization can be written as

$$(\tan x-1)(2\tan^2 x+\tan x+1)=0$$

$$2\tan^2 x+\tan x+1=\left(\sqrt{2}\tan x+\frac{1}{2\sqrt{2}}\right)^2+\frac{7}{8}>0,$$

Hence $\tan x-1=0$ which gives $x=n\pi+\frac{1}{4}\pi$

72 Ans $\frac{1}{2}n\pi+\frac{1}{4}\pi$ or $\frac{2}{3}n\pi+\frac{1}{6}\pi$, $n \in I$

Hint $2\sin^3 x-\sin x+\cos^2 2x=0$
or $\cos^2 2x-\sin x\,(1-2\sin^2 2x)=0$
or $\cos^2 2x-\sin x\cos 2x=0$
or $\cos 2x\,(\cos 2x-\sin x)=0$ etc

73 Ans $\frac{1}{8}(4n+1)\pi$ or $\frac{1}{4}n\pi-(-1)^n\left(\frac{1}{24}\pi\right)$, $n \in I$

Hint $(\tan 2x+\cot 2x)^2+2(\tan 2x+\cot 2x)-8=0$
Now put $\tan 2x+\cot 2x=t$ etc

74 $n\pi+\frac{1}{2}\pi$, $n \in I$

75 (b) The given equation can be written as
$3\sin x-4\sin^3 x-\sin x-4\sin x\cos x+3=0$
or $\sin x\,(2-4\cos x-4+4\cos^2 x)+3=0$

7 (i) $\sqrt{3}$ (ii) $\frac{1}{2}\sqrt{(2-\sqrt{2})}$, (iii) 1, (iv) $\frac{3}{4}$, (v) $-1/\sqrt{2}$

9 $\frac{5\sqrt{2}}{2}$ if $0 < A < \frac{1}{2}\pi$, and $\frac{31\sqrt{2}}{10}$ if $\frac{\pi}{2} < A < \pi$

[Note that $A \neq \frac{1}{2}\pi$, since $\sin A = \frac{4}{5}$ whereas $\sin \frac{\pi}{2} = 1$]

10 Adding both sides of given equalities, we get

$$\frac{a}{b} + \frac{A}{B} = \frac{\sin(x-\alpha)\cos(x-\beta) + \cos(x-\alpha)\sin(x-\beta)}{\sin(x-\beta)\cos(x-\beta)}$$

$$= \frac{\sin(2x-\alpha-\beta)}{\sin(x-\beta)\cos(x-\beta)} \quad (1)$$

Again multiplying both sides of given equalities,

$$\frac{aA}{bB} = \frac{\sin(x-\alpha)\cos(x-\alpha)}{\sin(x-\beta)\cos(x-\beta)}$$

Adding 1 to both sides,

$$\frac{aA+bB}{bB} = \frac{\sin(x-\alpha)\cos(x-\alpha) + \sin(x-\beta)\cos(x-\beta)}{\sin(x-\beta)\cos(x-\beta)}$$

$$= \frac{\frac{1}{2}\sin 2(x-\alpha) + \frac{1}{2}\sin 2(x-\beta)}{\sin(x-\beta)\cos(x-\beta)}$$

$$= \frac{\frac{1}{2}\; 2\sin(2x-\alpha-\beta)\cos(\alpha-\beta)}{\sin(x-\beta)\cos(x-\beta)}$$

Thus $\frac{aA+bB}{bB} = \frac{\sin(2x-\alpha-\beta)\cos(\alpha-\beta)}{\sin(x-\beta)\cos(x-\beta)}$ (2)

Now dividing (2) by (1), we get

$$\frac{aA+bB}{aB+bA} = \cos(\alpha-\beta)$$

11 First note that conditions given for x, α and β make both sides of the given equality meaningful. To prove the result, change left hand side of the given equality in terms of sine and cosine and write the right hand side as $(1-\cos\beta)/(1+\cos\beta)$. Now apply componendo and dividendo.

12 Hint We have $\cot B = \frac{\sin(A+C)}{2\sin A \sin C}$

$$= \frac{\sin A\cos C + \cos A \sin C}{2 \sin A \sin C}$$

or $\qquad 2\cot B = \cot C + \cot A$

14 (a) Hint We have $3\sin^2 A = 1 - 2\sin^2 B = \cos 2B$ and $6\sin A\cos A = 2\sin 2B$. Now divide.

(b) $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1-\tan\theta\tan\phi} = \frac{\tan\theta + \tan^3\theta}{1-\tan^4\theta}$

$$= \frac{\tan\theta}{1-\tan^2\theta} = \frac{1}{2}\tan 2\theta = \tan\alpha$$

$\theta + \phi = n\pi + \alpha$

or $\sin x\,[(2\cos x-1)^2-3]+3=0$

or $\sin x\,(2\cos x-1)^2+3\,(1-\sin x)=0$ (1)

since $0<x<\pi$ we have $0<\sin x\leqslant 1$

Hence $\sin x>0$ and $1-\sin x\geqslant 0$

so that $\sin x\,(2\cos x-1)^2\geqslant 0$

Hence (1) can hold only when we simulaneously have

$2\cos x-1=0$ and $1-\sin x=0$

or $\cos x=\frac{1}{2}$ and $\sin x=1$

But there is no real value of x which simulaneously satisfies both these equations Hence the origonal equation has no solution

76 $2n\pi$ or $n\pi+\frac{1}{2}\pi$ or $n\pi-\frac{1}{4}\pi,\ n\in I$

77 $x=(m+n)\pi+\frac{1}{6}\pi,\ y=(n-m)\pi-\frac{1}{6}\pi,$

or $x=(m+n)\pi-\frac{1}{6}\pi,\ y=(n-m)\pi+\frac{1}{6}\pi,\ m, n\in I$

Hint Add and subtract the given equations

78 $2n\pi+\frac{1}{2}\pi$

79 $(4n-1)\frac{\pi}{4},\ (2n+1)\frac{\pi}{2},\ n\in I$

80 Ans $n\pi+(-1)^n\frac{\pi}{8},\ n\pi+(-1)^n\frac{3\pi}{8}$

Hint The given equation is equivalent to the equation

$\sin x\,|\cos x|=\frac{1}{2\sqrt{2}}$

We consider two cases

(a) $\cos x>0$ We then have the equation

$\sin x\cos x=\frac{1}{2\sqrt{2}}$ or $\sin 2x=\frac{1}{\sqrt{2}}$ (1)

Solution of (1) is given by

$2x=n\pi+(-1)^n\left(\frac{1}{4}\pi\right)$ or $x=\frac{1}{2}n\pi+(-1)^n\left(\frac{1}{8}\right)\pi$ (2)

But out of these values, we have to select only those values which satesfy the condition $\cos x>0$ To do this we find those values of n for which the appropriate values of x lie in the first and fourth quadrants For $n=0, 1, 2, 3$ the values of x from (2) are

$\frac{1}{8}\pi,\ \frac{1}{2}\pi-\frac{1}{8}\pi,\ \pi+\frac{1}{8}\pi,\ \frac{3\pi}{2}-\frac{1}{8}\pi,$

16 Hint, $\cot 3A=\cot(2A-A)$ etc

17 (i) Hint $4x^2-16x+15<0 \Rightarrow (2x-3)(2x-5)<0$

$$\Rightarrow \frac{3}{2}<x<\frac{5}{2}$$

The only integral solution of the inequality is 2
Also the slope of the bisector of the first quadrant is $\tan 45°=1$
Hence $\tan \alpha=2$ and $\cos \beta=1$ by given condition
Now $\sin(\alpha+\beta)\sin(\alpha-\beta)=\sin^2\alpha-\sin^2\beta=\cos^2\beta-\cos^2\alpha$

$$=\cos^2\beta-\frac{1}{1+\tan^2\alpha}=1-\frac{1}{1+4}=\frac{4}{5}$$

18 Let $\theta=18°$ Then $\tan 3\theta=\tan(90°-2\theta)$ etc whence $\tan^2 18°=0.1056$

20 (a) Dividing the given relation by $\cos^2\alpha\cos^2\beta$, we get
$[\tan\alpha-\tan\beta+\cos 2\beta\tan\beta-\tan\alpha\tan\beta\sin 2\beta]^2$
$=4\tan\beta(\tan\alpha+\tan\beta)$

$[(\tan\alpha+\tan\beta)\cos 2\beta-\tan\alpha\cos 2\beta+\tan\alpha-\tan\beta$
$-\tan\alpha\tan\beta\sin 2\beta]^2=4\tan\beta(\tan\alpha+\tan\beta)$

or $[(\tan\alpha+\tan\beta)\cos 2\beta+\tan\alpha\ 2\sin^2\beta-\tan\beta-\tan\alpha\frac{\sin\beta}{\cos\beta}$
$2\sin\beta\cos\beta]^2=4\tan\beta(\tan\alpha+\tan\beta)$

or $[(\tan\alpha+\tan\beta)\cos 2\beta-\tan\beta]^2=4\tan\beta(\tan\alpha+\tan\beta)$ (1)

Now put $\tan\beta=x(\tan\alpha+\tan\beta)$ Then (1) reduces to

$$(\cos 2\beta-x)^2=4x$$

or $x^2-2x(\cos 2\beta+2)+\cos^2 2\beta=0$

$x=\frac{1}{2}\{2(\cos 2\beta+2)\pm\sqrt{[4(\cos 2\beta+2)^2-4\cos^2 2\beta]}\}$

$=2\cos^2\beta+1\pm\sqrt{(4+4\cos 2\beta)}$

$=2\cos^2\beta+1\pm 2\sqrt{2}\cos\beta$

Now since $x<1$ and $\cos\beta>0$, we must take $-$ve sign

Hence $x=2\cos^2\beta+1-2\sqrt{2}\cos\beta=(\sqrt{2}\cos\beta-1)^2$

$$\tan\alpha+\tan\beta=\frac{\tan\beta}{x}=\frac{\tan\beta}{(\sqrt{2}\cos\beta-1)^2}$$

or $\tan\alpha=\tan\beta\left[\frac{1}{(\sqrt{2}\cos\beta-1)^2}-1\right]$

(b) We have $\frac{4\sin^4 A}{a}+\frac{4\cos^4 A}{b}=\frac{4}{a+b}$

or $\frac{(1-\cos 2A)^2}{a}+\frac{(1+\cos 2A)^2}{b}=\frac{4}{a+b}$

or $b(a+b)(1-2\cos 2A+\cos^2 2A)$
$+a(a+b)(1+2\cos 2A+\cos^2 2A)=4ab$

i.e. $\frac{1}{8}\pi, \frac{3}{8}\pi, \frac{9}{8}\pi, \frac{11}{8}\pi$ (2)

The first two values of x given in (3) satisfy the condition

$$\cos x > 0$$

Hence the solution in this case is given by

$x=2m\pi+\frac{1}{8}\pi,\ x=2m\pi+\frac{3}{8}\pi,\ m \in I$

(b) $\cos x < 0$ Proceeding as in case (a), the solution is given by

$x=2m\pi+\frac{5\pi}{8},\ x=2m\pi+\frac{7\pi}{8},\ m \in I$

Thus we get four groups of solutions

$x=2m\pi+\frac{\pi}{8},\ x=2m\pi+\frac{3\pi}{8},\ x=m\pi+\frac{5\pi}{8},\ x=2m\pi+\frac{7\pi}{8}$

These four groups can be combined into two groups as follows

$x=k\pi+(-1)^k\frac{\pi}{8},\ x=k\pi+(-1)^k\frac{3\pi}{8},\ k \in I$

81 $2n\pi \pm \frac{1}{12}\pi,\ 2n\pi \pm \frac{5}{12}\pi,\ n \in I$

12 Ans No roots

Hint Re write the given equation as
$\sin 3x - \sin x - 2\sin 2x + 3 = 0$
or $2\cos 2x \sin x - 4\sin x\cos x + 3 = 0$
or $\sin x(2\cos 2x - 4\cos x) + 3 = 0$
or $\sin x(4\cos^2 x - 4\cos x - 2) + 3 = 0$
or $\sin x[(2\cos x - 1)^2 - 3] + 3 = 0$
or $\sin x(2\cos x - 1)^2 + 3(1 - \sin x) = 0$ (1)
Since $0 \leqslant x \leqslant \pi$, $\sin x \geqslant 0$ and $1 - \sin x \geqslant 0$
Obviously $(2\cos x - 1)^2 \geqslant 0$
Hence equation (1) is equivalent to the system of equations
$\sin x(2\cos x - 1)^2 = 0$ and $3(1 - \sin x) = 0$
From the second of these equations, we have
$\sin x = 1$ But then $\cos x = 0$ and
$\sin x(2\cos x - 1)^2 = 1 \neq 0$
Hence the given equation has no solution

83 Ans 3 roots Hint Write the equation as $\tan x = \pi/4 - x/2$
Draw the graph of the curve $y = \tan x$ and the straight line $y = \pi/4 - x/2$ in the interval $0 \leqslant x \leqslant 2\pi$ It can be easily

or $(a+b)^2 \cos^2 2A + 2(a^2-b^2)\cos 2A + (a-b)^2 = 0$

or $[(a+b)\cos 2A + (a-b)]^2 = 0$ or $\cos 2A = \dfrac{b-a}{b+a}$

Hence $\dfrac{1}{a^3}\sin^8 A + \dfrac{1}{b^3}\cos^8 A$

$$= \frac{1}{16a^3}(1-\cos 2A)^4 + \frac{1}{16b^3}(1+\cos 2A)^4$$

$$= \frac{1}{16a^3}\left(1-\frac{b-a}{b+a}\right)^4 + \frac{1}{16b^3}\left(1+\frac{b-a}{b+a}\right)^4$$

$$= \frac{16a^4}{16a^3(b+a)^4} + \frac{16b^4}{16b^3(b+a)^4} = \frac{1}{(a+b)^3}$$

21 Let $y = \cos\theta\,[(\sin\theta + \sqrt{(\sin^2\theta + \sin^2\alpha)}]$

Then $(y\sec\theta - \sin\theta)^2 = \sin^2\theta + \sin^2\alpha$

or $y^2\sec^2\theta - 2y\sec\theta\sin\theta - \sin^2\alpha = 0$

or $y^2\tan^2\theta - 2y\tan\theta + y^2 - \sin^2\alpha = 0$

which is a quadratic in $\tan\theta$. Since $\tan\theta$ is real, we must have

$4y^2 - 4y^2(y^2 - \sin^2\alpha) \geqslant 0$

or $1 - y^2 + \sin^2\alpha \geqslant 0$

or $y^2 \leqslant 1 + \sin^2\alpha$

whence $-\sqrt{(1+\sin^2\alpha)} \leqslant y \leqslant \sqrt{(1+\sin^2\alpha)}$

22 **Hint** $\sin(\alpha+\beta+\gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma$
$+ \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\cos\gamma$

$\sin\alpha + \sin\beta + \sin\gamma - \sin(\alpha+\beta+\gamma)$

$= \sin\alpha(1-\cos\beta\cos\gamma) + \sin\beta(1-\cos\gamma\cos\alpha)$
$+ \sin\gamma(1-\cos\alpha\cos\beta) + \sin\alpha\sin\beta\sin\gamma > 0$

$[\quad 0 < \alpha, \beta\ \gamma < \tfrac{1}{2}\pi]$

23 **Hint** Let $y = \dfrac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)} = \dfrac{(\tan\theta + \tan\alpha)(1+\tan\theta\tan\alpha)}{(\tan\theta - \tan\alpha)(1-\tan\theta\tan\alpha)}$

Now let $\tan\theta = x$ and $\tan\alpha = a$. Then

$y = \dfrac{(x+a)(1+xa)}{(x-a)(1-xa)}$ or $y(x-a)(1-xa) = (x+a)(1+xa)$

or $a(1+y)x^2 + (1+a^2)(1-y)x + a(1+y) = 0$

Since x is real we have

$(1+a^2)^2(1-y)^2 - 4a^2(1+y)^2 \geqslant 0$

or $\{(1+a^2)(1-y) - 2a(1+y)\}\{(1+a^2)(1-y) + 2a(1+y)\} \geqslant 0$

or $\{(1-a)^2 - (1+a)^2 y\}\{(1+a)^2 - (1-a)^2 y\} \geqslant 0$

or $\left\{\dfrac{(1-a)^2}{(1+a)^2} - y\right\}\left\{\left(\dfrac{1+a}{1-a}\right)^2 - y\right\} \geqslant 0$ (1)

seen that the line will cut the curve in three points Th abscissa of these points are the required roots of the give equation

84 Ans $n\pi, n\pi+(-1)^n(\pi/6), n \in I$

Hint Write the equation as

$$\sin^2 x-\sin x \sin^2 3x+\tfrac{1}{4}\sin^2 3x=0$$

or $(\sin x-\tfrac{1}{2}\sin^2 3x)^2+\tfrac{1}{4}\sin^2 3x\,(1-\sin^2 3x)=0$

or $(\sin x-\tfrac{1}{2}\sin^2 3x)^2+\dfrac{1}{16}\sin^2 6x=0$

This equation is evidently equivalent to the system of equation

$$2\sin x=\sin^2 3x \quad \sin 6x=0$$

From second of these equations, we have

$2\sin 3x\cos 3x=0$, i e $\sin 3x=0$ or $\cos 3x=0$

When $\sin 3x=0$, first equation gives $\sin x=0$ from which $x=n\pi$, $n \in I$, and when $\cos 3x=0$, then $\sin^2 3x=1$, and then from the first equation we get $\sin x=\tfrac{1}{2}$, whose solution is

$$n\pi+(-1)^n(\pi/6), n \in I$$

85 $\dfrac{1}{2}n\pi\pm\dfrac{1}{6}\pi$ $n \in I$

86 $\dfrac{1}{4}n\pi\pm\dfrac{1}{12}\pi, n \in I$

[Hint Write L H S as

$(\sin^2 2x+\cos^2 2x)^3-3\sin^2 2x\cos^2 2x(\sin^2 2x+\cos^2 2x)$]

87 $\dfrac{1}{4}n\pi+\dfrac{1}{8}\pi, n \in I$

Hint First prove that $16\sin^8 x+16\cos^8 x=2+12\cos^2 2x+2\cos^4 2x$ and then pass to $\cos 4x$]

88 $n\pi$, $n\pi-\dfrac{1}{4}\pi, n \in I$

89 $n\pi\pm\tfrac{1}{2}\cos^{-1}\left(-1+\dfrac{\sqrt{6}}{2}\right)$

90 We have

$\cos x-2\sin 2x-\cos 3x=(\cos x-\cos 3x)-2\sin 2x$

$=2\sin x\sin 2x-\sin 2x=2\sin 2x(\sin x-1)$

and $\quad 1-2\sin x-\cos 2x=(1-\cos 2x)-2\sin x$

$$=2\sin^2 x-2\sin x$$

$$=2\sin x(\sin x-1)$$

Thus the given equation is equivalent to the equation

$$|\sin 2x(\sin x-1)|=\sin x(\sin x-1)$$

Since $\left(\frac{1+a}{1-a}\right)^2 > \left(\frac{1-a}{1+a}\right)^2$, the inequality will hold if either

$$y < \left(\frac{1-a}{1+a}\right)^2 \quad \text{or} \quad y > \left(\frac{1+a}{1-a}\right)^2,$$

that is, if y does not lie between $\left(\frac{1-a}{1+a}\right)^2$ and $\left(\frac{1+a}{1-a}\right)^2$

But $\left(\frac{1-a}{1+a}\right)^2 = \left(\frac{1-\tan\alpha}{1+\tan\alpha}\right)^2 = \tan^2\left(\frac{1}{4}\pi - \alpha\right)$

and $\left(\frac{1+a}{1-a}\right)^2 = \tan^2\left(\frac{1}{4}\pi + \alpha\right)$ It follows that y cannot lie between $\tan^2\left(\frac{1}{4}\pi - \alpha\right)$ and $\tan^2\left(\frac{1}{4}\pi + \alpha\right)$

24 **Hint** (i) To prove $\cot 44° = \frac{3 + \cot 76° \cot 16°}{\cot 16° + \cot 76°}$

$$\text{R H S} = \frac{3 \sin 76° \sin 16° + \cos 76° \cos 16°}{\sin 76° \cos 16° + \cos 76° \sin 16°}$$

$$= \frac{2 \sin 76° \sin 16° + \sin 76° \sin 16° + \cos 76° \cos 16°}{\sin(\ 6 + 16°)}$$

$$= \frac{\{\cos(76° - 16°) - \cos(76° + 16\ )\} + \cos(76° - 16°)}{\sin 92°}$$

Put $\cos 60° = \frac{1}{2}$

$$= \frac{1 - \cos 92°}{\sin 92°} = \frac{2 \sin\ 46°}{2 \sin 46° \cos 46°} = \tan 46° = \cot 44°$$

(ii) $\tan 70° - \tan 20° = \frac{\sin(70° - 20°)}{\cos 70° \cos 20°} = \frac{2 \sin 50°}{\cos 90° + \cos 50°}$

$$= 2 \tan 50°$$

Now we have to prove $2 \tan 50° = 2 \tan 40° + 4 \tan 10°$
or $\tan 50° - \tan 40° = 2 \tan 10°$
which can be proved as above

25 **Hint** Denoting the given expression by y, we shall get
$x^2 (y - 1)\ \ 2x (y \cos\beta\ \ \cos\alpha) + y - 1 = 0$
Since x is real, $4 (y \cos\beta - \cos\alpha)^2 - 4 (y-1)^2 \geq 0$

$\Rightarrow (y \cos\beta - \cos\alpha - y - 1)(y \cos\beta - \cos\alpha + y - 1) \geq 0$

$\Rightarrow \{y \cos\beta - 1) + (1 - \cos\alpha)\}\{y(1 + \cos\beta) - (1 - \cos\alpha)\} \geq 0$

$\Rightarrow \left\{2 \sin^2\frac{\alpha}{2} - 2y \sin^2\frac{\beta}{2}\right\}\left\{2y \cos^2\frac{\beta}{2} - 2\cos^2\frac{\alpha}{2}\right\} \geq 0$

$\Rightarrow \left\{\left(\sin^2\frac{\alpha}{2}\Big/\sin^2\frac{\beta}{2}\right) - y\right\}\left\{y - \left(\cos^2\frac{\alpha}{2}\Big/\cos^2\frac{\beta}{2}\right)\right\} \geq 0$

$\Rightarrow \left(\sin\ \frac{\alpha}{2}\Big/\sin^2\frac{\beta}{2}\right) \leq y \leq \left(\cos\ \frac{\alpha}{2}\right)\Big/\left(\cos^2\frac{\beta}{2}\right)$

Now since $\sin x-1 \leqslant 0$, we have $|\sin x-1| = -(\sin x-1)$
Hence $-2|\sin x|\ |\cos x|(\sin x-1)=\sin x(\sin x\ 1)$
$\sin x-1=0$ or $-2|\sin x|\ |\cos x| = \sin x$
Solution of the first equation is $x=2n\pi+\frac{1}{2}\pi$, $n \in I$
To solve the equation $-2|\sin x|\ |\cos x| = \sin x$,
we consider two cases (i) $\sin x \geqslant 0$, (ii) $\sin x < 0$
In case (i), we have $-2 \sin x |\cos x| = \sin x$ from which we get $\sin x=0$ or $|\cos x| = -\frac{1}{2}$
First of these equations gives, $x=n\pi$, $n \in I$ and the second equation gives no solution
In case (ii), we get the equation $2 \sin x |\cos x| = \sin x$ and sin since $\sin x \neq 0$ ($\sin x < 0$) we have $|\cos x| = \frac{1}{2}$ or $\cos x = \pm\frac{1}{2}$ Its solution is

$$x=2n\pi\pm\frac{1}{3}\pi \quad x=2n\pi\pm\frac{2\pi}{3}$$

But from these values of x we must choose only those values for which $\sin x < 0$ In the first case, we have

$\sin(2n\pi\pm\frac{1}{3}\pi)=\pm\frac{\sqrt{3}}{2}$ and this means that we must take

$x=2n\pi-\frac{1}{3}\pi$ (In this case $\sin x < 0$)
Similarly in the second case we must take the solution as

$x=2n\pi-\frac{2\pi}{3}$ Thus we get the following answer

$$n\pi,\ 2n\pi+\frac{1}{2}\pi,\ 2n\pi-\frac{1}{3}\pi\ \ 2n\pi-\frac{2\pi}{3}$$

91 $A=2n\pi\pm\frac{1}{3}\pi$, $B=2m\pi\pm\frac{1}{3}\pi$ $m\ n \in I$
Hint Write the first equation as
$\frac{1}{2}(\cos A+\cos B)=\frac{1}{2}$ or $\cos A+\cos B=1$
Then solving this equation with $\cos A \cos B=1/4$ we get

$\cos A=\frac{1}{2}$ and $\cos B=\frac{1}{2}$ etc

92 (i) $x=(m+n)\pi\pm(1/6)\pi$ $y=(n-m)\pi\mp(1/6)\pi$ $m, n \in I$

(ii) $x=(2m+n)\pi+(-1)^n\frac{\pi}{2}-\frac{\pi}{3}$

$$y=(2m-n)\pi+(-1)^n\frac{\pi}{2}+\frac{\pi}{3},\ m, n \in I$$

Hint We have $\sin\frac{1}{2}(x+y)\cos\frac{1}{2}(x-y)=\frac{1}{2}$

and $\sin\frac{1}{2}(x+y)\sin\frac{1}{2}(x-y)=-\frac{\sqrt{3}}{2}$

Now putting $\frac{1}{2}(x+y)=s$ and $\frac{1}{2}(x-y)=t$ we get

26 The given relation is equivalent to

$1-\cos 24^\circ+1-\cos 42^\circ+1-\cos 78^\circ+1-\cos 96^\circ$
$=2+1-\cos 18^\circ+1-\cos 36^\circ$

or $(\cos 16^\circ-\cos 42^\circ)-(\cos 24^\circ+\cos 96^\circ)$
$+\cos 36^\circ-\cos 78^\circ=0$

or $2\sin 30^\circ \sin 12^\circ-2\cos 60^\circ \cos 36^\circ+\cos 36^\circ \quad \sin 12^\circ=0$

which is clearly true

27 Hint Dividing the given relation by $\cos^2\theta \cos^2\phi$, we get

$k^2(\tan\theta+\tan\phi)^2=\tan^2\theta(1+\tan^2\phi)+\tan^2\phi(1+\tan^2\theta)$

or $k^2(\tan\theta+\tan\phi)^2=(\tan\theta-\tan\phi)^2$
$-2\tan\theta\tan\phi(1+\tan\theta\tan\phi)$

or $\dfrac{\tan\theta-\tan\phi}{\tan\theta+\tan\phi}=\pm k$

$\dfrac{\tan\theta}{\tan\phi}=\dfrac{1\pm k}{1\mp k},$

by componendo and dividendo

29 (b) From the given conditions, it follows that

$0<\theta+\phi+\psi<3\pi/2,\ 0<\frac{1}{2}(\phi+\psi)<\frac{1}{2}\pi,\ \tan\frac{1}{2}\phi\tan\frac{1}{2}\psi<1$

(θ, ϕ, ψ being positive acute angles), and hance we can make the following transformations

$$\tan\tfrac{1}{2}(\phi+\psi)=\frac{\tan\frac{1}{2}\phi+\tan\frac{1}{2}\psi}{1-\tan\frac{1}{2}\phi\tan\frac{1}{2}\psi}=\frac{\frac{1}{3}\cot\frac{1}{2}\theta+\dfrac{2}{3\tan\frac{1}{2}\theta+\cot\frac{1}{2}\theta}}{1-\frac{1}{3}\cot\frac{1}{2}\theta\,\dfrac{2}{3\tan\frac{1}{2}\theta+\cot\frac{1}{2}\theta}}$$

$=\cot\frac{1}{2}\theta$ (after simplification)

i e $\tan\frac{1}{2}(\phi+\psi)=\tan\left(\dfrac{\pi}{2}-\dfrac{1}{2}\theta\right)$

or $\frac{1}{2}(\phi+\psi)=\dfrac{\pi}{2}-\frac{1}{2}\theta$ or $\theta+\phi+\psi=\pi$

[This is the only possible value since $0<\theta+\phi+\psi<3\pi/2$]

30 Ans $\cos^4(\beta-\alpha)-2\cos^2(\beta-\alpha)(b+a^2)+(b+a^2)^2$
$=4a^2b\sin^2(\beta-\alpha)$

Hint We have $\sin(\beta-\alpha)=\sin\{(\theta+\beta)-(\theta+\alpha)\}$ which gives

$\sin(\beta-\alpha)=\sqrt{(1-b)}\sqrt{(1-a^2)}+a\sqrt{b}$

or $\{\sin(\beta-\alpha)-a\sqrt{b}\}^2=(1-b)(1-a^2)$

or $\sin^2(\beta-\alpha)+a^2b-1+b+a^2-a^2b=2a\sqrt{b}\sin(\beta-\alpha)$

or $[(b+a^2)-\cos^2(\beta-\alpha)]^2=4a^2b\sin^2(\beta-\alpha)$

31 Subtracting the first equation from second, we get

$m(\cot x-\tan x)-n(\tan 2x+\cot 2x)=0$

$\sin s \cos t = 1/2$ and $\sin s \sin t = -\frac{\sqrt{3}}{2}$

Thus $\tan t = -\sqrt{3}$ on dividing

Hence $t = n\pi - \frac{1}{3}\pi, n \in I$

Substituting these values of t in (1) we get

$\sin s = (-1)^n$ whence $s = 2m\pi + (-1)^n \frac{\pi}{2}$

Thus $\frac{1}{2}(x+y) = 2m\pi + (-1)^n \frac{\pi}{2}$

and $\frac{1}{2}(x-y) = n\pi \quad \frac{1}{3}\pi$

Solving these equations for x and y, we get the solution

93 $x = n\pi \pm \frac{1}{12}\pi, y = m\pi \pm \frac{1}{6}\pi, m, n \in I$

94 (a) (i) Let $\cot^{-1}\left(-\frac{3}{4}\right) = \alpha$, then $\cot \alpha = -\frac{3}{4}$ and $\frac{1}{2}\pi < \alpha$

so that $\frac{1}{4}\pi < \frac{1}{2}\alpha < \frac{1}{2}\pi$ Now

$\sin\left[\frac{1}{2}\cot^{-1}\left(-\frac{3}{4}\right)\right] = \sin\frac{1}{2}\alpha = \sqrt{\left\{\frac{1}{2}\left(1-\cos\alpha\right)\right\}}$

$= \sqrt{\left\{\frac{1}{2}\left(1-\left(-\frac{3}{5}\right)\right)\right\}} = \frac{2}{\sqrt{5}}$

[Note that $\frac{1}{2}\pi < \alpha < \pi$ and $\cot \alpha = -\frac{3}{4} \Rightarrow \cos \alpha = -\frac{3}{5}$]

(ii) If $\cos^{-1}(-\frac{1}{3}) = \alpha$ then $\cos \alpha = -\frac{1}{3}$ and $0 < \alpha < \pi$

Since $\cos \alpha < 0$ we have $\frac{1}{2}\pi < \alpha < \pi$ and so $\cot \alpha < 0$

Hence $\cot \alpha = \frac{-1}{\sqrt{(3^2-1)}} = -\frac{1}{2\sqrt{2}} = -\frac{\sqrt{2}}{4}$

(b) (i) Let $\alpha = \tan^{-1}(-3)$ Then α obviously satisfies inequality $-\pi/2 < \alpha < 0$, whence $-\pi < 2\alpha < 0$ We n consider various ways to determine $2\tan^{-1}(-3) = 2\alpha$

(a) $-\pi < 2\alpha < 0 \Rightarrow \pi > -2\alpha > 0 \Rightarrow 0 < -2\alpha < \pi$

Then $\cos(-2\alpha) = \cos 2\alpha = \frac{1-\tan^2\alpha}{1+\tan^2\alpha} = \frac{1-9}{1+9} = -\frac{4}{5}$

$-2\alpha = \cos^{-1}(-4/5)$ or $2\alpha = -\cos^{-1}(-4/5)$

Thus $2\tan^{-1}(-3) = -\cos^{-1}(-4/5)$

(b) We have $-\pi < 2\alpha < 0 \Rightarrow 0 < \pi + 2\alpha < \pi$

Furthermore $\cos(\pi + 2\alpha) = -\cos 2\alpha = 4/5$

Hence $\pi + 2\alpha = \cos^{-1} 4/5$ that is, $2\alpha = -\pi + \cos^{-1}(4/5)$

Thus $2\tan^{-1}(-3) = -\pi + \cos^{-1}(4/5)$

or $\dfrac{m\cos 2x}{\sin x\cos x}=\dfrac{n}{\sin 2x\cos 2x}$

which gives $\cos 2x=\sqrt{\left(\dfrac{n}{2m}\right)}$

Now write the first equation as

$p\sin 2x=2m\sin^2 x+n\cos 2x$

or $p\sqrt{(1-\cos^2 2x)}=m-(m-n)\cos 2x$

or $p\sqrt{\left(1-\dfrac{n}{2m}\right)}=m-(m-n)\sqrt{\left(\dfrac{n}{2m}\right)}$

or $p\sqrt{(2m-n)}=m\sqrt{(2m)}-(m-n)\sqrt{n}$

32 We have, $\cos\alpha+\cos\beta=-\cos\gamma$, (1)

$\sin\alpha+\sin\beta=-\sin\gamma$

Squaring and adding (1) and (2),

$2+2\cos(\alpha-\beta)=1$ or $\cos(\alpha-\beta)=-\frac{1}{2}$,

which gives $\alpha-\beta=2n\pi\pm\dfrac{2\pi}{3}$, $n\in I$ (1)

Again dividing (1) by (2) and simplifying, we shall get (2)

$$\tan\left(\frac{\alpha+\beta}{2}\right)=\tan\gamma,$$

whence $\gamma=m\pi+\frac{1}{2}(\alpha+\beta)$, $m\in I$

Hence $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\frac{1}{2}(\alpha+\beta)$

$=\sin^2\alpha+\sin^2\left(\alpha-\dfrac{2\pi}{3}\right)+\sin^2\frac{1}{2}\left(2\alpha-\dfrac{2\pi}{3}\right)$

$=\sin^2\alpha+\left(-\frac{1}{2}\sin\alpha-\dfrac{\sqrt{3}}{2}\cos\alpha\right)^2+\left(\frac{1}{2}\sin\alpha-\dfrac{\sqrt{3}}{2}\cos\alpha\right)^2$

$=\sin^2\alpha+2\left(\dfrac{1}{4}\sin^2\alpha+\dfrac{3}{4}\cos^2\alpha\right)=\dfrac{3}{2}\left(\sin^2\alpha+\cos^2\alpha\right)=\dfrac{3}{2}$

[Note that we have taken +ve sign in (1) The same result will be obtained by taking −ve sign]

34 R H S $=2\sin^2\phi+2\sin\phi\,[\sin(2\theta+\phi)-\sin\phi]+\cos(2\theta+2\phi)$

$=[\cos 2\theta-\cos(2\theta+2\phi)]+\cos(2\theta+2\phi)=\cos 2\theta$

35 Ans $(x+y)(x^3+y^3)=(x+y)(x+y+1)(x+y-1)$

36 $-4\cos\dfrac{3}{2}A\cos\dfrac{3}{2}B\cos\dfrac{3}{2}C$

44 First multiply both sides of the given relations

$\dfrac{\cos\theta}{\cos\alpha}+\dfrac{\sin\theta}{\sin\alpha}=1$ and $\dfrac{\cos\phi}{\cos\alpha}+\dfrac{\sin\phi}{\sin\alpha}=1$

This will give

(c) We have $-\pi < 2\alpha < 0 \Rightarrow -\pi/2 < \pi/2+2\alpha < \pi/2$
and moreover $\tan(\pi/2+2\alpha) = -\cot 2\alpha$

$$= -\frac{1}{\tan 2\alpha} = -\frac{1-\tan^2\alpha}{2\tan\alpha}$$

$$= -\frac{1-9}{(-6)} = -\frac{4}{3}$$

And so $\pi/2+2\alpha = \tan^{-1}(-4/3)$ that is,

$$2\alpha = -\pi/2+\tan^{-1}(-4/3)$$

Thus $2\tan^{-1}(-3) = -\pi/2+\tan^{-1}(-4/3)$

Finally from (*a*), (*b*) and (*c*), we conclude

$2\tan^{-1}(-3) = -\cos^{-1}(-4/5) = -\pi+\cos^{-1}(4/5)$

$$= -\pi/2+\tan^{-1}(-4/3)$$

We have to prove

$\cos^{-1} 3/5 - 2\tan^{-1}\frac{1}{2} = \pi/2 - \cos^{-1} 4/5$ (1)

First observe that the three angles in (1) lie between 0 and $\pi/2$. It is clear that $0 < \cos^{-1} 3/5 < \pi/2$. Since if $\alpha = \cos^{-1} 3/5$, then $0 \leqslant \alpha \leqslant \pi$. But $\cos\alpha = 3/5$ is positive. It follows that $0 < \alpha < \pi/2$. Similarly $0 < \cos^{-1} 4/5 < \pi/2$ which implies that $0 > -\cos^{-1} 4/5 > -\pi/2$ and so $\pi/2 > \pi/2 - \cos^{-1} 4/5 > 0$ or $0 < \pi/2 - \cos^{-1} 4/5 < \pi/2$

Finally if $\tan^{-1}\frac{1}{2} = \beta$, then $-\pi/2 < \beta < \pi/2$ and $\tan\beta = \frac{1}{2}$. It follows that $0 < \beta < \pi/4$ and hence $0 < 2\beta < \pi/2$

Since all three angles lie between 0 and $\pi/2$, to prove their equality it suffices to show that some trigonometrical ratio say the cosine of angle has one and the same value. Thus we have $\cos\cos^{-1} 3/5 = 3/5$

$$\cos(2\tan^{-1}\tfrac{1}{2}) = \cos 2\beta = \frac{1-\tan^2\beta}{1+\tan^2\beta} = \frac{1-\frac{1}{4}}{1+\frac{1}{4}} = 3/5$$

and $\cos(\pi/2 - \cos^{-1} 4/5) = \sin\cos^{-1} 4/5 = \sin\sin^{-1} 3/5 = 3/5$

This completes the proof

Remark Observe carefully that the equality of some trigonometrical ratios does not alway imply the equality of the two angles. For example $\cos\sin^{-1}\frac{1}{2} = \cos\pi/6 = \sqrt{3}/2$ and $\cos\sin^{-1}(-\frac{1}{2}) = \cos(-\pi/6) = \sqrt{3}/2$, but $\sin^{-1}\frac{1}{2} \neq \sin^{-1}(-\frac{1}{2})$. However in certain cases the equality of some ratio may imply the equality of angles as the above example shows. But in such cases we must first ensure that the angles lie in a certain interval say $[0, \pi/2]$ or any other suitable interval

$$\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}+\frac{\sin(\theta+\phi)}{\sin\alpha\cos\alpha}=1 \quad (1)$$

Now subtracting the given relations and simplifying, we shall get

$$\tan\frac{\theta+\phi}{2}=\cot\alpha=\tan\left(\frac{\pi}{2}-\alpha\right)$$

This gives $\frac{\theta+\phi}{2}=n\pi+\frac{\pi}{2}-\alpha$

or $\theta+\phi=2n\pi+\pi-2\alpha$

Hence $\sin(\theta+\phi)=\sin(\pi-2\alpha)=\sin 2\alpha=2\sin\alpha\cos\alpha$

Now (1) gives,

$$\frac{\cos\theta\cos\phi}{\cos^2\alpha}+\frac{\sin\theta\sin\phi}{\sin^2\alpha}+1=0$$

45 $\text{L H S}=\frac{1}{4}[\{\cos(2\theta-\alpha-\beta)+\cos(\alpha-\beta)\}\{\cos(2\theta-\gamma-\delta)+\cos(\gamma-\delta)\}$
$+\{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)\}\{\cos(\gamma-\delta)-\cos(2\theta-\gamma-\delta)\}]$

$=\frac{1}{2}[\cos(2\theta-\alpha-\beta)\cos(2\theta-\gamma-\delta)+\cos(\alpha-\beta)\cos(\gamma-\delta)]$

$=\frac{1}{2}[\cos(\gamma+\delta)\cos(\alpha+\beta)+\cos(\alpha-\beta)\cos(\gamma-\delta)]$

$=\frac{1}{2}[(\cos\gamma\cos\delta-\sin\gamma\sin\delta)(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$
$+(\cos\alpha\cos\beta+\sin\alpha\sin\beta)(\cos\gamma\cos\delta+\sin\gamma\sin\delta)]$

$=\cos\alpha\cos\beta\cos\gamma\cos\delta+\sin\alpha\sin\beta\sin\gamma\sin\delta$

50 **Hint** L H S

$=2\sin(mA+mB)\cos(mA-mB)+2\sin mC\cos mC$

$=2\sin(m\pi-mC)\cos(mA-mB)+2\sin mC\cos[m\pi-m(A+B)]$

$=2(-1)^{m-1}\sin mC[\cos(mA-mB)-\cos(mA+mB)]$

$=4(-1)^{m-1}\sin mA\sin mB\sin mC$

52 $2n\pi\pm\frac{1}{6}\pi,\ 2n\pi+\frac{1}{5}\pi,\ n\in I$

53 $\frac{1}{2}n\pi+(-1)^n\left(\frac{\pi}{12}\right),\ n\in I,$

54 $n\pi+(-1)^{n+1}\left(\frac{1}{4}\pi\right),\ n\in I$

55 $\frac{1}{2}n\pi+\frac{1}{4}\pi\ \ n\in I$

56 $\frac{1}{5}n\pi+(-1)^n\left(\frac{1}{20}\pi\right)-\frac{2}{5},\ n\in I$

95 First note that 10 radian $=(3\pi+\alpha)$ radian where $0<\alpha<\frac{1}{2}\pi$
Hence $-\alpha=3\pi-10$ so that $-\frac{1}{2}\pi<3\pi-10<0$ (1)
Now we know that $\sin^{-1}\sin\theta=\theta$ provided $-\frac{1}{2}\pi\leqslant\theta\leqslant\frac{1}{2}\pi$ (2)

Hence arc $\sin(\sin 10)=\sin^{-1}\sin 10=\sin^{-1}\sin(3\pi+\alpha)$
$=\sin^{-1}(-\sin\alpha)=\sin^{-1}\sin(-\alpha)$
$=\sin^{-1}\sin(3\pi-10)=3\pi-10,$
by (1) and (2)

96 We have $\tan\cos^{-1}x=\tan\tan^{-1}\{\sqrt{(1-x^2)}/x\}=\sqrt{(1-x^2)}/x$
and $\sin\cot^{-1}(1/2)=\sin\sin^{-1}(2/\sqrt{5})=2/\sqrt{5}$
Hence the equation becomes $\sqrt{(1-x^2)}/x=2/\sqrt{5}$ or $9x^2=5$
$x=\pm\sqrt{5}/3$ But $x=-\sqrt{5}/3$ does not satisfy the equation since $\pi/2<\cos^{-1}(-\sqrt{5}/3)<\pi$ and so $\tan\cos^{-1}(-\sqrt{5}/3)$ will be negative whereas $\sin\cot^{-1}(1/2)$ is positive Hence $x=\sqrt{5}/3$ is the only solution

97 Ans 0

Hint If $\sin^{-1}x=\alpha$ and $\sin^{-1}(10x/13)=\beta$, then $2\alpha=\beta$ whence $\sin 2\alpha=\sin\beta$ which gives $2\sin\alpha\cos\alpha=\sin\beta$ or $2x\sqrt{(1-x^2)}$ $=10x/13\Rightarrow x=0,\ 12/13\quad -12/13$ But $x=\pm 12/13$ do not satisfy the given equation Hence $x=0$ is the only solution [Note that $x=12/13\Rightarrow 10x/13=120/169$ and $12/13>120/169$ whence $\alpha>\beta\Rightarrow 2\alpha>\beta$ Similarly for $x=-\frac{12}{13}$ can be proved]

98 Let $x=\cot\theta$ Then the equation takes the form
$\frac{2\pi}{3}=\cos^{-1}\{(\cot^2\theta-1)/(\cot^2\theta+1)+\tan^{-1}\{2\tan\theta/(1-\tan^2\theta)\}$
$=\cos^{-1}\cos 2\theta+\tan^{-1}\tan 2\theta=2n\pi\pm 2\theta+m\pi+2\theta,\ m\ n\in I$
$=(2n+m)\pi+4\theta$ [$-2\theta+2\theta$ does not give any solution]
$4\theta=2\pi/3-p\pi\ p\in I$
In this putting $p=0,-1,-2\ -3\ -4,$, we get
$\theta=\pi/6,\ 5\pi/12,\ 2\pi/3\ \ 11\pi/12\ \ 7\pi/6,$
and putting $p=1,\ 2\ \ 3\ \ 4$
$\theta=-\pi/12\ \ -\pi/3\ \ -7\pi/12\ \ -5\pi/6$
It can be easily seen that for all these values of θ only four values of $x=\cot\theta$ namely $\sqrt{3}\ \ 2-\sqrt{3},\ -1/\sqrt{3}$ and $-(2+\sqrt{3})$ are obtained It can be checked by substitution that last two values of x do not satisfy the equation For when $x=-1/\sqrt{3}$, then
$\cos^{-1}\{(x^2-1)/(x^2+1)\}=\cos^{-1}(-1/2)$

57 $2n\pi+\pi$, $n \in I$

58 $n\pi+\frac{1}{4}\pi$, $n \in I$, Hint Write the R H S as

$(\cos\theta-\sin\theta)^2+(\cos^2\theta-\sin^2\theta)$

59 $\frac{1}{4}n\pi$, $n \in I$

60 $n\pi$, $\frac{1}{2}n\pi \pm \frac{1}{21}\pi$, $n \in I$

Hint $\cos 4x=\cos^2 3x$ or $2\cos 4x=1+\cos 6x$

or $2(2\cos^2 2x-1)=1+(4\cos^3 2x-3\cos 2x)$

or $4\cos^3 2x-4\cos^2 2x-3\cos 2x+3=0$

or $(\cos 2x-1)(4\cos^2 2x-3)=0$ etc

61 $n\pi$, $n \in I$

62 (i) $n\pi+\frac{1}{2}\pi$, $n\pi+(-1)^n\left(\frac{1}{6}\pi\right)$, $2n\pi \pm\frac{2}{3}\pi$, $n \in I$

(ii) $n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$, $n \in I$

63 $\frac{1}{9}(6n+1)\pi$, $\frac{1}{3}(6n-1)\pi$, $n \in I$

Hint Divide both sides by 2 Then the given equation will take the form $\cos\left(\theta-\frac{1}{3}\pi\right)=\cos 2\theta$ etc

64 $n\pi-\frac{1}{4}\pi$, $n\pi$, $n \in I$

Hint Put $\tan x=t$ Then

$$\sin 2x=\frac{2\tan x}{1+\tan^2 x}=\frac{2t}{1+t^2}$$

Then given equation will take the form

$t^2(1+t)=0$ giving $\tan x=0$ or -1.

65 $n\pi\pm\frac{1}{6}\pi$, $n \in I$

Hint Put $\cos 2\theta=t$ Then $2\cos^2\theta=1+\cos 2\theta=1+t$

and $\tan^2\theta=\frac{1-\cos 2\theta}{1+\cos 2\theta}=\frac{1-t}{1+t}$ etc

66 $n\pi+\frac{1}{4}\pi$, $n\pi+\alpha$, $n \in I$, $\tan\alpha=2$ $0<\alpha<\frac{1}{2}\pi$

67 $n\pi+\frac{1}{4}\pi$, $n \in I$

68 $\frac{2}{3}n\pi+\frac{1}{6}\pi$ $n \in I$, $n\neq 3m-1$, $m \in I$

Hint First note that the equation will be meaningful if $\theta\neq\frac{1}{2}n\pi$ $(n \in I)$

$=2-\sqrt{3}$ and $\tan^{-1}\{2x/(x^2-1)\}=\tan^{-1}\sqrt{3}=\pi/3$,
so that in this case L H S $=2\pi/3+\pi/3=\pi\neq$ R H S. Similarly it can be seen that $x=-(2+\sqrt{3})$ does not satisfy the equation. Hence $x=\sqrt{3}$ and $x=2-\sqrt{3}$ are the only roots.

99 Ans $x=13$ 100 Ans $x=\sqrt{3}$

104 Let the lengths of three medians AD, BE, CF meeting at G, be p_1, p_2, p_3. Then $b^2+c^2=2AD^2+2CD^2=2p_1^2+2\cdot\frac{1}{4}a^2$,
whence $p_1^2=(2b^2+2c^2-a^2)/4$ with similar expressions for p_2^2 and p_3^2.
Also $GA=\frac{2}{3}p_1$, $GB=\frac{2}{3}p_2$ and $GC=\frac{2}{3}p_3$. Hence
$\cos\beta=(\frac{4}{9}p_1^2+\frac{4}{9}p_3^2-b^2)/(2.\frac{4}{9}p_1p_3)$
and $\frac{1}{2}\cdot\frac{2p_1}{3}\cdot\frac{2p_3}{3}\sin\beta=\triangle AGC=\frac{1}{3}\triangle ABC=\frac{1}{3}\triangle$

$\cot\beta=\{\frac{4}{9}(p_1^2+p_3^2)-b^2\}/(4\triangle/3)=(c^2+a^2-5b^2)/12\triangle$,
and so for $\cot\alpha$ and $\cot\gamma$

$\therefore$ $\cot\alpha+\cot\beta+\cot\gamma=-(a^2+b^2+c^2)/4\triangle$

Also $\cot A+\cot B+\cot C=\dfrac{2bc\cos A}{2bc\sin A}+\ldots+\ldots=\dfrac{b^2+c^2-a^2}{4\triangle}+\ldots+\ldots=\dfrac{(a^2+b^2+c^2)}{4\triangle}$. Hence the result.

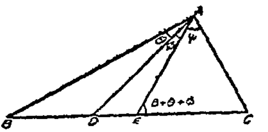
105 With usual notations, we have $\angle I_1I_2I_3=\pi-(\pi/2-A/2)-(\pi/2-C/2)=\pi/2-B/2$
and $I_1I_2=(r_1+r_2)/\cos(C/2)=s(\tan\frac{1}{2}A+\tan\frac{1}{2}B)/\cos\frac{1}{2}C$
$=s\cos\frac{1}{2}C/\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$

The required radius $=\dfrac{\frac{1}{2}I_1I_2\cdot I_2I_3\sin\angle I_1I_2I_3}{\frac{1}{2}(I_1I_2+I_2I_3+I_3I_1)}$ $\left[\text{Formula } r=\dfrac{S}{s}\right]$

$=\left(\dfrac{s}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}\right)\left(\dfrac{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}\right)$

$=\dfrac{R(\sin A+\sin B+\sin C)}{\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C}$ (Poove it yourself)

107 Here $t_1=\tan\theta$,
$t_2=\tan\phi$, $t_3=\tan\psi$
and $BD=DE=EC$
Using trigonometrical theorem
$(BE+EC)\cot(B+\theta+\phi)$
$=BE\cot(\theta+\phi)-EC\cot\psi$
or $3\cot(B+\theta+\phi)$



From (I) we can also say that $\theta=\tan^{-1}\frac{1}{2}(\cot\beta-\cot\alpha)$

(*b*) $h=a\tan\theta$

$h=(a+l)\tan\alpha$

$h=a+(l+l_1)\tan(\alpha-\beta)$

$a=h\cot\theta$ (1)

$(a+l)=h\cot\alpha$ (2)

$(a+l+l_1)=h\cot(\alpha-\beta)$ (3)

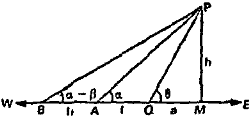


Fig 109

Multiply (2), by $(l+l_1)$ and (3), by l and subtract

$a(l+l_1-l)=h[(l+l_1)\cot\alpha-l\cot(\alpha-\beta)]$

$(h\cot\theta)\,l_1=h[(l+l_1)\cot\alpha-l\cot(\alpha-\beta)]$ by (I)

$$\tan\theta=\frac{l_1}{(l+l_1)\cot\alpha-l\cot(\alpha-\beta)}$$

51 Let $PQ=x$ or $QO=y$ (Unknown)

$\angle AQO=\theta$ say and $\angle APO=\phi$ say

then $\theta=\alpha+\phi$ $\therefore\alpha=\theta-\phi$

Now $c=y\tan\theta$ and $c=(x+y)\tan\phi$

$\alpha=\theta-\phi$ $\tan\alpha=\dfrac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}$

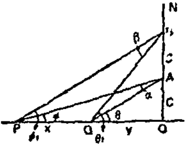


Fig. 110

$$=\frac{\dfrac{c}{y}-\dfrac{c}{x+y}}{1+\dfrac{c^2}{y(x+y)}}$$

$$\text{or }\tan\alpha=\frac{cx}{xy+y^2+c^2}$$

$xy+y^2+c^2=cx\cot\alpha$ (1)

Similarly taking $\angle BQO=\theta_1$ and $\angle BPO=\phi_1$ we have

$\theta_1=\beta+\phi_1$, or $\beta=\theta_1-\phi_1$

$2c=y\tan\theta_1$, $2c=(x+y)\tan\phi_1$ and proceeding as before we have $xy+y^2+(2c)^2=(2c)\,x\cot\beta$ (2)

Subtracting (1), from (2) we get

$3c^2=cx\,(2\cot\beta-\cot\alpha)$

$$PQ=x=\frac{3c}{2\cot\beta-\cot\alpha}$$

52 If the observer is at P and PA, PB are tangents drawn from P to the baloon, then $\angle APB=\alpha$,

$$\angle APO=\angle BPO=\alpha/2$$

Further we are given that the angle of elevation of the centre, i e, $\angle OPQ=\beta$ We have to find the height OQ of the centre O

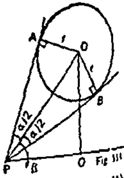


Fig 111

$$\frac{OA}{OP}=\sin\frac{\alpha}{2} \qquad OP=r \operatorname{cosec} \alpha/2 \qquad (1)$$

$$\text{Also } \frac{OQ}{OP}=\sin\beta \qquad OQ=OP\sin\beta=r\operatorname{cosec}\alpha/2\ \sin\beta \text{ by (1)}$$

53 Let C denote the centre of the plate AEB touching the wall at E Let CD $(=c)$ denote the height the candle $OC(=b)$ is the height of the plate above the ground Let MN represent the breadth of the shadow on the wall where it meets the ground OP $(=a)$ is the horizontal distance of MN from O so that OP is perpendicular to MN, and $PN=\frac{1}{2}MN$

Now from similar Δ's DCB and DON, we have

$$\frac{CB}{ON}=\frac{DC}{DO}, \text{ that is, } \frac{a}{ON}=\frac{c}{b+c}$$

$$\text{or} \quad ON=\frac{a(b+c)}{c}$$

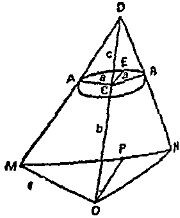


Fig 112

$$MN=2\ PN=2\sqrt{(ON^2-OP^2)}$$
$$=2\sqrt{\left\{\frac{a^2(b+c)^2}{c^2}-a^2\right\}}$$
$$=\frac{2a}{c}\sqrt{(b^2+2bc)}$$

54 Let the height PQ of the tower be h Since the angles of elevation of Q from each of the points A B, C is θ we have

$$PA=PB=PC=h\cot\theta \qquad (1)$$

Since P is equidistant from A B and C it is the circum centre of the ΔABC

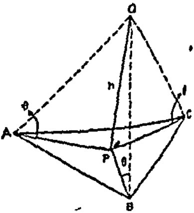


Fig 113

$=2\cot(\theta+\phi)-\cot\psi$ (1) [$BE=2EC$]

Again trigometrical theorem gives

$(DE+EC)\cot(B+\theta+\phi)=DE\cot\phi \quad EC\cot\psi$

or $2\cot(B+\theta+\phi)=\cot\phi-\cot\psi$ (2)

[$DE=EC$]

Eliminating $\cot(B+\theta+\phi)$ from (1) and (2), we get

$\frac{1}{3}[2\cot(\theta+\phi)-\cot\psi]=\frac{1}{2}[\cot\phi-\cot\psi]$

or $4\cot(\theta+\phi)-3\cot\phi=-\cot\psi$

or $\dfrac{4(\cot\theta\cot\phi-1)}{\cot\theta+\cot\phi}-3\cot\phi=-\cot\psi$

or $4\cot\theta\cot\phi-4 \quad 3\cot\theta\cot\phi-3\cot^2\phi$

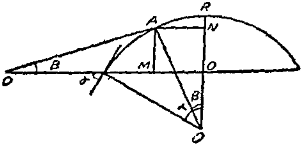
$=-\cot\theta\cot\psi-\cot\phi\cot\psi$

or $\cot\theta\cot\phi+\cot\theta\cot\psi+\cot\phi\cot\psi+\cot^2\phi=4(1+\cot^2\phi)$

or $(\cot\theta+\cot\phi)(\cot\phi+\cot\psi)=4(1+\cot^2\phi)$

or $\left(\dfrac{1}{t_1}+\dfrac{1}{t_2}\right)\left(\dfrac{1}{t_2}+\dfrac{1}{t_3}\right)=4\left(1+\dfrac{1}{t_2^2}\right)$

108 Let O represent the position of the observer and OPQ the horizontal line through O meeting the hill at P and the vertical through the centre C of the sphere in Q let OA be the tangent to the sphere from O touching it in A then



$\angle AOQ=\angle ACQ=\beta$ and $\angle PCQ=\alpha$

[Note that since the hill slopes at α at the bottom, tangent at P makes an angle α with OQ and hence the radius CP must also make an angle α with CQ]

Let r be the radius of the hill and AM the perpendicular to PQ Then

$$\tan\beta=\tan AOQ=\frac{AM}{OP+PQ-MQ}=\frac{QN}{OP+PQ-MQ}$$

$$=\frac{CN-CQ}{OP+PQ-MQ}$$

$$=\frac{r\cos\beta-r\cos\alpha}{a+r\sin\alpha-r\sin\beta}$$

$$I=\int_0^\infty \frac{(t^2-1)\,t}{(t^2+1)^2\,t}\,dt=\int_0^1 \frac{t^2-1}{(t^2+1)^2}\,dt+\int_1^\infty \frac{(t^2-1)}{(t^2+1)^2}\,dt$$

$$=I_1+I_2$$

Now in the second integral I_2 put $t=1/z$, $dt=-1/z^2\,dz$

Then $I_2=\int_1^0 \frac{(1/z^2-1)\,(\;\;1/z^2)\,dz}{(1/z^2+1)^2}=-\int_0^1 \frac{(z^2-1)\,dz}{(z+1)^2}=-I_1$

$I_1+I_2=0$

Hence $I=0$

209 Ans 0

[Hint Represent as the sum of two intgerals as in Q 221 and put $x=1/t$ in second integral]

210 (i) $(2k\,!)/(2^k\,k\,!)^2$

(ii) $\frac{1}{3}\pi$

215 $a=5$, $b=-6$, $c=3$,

216 Ans –

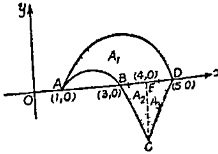
217 1 5

218 4 5

221, $4a^2$

222 Ans 73/6

Hint Solving the given equation we can easily find that $x_A=1$, $x_B=3$, $x_C=4$ and $x_D=5$ If A_1, A_2, A_3 are the areas as shown in the adjoining figure then



$$A_1=\int_1^5 (-x^2+6x-5)\,dx-\int_1^3 (-x^2+4x-3)\,dx=\frac{28}{3},$$

$$A_2=\int_3^4 (-x^2-4x+3)\,dx=4/3 \text{ and}$$

$$A_3=\int_4^5 (-3x+15)\,dx=3/2,$$ after simple calculations Hence the required area $A=A_1+A_2+A_3=73/6$

223 Ans $\frac{32}{3}-4\sqrt{3}+\frac{8}{3}\pi$

Algebra

224 Ans $\frac{a+b}{a}, \frac{a-b}{b}$

225 (i) Ans $4/(1+\sqrt{a}+a)$

(ii) $(1+a)\,(1+a^{1/3})\,(1-a)^{-1}-\sqrt{a}-1$

226 $-1<m<3$

Hence $\sin\beta\,(a+r\sin\alpha-r\sin\beta)=\cos\beta\,(r\cos\beta-r\cos\alpha)$

or $\quad a\sin\beta=r\,[\cos^2\beta-\cos\alpha\cos\beta-\sin\alpha\sin\beta+\sin^2\beta]$

$$=r\,[1-\cos(\alpha-\beta)]=2r\sin^2\left(\frac{\alpha-\beta}{2}\right)$$

or $\quad r=\dfrac{a\sin\beta}{2\sin^2\left(\frac{\alpha-\beta}{2}\right)}$

Height of the hill above the plane $=QR=CR-CQ$

$$=r-r\cos\alpha=2r\sin^2\frac{\alpha}{2}=\frac{a\sin\beta\sin^2\frac{\alpha}{2}}{\sin^2\left(\frac{\alpha-\beta}{2}\right)}$$

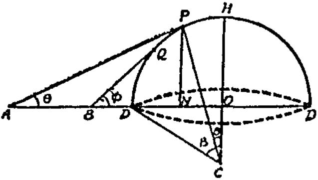
109 Do yourself

110 Ans 114 4123 metre.

111 Do yourself

112 Do yourself

113 Let A, B be the two points of observations Let AB meet the vertical through the centre C of the sphere in O and the hill itself in D,



Let the tangents from A and B touch the sphere at P and Q and let H be the highest point of the hill Draw PN prependicular to OAB If r is the radius of the hill, and β be the $\angle OCD$, we have

$$\tan\theta=\frac{PN}{NA}=\frac{CH-CO}{OD+DA-ON}$$

$$=\frac{r\cos\theta-r\cos\beta}{r\sin\beta+a-r\sin\theta}$$

or $\quad r\,[1-\cos(\theta-\beta)]=a\sin\theta$, after simplifications

$\therefore\quad 2r\sin^2\dfrac{\theta-\beta}{2}=a\sin\theta=2a\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$

227 Squaring, $a+b+c-3x=-2\Sigma\sqrt{\{(a-x)(b-x)\}}$ (1)

or $a+b+c+3x=6x-2\Sigma\sqrt{[(a-x)(b-x)]}$

Squaring again,

$$(a+b+c+3x)^2=4[3x-\Sigma\sqrt{((a-x)(b-x))}]^2$$
$$=4[9x^2-6x\Sigma\sqrt{((a-x)(b-x))}+\{\Sigma\sqrt{(a-x)(b-x)}\}^2]$$
$$=4[9x^2+3x(a+b+c-3x)+\Sigma(a-x)(b-x)+2\sqrt{([a-x)(b-x)(c-x)]}\{\Sigma\sqrt{(a-x)}\}] \text{ using (1)}]$$
$$=4[3x(a+b+c)-\Sigma ab-2x\Sigma a+3x^2+0]$$
$$=4(ab+bc+ca)+4x(a+b+c+3x)$$

or $(a+b+c+3x)^2-4x(a+b+c+3x)=4(ab+bc+ca)$

or $(a+b+c+3x)(a+b+c-x)=4(ab+bc+ca)$

229 Hint $|(1+i)z^3+iz| \leqslant |z^3|+|i||z|^3+|i||z|$
$$=|z|^3+|z|^3+|z|$$
$$<\frac{1}{8}+\frac{1}{8}+\frac{1}{2}=\frac{3}{4}\quad [\ |i|=1 \text{ and } |z|<\tfrac{1}{2}]$$

232 Hint Put $p=r\cos\theta$, $1=r\sin\theta$

233 Put $z=re^{i\theta}$ so that $|z|=r$ Then since $|z^2|=|z|^2$, the given equation is equivalent to

$r^2(\cos 2\theta+i\sin 2\theta)+r^2(\cos\theta+i\sin\theta)+r^2=0$

Since $r\neq 0$ equating real and imaginary parts to 0 we get

$\cos 2\theta+\cos\theta+1=0$ (1) and $\sin 2\theta+\sin\theta=0$ (2)

From (2), $\sin\theta(2\cos\theta+1)=0$ $\sin\theta=0$ or $\cos\theta=-\frac{1}{2}$

$\theta=0$ π, $2\pi/3$, $4\pi/3$ But $\theta=0$, π do not satisfy (1) whereas $\theta=2\pi/3$ $4\pi/3$, satisfy it

Hence roots are

$r[\cos(2\pi/3)+i\sin(2\pi/3)]$ and $r[\cos(4\pi/3)+i\sin(4\pi/3)]$

or $r\left(-\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)$ and $r\left(-\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)$ where r is any non zero real number Clearly $z=0$ is also a solution Hence the roots are $r\left(-\frac{1}{2}\pm\frac{\sqrt{3}i}{2}\right)$ where r is any real number

237 Ans $2(\cos 144°+i\sin 144°)$ $2(\cos 216°+i\sin 216°)$, $\sqrt[5]{31}(\cos 108°+i\sin 108°)$ $-\sqrt[5]{31}$, $\sqrt[5]{31}(\cos 252°+i\sin 252°)$

238, Ans $-\frac{1}{2}<m<\frac{1}{2}$

Hint If α β are the roots such that $\alpha>1$ and $\beta<1$, then $(\alpha-1)(\beta-1)<0$, i e $\alpha\beta-(\alpha+\beta)+1<0$

But $\alpha+\beta=m/(2m+1)$ and $\alpha\beta=(m-2)/(2m+1)$

$$\frac{m-2}{2m+1}-\frac{m}{2m+1}+1<0 \text{ i e } \frac{2m-1}{2m+1}<0$$

from which we have $-\frac{1}{2}<m<\frac{1}{2}$

or $\sqrt{r}\left[\sin\frac{\theta}{2}\cos\frac{\beta}{2}-\cos\frac{\theta}{2}\sin\frac{\beta}{2}\right]=\sqrt{a}\sqrt{\left(\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)}$

Dividing by $\sin\frac{\theta}{2}$, we get

$$\sqrt{r}\left[\cos\frac{\beta}{2}-\cot\frac{\theta}{2}\sin\frac{\beta}{2}\right]=\sqrt{\left(a\cot\frac{\theta}{2}\right)} \quad (1)$$

Similarly, we shall have

$$\sqrt{r}\left[\cos\frac{\beta}{2}-\cot\frac{\phi}{2}\sin\frac{\beta}{2}\right]\sqrt{\left(b\cot\frac{\phi}{2}\right)} \quad (2)$$

Subtracting (1) from (2), we get

$$\sqrt{r}\sin\frac{\beta}{2}\left[\cot\frac{\theta}{2}-\cot\frac{\phi}{2}\right]=\sqrt{\left(b\cot\frac{\phi}{2}\right)}-\sqrt{\left(a\cot\frac{\theta}{2}\right)}$$

Finally we have

$$OH=r(1-\cos\beta)=2r\sin^2\frac{\beta}{2}$$

$$=2\left[\frac{\sqrt{\left(b\cot\frac{\phi}{2}\right)}-\sqrt{\left(a\cot\frac{\theta}{2}\right)}}{\cot\frac{\theta}{2}-\cot\frac{\phi}{2}}\right]^2$$

Coordinate Geometry

114 $\left(\frac{\mu a_1+\lambda a_2+\nu a_3}{\lambda+\mu+\nu}, \frac{\mu b_1+\lambda b_2+\nu b_3}{\lambda+\mu+\nu}\right)$

115 (4, 2) 116 $y=\sqrt{3}(x-2)$, $B\left(\frac{1}{2}(4+\sqrt{2}), \frac{\sqrt{6}}{2}\right)$

117 $(-3, 2\sqrt{2})$ 118 $(a, 0)$, $\frac{1}{2}(a+b)$, $\frac{1}{2}(a+b)$, $(0, b)$, $(\frac{1}{2}(a-b), \frac{1}{2}(b-a))$

122 Hint, Take A as origin and axes along AB and AC

123 $(-7, 3)$

124 Consider the determinant $\Delta=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

If Δ' is the det formed by the cofactors of Δ then we know that

$$\Delta'=\Delta^2, \text{ i e } \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \quad (1)$$

Now solving $a_2x+b_2y+c_2=0$ and $a_3x+b_3y+c_3=0$, we get

239 If possible let $\alpha+i\beta$ be a root. Then $\alpha-i\beta$ is also a root. Substituting these values for x and subtracting the first result from the second, we get

$$\beta\left\{\frac{A^2}{(\alpha-a)^2+\beta^2}+\frac{B^2}{(\alpha-b)^2+\beta^2}+\frac{C^2}{(\alpha-c)^2+\beta^2}+\ldots+\frac{H^2}{(\alpha-h)^2+\beta^2}\right\}=0,$$

which is impossible unless $\beta=0$

244 Ans $\frac{1}{2}(x+y+z)\{(y-z)^2+(z-x)^2+(x-y)^2\}$

Hint Put $x^2-yz=a$, $y^2-zx=b$, $z^2-xy=c$. Then given exression$=\frac{1}{2}(a+b+c)\{(a-b)^2+(b-c)^2+(c-a)^2\}$
But $a+b+c=\frac{1}{2}\Sigma(x-y)^2$
and $\Sigma(a-b)^2=(x+y+z)^2\,\Sigma(x-y)^2$
Expression$=\frac{1}{4}(x+y+z)^2\,[\Sigma(x-y)^2]$

245 (i) Ans $|a|>2$

[Hint Factorizing, L H S $=(a+1)(1+x+x^2)(1-x+x^2)$
Hence the equation reduces to $(1+x+x^2)(x^2-ax+1)=0$
The roots of $1+x+x^2=0$ are imaginary and for real and distinct roots of $x^2-ax+1=0$ we must have
$a^2-4>0$ or $|a|>2$

(ii) Do yourself

247 Ans $a=\frac{19}{15}$, $b=-\frac{1}{135}$

[Hint The given expression will be an integral function of x if $3x^2+3x-1$ is a factor of x^5+ax^4+b. So we can write
$3(x^5+ax^4+b)\equiv(3x^2+3x-1)(x^3+px^2+qx-3b)$
Now compare the coeff s of like powers of x etc]

249 (i) Ans $-1<m<5$

[Hint Since $x^2+x+1>0$ for all real x, the given inequality can be written as
$-3(x^2+x+1)<x^2+mx+1<3(x^2+x+1)$
$4x^2+(m+3)x+4>0$ (1)
and $2x^2-(m-3)x+2>0$ (2)
The inequality (1) will hold for all x if
$(m+3)^2-64<0$ or $(m+11)(m-5)<0$ which implies that
$-11<m<5$ (3)
Similarly (2) will hold for all x if $-1<m<7$ (4)
Finally from (3) and (4), $-1<m<5$]

(ii) Ans $-1<a<2$

250 Here $\Delta=B^2-4AC=(2a^2+a+3)^2-4(a^2+a-2)(a^2-1)$
$=25a^2+10a+1$ (after simplifications)
$=(5a+1)^2$

$$\frac{x_1}{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}} = \frac{y_1}{\begin{vmatrix} c_2 & a_2 \\ c_3 & a_3 \end{vmatrix}} = \frac{1}{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}$$

i e $\frac{x_1}{A_1}=\frac{y_1}{B_1}=\frac{1}{C_1}$

Hence point of intersection of the above two lines is $(A_1/C_1, B_1/C_1)$ This gives one of the vertices of triangle Similarly other two vertices are $(A_2/C_2, B_2/C_2)$ and $(A_3/C_3, B_3/C_3)$ Hence area of the triangle

$$=\begin{vmatrix} A_1/C_1 & B_1/C_1 & 1 \\ A_2/C_2 & B_2/C_2 & 1 \\ A_3/C_3 & B_3/C_3 & 1 \end{vmatrix} = \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \div (C_1C_2C_3$$

$$= \begin{vmatrix} a_1 & a_1 & a_1 \\ a_2 & b_2 & b_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2 \div (a_2b_3-a_3b_2)(a_3b_1-a_1b_3)(a_1b_2-a_2b_1)$$

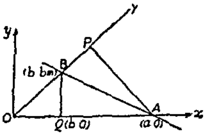
[using (1) and substituting the values of C_1 C_2 and C_3]

125 Proceed as in Q 124 Put $a_1=\cos\alpha$, $b_1=\sin\alpha$ $c_1=-p_1$ etc

126 Hint take the equation of sides as $x\cos\alpha+y\sin\alpha-p=0$ $x\cos\beta+y\sin\beta-q=0$ and $x\cos\gamma+y\sin\gamma-r=0$

129 $\left(-\frac{1}{2}, \frac{7}{2}\right)$

131 Take OX as x axis and O as origin Then equation to OY may be written as $y=mx$ where m is a constant Take the coordinates of A as $(a, 0)$ and that of B as $(b\ bm)$ Then coordinates of Q are evidently $(b, 0)$ To obtain the coordinates of P we write the equation of AP which is perpendicular to OY Hence its equation is ,



$$y-0=-\frac{1}{m}(x-a) \qquad (1)$$

Solving (1) with $y=mx$, we get the coordinates of P as

$\left(\frac{a}{1+m^2}, \frac{ma}{1+m^2}\right)$ (Note that here a, b are parameters)

Since Δ is a perfect square, roots are rational

The roots are given by $x = \dfrac{-(2a-a+3) \pm (5a+1)}{2(a+a-2)}$

$= -\dfrac{2(a-1)^2}{2(a-1)(a+2)}$ or $\dfrac{-2(a+2)(a+1)}{2(a-1)(a+2)}$

Since $a \neq 1$ and $a \neq -2$ the roots are

$x_1 = -\dfrac{a-1}{a+2}, x_2 = -\dfrac{a+1}{a-1}$

251 Since roots are real and different, we have
$a^2-4b > 0$ or $b < \frac{1}{4}a^2$ (1)
Again $\alpha+\beta = a$ and $\alpha\beta = b$ and $\alpha - \beta < c$ (Assuming $\alpha > \beta$)
Hence $(\alpha-\beta)^2 < c^2 \Rightarrow (\alpha+\beta)^2 - 4\alpha\beta < c^2 \Rightarrow a^2 - 4b < c^2$
$\Rightarrow \frac{1}{4}(a^2-c^2) < b$ (2)
From (1) and (2) $\frac{1}{4}(a^2-c^2) < b < \frac{1}{4}a^2$

252 Ans $\{(-1 \pm \sqrt{5})/2\}$

253 Ans 2, 1/16
[Hint Put $\log_2 x = t$ The equation will reduce to $\dfrac{t}{1+t} = \dfrac{2(2+t)}{3(3+t)}$ which on solving will give $\log_2 x = t = 1, -4$
$x = 2$ or 2^{-4}]

254 Ans $\pm 27/14$

255 Ans $x = 4, 5$

Hint First note that an equation of the form $[f(x)]^{\phi(x)} = 1$ will reduce to two equations
(1) $f(x) = 1$ or (2) $\varphi(x) = 0$
All those values of x will serve as solutions for which $f(x)$ and $\varphi(x)$ are defined and $f(x) > 0$
Here we have $|x-3| = 1$ which gives $x-3 = \pm 1$ or $x = 4, 2$ or $(x^2-8x+15)/(x-2) = 0$ which gives $x = 3, 5$ Of these 4 roots $x = 2, 3$ are extraneous since for $x = 2$ the exponent $\varphi(x)$ of the original equation is not defined and for $x = 3$, the base $f(x) = |x-3|$ is 0

256 Ans $x = 1$
[Hint First not that $x > 0$ $3x \neq 1$ $9x \neq 1$
Further setting $x \neq 1$ we pass to base x and get

$\dfrac{1}{\log_x 3x} = \dfrac{1}{\log_x 9x}$ or $\log_x 9x = \log_x 3x$

or $\log_x 3 + \log_x 3x = \log_x 3x$ or $\log_x 3 = 0$

which gives $3 = x^0 = 1$ But this is false Hence equation (2) has no solution But it would be wrong to infer from this that the original equation has no solution We have not yet considered the value $x = 1$, which evidently is a solution of the original equation This example shows that on passing to a new base involving the unknown a loss of the root of the original equation can occur Hence we must guard against such errors]

Now the equation of AB is

$$y-0=\frac{0-bm}{a-b}(x-a) \text{ or } xbm+y(a-b)-abm=0 \qquad (2)$$

Suppose it passes through the fixed point (x_1, y_1) Then $bmx_1+(a-b)y_1-abm=0$

We have to prove using (2) that PQ also passes through a fixed point The equation of PQ is

$$y-0=\frac{\frac{am}{1+m^2}-0}{\frac{a}{1+m^2}-b}(x-b) \quad \text{or} \quad y=\frac{am}{a-b(1+m^2)}(x-b)$$

or $amx-\{a-b(1+m^2)\}y-abm=0 \qquad (3)$

Substituting the value of abm from (2) in (3) the equation of PQ can be written as

$amx-\{a-b(1+m^2)\}y-bmx_1-(a-b)y_1=0$

or $a(mx-y-y_1)+b\{(1+m^2)y-mx_1+y_1\}=0$

which for all values of a and b passes through the intersection of the lines

$mx-y-y_1=0$ and $(1+m^2)y-mx_1+y_1=0$

These are fixed lines since $x_1\ y_1$, m are constants Hence PQ also passes through a fixed point

132 Hint If P is the circum-centre, G the centroid and Q the ortho-centre then we know that $P, G\ Q$ are collinear and $PG\ \ GQ=1\ \ 2$

133 Ans $\left(-\frac{1}{10}\ \frac{265}{112}\right)$, radius$=\frac{51}{112}$

135 Centroid $G\left(\frac{11}{3}, 5\right)$ circum centre $P\left(-\frac{27}{2}, \frac{39}{2}\right)$

Now to find the coordinates of the ortho-centre Q, use the property

$PG\ \ GQ=1\ \ 2$ This will give Q (38, −24)

136 Ans Centres are $(\frac{8}{3}, \frac{8}{3})$, (2, 12), (12 2) and (−3, −3) and radii are $\frac{8}{3}\sqrt{2}, 4\sqrt{2}, 4\sqrt{2}$ and $6\sqrt{2}$

137 $\left(4+\frac{1}{2}\sqrt{3}, \frac{3}{2}+\sqrt{3}\right)$, $\left(4+\frac{1}{6}\sqrt{3}, \frac{3}{2}+\frac{1}{3}\sqrt{3}\right)$

138 Ans $A(y-k)-B(x-h)=\pm(Ax+By+C)$

139 Let the line be $ax+by+c=0 \qquad (1)$

and the fixed points be $(x_1, y_1), (x_2, y_2), \ (x_n, y_n)$

Then according to the question

$$\sum_{i=1}^{n}\left[\frac{ax_i+by_i+c}{\sqrt{(a^2+b^2)}}\right]=0$$

257 $x=-6, 2, y=9, -3$

258 (a) $x=4/3, y=2/3$ 259 (i) $x=\pm 1, \pm 2, y=\pm 2, \pm 1$
(ii) $x=\pm 3, y=\pm 5, z=\pm 7$
(b) Ans $x=2$ $y=2$, $z=1$
Hint Subtract the first two equations

260 $x=1, -1, \pm \frac{7}{\sqrt{3}}, y=3 -3, \pm \frac{2}{\sqrt{3}}, z=5, -5, \pm \frac{11}{\sqrt{5}}$

261 Hint Let n be odd say $2m+1$ Then the required no of ways$=m+2\{1+2+3+\ \ +(m-1)\}=m+(m-1)m=m^2$
$=[\frac{1}{2}(n-1)]^2=\frac{1}{4}(n-1)^2$
When $n=2m$ (even), then
no of ways$=2[1+2+3+\ \ +(m-1)]=(m-1)m$
$=\frac{1}{2}(n-1)\frac{1}{2}n=\frac{1}{4}n(n-2)$

262 (i) $\frac{1}{6}n(n-4)(n-5)$, (ii) $\frac{1}{6}(n-2)(n-3)(n-4)$

263 $\frac{1}{6}n(n-4)(n-5)$ 266 $n(n+1)(n+2)\ \ (n+r-1)$

267 Ans ${}^{15}C_3-({}^4C_3+{}^5C_3+{}^6C_3)$

268 Ans 35 ways
[Hint Required number of ways is the coefficient of x^{14} in $(x^3+x^4+x^5+x^6+x^7)^4$]

269 We consider four cases

(i) When all the 10 balls are of one colour only
In this case the no of ways$={}^4C_1=4$

(ii) When ten balls contain two colours only
No of ways in this case
$=$coeff^t of x^{10} in ${}^4C_2(x+x^2+x^3+\ \ )^2$
$=$coeff^t of x^{10} in $6x^2(1-x)^{-2}$
$=$coeff^t of x^8 in $6[1+2x+3x^2+\ \ +(r+1)x^r+\ \ ]$
$=6\times 9=54$

(iii) When the ten balls contain three colours only
No of ways in this case
$=$coeff^t of x^{10} in ${}^4C_3(x+x^2+x^3+\ \ )^3$
$=$coeff^t of x^7 in $4(1-x)^{-3}$
$=$coeff^t of x^7 in $4[1+3x+\ \ \frac{1}{2}(r+1)(r+2)x^r+\ \ ]$
$=4\times\frac{1}{2}\times 8\times 9=144$

(iv) When 10 balls contain all the four colours
No of ways in this case
$=$coeff^t of x^{10} in $(x+x^2+x^3+\ \ )^4$
$=$coeff^t of x^6 in $(1-x)^{-4}$
$=$coeff^t of x^6 in $[1+4x+\ \ +\frac{(r+1)(r+2)(r+3)}{3!}x^r+\ \ ]$
$=\frac{7\times 8\times 9}{6}=84$

Hence the total no of ways$=4+54+144+84=286$

or $a \sum_{i=1}^{n} x_i + b \sum_{i=1}^{n} y_i + nc = 0$ (2)

Now multiplying equation (1) by n and subtracting (2) from it, we get

$a[nx - \Sigma x_i] + b[ny - \Sigma y_i] = 0$

or $(nx - \Sigma x_i) + \frac{b}{a}[ny + \Sigma y_i] = 0$

This equation is of the form $u + \lambda v = 0$ where $u = 0$ and $v = 0$ are fixed lines. Hence the line (1) always passes through a fixed point.

143 Ans (0 0), $(-C/3A, -C/3B)$, $(-B/2A, C/2B)$

145 $9x + 17y + 9 = 0$ 146 $15y = 5x + 14$

147 $3y + 5x = 34$, $5y - 3x = 34$ 148 $a = 3$ $b = 6$ $3y - x + 7 = 0$

149 $\left(\frac{3}{m} - 2, 3 - 2m\right)$, $xy - 3x + 2y = 0$ $xy - 6x + 6y = 0$

150 $\frac{1}{\sqrt{2}}$

151 Hint Find the circle through first three points and verify that the other two points also satisfy it. The circle on which the five points lie is

$x^2 + y^2 + 42x + 26y - 3615 = 0$ or $(x + 21)^2 + (y + 13)^2 = 65^2$

152 $x^2 + y^2 - 3x + 2 = 0$ $2x^2 + 2y^2 - 7x + \sqrt{3}y + 6 = 0$ and $2x^2 + 2y^2 - 5x - \sqrt{3}y + 3 = 0$

153 Hint If $(h\ k)$ is the centre then h will be the radius since circles touch y-axis. Then $(h - a)^2 + (k - 5a)^2 = h^2$ and $(h - 4a)^2 + (k - a)^2 = h^2$. This will give centres (h, k) of two circles as $A\left(\frac{5}{2}a\ \ 3a\right)$ and $B\left(\frac{205}{3}a\ \frac{29}{3}a\right)$ and radii as $\frac{5}{2}a$ and $\frac{205}{3}a$ respectively. Now angle between tangents is the same as angle between radii. So find slopes m_1 m of the radii AP and BP where P is $(4a, a)$ or $(a\ 5a)$ etc

154 Ans $lx^2 + my^2 - x - y = 0$ locus $x^2 + y^2 = 1$

155 Hint Equation of any tangent to $(x - a)^2 + y^2 = b^2$ is $y = m(x - a) + b\sqrt{(1 + m^2)}$ and perp tangent to $(x + a)^2 + y^2 = c^2$ is $y = -\frac{1}{m}(x + a) + c\sqrt{\left(1 + \frac{1}{m^2}\right)}$

These equations may be written as

$b\sqrt{(1 + m^2)} + (x - a)\,m - y = 0$ and $c\sqrt{(1 + m^2)}$

By cross multiplication,

From (iv), no of ways in which balls are of all the four colours = 84

270 Ans 505

271 The number of ways in which m identical balls can be distributed among $2m$ boxes is

$$^{2m}C_m = \frac{(2m)!}{(m!)^2}$$

So we have to prove that

$$\frac{4^m}{2\sqrt{m}} \leqslant \frac{(2m)!}{(m!)^2} \leqslant \frac{4^m}{\sqrt{(2m+1)}} \quad (1)$$

We prove each inequality by induction We first prove that

$$\frac{4^m}{2\sqrt{m}} \leqslant \frac{(2m)!}{(m!)^2} \quad \text{or} \quad \frac{4^{2m}}{4m} \leqslant \frac{\{(2m)!\}^2}{(m!)^4}$$

or $$\frac{m\,\{(2m)!\}^2}{4^{2m-1}\,(m!)^4} \geqslant 1 \quad (2)$$

Let $P(m) = \dfrac{m\,\{(2m)!\}^2}{4^{2m-1}\,(m!)^4}$

$P(1) = \dfrac{1\cdot 4}{3\cdot 1} = 1$ So (2) holds true for $m=1$

Now assume $$P(k) = \frac{k\,\{(2k)!\}^2}{4^{2k-1}\,(k!)^4} \geqslant 1 \quad (3)$$

Then $$P(k+1) = \frac{(k+1)\,\{(2k+2)!\}^2}{4^{2k+1}\,((k+1)!)^4}$$

$$= \frac{(k+1)\,(2k+2)^2\,(2k+1)^2\,\{(2k)!\}^2}{16\cdot 4^{2k-1}\,(k+1)^4\,(k!)^4}$$

$$= \frac{(2k+1)^2}{4(k+1)k}\cdot\frac{k\,\{(2k)!\}^2}{4^{2k-1}\,(k!)^4} = \frac{(2k+1)^2}{4\,(k+1)\,k}\,P(k) > 1$$

$$\left[\quad P(k) \geqslant 1 \text{ and } \frac{4k^2+4k+1}{4k^2+4k} > 1\right]$$

So (2) holds true for all natural numbers m

Similarly other inequality can be proved

272 We have

$(a^2+b^2)(c^2+d^2) = a^2c^2 + b^2c^2 + a^2d^2 + b^2d^2$
$= (ac+bd)^2 + (bc-ad)^2$

Hence the statement holds for two factors

Now assume that

$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots(a_m^2+b_m^2) = A^2+B^2$

Then multiplying both sides by $a_{m+1}^2+b^2_{m+1}$ we get

$(a_1^2+b_1^2)(a_2^2+b_2^2)\ldots(a_m^2+b_m^2)(a_{m+1}^2+b_{m+1}^2)$
$=(A^2+B^2)(a_{m+1}^2+b_{m+1}^2)$
$=A^2\,a_{m+1}^2+B^2\,a_{m+1}^2+A^2\,b_{m+1}^2+B^2\,b_{m+1}^2$
$=(A\,a_{m+1}+B\,b_{m+1})^2+(B\,a_{m+1}-A\,b_{m+1})^2$

Hence the statement holds for any no of factors

So by induction, it holds for any no of factors

$$\frac{\sqrt{(1+m^2)}}{-(x^2-a^2)-y^2}=\frac{m}{-yc+b\,(x+a)}=\frac{1}{-by-c\,(x-a)}$$
$$=\frac{\sqrt{(1+m^2)}}{\sqrt{[(bx-cy+ab)^2+(cx+by-ac)^2]}},$$

[By ratio and proportion]

Hence the required locus is

$(a^2-x^2-y^2)^2=(bx-cy+ab)^2+(cx+by-ac)^2$

Again equations of the bisectors of the two tangents is given by

$$\frac{m(x+a)-y+b\sqrt{(1+m^2)}}{\sqrt{(1+m^2)}}=\pm\frac{(x+a)+my-c\sqrt{(1+m^2)}}{\sqrt{(1+m^2)}}$$

Taking +ve sign, we shall get

$(m-1)\,x-(m+1)\,y+(b+c)\sqrt{(1+m^2)}-a\,(1+m)=0$

or $y+a=\frac{m-1}{m+1}\,x+\frac{b+c}{m+1}\,\sqrt{(1+m^2)}$

or $y+a=\frac{m-1}{m+1}\,x+\frac{(b+c)}{\sqrt{2}}\,\sqrt{\left[1+\left(\frac{m-1}{m+1}\right)^2\right]}$

or $y+a=m\,x+\frac{b+c}{\sqrt{2}}\,\sqrt{(1+m^2)}$ where $m=\frac{m-1}{m+1}$

This clearly touches the fixed circle $x^2+(y+a)^2=\frac{(b+c)^2}{2}$ whatever m may be

Similarly other bisector may be shown to touch another fixed circle

156 Hint Let AB and CD be two given rods which slide along x axis and y axis respectively so that $AB=a$ and $CD=b$

Since A, B C, D are concylic, a cicle will pass through them

Let the equation of this circle be

$$x^2+y^2-2hx-2k\,y+c=0 \qquad (1)$$

so that its centre is (h, k) We have to find the locus of $(h\ k)$

Solving (1) with $y=0$ (i e x axis), we get

$$x^2-2xh+c=0 \qquad (1)$$

If A and B be $(x_1\ 0)$ and $(x_2\ 0)$ respectively, then x_1 and x_2 are the roots of (2)

$x_1+x_2=2h$ and $x_1\,x_2=c$

Hence $a=AB=|\,x_1-x_2\,|=\sqrt{\{(x_1+x_2)^2-4\,x_1x_2\}}$
$=\sqrt{(4h^2-4c)}$

or $4h^2-4c=a^2$ (3)

Similarly solving (1) with $x=0$ and proceeding as above we shall get

$$4k^2-4c=b^2 \qquad (4)$$

Subtracting (1) from (3), and generalizing we get the locus of

276 Hint $a > \frac{ar}{1-r}$ if $1-r > r$ i e if $r < \frac{1}{2}$

277 Hint $x = a + (n-1)(b-a)$

and $\frac{1}{y} = \frac{1}{a} + (n-1)\left(\frac{1}{b} - \frac{1}{a}\right)$

or $x - a = (n-1)(b-a)$ and $\frac{b}{y} = \frac{b+(n-1)(a-b)}{a}$

Also $y - a = \frac{(b-a)\,a\,(n-1)}{b+(n-1)(a-b)}$

Hence $\frac{x-a}{y-a} = \frac{b+(n-1)(a-b)}{a} = \frac{b}{y}$

278 (a) Take the terms equidistant from the n^{th} term $\{a+(n-1)d\}$ Let $T_1, T_2, T_3,$ denote the successive products of the terms equidistant from this term Then

$T_1 = \{a+(n-2)d\}\{a+nd\} = a^2 + 2(n-1)ad + n(n-2)d^2$

$T_2 = \{a+(n-3)d\}\{a+(n+1)d\} = a^2 + 2(n-1)ad + (n+1)(n-3)d^2$

$T_3 = \{a+(n-4)d\}\{a+(n+2)d\} = a^2 + 2(n-1)ad + (n+2)(n-4)d^2$

$T_4 = \{a+(n-5)d\}\{a+(n+3)d\} = a^2 + 2(n-1)ad + (n+3)(n-5)d^2$

Now $T_2 - T_1 = -3d^2$ $T_3 - T_2 = -5d^2$ $T_4 - T_3 = -7d^2$ thus the differences $T_2 - T_1$, $T_3 - T_2$, $T_4 - T_3$, are in A P

(b) Let x, y, z respectively denote the hundreadth, the tenth and unit's digit of the number where x, y, z are in A P we then have

$x + z = 2y$ (1)

And since the number $100x + 10y + z$ is divisible by 45, we can write

$100x + 10y + z = 45k$ (2)

where k is an integer

From (1) and (2) we get

$y = \frac{1}{2}(x+z)$ (3)

and $k = \frac{1}{45}(100x + 5x + 5z + z) = \frac{105x + 6z}{45}$ (4)

We have to solve (3) and (4) in integers First note that x y, z can take the values 0, 1, 2 3 ,9 only Also from (4) it is clear that z can take the value 0 and 5 only otherwise k will not be an integer We take $z = 0$ first Then x can take the values 2, 4, 6, 8 only to make y a +ive integer Out of these 2, 4, 8 do not make k an integer

Hence $x = 6$ Then $y = 3$ so that the number in this case is 630

We now teke $z = 5$, then x can take the values 1 3 5, 7, 9 to make y a +ive integer Out of these values 3, 5, 9 do not make k an integer Hence $x = 1$, 7 only Then $y = 3, 6$

(h, k) as $4(x^2-y^2)=a^2-b^2$

157 **Hint** Take O as origin and one of the straight lines be OX (the x axis) and the other line OY inclined to OX at an angle α so that its equation is $y=x\tan\alpha$. Let (h, k) be the centre of the variable circle through O so that its equation may be written as

$x^2+y^2-2hx-2ky=0$

It intersects OX i.e. $y=0$ at P where $x^2-2hx=0$

so that $OP=2h$

It intersects OY i.e. $y=x\tan\alpha$ at Q where

$x^2+x^2\tan^2\alpha-2hx-2kx\tan\alpha=0$

This gives $x_1=0$

and $x_2=\dfrac{2h+2k\tan\alpha}{1+\tan^2\alpha}=2(h\cos\alpha+k\sin\alpha)\cos\alpha$

$OP=x_2=2(h\cos\alpha+k\sin\alpha)\cos\alpha$

Then $y_1=0$ and $y_2=x_2\tan\alpha$

$OQ=\sqrt{(x_2^2+y_2^2)}=x_2\sqrt{(1+\tan^2\alpha)}=x_2\sec\alpha$

$=2(h\cos\alpha+k\sin\alpha)$

By hypothesis $m\,OP+n\,OQ=1$

Putting the values of OP and OQ we get

$2m(h\cos\alpha+k\sin\alpha)\cos\alpha+2n(h\cos\alpha+k\sin\alpha)=1$

This shows that the centre (h, k) lies on the fixed straight line AB whose equation is

$2(m\cos\alpha+n)(x\cos\alpha+y\sin\alpha)=1$

Let ON be perpr to AB. Produce ON to O' such that $ON=NO'$. Then it is clear that AB is the perpendicular bisector of OO'. Hence the circle having its centre on AB and passing through O must also pass through O' which is also fixed

158 Ans $(5, 1), (-1, 5)$

159 $123x^2-64xy+3y^2-664x+226y+763=0$

160 Hint Take the vertices of the triangle as $(-a, 0)$, $(a, 0)$ and (α, β). Then the required locus is

$(x+a)^2+y^2+(x-a)^2+y^2+(x-\alpha)^2+(y-\beta)^2=0$

or $3x^2+3y^2-2x\alpha-2y\beta+2a^2+\alpha^2+\beta^2=0$, which is a circle

167 Hint First note that shortest distance from the circle is the distance from its centre minus the radius. Hence the required locus is

$\sqrt{\{(x-1)^2+(y+3)^2\}}-4=x-3$

or $(x-1)^2+(y+3)^2=(x+1)^2$ or $y^2+6y-4x+9=0$

Calculus

168 $\{1+\sqrt{2}+\sqrt{3}+2\sqrt{(2x)}+2\sqrt{(3x)}+2\sqrt{(6x)}+3x\sqrt{6}\}/(2\sqrt{x})$

Hence in this the numbers are 135 and 765
Thus the required numbers are 135 630 and 765

279 (i) Let a, b, c be in A P so that $b=\frac{a+c}{2}$

Then $\frac{a^n+c^n}{2} > \left(\frac{a+c}{2}\right)^n=b^n$ or $a^n+c^n > 2b^n$

(ii) Let a b c be in G P so that $b^2=ac$

Then $\frac{a^n+c^n}{2} > (a^n c^n)^{1/2}=(\sqrt{ac})^n=b^n$ or $a^n+c^n > 2b^n$

(iii) Let a b c be in H P so that $\sqrt{(ac)} > b$
[G M > H M]

Then $\frac{a^n+c^n}{2} > (a^n c^n)^{1/2}=(\sqrt{(ac)})^n > b^n$ or $a^n+c^n > 2b^n$

280 (a) The given equality can be written as
$(a_2^2-a_1 a_3)^2+(a_3^2-a_2 a_4)^2+(a_4^2-a_3 a_5)^2+ \quad =0$
Hence $a_2^2-a_1 a_3=0$ $a_3^2-a_2 a_4=0$ $a_4^2-a_3 a_5=0$,

$\frac{a_2}{a_1}=\frac{a_3}{a_2}=\frac{a_4}{a_3}=\frac{a_5}{a_4}=$

a_1 a_2 a_3 a_4, are in G P
(b) Ans $a(2+\sqrt{3})$
Hint The lengths of the successive segments of the infinite polygonal line which we have constructed form a decreasing G P By the length of the infinite polygonal line, we mean the sum of this G P

281 (i) $S_n=\frac{1}{2} n [2a+(n-1) d]$
If the sum be $(2n)^2$ or $4n^2$ we must have
$2a+(n-1) d=8n$ for all n
Then $d=8$ and $a=4$
Hence the series is
$4+12+20+$
It is easy to verify that the sum of n terms of this series is $4n^2$
(ii) In this case we have to prove that $(2n+1)$ is the sum of n terms of an A P of integers increased by unity This means that we have to prove that $(2n+1)^2-1$, is the sum of n terms of A P of integers Thus $(2n+1)^2-1=\frac{1}{2} n [2a+(n-1) d]$
or $8n+8=2a+(n-1) d$ when $a=d=8$
Hence the series in this case is
$8+16+24+$
Sum of this series to n terms can easily be seen to be $4n^2+4n$ required

282 Let $P^r=(2n-1)+(2n+1)+(2n+3)+$ to P terms

(i) $\sin \pi x > 0$ when $2m < x < 2m+1$
(ii) $\sin \pi x < 0$ when $2m+1 < x < 2m+2$ and
(iii) $\sin \pi x = 0$ when $x = 0, 1, 2, 3,$
(m is a + ive integer here)

Hence $f(x) = \lim_{t\to\infty} \frac{1 - 1/(1+\sin \pi x)^t}{1 + 1/(1+\sin \pi x)^t} = 1$ in case (i),

$f(x) = -1$ in case (ii) since $\lim_{t\to\infty} (1+\sin \pi x)^t = 0$ in this case

and $f(x) = 0$ in case (iii)
Thus $f(x) = 1$ when $2m < x < 2m+1$
and $f(x) = -1$ when $2m+1 < x < 2m+2$
and $f(x) = 0$ for all integral valves of x Rest do yourself,

191 Hint We first note that $y = -\sqrt{(x-2)^2} = -|x-2|$

Hence $y = \begin{cases} -(x-2) & \text{if } x \geqslant 2 \\ x-2 & \text{if } x \leqslant 2 \end{cases}$

Now draw the graph yourself

192 Here $p(x) = f(x) + f(1-x)$ and so $p'(x) = f'(x) - f'(1-x)$
Also clearly $1-x > x$ for $0 < x < \frac{1}{2}$ and $(1-x) < x$ for $\frac{1}{2} < x < 1$ Since $f''(x) < 0$ in $[0, 1]$ the function $f'(x)$ is decreasing in $[0 \; 1]$ Hence from the above discussion $p'(x) > 0$ for $0 < x < \frac{1}{2}$ and $p'(x) < 0$ for $\frac{1}{2} < x < 1$
It follows that $p(x)$ is increasing in $0 \leqslant x \leqslant \frac{1}{2}$ and decreasing in $\frac{1}{2} \leqslant x \leqslant 1$

195 $p = 1/4$, $b = 1/2$, $c = 1/4$

196 $\frac{\pi l}{\pi+4}$ in the shape of a circle, $\frac{4l}{\pi+4}$ in the shape of a square

198 Let the ladder be inclined to the ground at an angle θ Then it is easy to see that its length y is given by

$$y = h \operatorname{cosec} \theta + b \sec \theta$$

Now show that

$$\frac{dy}{d\theta} = 0 \text{ gives } \tan \theta = h^{1/3}/a^{1/3} \text{ and } \frac{d^2y}{d\theta^2} > 0$$

Hence Min $y = \frac{h\sqrt{(h^{2/3}+a^{2/3})}}{h^{1/3}} + \frac{a\sqrt{(h^{2/3}+a^{2/3})}}{a^{1/3}}$

$= (h^{2/3}+a^{2/3})\sqrt{(h^{2/3}+a^{2/3})}$

$= 16\sqrt{(16)} = 64m$ eters (using the given relation)

199 Ans $1-\sqrt{3}$, -1

200 (i) $\frac{\sqrt{2}}{5}(\tan^2 x + 5)\sqrt{(\tan x)} + c$

$=\frac{1}{2} P[2(2n\ \ 1)+(P-1)2]=P(2n+P-2)$
or $P^{r-1}-P+1=2n-1$

Hence the first term of the series is $P^{r-1}-P+1$
[The series whose sum is P^r is $A+(A+2)+(A+4)+$ to P terms]

286 Ans 6
Clearly the terms free of radicals are $T_1, T_{11}, T_{21}\ T_{31}, T_{41}, T_{51}$

290 Hint Let $S=3C_1+7C_2+11C_3+\ \ +(4n-5)C_{n-1}+(4n-1)C_n$ (1)
Since ${}^nC_r={}^nC_{n-r}$, we can write (1) as
$S=(4n-1)C_0+(4n-5)C_1+(4n-9)C_2+\ \ +3C_{n-1}$ (2)
Adding (1) and (2) we get

$2S=(4n-1)C_0+(4n-2)(C_1+C_2+C_3+\ \ +C_{n-1})+(4n-1)C_n$

$=2(4n\ \ 1)+(4n-2)(2^n-2)$ [$C_0=C_1=1$]

$S=1+(2n-1)2^n$

294 Operate C_3-C and C_2-C_1 and then again operate C_3-C_2 in the newly formed determinant

295 $\Delta=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}^2$

First det is obtained by multiplying

$$\Delta_1=\begin{vmatrix} 0 & c & b \\ c & 0 & a \\ b & a & 0 \end{vmatrix}$$

by itself row by row and second follows from the fact that it is the det formed by the cofactors of Δ_1 and so $\Delta=\Delta_1^2$

297 Write $\Delta=\begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 0 \end{vmatrix}\begin{vmatrix} b_1 & a_1 & 0 \\ b_2 & a_2 & 0 \\ b_3 & a_3 & 0 \end{vmatrix}$

298 Hint Consider the $(n+1)$ numbers $x_1, x_2\ x_3\ \ , x_{n+1}$ where $x_1=1$ and $x_2=x_3=x_4=\ \ =x_{n+1}=1+\frac{1}{n}$ and apply

A M > G M i e $\frac{x_1+x_2+\ \ +x_{n+1}}{n+1}>(x_1\,x_2\ \ x_{n+1})^{1/n+1}$

i e $\frac{1+n\left(1+\frac{1}{n}\right)}{n+1}>\left[\left(1+\frac{1}{n}\right)^n\right]^{1/n+1}$

or $\left(1+\frac{1}{n+1}\right)>\left[\left(1+\frac{1}{n}\right)^n\right]^{1/n+1}$

(ii) $ln\left\{\frac{1+e^x-\sqrt{(1+e^x+e^{2x})}}{1-e^x+\sqrt{(1+e^x+e^{-x})}}\right\}+c$

(iii) $ln\left\{\frac{x^2+1+\sqrt{(x^4+3x^2+1)}}{x}\right\}+c$

[Hint Divide by x^2 and then put $x+\frac{1}{x}=t$

(iv) $\frac{2}{3}\cdot\frac{(\tan^2 x+3)}{\sqrt{(\tan x)}}+c$

[Hint Divide by $\cos^4 x$]

(v) $\tan^{-1}\sqrt{(x^2-1)}-\frac{ln\ x}{\sqrt{(x^2-1)}}+c$

(vi) Ans $\frac{2}{3}\sin^{-1}(\cos^{3/2} x)$

[Hint Write N^r as $\sqrt{(\cos x)}\sin x$ and put $\cos^{3/2} x=z$]

(vii) $I=\int x\, e^{\sin x}\cos x\, dx-\int e^{\sin x}\sec x\tan x\, dx$

$=[x\, e^{\sin x}-\int e^{\sin x}dx]-[e^{\sin x}\sec x-\int e^{\sin x}\cos x\sec x\, dx]$

$=xe^{\sin x}-e^{\sin x}\sec x+c$

201 $\frac{11}{48}+\frac{5\pi}{64}$

203 **Ans** $(u-e\sin u)/(1-e^2)^{3/2}$

where $\sqrt{(1-e)}\tan\frac{1}{2}\theta=\sqrt{(1+e)}\tan\frac{1}{2}u$

[Hint Put $\tan\frac{1}{2}\theta=\{(1+e)/(1-e)\}\tan\frac{1}{2}u$]

204 **Ans** $n\,!$

205 **Hint** Put $\cos x=\frac{a\cos\theta+b}{a+b\cos\theta}$ Then given integral will reduce to $(a^2-b^2)^{-3/2}\int_0(a-b\cos x)\,dx$

Ans $(a^2-b^2)^{-3/2}\left[a\cos^{-1}\left(\frac{a\cos\theta+b}{a+b\cos\theta}\right)-\frac{b\sqrt{(a^2-b^2)}\sin\theta}{a-b\cos\theta}\right]$

206 **Ans** $5\pi/4$

[Hint Put $x=\cos\theta$ and integrate by parts]

207 Put $x=1/z$ in any one of the integrals say first

Then $I=\int_0^\infty\frac{dx}{1+x^4}=-\int_\infty^0\frac{z^2\,dz}{1+z^4}=\int_0^\infty\frac{z^2\,dz}{1+z^4}=\int_0^\infty\frac{x^2\,dx}{1+x^4}$

Hence $2I=\int_0^\infty\frac{dx}{1+x^4}+\int_0^\infty\frac{x^2\,dx}{1+x^4}=\int_0^\infty\frac{1+x^2}{1+x^4}\,dx$

$=\int_0\frac{1+1/x^2}{x^2+1/x^2}\,dx$

Now put $x-1/x=t$ etc

208 **Hint** Put $x^2-1=t^2$ so that $x\,dx=t\,dt$

or $\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n}$

299 Ans $-2 \leqslant x \leqslant 2$

300 Ans $-1 \leqslant a \leqslant 7$

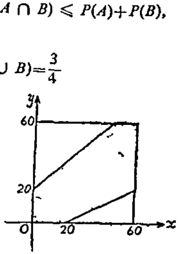
Since $1-x+x^2 > 0$ for all real x, the given inequality is equivalent to $2-ax-x^2 \leqslant 3-3x+3x^2$
or $4x^2+(a-3)\,x+1 \geqslant 0$
This inequality will hold for all x if $(a-3)^2-16 \leqslant 0$
or $|a-3| \leqslant 4$ or $-4 \leqslant a-3 \leqslant 4$ or $-1 \leqslant a \leqslant 7$

301 Hint $P(A)=1/3$, $P(A \cup B)=3/4$
Now $P(A \cup B) \leqslant P(A)+P(B)-P(A \cap B) \leqslant P(A)+P(B)$,
or $\frac{3}{4} < \frac{1}{3}+P(B)$ or $\frac{5}{12} \leqslant P(B)$

Again $B \subset A \cup B \Rightarrow P(B) \leqslant P(A \cup B)=\frac{3}{4}$

Hence $\frac{5}{12} \leqslant P(B) \leqslant \frac{3}{4}$

303 Ans $\frac{5}{9}$



We denote the time of arrival of P by x and that of Q by y. For the meeting between P and Q to take place, it is necessary and sufficient that $|x-y| \leqslant 20$
We represent x and y as cartesian coordinates in the plane, take one minute as unit of scale. All possible outcomes will be described as points of a square with side 60. Fovourable outcomes will be in the shaded figure. The required probability is the ratio of the area of the shaded figure to the area of the whole square.

Thus $P=\frac{60^2-40^2}{60^2}=\frac{5}{9}$

304 The sportsman's chance of missing at a distance k is $1-a^2/a\,k$, i.e. $1-1/k^2$, $k=2, 3, 4, \ldots, n$. (Values of $k > n$ are ruled out since the animal escapes if the sportsman misses at distance na.) Hence the probability of animal's escape is given by

$$p=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{n^2}\right)$$

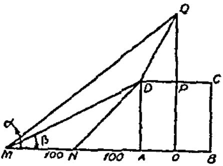
$$=\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{4}\right)\cdots\left(1-\frac{1}{n}\right)\right]\left[\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)\left(1+\frac{1}{4}\right)\cdots\left(1+\frac{1}{n}\right)\right]$$

$$=\left(\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdots\frac{n-1}{n}\right)\left(\frac{3}{2}\cdot\frac{4}{3}\cdot\frac{5}{4}\cdots\frac{n+1}{n}\right)$$

$$PA=PB=PC=\text{circumradius of } \triangle ABC=\frac{abc}{4\Delta} \tag{2}$$

Hence from (1), and (2), we obtain,

$h \cot \theta=\frac{abc}{4\Delta}$ or $h=\frac{abc \tan \theta}{4\Delta}$



55 Let $ABCD$ represent the vertical cross section of the tower through the middle so that side of the square is of length $AB=a$, say and height of the tower is $OP=h$ say Let height of flag staff $PQ=b$ The two points of observation are N and M where $AN=NM=100$ meter It is given that

$\tan \alpha=\frac{5}{9}$ and $\tan \beta=\frac{1}{2}$ Also since at N, the man sees the flag, the points N, D, Q are in a straight line

Now $\frac{1}{2}=\tan \beta=\frac{AD}{AM}=\frac{AD}{200}$ or $AD=100=OP$

$$h=OP=100 \tag{1}$$

$AD=AN=100$ so that $\angle AND=\angle NDA=45°$

Hence also $\angle ODP=45°$ so that $\angle DQP=45°$

$$\text{Therefore } PD=PQ \quad \text{or } \frac{a}{2}=b \tag{2}$$

Again $\frac{5}{9}=\tan \alpha=\frac{OQ}{OM}=\frac{h+b}{200+\frac{a}{2}}=\frac{100+b}{200+b}$ by (1) and (2)

or $1000+5b=900+9b$ or $b=25$ and so $a=2b=50$

Hence height of tower$=100$ meters side of square base$=50$ meter and height of the flag staff $=25$ meters

56 (a) Figure is self explanatory we have,

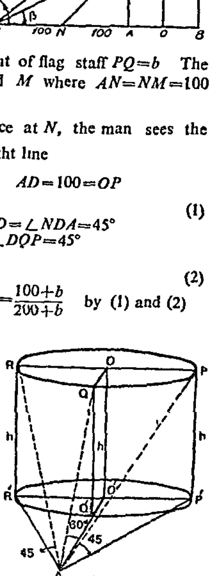


Fig 114

$AP=AR=h \cot 45°=h$ and

$AQ=h \cot 60°=\frac{h}{\sqrt{3}}$

Now in $\triangle AOP$, we have

$AO=AQ+QO=\frac{h}{\sqrt{3}}+r,$

$OP = r$, $AP = h$ and $\angle AOP = 90°$

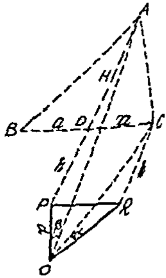
Hence $OA^2 + OP^2 = AP^2$ $\left(\frac{h}{\sqrt{3}} + r\right)^2 + r^2 = h^2$

or $h^2 - \sqrt{3}hr - 3r^2 = 0$

$$h = \frac{\sqrt{3}r \pm \sqrt{(3r^2 + 12r^2)}}{2}$$

or $\frac{h}{r} = \frac{\sqrt{3} + \sqrt{15}}{2} = \frac{\sqrt{3}(\sqrt{5}+1)}{2}$

[Note That here, we rejected −ve sign since h and r are positive quantities]



(b) Let ABC be isosceles triangle mounted on the pole PD and O be the position of the man on the ground so that $OP = d$ Let $BC = 2a$ and the height $AD = H$ Angles α and β are the angles subtended by the vertex C and A at O respectively (see the figure) CQ is parallel to DP so that $CQ = h$ and then $PQ = a$ Now $d = PO = (h + H) \cot \beta$

or $H = d \tan \beta - h$ Also $OQ = h \cot \alpha$

Also from right angled $\triangle OPQ$ we have $a = PQ = \sqrt{OQ^2 - OP^2} = \sqrt{(h^2 \cot^2 \alpha - d^2)}$

Hence area of $\triangle ABC = \frac{1}{2} BC \cdot AD = aH$

$$= (d \tan \beta - h)\sqrt{(h^2 \cot^2 \alpha - d^2)}$$

57 Let $ABCD$ $A'B'C'D'$ represent the tower on square base $ABCD$ Since the elevations of B and D at O are the same each equal to 45°, it follows that the position O of the observer must be on the diagonal CA of the base produced The elevation of A' at O is 60° Then from the figure it is clear that

$$OA = h \cot 60 = \frac{h}{\sqrt{3}}$$

$$OB = OD = h \cot 45° = h$$

and $\angle OAB = 135$

Also

Hence from $\triangle OAB$, we have

$$OB^2 = OA^2 + AB^2 - 2\, OA \cdot AB \cos 135°$$

or $h^2 = \frac{h^2}{3} + a^2 - 2 \frac{h}{\sqrt{3}} \cdot a \left(-\frac{1}{\sqrt{2}} \right)$

or $2h^2 - \sqrt{6}\, ah - 3a^2$

$$=\left(\frac{1}{n}\right)\left(\frac{n+1}{2}\right)=\frac{n+1}{2n}$$

Hence odds against the sportsman are

$n+1 \quad 2n \quad (n+1)$ or $n+1 \quad n-1$

305 The difference can be $m+1, m+2, \ldots, n-1$ The difference $n-r$ can come in r ways as in following pairs

$$(1, n-r+1), (2, n-r+2), \ldots (r, n).$$

Hence the total no of ways in which difference can be $n-1, n-2, \ldots, n-r, \ldots, m+1$ is

$$1+2+3+\ldots+(n-m-1)=\frac{(n-m-1)(n-m)}{2}$$

and the total no of ways of drawing to counters is

${}^nC_2$ i e $n(n-1)/2$

Hence the required probability

$$=\frac{(n-m+1)(n-m)}{2}\times\frac{2}{n(n-1)}=\frac{(n-m)(n-m-1)}{n(n-1)}$$

306 Hint Use Baye s theorem

Since all coins are equally likely to be selected we have

$$P(A_1)=P(A_2)=P(A_3)=1/3$$

If A denotes the event of obtaining two heads and one tail, then

$$P(A/A_1)={}^3C_2\left(\frac{1}{2}\right)^2\frac{1}{2}=\frac{3}{8},$$

$$P(A/A_2)={}^3C_2\left(\frac{2}{3}\right)^2\frac{1}{3}=\frac{4}{9},\ P(A/A_3)={}^3C_2\left(\tfrac{1}{3}\right)^2\left(\frac{2}{3}\right)=\frac{2}{9}$$

Hence $P(A_1/A)=\dfrac{P(A_1)\,P(A/A_1)}{P(A_1)\,P(A/A_1)+P(A_2)\,P(A/A_2)+P(A_3)\,P(A/A_3)}$

$$=\frac{\frac{1}{3}\cdot\frac{3}{8}}{\frac{1}{3}\cdot\frac{3}{8}+\frac{1}{3}\cdot\frac{4}{9}+\frac{1}{3}\cdot\frac{2}{9}}=\frac{9}{25}$$

Vectors

307 Ans -13

308 $-11/2$

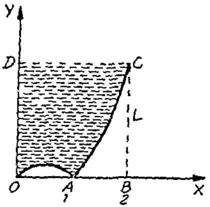
309 $(48, -64, 60)$

310 Ans $\lambda=-\frac{31}{12}, \mu=\frac{41}{12}$

313 Ans 38 units

314 Ans $-3i+11j+9k$

315 Ans $\sqrt{17}$ km/h, $\tan^{-1}(1/4)$ north of east



At $x=2$, $y=2(2-1)^2=2$ so that $OB=BC=2$

. The required area = shaded area

= Area of the square $OBCD$

$$-\int_0^2 y\,dx$$

$$=2^2-\int_0^2 x(x-1)^2\,dx \quad \text{Integrate by parts}$$

$$=4-\left[x\frac{(x-1)^3}{3}-\frac{1}{3}\int (x-1)^3\,1\,dx\right]$$

$$=4-\left[\frac{x}{3}(x-1)^3-\frac{(x-1)^4}{12}\right]_0^2$$

$$=4-\left[\frac{1}{3}(2-0)-\frac{1}{12}(1-1)\right]=4-\frac{2}{3}=\frac{10}{3} \text{ sq units}$$

11 Ans $h=50\sqrt{3}$

Hint If O is the mid point of BC and h, the height of the tower, then $OA=h\cot 45^\circ=h$ and $OB=OC=h\cot 60^\circ=h/\sqrt{3}$

Also $AB=AC=100$

Hence from right angled $\triangle AOB$ $AO^2+OC^2=AB^2$

or $h^2+\frac{1}{3}h^2=100^2$ or $h^2=\frac{100^2\times 3}{4}$, $h=50\sqrt{3}$m

12 Refer Q 16 and 17 P 408 and putting $(n-1)$ for n in the values of Σn, Σn^2 and Σn^3 as for $a=1$ $a-1=0$, for $a=n$, $a-1=n-1$

$$\sum_{a=1}^{n}\Delta_a=\begin{vmatrix} \frac{(n-1)n}{2} & n & 6 \\ \frac{1}{6}(n-1)\,n(2n-1) & 2n^2 & 4n-2 \\ \frac{1}{4}(n-1)^2 n^2 & 3n^3 & 3n^2-3n \end{vmatrix}$$

I. I T., Roorkee and M N R Papers

1989

MATHEMATICS

I I T 1989

PART 'A'

1 Find the values of a and b so that the function

$$f(x)=\begin{cases} x+a\sqrt{2}\sin x & 0<x<\pi/4, \\ 2x\cot x+b & \pi/4\leqslant x\leqslant\pi/2, \\ a\cos 2x-b\sin x & \pi/2<x\leqslant\pi \end{cases}$$

is continuous for $0<x\leqslant\pi$

2 If vectors $\vec{a}, \vec{b}, \vec{c}$ are coplanar, show that

$$\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \end{vmatrix}=\vec{0}$$

3 Evaluate $\int(\sqrt{\tan x}+\sqrt{(\cot x)})\,dx$

4 Using mathematical induction, prove that

$${}^mC_0\,{}^nC_k+{}^mC_1\,{}^nC_{k-1}+\quad+{}^mC_k\,{}^nC_0={}^{(m+n)}C_k$$

where m, n, k are positive integers, and ${}^pC_q=0$ for $p<q$

5 Draw a graph of the function

$$y=[x]+|1-x|,\ -1\leqslant x\leqslant 3$$

Determine the points, if any, where this function is not differentiable

Taking $\frac{(n-1)\,n}{12}$ common from c_1 it becomes identical with c_3 and hence $\Sigma\Delta_a=0$ i.e. constant

13 p_1 = probability of winning the best of three games

$= $ last two terms in the expansion of $(0.6+0.4)^3$

$= {}^3c_2\,(0.6)\,(0.4)^2+(0.4)^3$

$= 0.288+0.064=0.35200$

And p_2 = probability of winning the best of 5 games

$= $ last three terms in the expansion of $(0.6+0.4)^5$

$= {}^5c_3\,(0.6)^2\,(0.4)^3+{}^5c_4\,(0.6)\,(0.4)^4+(0.4)^5$

$=0.2304+0.0768+0.01024$

$=0.31744$

Since $p_1 > p_2$, the first option i.e. the option of best of three games has a higher probability of winning the match

14 We have

$$(1-x)^n=c_0-c_1x+c_2x^2-c_3x^3+\ldots+(-1)^n c_nx^n$$

Multiplying by x, we get

$$x\,(1-x)^n=c_0x-c_1x^2+c_2x^3-c_3x^4+\ldots+(-1)^n c_nx^{n+1}$$

Differentiating both sides w.r.t. x, we get

$$1\,(1-x)^n+x\,n\,(1-x)^{n-1}\,(-1)$$
$$=c_0-2^2c_1x+3c_2x^2-4c_3x^3+\ldots$$

Multiplying by x, we obtain

$$x\,(1-x)^n-nx^2\,(1-x)^{n-1}=c_0x-2c_1x^2+3c_2x^3-4c_3x^4+\ldots+(-1)^n c_n\,(n+1)\,x^{n+1}$$

Differentiating again, we have

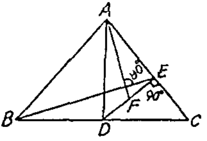
$$1\,(1-x)^n+x\,n(1-x)^{n-1}\,(-1)-2nx\,(1-x)^{n-1}-n(n-1)\,x^2\,(1-x)^{n-2}\,(-1)$$
$$=c_0-2c_1x+3^2\,c_2x^2-4^2\,c_3x^3+\ldots+(-1)^n c_n\,(n+1)^2x^n$$

Finally, putting $x=1$ we obtain

$$0=c_0-2^2c_1+3^2c_2-4^2c_3+\ldots+(-1)^n c_n\,(n+1)^2,$$

as required

15 Let the side BC be chosen along x axis its length be $2a$ and mid point D be origin so that the points B and C are $(-a, 0)$ and $(a, 0)$. Since the triangle is isosceles therefore median AD will be perpendicular to base BC so that AD is along y axis



6 If f and g are continuous functions on $[0, a]$ satisfying

$$f(x)=f(a-x)$$

and

$$g(x)+g(a-x)=2,$$

then show that

$$\int_0^a f(x)\,g(x)\,dx=\int_0^a f(x)\,d$$

7 In a triangle OAB, E is the midpoint of OB and D is a point on AB such that $AD : DB=2 : 1$ If OD and AE intersect at P, determine the ratio $OP : PD$ using vector methods

8 If $x=\sec\theta-\cos\theta$ and $y=\sec^n\theta-\cos^n\theta$ then show that

$$(x^2+4)\left(\frac{dy}{dx}\right)^2=n^2(y^2+4)$$

9 If $(m_i, 1/m_i)$, $m_i>0$, $i=1, 2, 3, 4$, are four distinct points on a circle, then show that

$$m_1m_2m_3m_4=1$$

10 Find all maxima and minima of the function

$$y=x(x-1)^2,\ 0\leqslant x\leqslant 2$$

Also determine the area bounded by the curve $y=x(x-1)^2$ the y-axis and the line $y=2$

11 ABC is a triangular park with $AB=AC=100\ m$ A television tower stands at the midpoint of BC The angles of elevation of the top of the tower at A, B, C, are $45°$, $60°$, $60°$, respectively Find the height of the tower

12. Let $\Delta_a=\begin{vmatrix} a-1 & n & 6 \\ (a-1)^2 & 2n^2 & 4n-2 \\ (a-1)^3 & 3n^3 & 3n^2-3n \end{vmatrix}$

Show that $\sum\limits_{a=1}^{n}\Delta_a=c$, a constant

13 Suppose the probability for A to win a game against B is 0 4 If A has an option of playing either a "best of 3 games" or a 'best of 5 games' match against B, which

and as such let A be $(0, h)$

Equation to AC is $\frac{x}{a}+\frac{y}{h}=1$ (Intercepts from)

Equation to $DE \perp$ to AC is $\frac{x}{h}-\frac{y}{a}=0$

Solving their we get the point E as $\left(\frac{ah^2}{a^2+h^2}, \frac{a h}{a^2+h^2}\right)$

F the mid point of DE is $\left(\frac{ah^2}{2(a^2+h^2)}, \frac{a^2h}{2(a^2+h^2)}\right)$

Now find m_1 the slope of BE and m_2 the slope of AF and show that $m_1m_2=-1$

16 (i) Ans False

Put $x^n=\tan\theta$ and $y^n=\tan\phi$

Then $\frac{x^ny^m}{(1+x^{2n})(1+y^{2m})}=\frac{\tan\theta\tan\phi}{\sec^2\theta\sec^2\phi}$

$=\sin\theta\cos\theta\sin\phi\cos\phi$

$=\frac{1}{4}\sin 2\theta\sin 2\phi$

$\leqslant\frac{1}{4}$, since $\sin 2\theta\leqslant 1$ and $\sin 2\phi\leqslant 1$

$\Rightarrow \sin 2\theta\sin 2\phi\leqslant 1$

(ii) *Ans True*

Since the centre $(3, -1)$ lies on the line

(iii) Ans False

Refer Q 53 (i) to (ii) P 807 $[a-b, b-c, c-a]=0$

(iv) Ans False

Since A and B are independent events, we have

$P(A\cap B)=P(A)\,P(B)$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=P(A)+P(B)-P(A)\,P(B)$

$=0\,2+0\,3-0\,2\times 0\,3$

$=0\,5-0\,06$

$=0\,44$

17 (i) $\Delta=16q^4-4(2q^4-r)=8q^4+4r=+\text{ive}$

q^4 is +ive and $r=\alpha^4+\beta^4=+\text{ive}$ two real roots

(ii) Ans *See Q 34 (b) P 336*

(iii) Ans (C),

If r is the radius of the circle, then

$\pi r^2=154$ or $r^2=154\times\frac{7}{22}$ $\left[\text{Taking } \pi=\frac{22}{7}\right]$

$r^2=49$ or $r=7$

option should A choose so that the probability of his winning the match is higher ? (No game ends in a draw)

14 Prove that

$$C_0-2^2C_1+3^2C_2- \quad +(-1)^n(n+1)^2C_n=0, \; n>2,$$

where $C_r={}^nC_r$

15 Let ABC be a triangle with $AB=AC$ If D is the mid-point of BC, E the foot of the perpendicular drawn from D to AC and F the midpoint of DE, prove that AF is perpendicular to BE

PART 'B'

16 This question contains four statements, each of which is either true or false Indicate your choice of the answer in the answer book by writting TRUE or FALSE, followed by a brief reasoning in one or two sentences, for each statement

(i) If x and y are positive real numbers and m n are any positive integers, then

$$\frac{x^n y^m}{(1+x^{2n})(1+y^{2m})}>\frac{1}{4}$$

(ii) The line $x+3y=0$ is a diameter of the circle

$$x^2+y^2-6x+2y=0$$

(iii) For any three vectors $\vec{a}$ $\vec{b}$ and $\vec{c}$,

$$(\vec{a}-\vec{b})\;(\vec{b}-\vec{c})\times(\vec{c}-\vec{a})=2\vec{a}\;\vec{b}\times\vec{c}$$

(iv) If the probability for A to fail in an examination is 0 2 and that for B is 0 3, then the probability that either A or B fails is 0 5

17 There are six parts in this question Four choices are given for each part and one of them is correct Indicate your choice of correct answer for each part in your answer book by writing one of the letters A, B, C, D, which ever is appropriate

(i) If α and β are the roots of $x^2+px+q=0$ and α^4, β^4 are the roots of $x^2-rx+s=0$, then the equation $x^2-4qx+2q^2-r=0$ has always

Also solving the equation of two given diameters, we get the coordinates of the centre as (1, −1)

Hence the equation of the circle is

$$(x-1)^2+(y+1)^2=7^2=49$$

or $$x^2+y^2-2x+2y=47$$

(iv) Ans (D)

$$a^2\alpha^2+b\alpha+c=0 \text{ and } a^2\beta^2-b\beta-c=0$$

Let $f(x)=a^2x^2+2bx+2c=0$

$$f(\alpha)=a^2\alpha^2+2(b\alpha+c)=a^2\alpha^2-2a^2\alpha^2=-a^2\alpha^2=-\text{ive}$$

$$f(\beta)=a^2\beta^2+2(b\beta+c)=a^2\beta^2+2a^2\beta^2=3a^2\beta^2=+\text{ive}$$

Since $f(\alpha)$ and $f(\beta)$ are of opposite sign then we know from theory of equations that a root γ of the equation $f(x)=0$ lies between α and β $\alpha<\gamma<\beta$

(v) Ans (A)

First note that a number will be divisible by 3, if the sum of its digits is divisible by 3

The sum of 5 digits 1, 2, 3 4 5 is 15 which is devisible by 3 *Hence all the five digit numbers formed by these digits* are divisible by 3

Their number$=5!=120$

With 0 the four non zero digits whose sum is divisible by 3 are 1, 2 4 and 5

[Note that there is no other combination of four digits out of 5 non zero digits whose sum is divisible by 3 as can be verified by writing all the fine combinations of 5 non zero digits taken 4 at a time]

The number of numbers in this case

$$=4\ 4\ 3\ 2\ 1=96$$

Hence the required number of numbers

$$=120+96=216$$

(vi) Ans (B)

The given equation can be written as

$$(\cos x-\sin x)-3(\cos 2x-\sin 2x)+\cos 3x-\sin 3x=0$$

or $$\cos x-\cos(\pi/2-x)-3[\cos 2x-\cos(\pi/2-2x)]+\cos 3x-\cos(\pi/2-3x)=0$$

(A) two real roots (C) two negative roots
(B) two positive roots (D) one positive and one negative root

(ii) If the two circles $(x-1)^2+(y-3)^2=r^2$ and $x^2+y^2-8x+2y+8=0$ intersect in two distinct points then,

(A) $2 < r < 8$ (B) $r < 2$
(C) $r=2$ (D) $r > 2$

(iii) The lines $2x-3y=5$ and $3x-4y=7$ are diameters of a circle of area 154 sq units Then the equation of this circle is

(A) $x^2+y^2+2x-2y=62$ (C) $x^2+y^2-2x+2y=47$
(B) $x^2+y^2+2x-2y=47$ (D) $x^2+y^2-2x+2y=62$

(iv) Let a, b, c be real numbers, $a \neq 0$ If α is a root of $a^2x^2+bx+c=0$, β is root of $a^2x^2-bx-c=0$ and $0 < \alpha < \beta$, then the equation $a^2x^2+2bx+2c=0$ has a root γ that always satisfies

(A) $\gamma=\frac{\alpha+\beta}{2}$ (C) $\gamma=\alpha$

(B) $\gamma=\alpha+\frac{\beta}{2}$ (D) $\alpha < \gamma < \beta$

(v) A five digit number divisible by 3 is to be formed using the numerals 0, 1, 2 3, 4, and 5 without repetition The total number of ways this can be done is

(A) 216 (B) 240 (C) 600 (D) 3125

(vi) The general solution of
$\sin x-3\sin 2x+\sin 3x=\cos x-3\cos 2x+\cos 3x$ is

(A) $n\pi+\frac{\pi}{8}$ (C) $(-1)^n\frac{n\pi}{2}+\frac{\pi}{8}$

(B) $\frac{n\pi}{2}+\frac{\pi}{8}$ (D) $2n\pi+\cos^{-1}\frac{3}{2}$

18. There are four parts in this question Each part has more than one correct answer Indicate all correct answers for each part by writing the corresponding letters *A, B, C, D* in the answer book

(i) Let $g(x)$ be a function defined on $[-1, 1]$ If the area of the equilateral triangle with two of its vertices at $(0, 0)$ and $[x, g(x)]$ is $\sqrt{3}/4$, then the function $g(x)$ is or $$2\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4}-x\right)-3\ 2\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4}-2x\right)+2\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4}-3x\right)=0$$

or $$\sin\left(\frac{\pi}{4}-x\right)+\sin\left(\frac{\pi}{4}-3x\right)-3\sin\left(\frac{\pi}{4}-2x\right)=0$$

or $$2\sin\left(\frac{\pi}{4}-2x\right)\cos x-3\sin\left(\frac{\pi}{4}-2x\right)=0$$

or $$\sin\left(\frac{\pi}{4}-2x\right)\ (2\cos x-3)=0$$

Since $\cos x\neq\frac{3}{2}$, we must have

$$\sin(\pi/4-2x)=0 \text{ or } \sin(2x-\pi/4)=0$$

The solution of this equation is

$$2x-\frac{\pi}{4}=n\pi$$

or $$x=\frac{n\pi}{2}+\frac{\pi}{8},\quad n\in I$$

18 (i) Ans (B) and (C)

The side of the equilateral triangle

$$=\sqrt{[(x-0)^2+\{g(x)-0\}^2]}=\sqrt{[x^2+g^2(x)]}$$

Area of the equilateral triangle

$$=\frac{\sqrt{3}}{4}(\text{side})^2=\frac{\sqrt{3}}{4}[x^2+g^2(x)]=\frac{\sqrt{3}}{4}\ (\text{given})$$

Hence $x^2+g^2(x)=1$ or $g^2(x)=1-x^2$

$g(x)=+\sqrt{(1-x^2)}$ or $g(x)=-\sqrt{(1-x^2)}$

Note That a function cannot have two *distinct* values at a point and so the possibility $g(x)=\pm\sqrt{(1-x^2)}$ *is ruled out*

(ii) First note that $x>0$ The given equation can be written as

$$\frac{3}{4}(\log_2 x)^2+\log_2 x-\frac{5}{4}=\log_x\sqrt{2}=\tfrac{1}{2}\log_x 2$$

or $$\frac{3}{4}t^2+t-\frac{5}{4}=\frac{1}{2}\ \frac{1}{t}\quad\text{where } t=\log_2 x$$

or $$3t^3+4t^2-5t-2=0$$

$$(t-1)(3t^2+7t+2)=0\quad\text{or}\quad(t-1)(t+2)(3t+1)=0$$

$$\log_2 x=t=1\ \ -2,\ -1/3$$

(A) $g(x)=\pm\sqrt{(1-x^2)}$ (C) $g(x)=-\sqrt{(1-x^2)}$
(B) $g(x)=\sqrt{(1-x^2)}$ (D) $g(x)=\sqrt{(1+x^2)}$

(ii) The equation

$$x^{3/4(\log_2 x)^2+\log_2 x-5/4}=\sqrt{2}$$

has

(A) at least one real solution
(B) exactly three real solutions
(C) exactly one irrational solution
(D) complex roots

(iii) If $f(x)=\frac{1}{2}x-1$, then, on the interval $[0, \pi]$,

(A) $\tan[f(x)]$ and $1/f(x)$ are both continuous
(B) $\tan[f(x)]$ and $1/f(x)$ are both discontinuous
(C) $\tan[f(x)]$ and $f^{-1}(x)$ are both continuous
(D) $\tan[f(x)]$ is continuous but $1/f(x)$ is not

(iv) If E and F are independent events such that $0 < P(E) < 1$, and $0 < P(F) < 1$ then

(A) E and F are mutually exclusive
(B) E and F^c (the complement of the event F) are independent
(C) E^c and F^c are independent
(D) $P(E \mid F)+P(E^c \mid F)=1$

19 This question contains eight incomplete statements Determine your answers to be inserted in the blanks so that the statements are complete Write these answers only in your answer book, strictly in the order in which the statements appear below (16)

(i) ABC is an isosceles triangle inscribed in a circle of radius r If $AB=AC$ and h is the altitude from A to BC then the triangle ABC has perimeter $P=2\{\sqrt{(2hr-h^2)}+\sqrt{(2hr)}\}$ and area $A=$ Also $\lim_{h\to 0}\frac{A}{P^3}=$

(ii) There are exactly two distinct linear functions, , and which map $[-1, 1]$ onto $[0\ 2]$

(iii) The value of $\int_{-2}^{2} |1-x^2|\, dx$ is

$$x=2,\ 2^{-3},\ 2^{-1/3}=2,\ \tfrac{1}{8},\ \frac{1}{2^{1/3}}$$

This we have exactly three real solutions Again we know that a cubic equations has at least one real solution *A* and *B* are correct

(iii) Ans (B)

First note that $[x]$ means the greatest integer not exceeding x Keeping this in mind, we find

$$[f(x)]=[\tfrac{1}{2}x-1]=-1, \text{ when } 0\leqslant x<2$$
$$=0, \quad \text{when } 2\leqslant x\leqslant \pi$$
$$\tan[f(x)]=\tan(-1)=-\tan 1,\ 0\leqslant x<2$$
$$=\tan 0=0,\ 2\leqslant x\leqslant \pi$$

The function $\tan[f(x)]$ is clearly discontinuous at $x=2$

Also the function $1/f(x)=1/(\tfrac{1}{2}x-1)$ is discontinuous at $x=2$

These two points are discontinuous at all other points of the interval $[0, \pi]$

The function $f^{-1}(x)$ is defined by

$$f^{-1}(x)=y \Leftrightarrow f(y)=x \Leftrightarrow \tfrac{1}{2}y-1=x$$
$$\Leftrightarrow y=2x+1$$

Thus $f^{-1}(x)=2x+1$ which is continuous on $[0, \pi]$ Hence (B) is the only correct answer

(iv) Ans (B), (C) and (D)

Since E F are independent, we have

$$P(E\cap F)=P(E)\,P(F) \qquad (1)$$

Now $P(E\cap F^c)=P(E)-P(E\cap F)$
$=P(E)-P(E)\,P(F)$ [by (1)]
$=P(E)\,[1-P(F)]$
$=P(E)\,P(F^c)$

E and F^c are independent

Again, $P(E^c\cap F^c)=P(E\cup F)^c=1-P(E\cup F)$
$=1-P(E)-P(F)+P(E\cap F)$
$=1-P(E)-P(F)+P(E)\,P(F)$ [by (1)]
$=P(E^c)-P(F)\,[1-P(E)]$
$=P(E^c)-P(F)\,P(E^c)$
$=P(E^c)\,[1-P(F)]=P(E^c)\,P(F^c)$

Hence E^c and F^c are also independent

(iv) If α, β, γ are the cube roots of p, $p < 0$, then for any x, y and z,

$$\frac{x\alpha+y\beta+z\gamma}{x\beta+y\gamma+z\alpha}=$$

(v) If a and b are real numbers between 0 and 1 such that the points $z_1=a+i$, $z_2=1+bi$ and $z_3=0$ form an equilateral triangle, then $a=$ and $b=$

(vi) A pair of fair dice is rolled together till a sum of either 5 or 7 is obtained. Then the probability that 5 comes before 7 is

(vii) The area of the triangle formed by the positive x axis and the normal and the tangent to the circle

$x^2+y^2=4$ at $(1,\sqrt{3})$ is

(viii) The greater of the two angles

$A=2\tan^{-1}(2\sqrt{2}-1)$ and $B=3\sin^{-1}(1/3)+\sin^{-1}(3/5)$ is

Solutions

I I T 1989

1 We apply the test of continuity at $x=\pi/4$ and $x=\pi/2$ to get the values of a and b. At $x=\pi/4$, we have

$$f(\pi/4)=2\cdot\pi/4\cot(\pi/4)+b=\pi/2+b,$$

$$f\left(\frac{\pi}{4}-0\right)=\lim_{h\to0}\left[\frac{\pi}{4}-h+a\sqrt{2}\sin\left(\frac{\pi}{4}-h\right)\right]$$

$$=\frac{\pi}{4}+a\sqrt{2}\cdot\frac{1}{\sqrt{2}}=a+\frac{\pi}{4}$$

and

$$f\left(\frac{\pi}{4}+0\right)=\lim_{h\to0}\left[2\left(\frac{\pi}{4}+h\right)\cot\left(\frac{\pi}{4}+h\right)+b\right]$$

$$=\frac{\pi}{2}\cdot1+b=\frac{\pi}{2}+b$$

For continuity at $x=\pi/4$, we have

$$\pi/2+b=a+\pi/4 \quad \text{or} \quad a-b=\pi/4 \qquad (1)$$

At $x=\pi/2$ we have

$$f\left(\frac{\pi}{2}\right)=2\cdot\frac{\pi}{2}\cot\frac{\pi}{2}+b=b,$$

$$f\left(\frac{\pi}{2}-0\right)=\lim_{h\to0}\left[2\left(\frac{\pi}{2}-h\right)\cot\left(\frac{\pi}{2}-h\right)+b\right]=b,$$

Finally, $P(E/F)+P(E^c/F)=\frac{P(E\cap F)}{P(F)}+\frac{P(E^c\cap F)}{P(F)}$

$=\frac{P(E\cap F)+P(E^c\cap F)}{P(F)}=\frac{P(F)}{P(F)}=1$

19 (i) Ans area $A=h\sqrt{(2hr-h^2)}$

and $\lim_{h\to 0}\frac{A}{P^3}=\frac{1}{128r}$

Here $AB=AC$, altitude $AD=h$ and radius of the circle$=r$, so that

$PA=PB=PC=r$

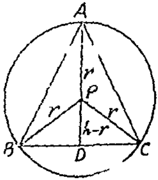
$PD=h-r,$

and $DB=\sqrt{[r^2-(h-r)^2]}$

$=\sqrt{(2rh-h^2)}$

Also $AC=\sqrt{[AD^2+DC^2]}$

$=\sqrt{[h^2+2rh-h^2]}=\sqrt{2rh}$



$P=$Perimeter of

$\Delta ABC=2\,(DC+AC)$

$=2\,[\sqrt{(2rh-h^2)}+\sqrt{(2rh)}]$

And area $A=AD\cdot DC=h\sqrt{(2hr-h^2)}$

Also $\lim_{h\to 0}\frac{A}{P^3}=\lim_{h\to 0}\frac{h\sqrt{(2rh-a^2)}}{8\,[\sqrt{(2hr-h^2)}+\sqrt{2hr}]^3}$

$=\lim_{h\to 0}\frac{h^{3/2}\sqrt{(2r-h)}}{8h^{3/2}\,[\sqrt{(2r-h)}+\sqrt{2r}]^3}$

$=\lim_{h\to 0}\frac{\sqrt{(2r-h)}}{8[\sqrt{(2r-h)}+\sqrt{2r}]^3}=\frac{\sqrt{2r}}{8\,[\sqrt{2r}+\sqrt{2r}]^3}$

(ii) Ans $y=x+1$ and $y=1-x$

(iii) Ans 4

$\int_{-2}^{2}|1-x^2|\,dx=2\int_0^2|1-x^2|\,dx$

$=2\left[\int_0^1|1-x^2|\,dx+\int_1^2|1-x^2|\,dx\right]$

$=2\left[\int_0^1(1-x^2)\,dx+\int_1^2(x^2-1)\,dx\right]$

by the definition of modulus function

$=2\left[\left(x-\frac{1}{3}x^3\right)_0^1+\left(\frac{1}{3}x^3-x\right)_1^2\right]$

$=2\,[(1-\frac{1}{3})-0+(\frac{1}{3}\cdot 8-2)-(\frac{1}{3}\cdot 1-1)]$

$=2\left[\frac{2}{3}+\frac{2}{3}+\frac{2}{3}\right]=4$

and $$f\left(\frac{\pi}{2}+0\right)=\lim_{h\to 0}\left[a\cos 2\left(\frac{\pi}{2}+h\right)-b\sin\left(\frac{\pi}{2}+h\right)\right]=-a-b$$

• For continuity at $x=\pi/2$, we have

$$b=-a-b \quad \text{or} \quad a+2b=0 \qquad (2)$$

Solving (1) and (2), we get $a=\pi/6$, $b=-\pi/12$

2 See Q 34 of problem set (B) on Vectors

3 $$I=\int\frac{\sin x+\cos x}{\sqrt{(\sin x\cos x)}}\,dx=\int\frac{\sqrt{2}\,(\sin x+\cos x)\,dx}{\sqrt{(\sin 2x)}}$$

$I=\sqrt{2}\sin^{-1}(\sin x-\cos x)$ See Q 30 set (F) P 688

4 First, let $m=n=1$, since k is a positive integer less than or equal to the smaller of m and n, we must have $k=1$ when $m=n=1$

So in this L H S $={}^{1}C_0\,{}^{1}C_1+{}^{1}C_1\,{}^{1}C_0=1+1=2$

and R.H S $={}^{1+1}C_1={}^{2}C_1=2$.

Hence the theorem holds for $m=n=1$

Now assume that the Theorem holds for any fixed positive integers m and n, that is, we assume

$$^{m}C_0\,{}^{n}C_k+{}^{m}C_1\,{}^{n}C_{k-1}+{}^{m}C_2\,{}^{n}C_{k-2}+\ldots+{}^{m}C_{k-1}\,{}^{n}C_1+{}^{m}C_k\,{}^{n}C_0={}^{m+n}C_k \quad \ldots(1)$$

Now $$^{m+1}C_0\,{}^{n+1}C_k+{}^{m+1}C_1\,{}^{n+1}C_{k-1}+{}^{m+1}C_2\,{}^{n+1}C_{k-2}+\ldots+{}^{m+1}C_{k-1}\,{}^{n+1}C_1+{}^{m+1}C_k\,{}^{n+1}C_0$$

$$=1\,({}^{n}C_{k-1}+{}^{n}C_k)+({}^{m}C_0+{}^{m}C_1)\,({}^{n}C_{k-2}+{}^{n}C_{k-1})+({}^{m}C_1+{}^{m}C_2)\,({}^{n}C_{k-3}+{}^{n}C_{k-2})+\ldots+({}^{m}C_{k-2}+{}^{m}C_{k-1})\,({}^{n}C_0+{}^{n}C_1)+({}^{m}C_{k-1}+{}^{m}C_k)\,1$$

$$=({}^{n}C_{k-1}+{}^{m}C_1\,{}^{n}C_{k-2}+{}^{m}C_2\,{}^{n}C_{k-3}+\ldots+{}^{m}C_{k-1}\,{}^{n}C_0)$$
$$+({}^{n}C_k+{}^{m}C_1\,{}^{n}C_{k-1}+{}^{m}C_2\,{}^{n}C_{k-2}+\ldots+{}^{m}C_{k-1}\,{}^{n}C_1+{}^{m}C_k)$$
$$+({}^{m}C_0\,{}^{n}C_{k-2}+{}^{m}C_1\,{}^{n}C_{k-3}+\ldots+{}^{m}C_{k-2}\,{}^{n}C_0)$$
$$+{}^{m}C_0\,{}^{n}C_{k-1}+{}^{m}C_1\,{}^{n}C_{k-2}+\ldots+{}^{m}C_{k-2}\,{}^{n}C_1+{}^{m}C_{k-1})$$

5 First note that $[x]$ means the greatest integer not exceeding x and $|x|$ is defined as $|x|=x$ if $x\geq 0$, $|x|=-x$ if $x<0$ Hence y is defined as following

$y=-1+1-x=-x$ if $-1\leq x<0$,
$y=0+1-x=1-x$ if $0\leq x<1$
$y=1+x-1=x$ if $1\leq x<2$,
$y=2+x-1=x+1$ if $2\leq x<3$
$y=2+x-1=x+1$ if $2\leq x<3$
and $y=[3]+|1-3|=3+2=5$ if $x=3$

(iv) Since $p<0$ $\quad p=-q$ where q is +ive

$p^{1/3}=-q^{1/3}(1)^{1/3}$ If $1, w, w^2$ be cube roots of unity,

then $\alpha=-q^{1/3}\,1,\ \beta=-q^{1/3}w,\ \gamma=-q^{1/3}w^2$

$$\text{given expression}=\frac{x+yw+zw^2}{xw+yw^2+z}$$

$$=\frac{1}{w}\,\frac{xw+yw^2+z}{xw+yw^2+z}\left(\because\ w^3=1\ \ \frac{1}{w}\right)=\frac{w^3}{w}=w^2$$

(v) Ans $a=b=2-\sqrt{3}$

Since the triangle with vertices $z_1=a+i,\ z_2=1+bi$ and $z_3=0$ is equilateral, we have

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1 \quad (\text{Q } 64 \text{ P } 40)$$

or $\quad (a+i)^2+(1+ib)^2+0=(a+i)(1+ib)+0+0$

or $\quad a^2-b^2+2i(a+b)=(a-b)+i(1+ab)$

Equating real and imaginary parts,

$$a^2-b^2=a-b \qquad (1)$$

and
$$2(a+b)=1+ab \qquad (2)$$

From (1), $(a-b)[(a+b)-1]=0$

$\Rightarrow$ Either $a=b$ or $a+b=1$

Taking $a=b$, we get from (2)

$$4a=1+a^2 \quad \text{or} \quad a^2-4a+1=0$$

$$a=\frac{4\pm\sqrt{(16-4)}}{2}=2\pm\sqrt{3}$$

Since $0<a<1$ and $0<b<1$,

we have $\quad a=b=2-\sqrt{3}$

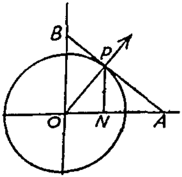
Taking $a+b=1$ or $b=1-a$, we get from (2),

$$2=1+a(1-a) \quad \text{or} \quad a^2-a+1=0$$

which gives imaginary values of a

Hence $a=b=2-\sqrt{3}$ is the required value of a and b

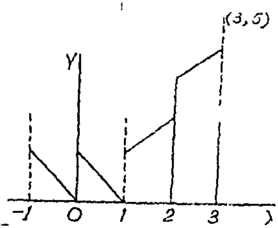
(vi) See Q 21 of problem set (B) on probability of our IIT Guide



(vii) The tangent at $P(1, \sqrt{3})$ is $x+y\sqrt{3}=4$ and normal is $\sqrt{3}x-y=0$. They form a triangle OPA with +ive x axis. Clearly $OA=4$ and $PN=\sqrt{3}$

$$\Delta=\tfrac{1}{2}\,OA\cdot PN=2\sqrt{3}$$

Hence the graph of the function is as shown



From the graph it is clear that the function y is not continuous at $x=0, 1, 2,$ and 3 and so it is non differentiable at these points. Note that y is continuous and differentiable on the right at $x=-1$ y is continuous and differentiable at all other points in the domain of definition

6 By property 4, we have

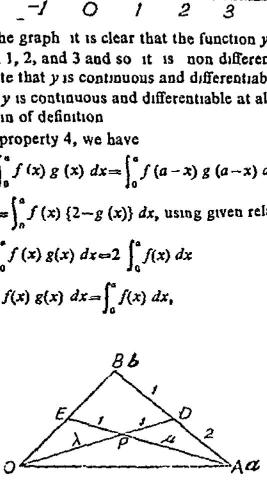
$$\int_0^a f(x)\, g(x)\, dx=\int_0^a f(a-x)\, g(a-x)\, dx$$

$$=\int_0^a f(x)\, \{2-g(x)\}\, dx, \text{ using given relations}$$

$$2\int_0^a f(x)\, g(x)\, dx=2\int_0^a f(x)\, dx$$

or $$\int_0^a f(x)\, g(x)\, dx=\int_0^a f(x)\, dx,$$

as required



7 With O as origin let **a** and **b** be the position vectors of A and B respectively Then the position vector of E, the mid point

(viii) Ans A

We have

$$A=2\tan^{-1}(2\sqrt{2}-1)=2\tan^{-1}(2\times 1\ 414-1)$$
$$=2\tan^{-1}(1\ 828)>2\tan^{-1}\sqrt{3}=2\frac{\pi}{3}=\frac{2\pi}{3}$$
$$[\because\ \sqrt{3}=1\ 732<1\ 82$$

Thus $A>\frac{2\pi}{3}$

For the R H S, we first note that

$$\sin^{-1}\tfrac{1}{3}<\sin^{-1}\tfrac{1}{2}=\frac{\pi}{6},\text{ so that}$$

$$0<3\sin^{-1}\tfrac{1}{3}<\frac{\pi}{2}$$

Now $3\sin^{-1}\frac{1}{3}=\sin^{-1}\left(3\times\frac{1}{3}-4\times\frac{1}{27}\right)$

$$=\sin^{-1}\frac{23}{27}=\sin^{-1}(\ 851)<\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$$

$$\left[\quad\frac{\sqrt{3}}{2}=0\ 866\right]$$

Thus $3\sin^{-1}\frac{1}{3}<\frac{\pi}{3}$

Also $\sin^{-1}\frac{3}{5}=\sin^{-1}(0\ 06)<\sin^{-1}\frac{\sqrt{3}}{2}=\frac{\pi}{3}$

$$B<\frac{\pi}{3}+\frac{\pi}{3}=\frac{2\pi}{3}$$

Thus $A>\frac{2\pi}{3}$ and $B<\frac{2\pi}{3}$

$$A>B$$

Roorkee Papers 1989

1 Solve the equation

$$2^{|x+1|}-2^{x}=\left|2^{x}-1\right|+1$$

2. The sum of an infinite geometric progression is 2 and the sum of the geometric progression made from the cubes of the terms of this infinite series is 24 Then find the series.

3 In an examination, the maximum marks for each of the three papers are 50 each Maximum marks for the fourth

of OB, is $b/2$. Again, since $AD : DB = 2 : 1$, the position vector of D is $\frac{1\mathbf{a}+2\mathbf{b}}{1+2}$ i.e. $\frac{\mathbf{a}+2\mathbf{b}}{3}$. Now let $OP : PD = \lambda : 1$ and

$$AP : PE = \mu : 1$$

Then equating the position vectors of P, we get

$$\frac{\lambda \frac{\mathbf{a}+2\mathbf{b}}{3}+1 \cdot 0}{\lambda+1} = \frac{\mu \frac{\mathbf{b}}{2}+1 \cdot \mathbf{a}}{\mu+1}$$

or $$\left(\frac{\lambda}{3(\lambda+1)}-\frac{1}{\mu+1}\right)\mathbf{a}+\left\{\frac{2\lambda}{3(\lambda+1)}-\frac{\mu}{2(\mu+1)}\right\}\mathbf{b}=0$$

Since $\mathbf{a}$ and $\mathbf{b}$ are non collinear vectors, and if there exists a relation of the form $x\mathbf{a}+y\mathbf{b}=0$ then we must have $x=0$, $y=0$.

Solving these we get $\mu=4$ and $\lambda=3/2$. $\therefore OP : PD = 3 : 2$

8. We have to prove that $\left(\frac{dy}{dx}\right)^2 = n^2 \frac{(y^2+4)}{x^2+4}$

using $(a-b)^2+4ab=(a+b)^2$

$$\text{R H S} = n^2 \frac{(\sec^n\theta+\cos^n\theta)^2}{(\sec\theta+\cos\theta)^2} \qquad \text{(I)}$$

Now $\frac{dx}{d\theta}=\sec\theta\tan\theta+\sin\theta=\tan\theta(\sec\theta+\cos\theta)$

$$\frac{dy}{d\theta}=n(\sec^{n-1}\theta\sec\theta\tan\theta+n\cos^{n-1}\theta\sin\theta)$$

$$=n\tan\theta(\sec^n\theta+\cos^n\theta)$$

$\left(\frac{dy}{dx}\right)^2$ = as given by (I) = L H S

9. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Since $(m_i, 1/m_i)$ lies on it, we have

$$m_i^2+\frac{1}{m_i^2}+2gm_i+\frac{2f}{m_i}+c=0$$

or $$m_i^4+2gm_i^3+cm_i^2+2fm_i+1=0$$

Since roots of this equation in m_i are m_1, m_2, m_3, m_4, we have

$$m_1 m_2 m_3 m_4 = \frac{\text{constant term}}{\text{coeff of } m_i^4} = \frac{1}{1} = 1$$

10. It is easy to find that y is Max. at $x=\frac{1}{2}$ and min. at $x=1$

The shape of the curve is as shown

paper are 100 Find the number of ways in which the candidate can score 60% marks in the aggregate

4 A five digit number is formed by the digits 1, 2, 3, 4, 5 without repetition Find the probability that the number formed is divisible by 4

5 Let A and B be two complex numbers such that

$$\frac{A}{B}+\frac{B}{A}=1$$

Prove that the origin and the two points represented by A and B form vertices of an equilateral triangle

6 Find a and b such that the inequality

$$a \leqslant 3\cos x+5\sin\left(x-\frac{\pi}{6}\right) \leqslant b \text{ hold good for all } x$$

7 Show, without using tables or calculators, that
$\tan 9^\circ-\tan 27^\circ-\tan 63^\circ+\tan 81^\circ=4$

8 A, balloon is observed simultaneously from three points A B and C on a straight road directly under it. The angular elevation at B is twice that at A and the angular elevation at C is thrice that at A If the distance between A and B is 200 meters and the distance between B and C is 100 meters, find the height of the balloon

9 Solve the equation

$$4\sin^4 x+\cos^4 x=1$$

10 Find the equation of the circles passing through the point (2, 8) touching the lines $4x-3y-24=0$ and $4x+3y-42=0$ and having x coordinate of the centre of the circle less than or equal to 8

11 The extremities of the diagonal of a square are (1, 1) (−2, −1) Obtain the other two vertices and the equation of the other diagonal

12 A variable straight line passes through the points of intersection of the line $x+2y=1$ and $2x-y=1$ and meets the coordinate axes in A and B Find the locus of the middle point of AB

13 The abscissa of the two points A and B are the roots of the equation $x^2+2x-a^2=0$ and the ordinates are the roots of the equation $y^2+4y-b^2=0$, Find the equation of the circle with AB as its diameter Also find the coordinates of the centre and the length of the radius of the circle

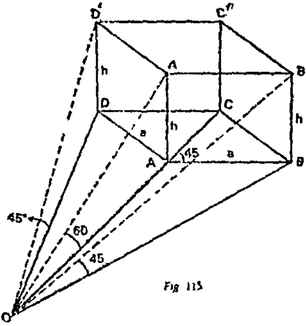


Fig 115

$$h=\frac{\sqrt{6}a\pm\sqrt{(6a^2+24a^2)}}{4}$$

or $$\frac{h}{a}=\frac{\sqrt{6}\,(\sqrt{5}+1)}{4}$$

rejecting the negative sign

58 Let P denote the position of spire and A, B, C the positions of consecutive mile stones so that

$AB=BC=1$ mile

[Note The total angle at $C=60°$ in figure

Let $\angle PBA=\theta$ Then by trigonometrical theorem, we have

$(1+1)\cot\theta$

$=1\cot 30°-1\cot 45°$

or $\cot\theta=(\sqrt{3}-1)/2$ (1)

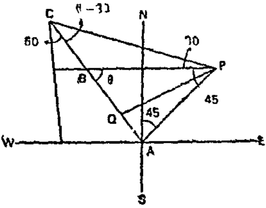


Fig 116

If PQ is perpendicular to the road ABC, then PQ is the shortest distance of P from the road Now from $\triangle CBP$, we have

$$\frac{BP}{\sin(\theta-30°)}=\frac{BC}{\sin 30°}=\frac{1}{\sin 30°}=2$$

or $BP=2\sin(\theta-30^\circ)=2[\sin\theta\cos 30^\circ-\cos\theta\sin 30^\circ]$

$$=2\left[\frac{\sqrt{3}}{2}\sin\theta-\tfrac{1}{2}\cos\theta\right]=\sqrt{3}\sin\theta-\cos\theta$$

Then $PQ=PB\sin\theta=(\sqrt{3}\sin\theta-\cos\theta)\sin\theta$

$$=(\sqrt{3}-\cot\theta)\sin^2\theta=\frac{\sqrt{3}-\cot\theta}{1+\cot^2\theta}=\frac{\sqrt{\{(3-(\sqrt{3}-1)\}/2}}{1+\{(\sqrt{3}-1)/2\}}$$

$$=\frac{(\sqrt{3}+1)/2}{(4-\sqrt{3})/2}=\frac{(\sqrt{3}+1)(\sqrt{3}+4)}{16-3}=\frac{7+5\sqrt{3}}{13}$$

59 Do yourself Similar to problem 53

60 Let OP represent the tower and PQ the flag staff Let R, S and M be the points of observation, so that $\angle QRP=\alpha=\angle QSP$, $\angle QMP=\beta$, and $RM=a=MS$

Since $\angle QRP=\angle QSP$, a circle will pass through the four points P, Q, R and S Let C be the centre of this circle, and from C draw CL and CM perpendiculars to PQ and RS, bisecting them in I and M respectively Join CQ, CP and CR

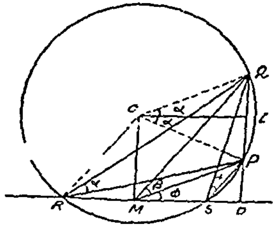
Let the $\angle PMO=\phi$ We then have

$\angle PCQ=2\alpha$, so that

$\angle PCL=\alpha=\angle LCQ$

$$CL=\frac{x}{2}\cot\alpha=MO,$$

where $PQ=x$, and $CR=CQ=\frac{1}{2}x\operatorname{cosec}\alpha$



Also $CM=\sqrt{(CR^2-RM^2)}=\sqrt{\left(\frac{x^2}{4}\operatorname{cosec}^2\alpha-a^2\right)}=IO$

Now $\tan(\beta+\phi)=\frac{OQ}{OM}=\frac{OI+LQ}{CL}=\frac{CM+x/2}{CL}$

14 Draw the graph of the function

$f(x)=x-|x-x^2|, -1\leq x\leq 1$

and discuss its continuity or discontinuity in the interval $-1\leq x\leq 1$ Graph to be drawn on a page of your answerscript

15 It is desired to construct a cylindrical vessel of capacity 500 cubic meters open at the top What should be the dimensions of the vessel so that the material used is minimum, given that the thickness of the material used is 2 cms

16 The area of a circular plot of land in the form of a unit circle is to be divided into two equal parts by the arc of a circle whose centre is on the circumference of the circular plot Show that the radius of the circular arc is $2\cos\theta$, whose θ is given by

$$\frac{\pi}{2}=\sin 2\theta-2\theta\cos 2\theta$$

17 Determine the area of that portion of the circle $x^2+y^2=64$ which is exterior to the parabola $y^2=12x$

18 It is given that

$$\bar{x}=\frac{\bar{b}\times\bar{c}}{[\bar{a}\ \bar{b}\ \bar{c}]},\ \bar{y}=\frac{\bar{c}\times\bar{a}}{[\bar{a}\ \bar{b}\ \bar{c}]},\ \bar{z}=\frac{\bar{a}\times\bar{b}}{[\bar{a}\ \bar{b}\ \bar{c}]}$$

where $\bar{a}$ $\bar{b}$, $\bar{c}$ are non coplanar vectors Show that $\bar{x}$ $\bar{y}$, $\bar{z}$ also form a non coplanar system Find the value of

$\bar{x}\cdot(\bar{a}+\bar{b})+\bar{y}\cdot(\bar{b}+\bar{c})+\bar{z}\cdot(\bar{c}+\bar{a})$

19 It is given that

$\bar{r}\times\bar{b}=\bar{c}\times\bar{b}$, $\bar{r}\cdot\bar{a}=0$ and $\bar{a}\cdot\bar{b}\neq 0$

What is the geometrical meaning of these equations separately ?

If the above three statements hold good simultaneously, determine the vector $\bar{r}$ in terms of $\bar{a}$, $\bar{b}$ and $\bar{c}$

Solutions

1. We consider three cases

(a) $x\leq -1$, (b) $-1<x\leq 0$, (c) $x>0$

In case (a), the equation takes the form

$2^{-(x+1)}-2^{x}=1-2^{x}+1$

or $2^{-(x+1)}=2^{1}$ $-(x+1)=1$

5 Simple

6 See Q 13 P 844

7 Q 6 (a) P 808

8 See Q 49 P 160

9

10 Q 43 (a) P 212

11 Q 21 P 112

12 $\alpha=\cos\theta+i\sin\theta,\ \beta=\cos\theta-i\sin\theta$

$\alpha^n=\cos n\theta+i\sin n\theta,\ \beta^n=\cos n\theta-i\sin\theta$

$S=2\cos n\theta,\ P=1 \qquad x^2-Sx+P=0$

or $\quad x^2-2\cos n\theta\, x+1=0$ is the required equation

Hence $x = -2$

Since $x = -2$ satisfies (a), it is a solution of the given equation

In case (b), the equation reduces to

$2^{x+1} - 2^x = 1 - 2^x + 1$ or $2^{x+1} = 2^1$

$x + 1 = 1$ or $x = 0$

Since $x = 0$ satisfices (b), it is a solution of the given equation

In case (c), the equation becomes

$2^{x+1} - 2 = x^x - 1 + 1$ or $2^{x+1} = 2\,2^x = 2^{x+1}$,

which is an identity Hence (c) implies that all real values of $x > 0$ constitute the solutions in this case

From the above discussion, the required solution is

$x = -2$ and $x \geqslant 0$

2 $\quad \dfrac{a}{1-r} = 2$ and $\dfrac{a^2}{(1-r^2)} = 24$ where $r < 1$

Eliminating a we get, $\dfrac{1-r^3}{(1-r)^3} = \dfrac{8}{24}$

or $3(1+r+r^2) = (1-r)^2 = 1 - 2r + r^2$

or $2r^2 + 5r + 2 = 0$ or $(2r+1)(r+2) = 0$ $\quad r = -\frac{1}{2}$

Putting the value of r we get $a = 3$

$3 - 3/2 + 3/4 - 3/8$ is the series

3 Aggregate of marks $= 50 \times 3 + 100 = 250$

60% of the aggregate $= \dfrac{3}{5} \times 250 = 150$

Now the number of ways of getting 150 marks in the aggregate

$=$ coeffi t of x^{150} in $(x^0 + x^1 + x^2 + \quad + x^{50})^3$
$(x^0 + x^1 + x^2 + x^3 + \quad + x^{100})$

$= \quad ' \quad ' \left(\dfrac{1-x^{51}}{1-x}\right)^3 \left(\dfrac{1-x^{101}}{1-x}\right)$

$= \quad ' \quad ' (1-x^{51})^3 (1-x^{101}) (1-x)^{-4}$

$= \quad ' \quad ' (1 - 3x^{51} + 3x^{102} - x^{153}) (1 - x^{101}) (1-x)^{-4}$

$= \quad ' \quad ' \left(1 - 3x^{51} \quad + 3x^{102} + 3x^{152} \quad\right)$

$\times \left[1 + 4x + \quad + \dfrac{(r+1)(r+2)(r+3)}{6} x^r + \quad \right]$

$$=1\ \frac{151\ 152\ 153}{6}-3\ \frac{100\ 101\ 102}{6}-1\ \frac{50\ 51\ 52}{6}+3\ \frac{49\ 50\ 51}{6}$$

$=151\ 76\cdot51-100\ 101\ 51-50\ 17\ 26+49\cdot25\ 51$

$=51\ (151\ 76-100\ 101)+17\ (49\ 25\ 3-50\ 26)$

$=51\ (11476-10100)+17\ (3675-1300)$

$=51\ 1376+17\ 2375=70176+40375$

$=110551$

4. n=Total number of five digit numbers$=5\,!=120$ Now a number will be divisible by 4 if the last two digits are divisible by 4 Therefore the last two digits can be 12, 24, 32, 52, that is, they can be filled in 4 ways Corresponding to each of these ways, there are $3! = 6$ ways of filling the remaining three places

Hence m=favourable no of ways$=4\times6=24$

The required probability$=\frac{24}{120}=\frac{1}{5}$

5 Refer Q 64 P 40, if z_1, z_2, z_3 be the vertices of the Δ then it will be equilateral if

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$$

Since one vertex is at the origin, therefore putting $z_3=0$ we have $\quad z_1^2+z_2^2=z_1z_2 \quad$ or $\frac{z_1}{z_2}+\frac{z_2}{z_1}=1$ Proved

6 $3\cos x+5\sin(x-\pi/6)$

$=3\cos x+5\sin x\cos\pi/6-5\cos x\sin\pi/6$

$=\frac{1}{2}\cos x+\frac{5\sqrt{3}}{2}\sin x$

$\therefore\quad a=-\sqrt{\left[\left(\frac{1}{2}\right)^2+\left(\frac{5\sqrt{3}}{2}\right)^2\right]}=-\sqrt{(19)}$

and $\quad b=+\sqrt{19}$

7 See Q 33 P 247

If θ be the angle which OP makes with RQP produced, then in the $\triangle OPQ$, $(2c)^2=c^2+(9a)^2-2c\,9a\cos(\pi-\theta)$
and in $\triangle OPR$, $(3c)^2=c^2+(16a)^2-2c\,16a\cos(\pi-\theta)$

Hence $c^2=42a^2$ and $\cos\theta=\dfrac{5a}{2c}$,

so that $\sec\theta=\dfrac{2c}{5a}=\dfrac{2}{5}\;\dfrac{\sqrt{(42)}\,a}{a}=\dfrac{2}{5}\sqrt{(42)}=2\ 6$ nearly

66 Let h be the height of the tower Then $DA=h\cot\alpha$, $DB=h\cot\beta$, and $DC=h\cot\gamma$ (Draw the figure yourself)
Now by Ptolemys theorem (6, ch 6), we get
$AB\ CD+AD\ BC=AC\ BD$ $\quad AB=BC$
$AB\ h\cot\gamma+h\cot\alpha\ AB=h\cot\beta\ 2\ AB\cos\theta$, [
or $\cot\alpha+\cot\gamma=2\cos\theta\cot\beta$

67 (a) Let A,B,C,D be the points on the ground directly below the position of the bird at the time of observations from a point O Then $OA=h\cot\alpha$, $OB=h\cot\beta$, $OC=h\cot\gamma$, and $OD=h\cot\delta$, where h is the height of the bird above the ground (Draw the figure yourself)

Since the speed of the bird is constant, we have
$AB=BC=CD=a$, say Let $\angle OAD=\theta$ Then
in $\triangle OAB$, $h^2\cot^2\beta=h^2\cot^2\alpha+a^2-2ah\cot\alpha\cos\theta$ (1)
in $\triangle OAC$ $h^2\cot^2\gamma=h^2\cot^2\alpha+4a^2-4ah\cot\alpha\cos$ (2)
and in $\triangle OAD$, $h^2\cot^2\delta=h^2\cot^2\alpha+9a^2-6ah\cot\alpha\cos$
Subtracting (2) from (1), $h^2(\cot^2\beta-\cot^2\gamma)$
$=-3a^2+2ah\cot\alpha\cos\theta$

Also from (3), $h^2(\cot^2\alpha-\cot^2\delta)=-9a^2+6ah\cos$
Finally from (4) and (5), we obtain
$\cot^2\alpha-\cot^2\delta=3(\cot^2\beta-\cot^2\gamma)$

(b) Do yourself

8 Refer Q 17 P 111 Put $a=200$ and $b=100$, $\therefore h=100\sqrt{3}$

9 $$4\sin^4 x+\cos^4 x=1$$

or $$(2\sin^2 x)^2+\tfrac{1}{4}(2\cos^2 x)^2=1$$

or $$(1-\cos 2x)^2+\tfrac{1}{4}(1+\cos 2x)^2=1$$

or $$4-8\cos 2x+4\cos^2 2x+1+2\cos 2x+\cos^2 2x=4$$

or $$5\cos^2 2x-6\cos 2x+1=0$$

or $$(\cos 2x-1)(5\cos 2x-1)=0$$

$\therefore$ $\cos 2x=1$, whence $2x=2n\pi$ or $x=n\pi$

or $\cos 2x=1/5=\cos\alpha$, say,

which gives $2x=2n\pi\pm\alpha$ or $x=n\pi\pm\dfrac{\alpha}{2}$,

where $\cos\alpha=1/5$, $0<\alpha<\dfrac{\pi}{2}$

$x=n\pi$ or $x=n\pi\pm 2/\alpha$. Ans

10 The bisectors of the given tangents are $y=3$ and $x=$ [illegible] on which will lie the centre of the centre Since $x\leqslant 8$, $x=\dfrac{66}{8}$ is rejected Hence let the centre be $(h, 3)$ and radius be taken as a The circle posses through $(2, 8)$ $(h-2)^2+(3-8)^2=a^2$

or $$(h-2)^2+25=a^2 \quad (1)$$

The condition of tangency $p=r$ for any tangent gives
$$4h-33=\pm 5a \quad (2)$$

Eliminating (a) between (1) and (2) we get
$$9h^2+164h-364=0 \quad\text{or}\quad (h-2)(9h+182)=0$$

$\therefore$ $h=2$ and from (2), $a=5$
$$(x-2)^2+(y-3)^2=25$$

or $$x^2+y^2-4x-6y-12=0$$

11 Do yourself as in Q 26 P 278 The other two vertices are $(\tfrac{1}{2}, -3/2)$ and $(-3/2, 3/2)$ and $6x+4y+3=0$ in the diagonal

12 Proceed as in Q. 12 (a, b) P 308 or Q 13 (a)

Ans. $10xy=x+3y$

13 Proceed as in Q 53 P 338
$x^2+y^2+2x+4y-(a^2+b^2)=0$, $(-1, -2)$, $\sqrt{(a^2+b^2+5)}$

Similarly we can say that

$\cos^{-1}(\cos\theta)=\theta$, $\cos(\cos^{-1}x)=x$, $0\leqslant\theta\leqslant\pi$, $-1\leqslant x\leqslant 1$

$\tan^{-1}(\tan\theta)=\theta$, $\tan(\tan^{-1}x)=x$, $-\frac{\pi}{2}\leqslant\theta\leqslant\frac{\pi}{2}$, $x\in R$

$\cot^{-1}\cot\theta=\theta$, $(0<\theta<\pi)$

2 $\quad \sin^{-1}x=\operatorname{cosec}^{-1}\left(\frac{1}{x}\right)$ or $\operatorname{cosec}^{-1}x=\sin^{-1}\left(\frac{1}{x}\right)$

If $x=\sin\theta$, then $\theta=\sin^{-1}x$ (1)

$\frac{1}{x}=\frac{1}{\sin\theta}$ or $\operatorname{cosec}\theta=\frac{1}{x}$ $\qquad \theta=\operatorname{cosec}^{-1}\frac{1}{x}$ (2)

Hence from (1) and (2) on equating the value of θ,

$$\sin^{-1}x=\operatorname{cosec}^{-1}1/x$$

Again if $x=\operatorname{cosec}\theta$ then $\theta=\operatorname{cosec}^{-1}x$ (3)

Also $\frac{1}{x}=\frac{1}{\operatorname{cosec}\theta}=\sin\theta$, $\qquad \theta=\sin^{-1}\frac{1}{x}$ (4)

Hence from (3) and (4) equating the value of θ, we get

$$\operatorname{cosec}^{-1}x=\sin^{-1}1/x$$

Similarly we can say that

$$\cos^{-1}x=\sec^{-1}1/x \text{ and } \sec^{-1}x=\cos^{-1}1/x$$

$$\tan^{-1}x=\cot^{-1}1/x \text{ and } \cot^{-1}x=\tan^{-1}1/x$$

§ 3 To find the value of one inverse function in terms of another inverse function

If $\sin\theta=x$ then $\theta=\sin^{-1}x=\operatorname{cosec}^{-1}(1/x)$ (1)

$\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{(1-x^2)}$

$$\theta=\cos^{-1}\sqrt{(1-x^2)}=\sec^{-1}\frac{1}{\sqrt{(1-x^2)}} \qquad (2)$$

$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{x}{\sqrt{(1-x^2)}}$

$$\theta=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}=\cot^{-1}\frac{\sqrt{(1-x^2)}}{x} \qquad (3)$$

Equating the value of θ from (1) (2) and (3) we get

$$\sin^{-1}x=\cos^{-1}\sqrt{(1-x^2)}=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}=\cot^{-1}\frac{\sqrt{(1-x^2)}}{x}$$

$$=\sec^{-1}\frac{1}{\sqrt{(1-x^2)}}=\operatorname{cosec}^{-1}\frac{1}{x} \qquad (4)$$

Similarly we can say that

$$\cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}=\tan^{-1}\frac{\sqrt{(1-x^2)}}{x}=\cot^{-1}\frac{x}{\sqrt{(1-x^2)}}$$

$$=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\frac{1}{\sqrt{(1-x^2)}}$$

14. Using the definition of the modulus function $|x| = x$ if $x \geqslant 0$ and $|x| = -x$ if $x < 0$, we can write the given function in the following form

$$y = f(x) = x - |x - x^2| = x - |x|\,|1 - x|$$

$$= x - (-x)(1-x) = 2x - x^2 \text{ if } -1 \leqslant x \leqslant 0$$

and $y = x - x(1-x) = x^2$ if $0 < x \leqslant 1$

Now $y = 2x - x^2$ can be written as

$$(x-1)^2 = -(y-1),$$

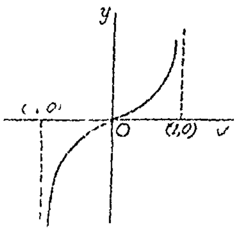
which is a parabola

Thus we have

$$(x-1)^2 = -(y-1) \text{ for } -1 \leqslant x \leqslant 0$$

and $$y = x^2 \text{ for } 0 < x \leqslant 1$$

The portions of these two parabolas under the specified limits are shown below



From the graph, it is clear that the function $f(x)$ is continuous at all points of the given interval. It can also be proved analytically very easily

15. Let r and h be the internal radius and height of the cylinderical vessel and let V denote the volume of the material, then according to the given conditions in the question, we have

$$\pi r^2 h = 500 \qquad (1)$$

and $$V = \pi(r+2)^2(h+2) - \pi r^2 h$$

§ 6 $\sin^{-1}(-x) = -\sin^{-1} x$

$\cos^{-1}(-x) = \pi - \cos^{-1}(x)$ (Note)

$\tan^{-1}(-x) = -\tan^{-1} x$

$\cot^{-1}(-x) = \pi - \cot^{-1}(x)$

Put $-x = \sin\theta \qquad \theta = \sin^{-1}(-x)$ (1)

$x = -\sin\theta = \sin(-\theta)$

$-\theta = \sin^{-1} x$ or $\theta = -\sin^{-1} x$ (2)

Hence from (1) and (2) we get $\sin^{-1}(-x) = -\sin^{-1} x$

Similarly $\tan^{-1}(-x) = -\tan^{-1}(x)$

Again put $-x = \cos\theta \qquad \theta = \cos^{-1}(-x)$

$x = -\cos\theta = \cos(\pi - \theta)$

$\pi - \theta = \cos^{-1} x$ or $\theta = \pi - \cos^{-1} x$ (4)

Hence from (3) and (4) we get $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$

§ 7 $2\sin^{-1} x = \sin^{-1} 2x\sqrt{(1-x^2)}$

$2\cos^{-1} x = \cos^{-1}(2x^2-1)$

$2\tan^{-1} x = \tan^{-1}\dfrac{2x}{1-x^2}$

Put $x = \sin\theta \qquad \theta = \sin^{-1} x$

Now $\sin 2\theta = 2\sin\theta\cos\theta = 2\sin\theta\sqrt{(1-\sin^2\theta)} = 2x\sqrt{(1-x^2)}$

$2\theta = \sin^{-1} 2x\sqrt{(1-x^2)}$

or $2\sin^{-1} x = \sin^{-1} 2x\sqrt{(1-x^2)}$

The other relations may be similarly proved

8 $2\tan^{-1} x = \sin^{-1}\dfrac{2x}{1+x^2} = \cos^{-1}\dfrac{1-x^2}{1+x^2} = \tan^{-1}\dfrac{2x}{1-x^2}$

Put $x = \tan\theta \qquad \tan^{-1} x = \theta$

$2\theta = \sin^{-1}\dfrac{2\tan\theta}{1+\tan^2\theta} = \cos^{-1}\dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \tan^{-1}\dfrac{2\tan\theta}{1-\tan^2\theta}$

or $2\theta = \sin^{-1}\sin 2\theta = \cos^{-1}\cos 2\theta = \tan^{-1}\tan 2\theta$

or $2\theta = 2\theta = 2\theta = 2\theta$

9 $3\sin^{-1} x = \sin^{-1}(3x-4x^3) \quad 3\cos^{-1} x = \cos^{-1}(4x^3-3x)$

$3\tan^{-1} x = \tan^{-1}\dfrac{3x-x^3}{1-3x^2}$

10 $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\{x\sqrt{(1-y^2)} \pm y\sqrt{(1-x^2)}\}$

$\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\{xy \mp \sqrt{(1-x^2)}\sqrt{(1-y^2)}\}$

Put $x = \sin\theta \quad y = \sin\phi \qquad \sin^{-1} x = \theta,\ \sin^{-1} y = \phi$

L H S $= \theta \pm \phi$

R H S $= \sin^{-1}\{\sin\theta\sqrt{(1-\sin^2\phi)} \pm \sin\phi\sqrt{(1-\sin^2\theta)}\}$

$= \sin^{-1}\{\sin\theta\cos\phi \pm \cos\theta\sin\phi\}$

$= \sin^{-1}\sin(\theta\pm\phi) = \theta\pm\phi$

$$=(r+2)^2\left\{\frac{500}{\pi r^2}+2\right\}-500 \text{ from(1)}$$

$$\text{Now } \frac{dV}{dr}=2\pi\,(r+2)\left\{\frac{500}{\pi r^2}+2\right\}+\pi\,(r+2)^2\left(-\frac{1000}{\pi r^3}\right)=0$$

for maxima and minimum

Since $r\neq-2$, we have

$$\frac{250}{\pi r^2}+1-\frac{250\,(r+2)}{\pi r^3}=0$$

or $$250r+\pi r^3-250\,(r+2)=0$$

or $$r=\left(\frac{500}{\pi}\right)^{1/3}$$

Then (1) will give, $h=\left(\frac{500}{3}\right)^{1/3}$

Also it can be easily seen that

$$\frac{d^2V}{dr^2}>0 \text{ for } r=(500/\pi)^{1/3}$$

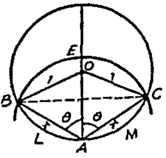
Hence V is minimum when

$$r=\left(\frac{500}{\pi}\right)^{1/3}=h$$

16 Let the centre of the unit circle be at O and let A be that of the circle whose arc BEC divides the unit circle in two equal parts so that

area $ALBECM=\frac{1}{2}$ area of unit circle

$$=\tfrac{1}{2}\pi\,1^2=\tfrac{1}{2}\pi \qquad (1)$$



(j) The value of $\tan\left[\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{2}{3}\right]$ is

(a) $\frac{6}{17}$ (b) $\frac{7}{16}$ (c) $\frac{17}{6}$ (d) None of these (IIT 83)

(k) $\tan\frac{1}{2}\left(\cos^{-1}\frac{\sqrt{5}}{3}\right)$ (Roorkee 83)

3 Prove that

(a) $2\tan^{-1}\frac{1}{5}+\sec^{-1}\frac{5\sqrt{2}}{7}+2\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

(b) $\cos^{-1}\frac{12}{13}+2\cos^{-1}\sqrt{\left(\frac{64}{65}\right)}+\cos^{-1}\sqrt{\left(\frac{49}{50}\right)}=\cos^{-1}\frac{1}{\sqrt{2}}$

4 If $\cos^{-1}p+\cos^{-1}q+\cos^{-1}r=\pi$, then prove that

$$p^2+q^2+r^2+2pqr=1$$

5 If $\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\pi$ or $\pi/2$ show that

$$x+y+z=xyz \quad \text{or} \quad xy+yz+zx=1$$

6 (a) If $\sin^{-1}\frac{2p}{1+p^2}-\cos^{-1}\frac{1-q^2}{1+q^2}=\tan^{-1}\frac{2x}{1-x^2}$, then prove that

$$x=\frac{p-q}{1+pq}$$

(b) If $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$ then x is equal to

(i) $\frac{a-b}{1+ab}$ (ii) $\frac{b}{1+ab}$ (iii) $\frac{b}{1-ab}$ (iv) $\frac{a+b}{1-ab}$

(M N R 88)

7 Prove that $\sin\left[\tan^{-1}\frac{1-x}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right]=1$

8 Prove that $\tan\left[\frac{1}{2}\sin^{-1}\frac{2a}{1+a^2}+\frac{1}{2}\cos^{-1}\frac{1-a}{1+a^2}\right]=\frac{2a}{1-a^2}$

(a) If $\cos^{-1}p/a+\cos^{-1}q/b=\alpha$ then

$$\frac{p^2}{a^2}-\frac{2pq}{ab}\cos\alpha+\frac{q^2}{b^2}=\sin^2\alpha$$

(b) If $\cos^{-1}\frac{x}{2}+\cos^{-1}\frac{y}{3}=\theta$ then prove that

$$9x^2-12xy\cos\theta+4y^2=36\sin^2\theta$$ (Roorkee 88)

10 Prove that

$$\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$$

11 (a) $\tan^{-1}\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\alpha$ then prove that

$$x^2=\sin 2\alpha$$

Let its radius be r so that

$$AC=AB=r \quad \text{Also } OA=OB=OC=1$$

If $\angle OAC=\theta$, then $\angle OCA=\angle OBA=\theta$

and $\angle AOC=\pi-2\theta$

Now $r=AC=2OA\cos\theta=2\cdot 1\cos\theta=2\cos\theta$

Equation to determine θ is found as follows,

Area $ALBECM$

$=2$ [Area of sector ACE+Area of segment AMC]

$=2$ [Area of sector ACE+Area of sector $OAMC$ − area of $\triangle OAC$]

$$=2\left[\tfrac{1}{2}\theta r^2+\tfrac{1}{2}(\pi-2\theta)1^2-\tfrac{1}{2}\cdot1\cdot1\sin(\pi-2\theta)\right]$$

$$=\theta\,4\cos^2\theta+\pi-2\theta-\sin 2\theta \qquad [\because r=2\cos\theta]$$

$$=2\theta(1+\cos 2\theta)+\pi-2\theta-\sin 2\theta=2\theta\cos 2\theta-\sin 2\theta+\pi \qquad (2)$$

From (1) and (2), we get

$$\frac{\pi}{2}=2\theta\cos 2\theta-\sin 2\theta+\pi \text{ or } \frac{\pi}{2}=\sin 2\theta-2\theta\cos 2\theta$$

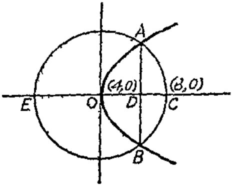
Note The radius r can also be found as follows

The circum radius of $\triangle ABC$ whose sides are r, r and $2r\sin\theta$ is 1 so that

$$1=\frac{r\,r\,2r\sin\theta}{4\text{ Area of }\triangle ABC}=\frac{2r^3\sin\theta}{4\cdot\frac{1}{2}\,2r\sin 2\theta}=\frac{r}{2\cos\theta}$$

$$r=2\cos\theta$$

17 The abscissae of points of intersection of the circle $x^2+y^2=64$ and the parabola $y^2=12x$ are given by



Prove that

20 $\tan^{-1}\frac{c_1x-y}{c_1y+x}+\tan^{-1}\frac{c_2-c_1}{1+c_2c_1}+\tan^{-1}\frac{c_3-c_2}{1+c_3c_2}+ \quad +\tan^{-1}\frac{1}{c_n}$
$=\tan^{-1}\frac{x}{y}$ (Dhanbad 88)

21 $\cot^{-1}\frac{pq+1}{p-q}+\cot^{-1}\frac{qr+1}{q-r}+\cot^{-1}\frac{rp+1}{r-p}=0$

22 $\tan^{-1}\sqrt{\left\{\frac{x(x+y+z)}{yz}\right\}}+\tan^{-1}\sqrt{\left\{\frac{(x+y+z)}{zx}\right\}}$
$+\tan^{-1}\sqrt{\left\{\frac{z(x+y+z)}{x}\right\}}=\pi$

23 $\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$
where $r^2=x^2+y^2+z^2$

Solve for x the equations 24-27

24 $\sec^{-1}\frac{x}{a}-\sec^{-1}\frac{x}{b}=\sec^{-1}b-\sec^{-1}a$

25 $\tan^{-1}\frac{a}{x}+\tan^{-1}\frac{b}{x}+\tan^{-1}\frac{c}{x}+\tan^{-1}\frac{d}{x}=\frac{\pi}{2}$

26 $\tan^{-1}\frac{1}{a-1}=\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{a^2-x+1}$

27 $\cot^{-1}x+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$

28 Find the angles (i) $\sin^{-1}\sin 10$ (ii) $\sin^{-1}\sin 5$ (iii) $\cos^{-1}\cos 10$
(iv) $\tan^{-1}\tan(-6)$

Solutions

1 (a) $\sin[\pi/3-\sin^{-1}(-\frac{1}{2})]$
$=\sin\left(\frac{\pi}{3}+\sin^{-1}\frac{1}{2}\right)=\sin\left(\frac{\pi}{3}+\frac{\pi}{6}\right)=\sin\frac{\pi}{2}=1$
$\sin^{-1}(-x)=-\sin^{-1}x$

(b) Let $\alpha=\cos^{-1}\cos\left(\frac{7\pi}{6}\right)$

Here $\frac{7\pi}{6}=2\pi-\frac{5\pi}{6}$ so that $\frac{5\pi}{6}$ lies between 0 and π

$\alpha=\cos^{-1}\cos\left(2\pi-\frac{5\pi}{6}\right)=\cos^{-1}\cos\left(\frac{5\pi}{6}\right)=\frac{5\pi}{6}$ Ans

(c) Just as (a) part

(d) $\cos^{-1}\cos\frac{4\pi}{3}=\cos^{-1}\cos\left(2\pi-\frac{2\pi}{3}\right)=\cos^{-1}\cos\frac{2\pi}{3}=\frac{2\pi}{3}$

(e) Let $x=\tan^{-1}\tan\left(\frac{3\pi}{4}\right)$ so that x must lie between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

$$x^2+12x-64=0 \quad \text{or} \quad (x+16)(x-4)=0$$

$x=4$ [We do'nt take $x=-16$ for which the values of y come out to be imaginary]

We have to find the area of the circle outside the parabola, i.e. we have to find the area $AEBOA$ shown shaded in the figure. For this, first we find the area $AOBCA$

Now the area $AOBCA=2$ Area $AODCA$

$$=2\left[\int_0^4 y_1\,dx+\int_4^8 y_2\,dx\right]$$

where $y_1^2=12x$ and $y_2^2=64-x^2$

$$=2\left[\int_0^4 2\sqrt{3}x^{1/2}\,dx+\int_4 \sqrt{(64-x^2)}\,dx\right]$$

$$=4\sqrt{3}\,\frac{2}{3}\left[\;x^{3/2}\;\right]_0^4+2\left[\tfrac{1}{2}x\sqrt{(64-x^2)}+\tfrac{1}{2}\;64\sin^{-1}\frac{x}{8}\right]$$

$$=\frac{8\sqrt{3}}{3}+\left[\;0-4\;\sqrt{(64-16)}+64\left\{\sin^{-1}1-\sin^{-1}\frac{1}{2}\right\}\right]$$

$$=\frac{64\sqrt{3}}{3}-16\sqrt{3}+64\left(\frac{\pi}{2}-\frac{\pi}{6}\right)$$

$$=\frac{16\sqrt{3}}{3}+\frac{64}{3}\pi$$

Hence the required area $AEBOA=\pi\,8^2-\left\{\frac{16\sqrt{3}}{3}+\frac{64}{3}\pi\right\}$

$$=\frac{120}{3}\pi-\frac{16\sqrt{3}}{3}=\frac{16}{3}(8\pi-\sqrt{3})$$

18 Refer property (3), P 840 $x=\mathbf{a}'$, $y=\mathbf{b}'$, $z=\mathbf{c}'$

$$[xyz]=[\mathbf{a}'\;\mathbf{b}'\;\mathbf{c}']=\frac{1}{[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]}\neq 0$$

because $[\mathbf{a}\;\mathbf{b}\;\mathbf{c}]\neq 0$ as a, b, c are non coplanar

x, y, z are non coplanar

Again $\quad \Sigma x\;(\mathbf{a}+\mathbf{b})=\Sigma\mathbf{a}'\;(\mathbf{a}+\mathbf{b})$

$$=\Sigma\,\mathbf{a}'\,\mathbf{a}+\Sigma\mathbf{a}'\,\mathbf{b}=1+1+1+0+0+0=3$$

by Prop 1 and 2 P 839

19 **Geometrical Meaning** Let A, B, C and P be the points whose position vectors are **a**, **b**, **c** and **r** respectively

Now $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1} [x\sqrt{(1-y^2)} \pm y\sqrt{(1-x^2)}]$

$$\text{L H S} = \sin^{-1}\left[\frac{4}{5}\sqrt{\left(1-\frac{25}{169}\right)}+\frac{5}{13}\sqrt{\left(1-\frac{16}{25}\right)}\right]$$

$$=\sin^{-1}\left(\frac{4}{5}\cdot\frac{12}{13}+\frac{5}{13}\cdot\frac{3}{5}\right)=\sin^{-1}\left(\frac{48}{65}+\frac{15}{65}\right)=\sin^{-1}\frac{63}{65}$$

$$\text{R H S} = \sin^{-1}\left[1\cdot\sqrt{\left(1-\frac{16^2}{65^2}\right)}-\frac{16}{65}\sqrt{(1-1)}\right]$$

$$=\sin^{-1}\sqrt{\left(\frac{65^2-16^2}{65^2}\right)}=\sin^{-1}\sqrt{\left(\frac{(65+16)(65-16)}{65^2}\right)}$$

$$=\sin^{-1}\frac{\sqrt{(81\times 49)}}{65}=\sin^{-1}\frac{9\times 7}{65}=\sin^{-1}\frac{63}{65}=\text{L H S}$$

(*b*) Do yourself

(*c*) Do yourself

(*d*) Since $\sin^{-1}\frac{12}{13}=\tan^{-1}\frac{12}{5}$ and $\cos^{-1}\frac{4}{5}=\tan^{-1}\frac{3}{4}$, we have

$\text{L H S} = \tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}+\tan^{-1}\frac{63}{16}$ and since $\frac{12}{5}\times\frac{3}{4}>1$,

we have

$$\tan^{-1}\frac{12}{5}+\tan^{-1}\frac{3}{4}=\pi+\tan^{-1}\frac{\frac{12}{5}+\frac{3}{4}}{1-\frac{12}{5}\cdot\frac{3}{4}}=\pi+\tan^{-1}\left\{-\frac{63}{16}\right\}$$

$$=\pi-\tan^{-1}\frac{63}{16} \quad [\because \tan^{-1}(-x)=-\tan^{-1}x]$$

Hence $\text{L H S} = \pi-\tan^{-1}\frac{63}{16}+\tan^{-1}\frac{63}{16}=\pi=\text{R H S}$

(*e*) $\operatorname{cosec}^{-1} x=\cot^{-1}\sqrt{(x^2-1)}$

$$\text{L H S} = \cot^{-1} 9+\cot^{-1}\sqrt{\left(\frac{41}{16}-1\right)}=\cot^{-1} 9+\cot^{-1}\frac{5}{4}$$

$$=\tan^{-1}\frac{1}{9}+\tan^{-1}\frac{4}{5}=\tan^{-1}\frac{\frac{1}{9}+\frac{4}{5}}{1-\frac{1}{9}\cdot\frac{4}{5}}=\tan^{-1}\frac{41}{41}=\tan^{-1} 1=\frac{\pi}{4}$$

(*f*) Do yourself

(*g*) $\tan^{-1} 2+\tan^{-1} 3=\pi+\tan^{-1}\frac{2+3}{1-2\cdot 3}$ $\quad xy=2\cdot 3=6>1$

First note that a b $\neq 0 \Rightarrow$ a $\neq 0$ b $\neq 0$ and a is not perpendicular to b

Now $r \times b = c \times b \Rightarrow (r-c) \times b = 0$

$\Rightarrow$ Either r c $= 0$ or r $-$ c $\parallel$ b [b $\neq 0$]

But r $-$ c $= 0$ means that points P and C coincide and r $-$ c $\parallel$ b implies that $\overrightarrow{CP}$ is parallel to $\overrightarrow{OB}$, i e figure $OBPC$ is a trapezium

Again r a $= 0 \Rightarrow$ Either r $= 0$ or r $\perp$ a

[Note that a $\neq 0$]

The value of r in terms of a, b and c is the same as found in Q 68 problem set B of vectors of our I I T Guide

M N R Papers 1989

1 Evaluate

$$\lim_{x \to \pi/2} \tan x \log_e (\sin x)$$

2 Differentiate $\tan^{-1} \{[\sqrt{(1+x^2)}-1]/x\}$ with respect to $\tan^{-1}(x)$

3 Find the maximum value of the function

$$40 / (3x^4 + 8x^3 - 18x^2 + 60)$$

4 Evaluate

$$\int_0^{\frac{1}{2}\pi} (\cos x \, dx) / (1 + \cos x + \sin x)$$

5 Find the area of a loop between the curve $y = a \sin x$ and x axis

6 Two forces $-i+2j-k$ and $2i-5j+6k$ act on a particle whose position vector is $4i-3j-2k$ and displaces it to another point whose position vector is $6i+j-3k$ Find the total work done by the force

7 Prove that four points whose position vectors are

$(4i+5j+k)$, $(j+k)$, $(3i+9j+4k)$ and $(-4i+4j+4k)$

are coplanar

8 The $(m+1)$th, $(n+1)$th, and $(r+1)$th terms of an arithmetic series are in geometric progression and m, n, and r are in harmonic progression Show that the ratio of the first term and common difference of the arithmetic progression is $(-n/2)$

$$=\tan^{-1}\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\cdot\frac{1}{7}}=\tan^{-1}\frac{25}{25}=\tan^{-1}1=\frac{\pi}{4}$$

(*b*) $\cos^{-1}x=\tan^{-1}\frac{\sqrt{(1-x^2)}}{x}$

$$\cos^{-1}\frac{12}{13}=\tan^{-1}\frac{\sqrt{(1-\frac{144}{169})}}{\frac{12}{13}}=\tan^{-1}\frac{5}{12}$$

$$\cos^{-1}\sqrt{\left(\frac{49}{50}\right)}=\tan^{-1}\frac{\sqrt{(1-\frac{49}{50})}}{\sqrt{(\frac{49}{50})}}=\tan^{-1}\frac{1}{7}$$

Similarly $\cos^{-1}\sqrt{\frac{64}{65}}=\tan^{-1}\frac{1}{8}$, $\cos^{-1}1/\sqrt{2}=\tan^{-1}1=\pi/4$

Thus we have to prove that

$$\tan^{-1}\tfrac{5}{12}+2\tan^{-1}\tfrac{1}{8}+\tan^{-1}\tfrac{1}{7}=\tan^{-1}1$$

or $\tan^{-1}\frac{\frac{5}{12}+\frac{1}{7}}{1-\frac{5}{12}\cdot\frac{1}{7}}=\tan^{-1}1-\tan^{-1}\frac{2\cdot\frac{1}{8}}{1-\frac{1}{64}}$

$\tan^{-1}\frac{47}{79}=\tan^{-1}1-\tan^{-1}\frac{16}{63}$

or $\tan^{-1}\frac{47}{79}=\tan^{-1}\frac{1-(16/63)}{1+(16/63)}=\tan^{-1}\frac{47}{79}$

Hence proved.

4 $\cos^{-1}p+\cos^{-1}q+\cos^{-1}r=\pi$

$\cos^{-1}[pq-\sqrt{(1-p^2)}\,(1-q^2)]=\pi-\cos^{-1}r=\cos^{-1}(-r)$

$$pq-\sqrt{[(1-p^2)\,(1-q^2)]}=-r$$

or $(pq+r)^2=(1-p^2)\,(1-q^2)$

or $p^2q^2+r^2+2pqr=1-p^2-q^2+p^2q^2$

or $p^2+q^2+r^2+2pqr=1$

5 L H S $=\tan^{-1}\frac{(x+y)}{1-xy}+\tan^{-1}z$

$$=\tan^{-1}\frac{\frac{x+y}{1-xy}+z}{1-\frac{x+y}{1-xy}\,z}=\tan^{-1}\frac{x+y+z-xyz}{1-xy-yz-zx}=\pi \text{ or }$$

$$\frac{x+y+z-xyz}{1-xy-yz-zx}=\tan\pi \text{ or } \tan\frac{\pi}{2}=0 \text{ or } \infty$$

Hence $x+y+z-xyz=0$ *i e* $x+y+z=xyz$

or $1-xy-yz-zx=0$ *i e* $xy+yz+zx=1$

6 (*a*) By result 8 page 160 the given equation is

$$2\tan^{-1}p-2\tan^{-1}q=2\tan^{-1}x$$

$$\text{or } \tan^{-1}p-\tan^{-1}q=\tan^{-1}x$$

or $\tan^{-1}\frac{p-q}{1+pq}=\tan^{-1}x$, $\quad x=\frac{p-q}{1+pq}$

9 An article manufactured by a company consists of two parts X and Y In the process of manufacture of the part X, 9 out of 104 parts may be defective Similarly, 5 out 100 are likely to be defective in the manufacture of the part Y Calculate the probability that the assembled product will not be defective

10 If the radii of the externally inscribed circles of a triangle are in arithmetic progression then prove that

$$(a-b)(s-c)=(b-c)(s-a),$$

where a, b, c are the sides of the triangle and $s=(a+b+c)/2$

11 AB is a vertical flag whose end A is on a horizontal plane through A C is mid point of AB and P is a point on horizontal plane through A The portion CB subtends an angle β at P If $nAB=AP$, then show that $\tan\beta=n/(2n^2+1)$

12 Construct an equation whose roots are the nth powers of the roots of the equation

$$x^2-2x\cos\theta+1=0$$

Solution of M N R Paper 1989

1 $$I=\underset{x\to\frac{\pi}{2}}{\text{Lt}}\ \frac{\log_e \sin x}{\cot x} - \frac{0}{0}$$

$$=\text{Lt}\ \frac{\frac{1}{\sin x}\cos x}{-\text{cosec}^2 x}=\text{Lt}\ -\cos x\sin x=0$$

2 Q 14 P 414

3 $z=\frac{1}{y}=\frac{1}{40}(3x^4+8x^3-18x^2+60)$

$x=-3, 0, 1$ z is Min Max, Min

or y is Max, Min, Max

4 By application of prop IV

$$2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{1+\cos x+\sin x}dx=\int_0^{\pi/2}\left[1-\frac{1}{1+\cos x+\sin x}\right]dx$$

$$=\Big[x\Big]_0^{\pi/2}-\int_0^{\pi/2}\frac{1}{2\cos^2 x/2+2\sin x/2\cos x/2}dx$$

Divide above and below by $\cos^2\frac{x}{2}$

$$I=\frac{\pi}{2}-\int_0^{\pi/2}\frac{1}{2}\frac{\sec^2 x/2\,dx}{1+\tan x/2}\qquad \text{Put } 1+\tan\frac{x}{2}=t$$

$$I=\frac{\pi}{2}-\int_1^2\frac{1}{t}\,dt=\frac{\pi}{2}-\Big[\log t\Big]_1^2=\frac{\pi}{2}-\log 2$$

$\frac{1-x}{1+x}=\frac{1-\sin 2\alpha}{1+\sin 2\alpha}$, $\quad \cos^2\alpha+\sin^2\alpha=1,\ 2\sin\alpha\cos\alpha=\sin 2\alpha$.

Hence $x^2=\sin 2\alpha$ from above

Alternative Put $x^2=\cos\theta$ Rest do yourself !

or put $x^2=\sin 2\alpha$

$\sqrt{(1\pm x)}=\sqrt{(1\pm\sin 2\alpha)}=\cos\alpha\pm\sin\alpha$ etc

(*b*) $1\pm\sin x=\cos^2\frac{x}{2}+\sin^2\frac{x}{2}\pm 2\sin\frac{x}{2}\cos\frac{x}{2}$

$=\left(\cos\frac{x}{2}\pm\sin\frac{x}{2}\right)^2$

Hence given expression $=\cot^{-1}\left(-\cot\frac{x}{2}\right)$

$=\cot^{-1}\left[\cot\left(\pi-\frac{x}{2}\right)\right]$

$=\pi-\frac{1}{2}x$ provided $0<x<2\pi$

12 We know that $2\tan^{-1}x=\cos^{-1}\frac{1-x^2}{1+x^2}$

Choose $x=\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\theta/2$

$$2\tan^{-1}\left[\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\theta/2\right]=\cos^{-1}\frac{1-\frac{a-b}{a+b}\tan^2\theta/2}{1+\frac{a-b}{a+b}\tan^2\theta/2}$$

$=\cos^{-1}\frac{a(1-\tan^2\theta/2)+b(1+\tan^2\theta/2)}{a(1+\tan^2\theta/2)+b(1-\tan^2\theta/2)}$

$=\cos^{-1}\frac{a(\cos^2\theta/2-\sin^2\theta/2)+b(\cos^2\theta/2+\sin^2\theta/2)}{a(\cos^2\theta/2+\sin^2\theta/2)+b(\cos^2\theta/2-\sin^2\theta/2)}$

$=\cos^{-1}\frac{a\cos\theta+b}{a+b\cos\theta}$

13 (*a*) $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}1-\tan^{-1}\frac{x+1}{x+2}$

or $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1-\frac{x+1}{x+2}}{1+\frac{x+1}{x+2}}$

or $\tan^{-1}\frac{x-1}{x-2}=\tan^{-1}\frac{1}{2x+3}$

$\frac{x-1}{x-2}=\frac{1}{2x+3}$ or $(x-1)(2x+3)=x-2$

$2x^2+x-3=x-2$ or $2x^2=1$ or $x=\pm 1/\sqrt{2}$

(*b*) The given question can be written as and $\tan\phi = \frac{OP}{OM} = \frac{OL-PL}{CL} = \frac{CM-x/2}{CL}$

Hence $\tan\beta = \tan\{(\beta+\phi)-\phi\} = \frac{\tan(\beta+\phi)-\tan\phi}{1+\tan(\beta+\phi)\tan\phi}$

or $\tan\beta = \frac{\frac{CM+x/2}{CL} - \frac{CM-x/2}{CL}}{1+\frac{(CM+x/2)(CM-x/2)}{CL^2}} = \frac{x\,CL}{CL^2+CM^2-x^2/4}$

or $CL^2+CM^2-\frac{x^2}{4} = x\,CL\cot\beta$

$\frac{x^2}{4}\cot^2\alpha + \frac{x^2}{4}\operatorname{cosec}^2\alpha - a^2 - \frac{x^2}{4} = x\cot\beta\,\frac{x}{2}\cot\alpha$

$\frac{x^2}{4}\cot^2\alpha + \frac{x^2}{4}(\operatorname{cosec}^2\alpha - 1) - \frac{x^2}{2}\cot\alpha\cot\beta = a^2$

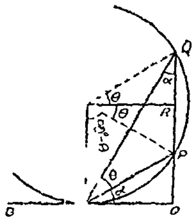
or $\frac{x^2}{2}[\cot^2\alpha - \cot\alpha\cot\beta] = a^2$

$\frac{x^2}{2}\cot\alpha\,(\cot\alpha-\cot\beta) = a^2$

or $\frac{x^2}{2}\,\frac{\cos\alpha}{\sin\alpha}\,\frac{\sin(\beta-\alpha)}{\sin\alpha\sin\beta} = a^2$

or $x = a\sin\alpha\sqrt{\left(\frac{2\sin\beta}{\cos\alpha\sin(\beta-\alpha)}\right)}$

61 Let OP and PQ represent the height of the house and the flag staff and the direction of the person is from D to O. Clearly the flag staff PQ subtends the greatest angle at a point A at which a circle through P, Q touches DO.



Hence $\angle PAQ = \theta$ and $AO = d$

First Method Let $\angle OAP = \alpha$,
Then in the alternate segment, $\angle AQP = \alpha$ so that $2\alpha+\theta = 90°$

Now $PQ = OQ - OP = d\{\tan(\theta+\alpha) - \tan\alpha\}$

$= d\left\{\frac{\sin(\theta+\alpha)}{\cos(\theta+\alpha)} - \frac{\sin\alpha}{\cos\alpha}\right\} = d\,\frac{\sin(\theta+\alpha-\alpha)}{\cos\alpha\cos(\theta+\alpha)}$

$$\frac{6x+2}{(2x-1)(3x+1)-1}=\frac{2}{x^2}$$

or $x^2(3x+1)=8x^2+6x$

$x=0,$

or $3x^2+x=8x+6$

or $3x^2-7x-6=0$ or $(3x+2)(x-3)=0$

$x=3,\ -2/3$

The value $x=-2/3$ is rejected as it makes L H S –ive and R H S is + ive

(*f*) $\tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}3x-\tan^{-1}x$

or $\tan^{-1}\frac{(x-1)+(x+1)}{1-(x^2-1)}=\tan^{-1}\frac{3x-x}{1-3x.x}$

or $\frac{2x}{2-x^2}=\frac{2x}{1+3x^2}$

$x+3x^3=2x-x^3$ or $4x^3-x=0$

or $x(4x^2-1)=0$ $x=0,\ \pm\frac{1}{2}$

(g) The given equation can be written as

$$2\tan^{-1}\frac{1}{2+\sqrt{3}}+\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{1}{x}$$

or $\tan^{-1}\frac{2\frac{1}{2+\sqrt{3}}}{1-\left(\frac{1}{2+\sqrt{3}}\right)^2}+\tan^{-1}\frac{\frac{1}{2+\sqrt{3}}-\frac{1}{3}}{1+\frac{1}{3}\frac{1}{2+\sqrt{3}}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{2(2+\sqrt{3})}{6+4\sqrt{3}}+\tan^{-1}\frac{1-\sqrt{3}}{7+3\sqrt{3}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{2(2+\sqrt{3})}{2\sqrt{3}(2+\sqrt{3})}+\tan^{-1}\frac{1-\sqrt{3}}{7+3\sqrt{3}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{1}{\sqrt{3}}+\tan^{-1}\frac{1-\sqrt{3}}{7+3\sqrt{3}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{\frac{1}{\sqrt{3}}+\frac{1-\sqrt{3}}{7+3\sqrt{3}}}{1-\frac{1}{\sqrt{3}}\frac{1-\sqrt{3}}{7+3\sqrt{3}}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{7+3\sqrt{3}+\sqrt{3}-3}{7\sqrt{3}+9-1+\sqrt{3}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{4+4\sqrt{3}}{8+8\sqrt{3}}=\tan^{-1}\frac{1}{x}$

or $\tan^{-1}\frac{1}{2}=\tan^{-1}\frac{1}{x}$ $\frac{1}{2}=\frac{1}{x}$ or $x=2$

4

Inverse Circular Functions

§ 1 (a) Definitions

The reader is familiar with the definition of inverse of a function However for his convenience, we reproduce the definition here as follows

$$f: A \to I$$

A function is said to be invertible if f is *bijective* and thus inverse function $f^{-1}: B \to A$ is defined by

$$f^{-1}(y)=x \text{ if } f(x)=y,\ x \in A,\ y \in B$$

More generally let A be the domain of definition an *injective* function f and B the set of its values By definition, from $y=f(x)$ $x \in A$, it follows that $x=f^{-1}(y)$, $y \in B$ If $x_1 \neq x_2$, then $y_1 \neq y_2$ and conversely [here $y_1=f(x_1)$, $y_2=f(x_2)$] It is evident that the domain of definition of f^{-1} is the set of values of f, and the set of values of f^{-1} is the domain of definition of the function f

§ 2 General and Principal Values of Inverse Circular Functions (Notation and Meaning)

From the properties of the function $y=\sin x$, for $-1 \leqslant a \leqslant 1$ there are infinitely many angles x which satisfy the equation $\sin x=a$ This infinite number of angles is symbolically denoted by $\text{Sin}^{-1} a$ that is, we use capital S to denote this infinite set of values of the angles There is one value among these values, which lies in the interval $[-\pi/2, \pi/2]$ This value is sometimes called the principal value of the angle and is denoted by $\sin^{-1} x$ (note that we use small s here) Thus $\sin^{-1} x$ *is the angle whose sine is equal to a and which lies in the interval* $[-\pi/2, \pi/2]$ In compact form We may write this definition as follows

$$\alpha=\sin^{-1} a \text{ if (1) } \sin \alpha=a \text{ and (2) } -\pi/2 \leqslant \alpha \leqslant \pi/2$$

and $\text{Sin}^{-1} a=n\pi+(-1)^n \alpha$ where $\alpha=\sin^{-1} a$

Similarly the definitions of the other inverse trigonometrical functions can be given We tabulate these definitions below

16 (i) Let $\sin^{-1}\frac{3x}{5}=\theta$, $\sin^{-1}\frac{4x}{5}=\phi$, $\sin^{-1}x=\psi$

$$\sin\theta=\frac{3x}{5}, \sin\phi=\frac{4x}{5}, \sin\psi=x$$

Now the given equation can be written as

$\theta+\phi=\psi$ whence $\sin(\theta+\phi)=\sin\psi$

or $\sin\theta\cos\phi+\cos\theta\sin\phi=\sin\psi$

or $\frac{3x}{5}\sqrt{\left[\left(1-\frac{16}{25}x^2\right)\right]}+\frac{4x}{5}\sqrt{\left[\left(1-\frac{9x^2}{25}\right)\right]}=x$

or $x\,[3\sqrt{(25-16x^2)}+4\sqrt{(25-9x^2)}]=25x$

One root is $x=0$ and the other roots are given by

$$4\sqrt{(25-9x^2)}=25-3\sqrt{(25-16x^2)}$$

Squaring $16\ (25-9x^2)=625-150\sqrt{(25-16x^2)}+9\,(25-16x^2)$

or $150\sqrt{(25-16x^2)}=450$ whence $25-16x^2=9$

$x=\pm 1$

A check shows that $x=0, 1, -1$ are all roots of the given equation

(ii) The given equation can be written as

$\sin^{-1}(1-x)=\pi/2+2\sin^{-1}x$ Taking sine, we $1-x$
$=\cos(2\sin^{-1}x)=1-2(\sin\sin^{-1}x)^2$ or $1-x=1-2x^2$

Hence $x=2x^2$

$x=0$ or $\frac{1}{2}$ A check shows that $x=\frac{1}{2}$ is not a root

Hence $x=0$ is the only root

17 $2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}$.

$$\text{L H S}=\tan^{-1}\frac{2\tan\alpha/2\tan(\pi/4-\beta/2)}{1-\tan^2\alpha/2\tan^2(\pi/4-\beta/2)}$$

Now change in terms of sine and cos

$$=\tan^{-1}\frac{2\sin\alpha/2\cos\alpha/2\sin(\pi/4-\beta/2)\cos(\pi/4-\beta/2)}{\cos^2\alpha/2\cos^2(\pi/4-\beta/2)-\sin^2\alpha/2\sin^2(\pi/4-\beta/2)}$$

$$=\tan^{-1}\frac{\sin\alpha\sin(\pi/2-\beta)}{2[\cos(\alpha/2-\beta/2+\pi/4)\cos(\alpha/2+\beta/2-\pi/4)]}$$

$$=\tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\cos(\pi/2-\beta)}=\tan^{-1}\frac{\sin\alpha\cos\beta}{\cos\alpha+\sin\beta}$$

18 First note that $\cot A>1$ if
$0<A<\pi/4$ and $\cot A<1$ if
$\pi/4<A<\pi/2$ Hence

$$\tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{(1+x^2)}} = \cos^{-1} \frac{1}{\sqrt{(1+x^2)}} = \cot^{-1} \frac{1}{x}$$
$$= \sec^{-1} \sqrt{(1+x^2)} = \operatorname{cosec}^{-1} \frac{\sqrt{(1+x^2)}}{x} \qquad (6)$$

Note that the equations in (4) and (5) are valid only when $0 \leqslant x \leqslant 1$ and the equations in (6) hold when $x \geqslant 0$

§ 4 $\sin^{-1} x + \cos^{-1} x = \pi/2, (-1 \leqslant x \leqslant 1)$

$\tan^{-1} x + \cot^{-1} x = \pi/2 \quad x \in R$

$\sec^{-1} x + \operatorname{cosec}^{-1} x = \pi/2 \quad x \leqslant -1$ or $x \geqslant 1$

First Method Let $\sin^{-1} x = \theta$ and $\cos^{-1} x = \phi$ so that

$$-\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2} \text{ and } 0 \leqslant \phi \leqslant \pi$$

Hence $-\frac{\pi}{2} \leqslant \theta + \phi \leqslant \frac{3\pi}{2}$

Now $\sin(\theta+\phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$$= x^2 + \sqrt{(1-x^2)}\sqrt{(1-x^2)} = x^2 + 1 - x^2 = 1$$

But there is only one angle between $-\pi/2$ and $3\pi/2$ whose sine is 1, namely $\pi/2$ Hence $\sin^{-1} x + \cos^{-1} x = \pi/2$

Second Method We have to prove $\sin^{-1} x = \pi/2 - \cos^{-1} x$

But $\sin(\pi/2 - \cos^{-1} x) = \cos\cos^{-1} x = x \qquad (1)$

And since $0 \leqslant \cos^{-1} x \leqslant \pi$ we have

$-\pi/2 \leqslant \pi/2 - \cos^{-1} x \leqslant \pi/2$ Hence (1) implies that

$\pi/2 - \cos^{-1} x = \sin^{-1} x$

Prove the other results yourself

§ 5 (a) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}$, if $xy < 1$

and (b) $\tan^{-1} x + \tan^{-1} y = \pi + \tan^{-1} \frac{x+y}{1-xy}$, if $xy > 1$

(c) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}$

Put $x = \tan\theta$, $y = \tan\phi$ $\quad \theta = \tan^{-1} x \quad \phi = \tan^{-1} y$

L H S $= \theta + \phi$

Now $\tan(\theta+\phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = \frac{x+y}{1-xy}$

$$\theta + \phi = \tan^{-1} \frac{x+y}{1-xy} = \text{R H S}$$

Again putting $y = x$ we get $2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$

The second part may be proved similarly

$$=\tan^{-1}\frac{\sqrt{\left(\frac{rz}{xy}\right)}(x+y)}{-(x+y)}=\pi+\tan^{-1}\left\{-\sqrt{\left(\frac{rz}{xy}\right)}\right\}$$

$$=\pi-\tan^{-1}\sqrt{\left(\frac{rz}{xy}\right)}$$

$$\tan^{-1}\left(\frac{rx}{yz}\right)+\tan^{-1}\sqrt{\left(\frac{ry}{zx}\right)}+\tan^{-1}\sqrt{\left(\frac{rz}{xy}\right)}=\pi$$

23 We have to prove that

$$\tan^{-1}\frac{yz}{xr}+\tan^{-1}\frac{zx}{yr}+\tan^{-1}\frac{xy}{zr}=\frac{\pi}{2}$$

or $$\tan^{-1}\frac{\frac{z}{r}\left(\frac{x^2+y^2}{xy}\right)}{1-\frac{z^2}{r^2}}=\pi/2-\tan^{-1}\frac{xy}{zr}$$

or $$\tan^{-1}\frac{\frac{zr}{xy}(x^2+y^2)}{x^2+y^2}=\cot^{-1}\frac{xy}{zr}$$

or $\tan^{-1}\frac{zr}{xy}=\tan^{-1}\frac{zr}{xy}$ which is true

24 From the given equation we have

$$\cos^{-1}\frac{a}{x}+\cos^{-1}\frac{1}{a}=\cos^{-1}\frac{1}{b}+\cos^{-1}\frac{b}{x}$$

or $$\cos^{-1}\left[\frac{a}{x}\ \frac{1}{a}-\left\{\sqrt{\left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)}\right\}\right]$$

$$=\cos^{-1}\left[\frac{1}{b}\ \frac{b}{x}-\sqrt{\left(1-\frac{1}{b^2}\right)\left(1-\frac{b^2}{x^2}\right)}\right]$$

or $$\frac{1}{x}-\frac{\sqrt{(x^2-a^2)}\sqrt{(a^2-1)}}{ax}=\frac{1}{x}-\frac{\sqrt{(b^2-1)}\sqrt{(x^2\ \ b^2)}}{bx}$$

or $b^2(a^2-1)(x^2-a^2)=a^2(b^2-1)(x^2-b^2)$

or $x^2(a^2b^2-b^2-a^2b^2+a^2)=a^2b^2(a^2-1-b^2+1)$

or $x^2(a^2-b^2)=a^2b^2(a^2-b^2)$ $\qquad x=ab$

25 $$\tan^{-1}\frac{(a+b)x}{x^2-ab}=\pi/2-\tan^{-1}\frac{(c+d)x}{x^2-cd}$$

or $$\tan^{-1}\frac{(a+b)x}{x^2-ab}=\cot^{-1}\frac{(c+d)x}{x^2-cd}=\tan^{-1}\frac{x^2-cd}{(c+d)x}$$

$(x^2-ab)(x^2-cd)=(a+b)(c+d)x^2$

or $x^4-x^2\,\Sigma ab+abcd=0$ $\qquad$ (1)

where $\Sigma ab=ab+ac+ad+bc+bd+cd$

x si given by equation (1)

Problem Set

1 Evaluate the following

(a) $\sin\left[\frac{\pi}{3}-\sin^{-1}\left(-\frac{1}{2}\right)\right]$ (b) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(c) $\sin\left[\frac{\pi}{2}-\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)\right]$ (d) $\cos^{-1}\left(\cos\frac{4\pi}{3}\right)$

(e) $\tan^{-1}\left(\tan\frac{3\pi}{4}\right)$ (f) $\tan^{-1}\left(\tan\frac{2\pi}{3}\right)$

(g) $\cos\left[\cos^{-1}\left(\frac{-\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$ (h) $\sin\left(\cos^{-1}\frac{3}{5}\right)$

(i) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ or $\cos(\tan^{-1}x)$

(j) $\sin(\cot^{-1}x)$ (M N R 81)

(k) $\sin(2\sin^{-1}0.8)$ (M N R 83)

(M N R 80)

(l) The principal value of $\sin^{-1}\left[\sin\left(\frac{2\pi}{3}\right)\right]$ is

(A) $-\frac{2\pi}{3}$ (B) $\frac{2\pi}{3}$ (C) $\frac{4\pi}{3}$ (D) $\frac{5\pi}{3}$

(E) None of these

(m) $\cos^{-1}\frac{1}{2}+2\sin^{-1}\frac{1}{2}$ is equal to (I I, T 86)

(A) $\frac{\pi}{4}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{2\pi}{3}$

(M N R 85)

2 Prove (a) $\sin^{-1}\frac{4}{5}+\sin^{-1}\frac{5}{13}+\sin^{-1}\frac{16}{65}=\frac{\pi}{2}$

(b) $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\cos^{-1}\frac{36}{85}$

(c) $\sin^{-1}\frac{3}{5}+\cos^{-1}\frac{12}{13}=\cos^{-1}\frac{33}{65}$

(d) $\sin^{-1}\frac{12}{13}+\cos^{-1}\frac{4}{5}+\tan^{-1}\frac{63}{16}=\pi$

(e) $\cot^{-1}9+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=\frac{\pi}{4}$

(f) $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{13}=\tan^{-1}\frac{2}{9}$

(g) $\tan^{-1}2+\tan^{-1}3=3\pi/4$

(h) $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{8}=\frac{\pi}{4}$

(i) $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{70}+\tan^{-1}\frac{1}{99}=\pi/4$ (Roorkee 81)

(ii) Arguing as in (i), we first see that the required angle $\sin^{-1}\sin 5$ is β (See the above figure). We first find the angle γ such that 5 and γ are symmetrical about the point $3\pi/2$ so that $5-3\pi/2=3\pi/2-\gamma$ or $\gamma=3\pi-5$. Now to find β we see that β and γ are symmetrical about the point $\pi/2$ so that $\gamma-\pi/2=\pi/2-\beta$ whence

$$\beta=\pi-\gamma=\pi-(3\pi-5)=5-2\pi.$$

(iii) Do yourself. Ans. $4\pi-10$

(iv) Do yourself. Ans. $2\pi-6$

(*b*) The value of $\cot^{-1}\left\{\dfrac{\sqrt{(1-\sin x)}+\sqrt{(1+\sin x)}}{\sqrt{(1-\sin x)}-\sqrt{(1+\sin x)}}\right\}$ is

(*i*) $\pi-x$, (*ii*) $2\pi-x$, (*iii*) $\dfrac{x}{2}$ (*iv*) $2\pi-\dfrac{x}{2}$

(M N R 86)

12 $2\tan^{-1}\left(\sqrt{\left(\dfrac{a-b}{a+b}\right)}\tan\dfrac{\theta}{2}\right)-\cos^{-1}\dfrac{a\cos\theta+b}{a+b\cos\theta}$

13 Solve for x the following

(*a*) $\tan^{-1}\dfrac{x-1}{x-2}+\tan^{-1}\dfrac{x+1}{x+2}=\dfrac{\pi}{4}$

(*b*) $\tan^{-1}\dfrac{1}{4}+2\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{6}+\tan^{-1}\dfrac{1}{x}=\dfrac{\pi}{4}$

(*c*) $\sin^{-1}x+\sin^{-1}2x=\pi/3$ **(Roorkee 80)**

(*d*) $\tan^{-1}2x+\tan^{-1}3x=\pi/4$ **(Roorkee 78)**

(*e*) $\tan^{-1}\dfrac{1}{2x+1}+\tan^{-1}\dfrac{1}{4x+1}=\tan^{-1}\dfrac{2}{x^2}$

(*f*) $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$

(*g*) $3\tan^{-1}\dfrac{1}{2+\sqrt{3}}-\tan^{-1}\dfrac{1}{x}=\tan^{-1}\dfrac{1}{3}$

(*h*) $3\sin^{-1}\dfrac{2x}{1-x^2}-4\cos^{-1}\dfrac{1-x}{1+x^2}+2\tan^{-1}\dfrac{2x}{1-x^2}=\dfrac{\pi}{3}$

(*i*) $\sin\left(\frac{1}{5}\cos^{-1}x\right)=1$

14 Find whether $x=2$ satisfies the equation

$$\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}=\tan^{-1}(-7)$$

15 Solve for x, $\sin[2\cos^{-1}\cot(2\tan^{-1}x)]=0$

16 Solve (*i*) $\sin^{-1}\dfrac{3x}{5}+\sin^{-1}\dfrac{4x}{5}=\sin^{-1}x$

(*ii*) $\sin^{-1}(1-x)-2\sin^{-1}x=\pi/2$

17 Prove that $2\tan^{-1}\left\{\tan\dfrac{\alpha}{2}\tan\left(\dfrac{\pi}{4}-\dfrac{\beta}{2}\right)\right\}$

$$=\tan^{-1}\left[\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}\right]$$

18 Prove that $\tan^{-1}(\frac{1}{2}\tan 2A)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=0$ if $\pi/4<A<\pi/2$

and $=\pi$ if $0<A<\frac{1}{4}\pi$

19 Solve the equation

$$\tan^{-1}2x+\tan^{-1}3x=n\pi+\frac{3\pi}{4}$$

(M N R 86)

10 $S=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\frac{1}{2}ab\sin C$

11 (i) $\Sigma (p-q)=(p-q)+(q-r)+(r-p)=0$

(ii) $\Sigma p(q-r)=p(q-r)+q(r-p)+r(p-q)=0$

(iii) $\Sigma (p-a)(q-r)=\Sigma p(q-r)-a\Sigma (q-r)=0$

Problem Set

In any $\triangle ABC$ prove the following

1 $(b+c)\cos A+(c+a)\cos B+(a+b)\cos C=a+b+c$

2 $a(b\cos C-c\cos B)=b^2-c^2$ (Roorkee 73)

3 $a\cos\frac{1}{2}(B-C)=(b+c)\sin\frac{1}{2}A$

4 $\dfrac{b^2-c^2}{a^2}=\dfrac{\sin(B-C)}{\sin(B+C)}$ (Roorkee 1969)

5 $a\sin(A/2+B)=(b+c)\sin A/2$

6 $(b+c-a)(\cot B/2+\cot C/2)=2a\cot A/2$

7 (a) $\{\cot A/2+\cot B/2\}\{a\sin^2 B/2+b\sin^2 A/2\}=c\cot C/2$ (Roorkee 88)

(b) $(a+b+c)(\tan A/2+\tan B/2)=2c\cot C/2$

8 $\dfrac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\dfrac{a^2+b^2}{a^2+c^2}$

9 $a(\cos B\cos C+\cos A)=b(\cos C\cos A+\cos B)$
$=c(\cos A\cos B+\cos C)$

10 $(b^2+c^2-a^2)\tan A=(c^2+a^2-b^2)\tan B=(a^2+b^2-c^2)\tan C$

11 (i) $\dfrac{c}{a-b}=\dfrac{\tan\frac{1}{2}A+\tan\frac{1}{2}B}{\tan\frac{1}{2}A-\tan\frac{1}{2}B}$ (ii) $\dfrac{c}{a+b}=\dfrac{1-\tan\frac{1}{2}A\tan\frac{1}{2}B}{1+\tan\frac{1}{2}A\tan\frac{1}{2}B}$

(iii) $\dfrac{a-b}{a+b}=\cot\dfrac{A+B}{2}\tan\dfrac{A-B}{2}$

Also prove that the area of the triangle is

$$\frac{1}{2}a^2\frac{\sin B\sin C}{\sin A}$$ (Roorkee 75)

(iv) $\cos^2\left(\dfrac{B-C}{2}\right)\Big/(b+c)^2+\sin^2\left(\dfrac{B-C}{2}\right)\Big/(b-c)^2=\dfrac{1}{a^2}$

12 $c^2=(a-b)^2\cos^2\frac{1}{2}C+(a+b)^2\sin^2\frac{1}{2}C$

13 $\dfrac{a^2\sin(B-C)}{\sin B+\sin C}+\dfrac{b^2\sin(C-A)}{\sin C+\sin A}+\dfrac{c^2\sin(A-B)}{\sin A+\sin B}=0$

14 $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$

15 $(b^2-c^2)\cot A+(c^2-a^2)\cot B+(a^2-b^2)\cot C=0$

16 $a\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)+b\sin\frac{1}{2}B\sin\frac{1}{2}(C-A)$
$+c\sin\frac{1}{2}C\sin\frac{1}{2}(A-B)=0$

17 $a^2(\cos^2 B-\cos^2 C)+b^2(\cos^2 C-\cos^2 A)$
$+c^2(\cos^2 A-\cos^2 B)=0$

$$\text{Hence } x=\tan^{-1}\tan\left(\pi-\frac{\pi}{4}\right)=\tan^{-1}\tan\left(-\frac{\pi}{4}\right)=-\frac{\pi}{4}$$

(*f*) Ans $\quad -\frac{\pi}{3}$,

(*g*) $\cos\left[\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]$ Now $\cos^{-1}(-x)=\pi-\cos^{-1}x$,

$$=\cos\left[\pi-\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\frac{\pi}{6}\right]=\cos\left[\pi-\frac{\pi}{6}+\frac{\pi}{6}\right]=\cos\pi=-1$$

Alternative Since $0\leqslant\cos^{-1}x\leqslant\pi$,

we have $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)=\frac{5\pi}{6}$ etc.

(*h*) $\sin\cos^{-1}(3/5)$,

$$=\sin\sin^{-1}\sqrt{(1-\tfrac{9}{25})}=\sin\sin^{-1}\tfrac{4}{5}=\tfrac{4}{5}$$

[Here we have used the formula $\cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}$]

(*i*) $\cos(\tan^{-1}\tfrac{3}{4})$ Now $\tan^{-1}x=\cos^{-1}\frac{1}{\sqrt{(1+x^2)}}$ (I)

$$\text{L.H S}=\cos\left(\cos^{-1}\frac{1}{\sqrt{(1+\frac{9}{16})}}\right)$$

$$=\cos(\cos^{-1}\tfrac{4}{5})=\tfrac{4}{5}, \qquad \text{by (I)}$$

For the second part, we have

$$\cos\tan^{-1}x=\cos\cos^{-1}\frac{1}{\sqrt{(1+x^2)}}=\frac{1}{\sqrt{(1+x^2)}}$$

(*j*) By § 3, $\cot^{-1}x=\sin^{-1}\frac{1}{\sqrt{(1+x^2)}}$

$$\sin(\cot^{-1}x)=\frac{1}{\sqrt{(1+x^2)}}$$

(*k*) $2\sin^{-1}x=\sin^{-1}2x\sqrt{(1-x^2)}$

$$\sin(2\sin^{-1}0\,8)=2\,\frac{8}{10}\sqrt{\left(1-\frac{64}{100}\right)}=0\,96$$

(*l*) $\sin\frac{2\pi}{3}=\sin\left(\pi-\frac{\pi}{3}\right)=\sin\frac{\pi}{3}$ $\qquad \sin^{-1}\sin\left(\frac{2\pi}{3}\right)=\frac{\pi}{3}$

Hence (E) Note that the value of $\sin^{-1}x$ lies between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$

(*m*) *Ans* (*d*) We have $\cos^{-1}\tfrac{1}{2}+2\sin^{-1}\tfrac{1}{2}$

$$=\frac{\pi}{3}+2\times\frac{\pi}{6}=\frac{2\pi}{3}$$

2 (*a*) $\sin^{-1}\tfrac{4}{5}+\sin^{-1}\tfrac{5}{13}=\sin^{-1}1-\sin\tfrac{16}{65}$, $\qquad \sin^{-1}1=\pi/2$

$$\tau=\sqrt{\left\{\frac{1-\cos\alpha}{7-\cos\alpha}\right\}},$$

38 If in a triangle ABC, $\angle C=60°$, then prove that

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$$ (I I T 75)

(ii) If in a $\triangle ABC$, $AC=3$, $BC=4$ and medians AD and BE are perpendicular, prove that $\cos C=5/6$. Hence find the area of the triangle

39 (i) If $\cos A=\sin B/(2\sin C)$, prove that $\triangle ABC$ is isosceles

(ii) If $a\cos A=b\cos B$, prove that $\triangle ABC$ is either isosceles or right angled

(iii) If in a triangle $\frac{a^2-b^2}{a^2+b^2}=\frac{\sin(A-B)}{\sin(A+B)}$ prove that it is either a right angled or an isosceles triangle (Roorkee 87)

(iv) If the angles of a triangle are 30° and 45° and the included side is $(\sqrt{3}+1)$ *cms*, the area of the triangle is $\frac{1}{2}(\sqrt{3}+1)$ (I I T 88)

40 (i) If $b+c=3a$, prove that $\cot\frac{1}{2}B\cot\frac{1}{2}C=2$, (Roorkee 86)

(ii) If $c+a=2b$, prove that $\cot\frac{1}{2}A\cot\frac{1}{2}C=3$,

(iii) If $\angle A=45°$, $\angle B=75°$, prove that $a+c\sqrt{2}=2b$ (I I T 61)

41 If in $\triangle ABC$ $a\tan A+b\tan B=(a+b)\tan\frac{1}{2}(A+B)$, prove that $A=B$

42 If in $\triangle ABC$, $\cos^2 A+\cos^2 B+\cos^2 C=1$, prove that the triangle is right angled

43 If in $\triangle ABC$ $\cot A+\cot B+\cot C=\sqrt{3}$, prove that the triangle is equilateral

44 If in $\triangle ABC$ $c(a+b)\cos\frac{1}{2}B=b(a+c)\cos\frac{1}{2}C$, prove that the triangle is isosceles

45 If in $\triangle ABC$, $\cos A+2\cos B+\cos C=2$, prove that the sides of the triangle are in A P

46 If the sides a, b, c of $\triangle ABC$ are in A P prove that

(i) $\cot\frac{1}{2}A$, $\cot\frac{1}{2}B$, $\cot\frac{1}{2}C$ are in A P

(ii) $\cos A\cot\frac{1}{2}A$, $\cos B\cot\frac{1}{2}B$, $\cos C\cot\frac{1}{2}C$ are in A P

(iii) $\cos\frac{A-C}{2}=2\sin\frac{B}{2}$, (iv) $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{3b}{2}$

47 If a^2, b^2, c^2 are in A P prove that $\cot A$, $\cot B$ and $\cot C$ are also in A P

$=\pi+\tan^{-1}\{5/(-5)\}=\pi+\tan^{-1}(-1)=\pi-\tan^{-1}1$
$=\pi-\pi/4=3\pi/4$

(*h*) L.H.S $=\tan^{-1}\dfrac{\frac{1}{2}+\frac{1}{5}}{1-\frac{1}{2}\cdot\frac{1}{5}}+\tan^{-1}\dfrac{1}{8}$

$=\tan^{-1}\dfrac{7}{9}+\tan^{-1}\dfrac{1}{8}=\tan^{-1}\dfrac{7/9+1/8}{1-(7/9)(1/8)}=\tan^{-1}\dfrac{65}{65}=\tan^{-1}1=\dfrac{\pi}{4}$

(*i*) $2\tan^{-1}\dfrac{1}{5}=\tan^{-1}\dfrac{2/5}{1-(1/25)}=\tan^{-1}\dfrac{5}{12}$

$4\tan^{-1}\dfrac{1}{5}=2\tan^{-1}\dfrac{5}{12}=\tan^{-1}\dfrac{2(5/12)}{1-(25/144)}=\tan^{-1}\dfrac{120}{119}$

L H S $=\tan^{-1}\dfrac{120}{119}-\left(\tan^{-1}\dfrac{1}{70}-\tan^{-1}\dfrac{1}{99}\right)$

$=\tan^{-1}\dfrac{120}{119}-\tan^{-1}\dfrac{1/70-1/99}{1+(1/99)(1/70)}=\tan^{-1}\dfrac{120}{119}-\tan^{-1}\dfrac{29}{6931}$

$=\tan^{-1}\dfrac{\frac{120}{119}-\frac{1}{239}}{1+\frac{120}{119}\cdot\frac{1}{239}}=\tan^{-1}\dfrac{120\cdot239-119}{119\cdot239+120}$

$=\tan^{-1}\dfrac{119\cdot239+(239-119)}{119\cdot239+120}$, writing $120=119+1$

$=\tan^{-1}1=\pi/4$

(*j*) $\cos^{-1}4/5=\tan^{-1}3/4$ and $\tan(\tan^{-1}3/4+\tan^{-1}2/3)$

$\tan\tan^{-1}\dfrac{\frac{3}{4}+\frac{2}{3}}{1-\frac{3}{4}\cdot\frac{2}{3}}=\dfrac{17}{6}$ Hence (*c*) is the answer

(*k*) Let $\cos^{-1}\dfrac{\sqrt{5}}{3}=\alpha$ $\quad\cos\alpha=\dfrac{\sqrt{5}}{3},\ 0<\alpha<\dfrac{\pi}{2}$

We have to find the value of

$$\tan\alpha/2=\sqrt{\left\{\frac{1-\cos\alpha}{1+\cos\alpha}\right\}}$$

or $\tan\alpha/2=\sqrt{\left(\dfrac{3-\sqrt{5}}{3+\sqrt{5}}\right)}=\sqrt{\dfrac{(3-\sqrt{5})^2}{9-5}}$

or $\tan\alpha/2=\frac{1}{2}(3-\sqrt{5})$.

3 (*a*) L H S $=2\left(\tan^{-1}\dfrac{1}{5}+\tan^{-1}\dfrac{1}{8}\right)+\tan^{-1}\sqrt{\left[\left(\dfrac{5\sqrt{2}}{7}\right)^2-1\right]}$

$=2\tan^{-1}\dfrac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\cdot\frac{1}{8}}+\tan^{-1}\dfrac{1}{7}$, $\qquad\sec^{-1}x=\tan^{-1}\sqrt{(x^2-1)}$

$=2\tan^{-1}\dfrac{13}{39}+\tan^{-1}\dfrac{1}{7}=2\tan^{-1}\dfrac{1}{3}+\tan^{-1}\dfrac{1}{7}$

$=\tan^{-1}\dfrac{2\cdot\frac{1}{3}}{1-\frac{1}{9}}+\tan^{-1}\dfrac{1}{7}=\tan^{-1}\dfrac{3}{4}+\tan^{-1}\dfrac{1}{7}$

$$\alpha^{-2}+\beta^{-2}+\gamma^{-2}=(\cot A+\cot B+\cot C)/\Delta$$ (IIT 77)

where Δ is the area of the triangle

59 Let O be a point inside a triangle ABC such that $\angle OAB=\angle OBC=\angle OCA=\omega$, then show that

(i) $\cot\omega=\cot A+\cot B+\cot C$

(ii) $\operatorname{cosec}^2\omega=\operatorname{cosec}^2 A+\operatorname{cosec}^2 B+\operatorname{cosec}^2 C$

60 $ABCD$ is a trapezium such that $AB \parallel CD$ and CB is $\perp$ to them. If $\angle ADB=\theta$, $BC=p$ and $CD=q$, show that

$$AB=\frac{(p^2+q^2)\sin\theta}{p\cos\theta+q\sin\theta}$$ (IIT 68)

61 Let A, B be two points on one bank of a straight river, and C, D two points on the other bank, the directions from A to B along the river being the same as from C to D. If $AB=a$, $\angle CAD=\alpha$, $\angle DAB=\beta$, $\angle CBA=\gamma$, prove that

$$CD=\frac{a\sin\alpha\sin\gamma}{\sin\beta\sin(\alpha+\beta+\gamma)}$$ (IIT 68)

62 With usual notations, if in a triangle, ABC

$\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$ then prove that

$\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$ (IIT 84)

63 If in a triangle ABC

$\cos A\cos B+\sin A\sin B\sin C=1$ show that (IIT 86)

$a:b:c=1:1:\sqrt{2}$

64 For a triangle ABC, it is given that

$\cos A+\cos B+\cos C=3/2$ prove that (IIT 84)

the triangle is equilateral

Solutions

1 Re writing the left-hand side we get

L H S $=(b\cos C+c\cos B)+(c\cos A+a\cos C)$

$+(a\cos B+b\cos A)=a+b+c$

2 L H S $=a(b\cos C-c\cos B)$

$$=a\left\{b\,\frac{a^2+b^2-c^2}{2ab}-c\,\frac{c^2+a^2-b^2}{2ca}\right\}$$

$$=\tfrac{1}{2}(a^2+b^2-c^2-c^2-a^2+b^2)=b^2-c^2$$

Alt On putting $a=b\cos C+c\cos B$,

L.H S $=b^2\cos^2 C-c^2\cos^2 B=b^2(1-\sin^2 C)-c^2(1-\sin^2 B)$

$=b^2-c^2$ $\quad$ $b\sin C=c\sin B$ by sine formula (*b*) (iv) is correct

7 $\sin\left[\tan^{-1}\frac{1-x^2}{2x}+\cos^{-1}\frac{1-x^2}{1+x^2}\right]=1$

Put $x=\tan\theta$

$$\text{L H S}=\sin\left[\tan^{-1}\frac{1-\tan^2\theta}{2\tan\theta}+\cos^{-1}\frac{1-\tan^2\theta}{1+\tan^2\theta}\right]$$

$$=\sin[\tan^{-1}\cot 2\theta+\cos^{-1}\cos 2\theta]$$

$$=\sin[\tan^{-1}\tan(\pi/2-2\theta)+2\theta]$$

$$=\sin[\pi/2-2\theta+2\theta]=\sin\pi/2=1$$

8 $\text{L H S}=\tan[\tfrac{1}{2}\,2\tan^{-1}a+\tfrac{1}{2}\,2\tan^{-1}a]$

$$=\tan\left[2\tan^{-1}a\right]=\tan\tan^{-1}\frac{2a}{1-a^2}=\frac{2a}{1-a^2}$$

9 (*a*) We have

$$\cos^{-1}\left[\frac{p}{a}\ \frac{q}{b}-\sqrt{\left(1-\frac{p^2}{a^2}\right)}\sqrt{\left(1-\frac{q^2}{b^2}\right)}\right]=\alpha$$

or $\frac{pq}{ab}-\sqrt{\left(1-\frac{p^2}{a^2}\right)\left(1-\frac{q^2}{b^2}\right)}=\cos\alpha$

$$\left(\frac{pq}{ab}-\cos\alpha\right)^2=1-\frac{p^2}{a^2}-\frac{q^2}{b^2}+\frac{p^2q^2}{a^2b^2}$$

or $\frac{p^2q^2}{a^2b^2}+\cos^2\alpha-\frac{2pq}{ab}\cos\alpha=1-\frac{p^2}{a^2}-\frac{q^2}{b^2}+\frac{p^2q^2}{a^2b^2}$

or $\frac{p^2}{q^2}-\frac{2pq}{ab}\cos\alpha+\frac{q^2}{b^2}=1-\cos^2\alpha=\sin^2\alpha$ Proved

(*b*) Proceed as above

10 Let $\cos^{-1}a/b=\theta$ so that $\cos\theta=a/b$

$$\text{L H S}=\tan(\pi/4+\theta/2)+\tan(\pi/4-\theta/2)$$

$$=\frac{1+\tan\theta/2}{1-\tan\theta/2}+\frac{1-\tan\theta/2}{1+\tan\theta/2}$$

$$=\frac{(1+\tan\theta/2)^2+(1-\tan\theta/2)^2}{1-\tan^2\theta/2}=2\,\frac{(1+\tan^2\theta/2)}{1-\tan^2\theta/2}$$

$$=2\,\frac{(\cos^2\theta/2+\sin^2\theta/2)}{\cos^2\theta/2-\sin^2\theta/2}=2\,\frac{1}{\cos\theta}=2\cdot\frac{1}{a/b}=2b/a$$

11 (*a*) From the given question

$$\frac{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}=\tan\alpha=\frac{\sin\alpha}{\cos\alpha}$$

Apply Componendo and Dividendo

$\frac{2\sqrt{(1+x^2)}}{2\sqrt{(1-x^2)}}=\frac{\cos\alpha+\sin\alpha}{\cos\alpha-\sin\alpha}$ Square

9 $a(\cos B\cos C+\cos A)=a\{\cos B\cos C-\cos(B+C)\}$
$=a\{\cos B\cos C-\cos B\cos C+\sin B\sin C\}$
$=a\sin B\sin C=k\sin A\sin B\sin C$
Similarly $b(\cos C\cos A+\cos B)=c(\cos A\cos B+\cos C)$
$=k\sin A\sin B\sin C$

10 Do yourself

11 (i) $\dfrac{c}{a-b}=\dfrac{k\sin C}{k\sin A-k\sin B}=\dfrac{2\sin\frac{C}{2}\cos\frac{C}{2}}{2\cos\frac{A+B}{2}\sin\frac{A-B}{2}}$

$=\dfrac{\sin\frac{C}{2}\sin\frac{A+B}{2}}{\sin\frac{C}{2}\sin\frac{A-B}{2}}\left[\cos\frac{A+B}{2}=\sin\frac{C}{2}\text{ and }\cos\frac{C}{2}=\sin\frac{A+B}{2}\right]$

$=\dfrac{\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2}}{\sin\frac{A}{2}\cos\frac{B}{2}-\cos\frac{A}{2}\sin\frac{B}{2}}=\dfrac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}$

[Dividing both N^r and D^r by $\cos A/2\cos B/2$]

(ii) Do yourself (iii) Do yourself

(iv) L H S = $\cos\left(\frac{B-C}{2}\right)\Big/ k(\sin B+\sin C)$

$+\sin\left(\frac{B-C}{2}\right)\Big/ k^2(\sin B+\sin C)^2$

$=1/4k\ \sin^2\left(\frac{B+C}{2}\right)+1/4k^2\cos^2\left(\frac{B+C}{2}\right)$

$=\left\{\cos\left(\frac{B+C}{2}\right)+\sin^2\left(\frac{B+C}{2}\right)\right\}\Big/k^2\left(2\sin\frac{B+C}{2}\cos\frac{B+C}{2}\right)^2$

$=1/k^2\sin^2(B+C)=1/k^2\sin^2 A=1/a^2$

12 R H S $=\frac{1}{2}[(a-b)^2(1+\cos C)+(a+b)^2(1-\cos C)]$
$=\frac{1}{2}[(a-b)^2+(a+b)^2-\{(a+b)^2-(a-b)^2\}\cos C]$

$=\frac{1}{2}[2(a^2+b^2)-4ab\cos C]=(a^2+b^2)-2ab\dfrac{a^2+b^2-c^2}{2ab}$

$=a^2+b^2-a^2-b^2+c^2=c^2$

13 $\dfrac{a^2\sin(B-C)}{\sin B-\sin C}=\dfrac{k^2\sin A\sin(B-C)}{\sin B-\sin C}$

$$\left(\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{1}{6}\right)+2\tan^{-1}\frac{1}{5}=\tan^{-1}1-\tan^{-1}\frac{1}{x}$$

$$\tan^{-1}\frac{\frac{1}{4}+\frac{1}{6}}{1-\frac{1}{4}\cdot\frac{1}{6}}+\tan^{-1}\frac{2\cdot\frac{1}{5}}{1-\frac{1}{25}}=\tan^{-1}\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$$

or $\tan^{-1}\dfrac{10}{23}+\tan^{-1}\dfrac{5}{12}=\tan^{-1}\dfrac{x-1}{x+1}$

or $\tan^{-1}\dfrac{\frac{10}{23}+\frac{5}{12}}{1-\frac{10}{23}\cdot\frac{5}{12}}=\tan^{-1}\dfrac{x-1}{x+1}$ or $\dfrac{235}{226}=\dfrac{x-1}{x+1}$

Hence $(235-226)\,x=-235-226$

or $9x=-461$ $\quad x=-\dfrac{461}{9}$

(c) $\sin^{-1}x+\sin^{-1}2x=\pi/3=\sin^{-1}\dfrac{\sqrt{3}}{2}$

$$\sin^{-1}x-\sin^{-1}\frac{\sqrt{3}}{2}=-\sin^{-1}2x$$

$$=\sin^{-1}\left[x\sqrt{\left(1-\frac{3}{4}\right)}-\frac{\sqrt{3}}{2}\sqrt{(1-x^2)}\right]=\sin^{-1}(-2x)$$

$$\frac{x}{2}-\frac{\sqrt{3}}{2}\sqrt{(1-x)}=-2x$$

$5x=\sqrt{3}\sqrt{(1-x^2)}$ Square

$25x^2=3-3x^2$ or $28x^2=3,$ $\quad x=\pm\dfrac{\sqrt{3}}{2\sqrt{7}}$

Clearly from the given equation the value of x must be +ive

$$x=\tfrac{1}{2}\sqrt{\left(\frac{3}{7}\right)}$$

(*d*) $\tan^{-1}\dfrac{2x+3x}{1-2x\cdot 3x}=\dfrac{\pi}{4}=\tan^{-1}1$

$\dfrac{5x}{1-6x^2}=1$ or $5x=1-6x^2$

$6x^2-5x-1=0$ or $(6x-1)(x+1)+0$

$x=1/6,\ -1$

We will eject the value $x=-1$ (why ?) and hence $x=1/6$

(*e*) $\tan^{-1}\dfrac{\frac{1}{2x+1}+\frac{1}{4x+1}}{1-\frac{1}{2x+1}\cdot\frac{1}{4x+1}}=\tan^{-1}\dfrac{2}{x^2}$

20 $\frac{a \sin (B-C)}{b^2-c^2}=\frac{k \sin A \sin (B-C)}{k\ (\sin^2 B-\sin^2 C)}=\frac{1}{k}\ \frac{\sin (B+C) \sin (B-C)}{\sin^2 B-\sin^2 C}$

$=1/k \qquad [\sin (B+C) \sin (B-C)=\sin^2 B-\sin^2 C]$

Similarly $\frac{b \sin (C-A)}{c^2-a^2}=\frac{c \sin (A-B)}{a^2-b^2}=\frac{1}{k}$

21 Do yourself

22 $a^3 \sin (B-C)=k^3 \sin^3 A \sin (B-C)$

$=k^3 \sin^2 A \sin (B-C) \sin (B-C) \quad [\quad \sin A=\sin (B+C)]$

$=k^3 \sin^2 A (\sin^2 B-\sin^2 C)$

Hence L H S $=k^3 [\sin^2 A (\sin^2 B-\sin^2 C)+\sin^2 B (\sin^2 C-\sin^2 A)+\sin^2 C (\sin^2 A-\sin^2 B)]=0$

23 Do yourself

24 $1-\tan \frac{1}{2}A \tan \frac{1}{2}B=1-\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}} \sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}}$

$=1-\frac{s-c}{s}=\frac{c}{s}=\frac{2c}{2s}=\frac{2c}{a+b+c}$

25 $2\,abc \cos A/2 \cos \frac{1}{2}B \cos \frac{1}{2}C$

$=2\,abc \sqrt{\left\{\frac{s(s-a)}{bc}\right\}} \sqrt{\left\{\frac{s(s-b)}{ca}\right\}} \sqrt{\left\{\frac{s(s-c)}{ab}\right\}}$

$=2\,s \sqrt{\{s(s-a)(s-b)(s-c)\}}=(a+b+c)\,S$

26 $bc \cos^2 \frac{1}{2} A=bc\ \frac{s(s-a)}{bc}=s\ (s-a),$

L H S $=s(s-a)+s(s-b)+s(s-c)$

$=3s^2-s(a+b+c)=3s^2-s\ 2s$

$=3s^2-2s^2=s^2$

27 $\frac{b-c}{a} \cos^2 \frac{1}{2} A=\frac{b-c}{a} \frac{s(s-a)}{bc}=\frac{s(s-a)(b-c)}{abc}$

L H S $=\frac{s}{abc} [(s-a)(b-c)+(s-b)(c-a)+(s-c)(a-b)]$

$=\frac{s}{abc} [s\{b-c+c-a+a-b\}-\{a(b-c)+b(c-a)+c(a-b)\}]$

$=\frac{s}{abc}\times 0=0$

28 Putting $\tan \frac{1}{2} A=\sqrt{\left(\frac{(s-b)(s-c)}{s(s-a)}\right)}$ etc we have

L H S $=\left[\frac{\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}}{(a-b)(a-c)}+\frac{\sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}}}{(b-c)(b-a)}+\frac{\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}}{(c-a)(c-b)}\right]$

(*h*) We have

$3(2\tan^{-1}x)-4(2\tan^{-1}x)+2(2\tan^{-1}x)=\pi/3$

or $2\tan^{-1}x=\pi/3$ $\quad\tan^{-1}x=\pi/6$ or $x=\tan\pi/6=1/\sqrt{3}$

(*i*) Let $\frac{1}{5}\cos^{-1}x=\alpha$ $\quad$ Since $0\leqslant\cos^{-1}x\leqslant\pi$, we have $0\leqslant\alpha\leqslant\pi/5$ Hence $\sin\alpha\neq1$, that is, $\sin(\frac{1}{5}\cos^{-1}x)=1$ has no solution

14 $\quad \tan^{-1}\dfrac{\dfrac{x+1}{x-1}+\dfrac{x-1}{x}}{1-\dfrac{x+1}{x-1}\,\dfrac{x-1}{x}}=\tan^{-1}(-7)$

$$\frac{2x^2-x+1}{1-x}=-7$$

or $2x^2-x+1=-7+7x$ or $2x^2-8x+8=0$

or $x^2-4x+4=0$ or $(x-2)^2=0$, $\quad x=2$

But if we put $x=2$ in the given equation the L H S is +ive and R H S is −ive Hence $x=2$ does not satisfy We will have to write the equation as $\tan^{-1}\dfrac{x+1}{x-1}+\tan^{-1}\dfrac{x-1}{x}=\pi+\tan^{-1}(-7)$

Now $x=2$ will make both sides +ive

15 $\quad 2\tan^{-1}x=\tan^{-1}\dfrac{2x}{1-x^2}=\cot^{-1}\dfrac{1-x^2}{2x}$

$$\text{L H S}=\sin\left[2\cos^{-1}\frac{1-x^2}{2x}\right]=0$$

Now $2\cos^{-1}z=\cos^{-1}(2z^2-1)$

$$2\cos^{-1}\frac{1-x^2}{2x}=\cos^{-1}\left[2\frac{(1-x^2)^2}{4x^2}\quad 1\right]$$

$$=\cos^{-1}\left[\frac{x^4-4x^2+1}{2x^2}\right]$$

$$\text{L H S}=\sin\left[\cos^{-1}\frac{x^4-4x^2+1}{2x^2}\right]=0$$

Again $\sin\cos^{-1}t=\sin\sin^{-1}\sqrt{(1-t^2)}=0$ $\quad 1-t^2=0$

Hence from (1) we have

$$1-\left[\frac{x^4-4x^2+1}{2x^2}\right]^2=0$$

$(x^4-4x^2+1)^2-(2x^2)^2=0$

or $(x^4-4x^2+1-2x^2)(x^4-4x^2+1+2x^2)=0$

$(x^4-2x^2+1)(x^4-6x^2+1)=0$

From 1st factor $(x^2-1)^2=0$, $\quad x=\pm1$

From 2nd factor $x^4-6x^2+9=-1+9$ or $(x^2-3)^2=8$

$x^2=3\pm2\sqrt{2}=(1\pm\sqrt{2})^2$ $\quad x=\pm(1\pm\sqrt{2})$

$= k^3 [\sin^2 A \sin B \cos B + \sin^2 A \sin C \cos C$
$+ \sin^2 B \sin C \cos C + \sin^2 B \sin A \cos A$
$+ \sin^2 C \sin A \cos A + \sin^2 C \sin B \cos B]$

$= k^3 [\sin A \sin B (\sin A \cos B + \cos A \sin B)$
$+ \sin B \sin C (\sin B \cos C + \cos B \sin C)$
$+ \sin C \sin A (\sin C \cos A + \cos C \sin A)]$

$= k^3 [\sin A \sin B \sin (A+B) + \sin B \sin C \sin (B+C)$
$+ \sin C \sin A \sin (C+A)]$

$= k^3 [\sin A \sin B \sin C + \sin B \sin C \sin A + \sin C \sin A \sin B]$

$= 3k \sin A \, k \sin B \, k \sin C = 3\,abc$

31 $a^2 - 2ab \cos (60° + C) = a^2 - 2ab (\cos 60° \cos C - \sin 60° \sin C)$

$= a^2 - 2ab \left(\frac{1}{2} \cos C - \frac{\sqrt{3}}{2} \sin C \right)$

$a^2 - ab \, \frac{a^2 + b^2 - c^2}{2ab} + \sqrt{3} ab \sin C$

$= \frac{a^2 - b^2 + c^2}{2} + \sqrt{3} k \, abc$

Similarly we can prove that

$c^2 - 2bc \cos (60° + A) = \frac{c^2 - b^2 + a^2}{2} + \sqrt{3} k \, abc$

32 $\sin^3 A \cos (B-C) + \sin^3 B \cos (C-A) + \sin^3 C \cos (A-B)$

$= \frac{1}{4} [2 \sin^2 A \, 2 \sin (B+C) \cos (B-C)$
$+ 2 \sin^2 B \, 2 \sin (C+A) \cos (C-A)$
$+ 2 \sin^2 C \, 2 \sin (A+B) \cos (A-B)]$

[$\because \sin A = \sin (B-C)$ etc]

$= \frac{1}{4} [(1 - \cos 2A)(\sin 2B - \sin 2C) + (1 - \cos 2B)$
$(\sin 2C + \sin 2A) + (1 - \cos 2C)(\sin 2A + \sin 2B)]$

$= \frac{1}{4} [2 (\sin 2A + \sin 2B + \sin 2C) - \sin (2A + 2B) - \sin (2B + 2C)$
$- \sin (2C + 2A)]$

[$\because \sin 2A \cos 2B + \cos 2A \sin 2B = \sin (2A + 2B)$ etc]

$= \frac{1}{4} [2 (\sin 2A + \sin 2B + \sin 2C) + \sin 2C + \sin 2A + \sin 2B]$

[$\because \sin (2A + 2B) = \sin (2\pi - 2C) = -\sin 2C$ etc]

$= \frac{3}{4} \, 4 \sin A \sin B \sin C$ [See identities Page 58]

$= 3 \sin A \sin B \sin C$

33 Let A, $2A$, $3A$ be the angles

$A + 2A + 3A = 180°$ or $A = 30$

The angles are 30°, 60° and 90

If a, b, c denote the sides opposite to these angles then

$$\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)=\pi+\tan^{-1}\frac{\cot A+\cot^3 A}{1-\cot^4 A}$$

if $0 < A < \frac{\pi}{4}$

and $=\tan^{-1}\frac{\cot A+\cot^3 A}{1-\cot^4 A}$,

if $\frac{\pi}{4} < A < \frac{\pi}{2}$

Also $\frac{\cot A+\cot^3 A}{1-\cot^4 A}=\frac{\cot A \operatorname{cosec}^2 A \sin^4 A}{\sin^4 A-\cos^4 A}$

$$=\frac{\cos A \sin A}{(\sin^2 A+\cos^2 A)(\sin^2 A-\cos^2 A)}$$

$$=-\frac{\sin 2A}{2\cos 2A}=-\tfrac{1}{2}\tan 2A$$

Hence $\tan^{-1}(\tfrac{1}{2}\tan 2A)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\tan^{-1}(\tfrac{1}{2}\tan 2A)+\pi+\tan^{-1}(-\tfrac{1}{2}\tan 2A)$ if $0 < A < \pi/4$

$=\tan^{-1}(\tfrac{1}{2}\tan 2A)+\pi-\tan^{-1}(\tfrac{1}{2}\tan 2A)=\pi$ if $0 < A < \pi/4$

and $\tan^{-1}(\tfrac{1}{2}\tan 2A)+\tan^{-1}(\cot A)+\tan^{-1}(\cot^3 A)$

$=\tan^{-1}(\tfrac{1}{2}\tan 2A)+\tan^{-1}(-\tfrac{1}{2}\tan 2A)$ if $\pi/4 < A < \pi/2$

$=\tan^{-1}(\tfrac{1}{2}\tan 2A)-\tan^{-1}(\tfrac{1}{2}\tan 2A)=0$ if $\pi/4 < A < \pi/2$

19 L.H.S $=\tan^{-1}\frac{2x+3x}{1-2x\cdot 3x}=\frac{3\pi}{4}$

or $\frac{5x}{1-6x^2}=\tan\frac{3\pi}{4}=-1$ or $6x^2-5x-1=0$

or $(x-1)(6x+1)=0$ $\qquad x=1, -1/6$

20 $T_1=\tan^{-1}\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{x}{y}\cdot\frac{1}{c_1}}=\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}$ etc

$$\text{L.H.S}=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)$$

$$+\cdots+\tan^{-1}\frac{1}{c_n}=\tan^{-1}\frac{x}{y}$$

21 $\cot^{-1}x=\tan^{-1}\frac{1}{x}$ $\qquad T_1=\tan^{-1}p-\tan^{-1}q$

22 Put $x+y+z=r$

$$\tan^{-1}\sqrt{\left(\frac{rx}{yz}\right)}+\tan^{-1}\sqrt{\left(\frac{ry}{zx}\right)}=\tan^{-1}\frac{\sqrt{\left(\frac{r}{xyz}\right)}(x+y)}{\left(1-\frac{r}{z}\right)} \ldots$$

$$2\sin\frac{B}{2}=\cos\left(\frac{90°}{2}\right)=\cos 45°=\frac{1}{\sqrt{2}}\quad\text{(from 2)}$$

Hence $\sin\frac{B}{2}=\frac{1}{2\sqrt{2}}$ and $\cos\frac{B}{2}=\sqrt{\left(1-\frac{1}{8}\right)}=\frac{\sqrt{7}}{2\sqrt{2}}$

$$\sin B=2\sin\frac{B}{2}\cos\frac{B}{2}=2\cdot\frac{1}{2\sqrt{2}}\cdot\frac{\sqrt{7}}{2\sqrt{2}}=\frac{\sqrt{7}}{4}$$

$$\sin A+\sin C=2\sin B=\frac{\sqrt{7}}{2}\quad(3)$$

Also $\sin A-\sin C=2\cos\frac{A+C}{2}\sin\frac{A-C}{2}=2\sin\frac{B}{2}\sin 45°$

$$=2\cdot\frac{1}{2\sqrt{2}}\cdot\frac{1}{\sqrt{2}}=\frac{1}{2}\quad(4)$$

Adding and subtracting (3) and (4), we get

$$2\sin A=\frac{\sqrt{7}+1}{2},\ i.e.\ \sin A=\frac{\sqrt{7}+1}{4},$$

and $2\sin C=\frac{\sqrt{7}-1}{2}$, i.e. $\sin C=\frac{\sqrt{7}-1}{4}$

Hence $a:b:c=\sin A:\sin B:\sin C=\sqrt{7}+1:\sqrt{7}:\sqrt{7}-1$

37 Let A be the least and C the greatest angle, then since

$a:b:c$ $s=a+b+c$ $\frac{a+b+c}{2}=1-x+1+1+x$

$$\cos\frac{\alpha}{2}=\cos\frac{C-A}{2}=\cos\frac{C}{2}\cos\frac{A}{2}+\sin\frac{C}{2}\sin\frac{A}{2}$$

$$=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}+\sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}}\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$$

$$=\sqrt{\left\{\frac{\frac{3}{2}\left(\frac{3}{2}-1-x\right)}{(1-x)1}\right\}}\sqrt{\left\{\frac{\frac{3}{2}\left(\frac{3}{2}-1+x\right)}{1(1+x)}\right\}}+\sqrt{\left\{\frac{\left(\frac{3}{2}-1+x\right)\left(\frac{3}{2}-1\right)}{(1+x)1}\right\}}\sqrt{\left\{\frac{\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-1-x\right)}{1(1+x)}\right\}}$$

$$=\frac{3}{4}\sqrt{\left(\frac{1-4x}{1-x}\right)}+\frac{1}{4}\sqrt{\left(\frac{1-4x^2}{1-x^2}\right)}=\sqrt{\left(\frac{1-4x^2}{1-x}\right)}$$

$$\cos\alpha=2\cos^2\frac{\alpha}{2}-1=\frac{2(1-4x^2)}{1-x^2}-1=\frac{1-7x}{1-x^2}$$

$$x^2=\frac{1-\cos\alpha}{7-\cos\alpha}\quad\text{or}\quad x=\sqrt{\left(\frac{1-\cos\alpha}{7-\cos\alpha}\right)}$$

38 (a) Since $C=60°$ we have

$c^2=a^2+b^2-2ab\cos C=a^2+b^2-2ab\cos 60°$

26 $\tan^{-1}\dfrac{1}{a-1}=\tan^{-1}\dfrac{\dfrac{1}{x}+\dfrac{1}{a^2-x+1}}{1-\dfrac{1}{x(a^2-x+1)}}$

$$\frac{1}{a-1}=\frac{a^2+1}{(a^2+1)x-x^2-1}$$

$$x^2-x(a^2+1)+1+(a-1)(a^2+1)=0$$

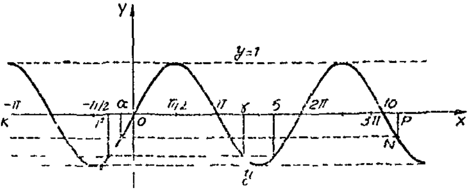
or $(x^2-a^2)-(a^2+1)(x-a)=0$

or $(x-a)(x+a-a^2-1)=0$ $\quad x=a$ or a^2-a+1

27 $\sin^{-1}\dfrac{1}{\sqrt{5}}=\tan^{-1}\dfrac{1}{2}$ and $\tan\dfrac{\pi}{4}=1$

Ans $x=3$

28 (i) By definition $\sin^{-1}\sin 10=\alpha$ say, is an angle that satisfies two conditions $\sin\alpha=\sin 10$ and $-\pi/2\leqslant\alpha\leqslant\pi/2$ In such cases, the required angle α can be found easily by means of the graph of the function $y=\sin x$ (See the figure below) Plot the number 10 on x-axis at the



point P and find $\sin 10$ geometrically (It is the ordinate $-PN$) and then draw the horizontal line (i e the line $y=\sin 10$) The abscissa of one of the points of intersection of this straight line with the graph lies at Q in the interval $[-\pi/2, \pi/2]$ and its sine is equal to $\sin 10$ The abscissa of Q say α is the desired angle It can be found easily by geometrical reasoning It is easy to see that the points α and 10 are symmetric about the point $3\pi/2$ so that $10-3\pi/2=3\pi/2-\alpha$ whence $\alpha=3\pi-10$

(ii) Since $a \cos A = b \cos B$, we have

$k \sin A \cos A = k \sin B \cos B$ or $\sin 2A = \sin 2B$

$2A = 2B$ or $\pi - 2B$

$A = B$, i.e. the $\triangle ABC$ is isosceles

or $A + B = \pi/2$ so that $C = \pi/2$, i.e. $\triangle ABC$ is right angled at C

(iii) $\dfrac{\sin (A-B)}{\sin (A+B)} = \dfrac{k^2 (\sin^2 A - \sin^2 B)}{k^2 (\sin^2 A + \sin^2 B)}$ by sine formula

or $\dfrac{\sin (A-B)}{\sin C} = \dfrac{\sin (A-B) \sin (A+B)}{\sin^2 A + \sin^2 B}$

or $\sin (A-B) \left[\dfrac{1}{\sin C} - \dfrac{\sin C}{\sin^2 A + \sin^2 B}\right] = 0$

Either $\sin (A-B) = 0$ $\quad A = B$ i.e. isosceles

or $\sin^2 A + \sin^2 B = \sin^2 C$ $\quad a^2 + b^2 = c^2$ $\quad$ rt angled

(iv) Let $\angle A = 30°$, $\angle B = 45°$ so that $AB = c = (\sqrt{3}+1)$ $\quad$ (1)

Also $\angle C = 180° - (30° + 45°) = 105°$ $\quad \sin C = \sin (90° + 15°)$
$= \cos 15°$

or $\sin C = \cos (45° - 30) = \dfrac{1}{2\sqrt{2}} (\sqrt{3}+1)$ $\quad$ (2)

Now by sine formula $\dfrac{b}{\sin B} = \dfrac{c}{\sin C}$ $\quad b = \sin B \, 2\sqrt{2} = 2$

$\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} 2 (\sqrt{3}+1) \frac{1}{2} = \frac{1}{2} (\sqrt{3}+1)$

40 (i) Since $b + c = 3a$, we have $\sin B + \sin C = 3 \sin A$

or $2 \sin \dfrac{B+C}{2} \cos \dfrac{B-C}{2} = 6 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$

or $\cos \dfrac{B-C}{2} = 3 \sin \dfrac{A}{2} \left[\quad \sin \dfrac{B+C}{2} = \cos \dfrac{A}{2} \right]$

or $\cos \dfrac{B-C}{2} = 3 \cos \dfrac{B+C}{2}$

or $\cos B/2 \cos C/2 + \sin B/2 \sin C/2$

$= 3 \cos B/2 \cos C/2 - 3 \sin B/2 \sin C/2$

Dividing by $\sin B/2 \sin C/2$, we get

$\cot B/2 \cot C/2 + 1 = 3 \cot B/2 \cot C/2 - 3$

or $\cot B/2 \cot C/2 = 2$

(ii) Do yourself

(iii) As $\angle A = 45°$, $\angle B = 75$, we have

$\angle C = 180° - (45° + 75°) = 60°$

Hence $a + c\sqrt{2} = k (\sin A + \sqrt{2} \sin C)$

$= k (\sin 45° + \sqrt{2} \sin 60°)$

$= k \left[\dfrac{1}{\sqrt{2}} + \sqrt{2} \dfrac{\sqrt{3}}{2}\right] = \dfrac{\sqrt{3}+1}{\sqrt{2}} k$

5

Properties of Triangle

§ 1 Relations Between Sides and Trigonometrical Ratios of the Angles of any Triangle

Basic Formulae We shall denote by a, b, c the lengths of sides BC, CA and AB of a triangle ABC. Semi perimeter of the triangle will be denoted by s and its area by S or Δ.

1 $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$

2 $\cos A=\frac{b^2+c^2-a^2}{2bc}$, $\cos B=\frac{c^2+a^2-b^2}{2ca}$, $\cos C=\frac{a^2+b^2-c^2}{2ab}$

3 $\sin\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\right\}}$, $\sin\frac{B}{2}=\sqrt{\left\{\frac{(s-c)(s-a)}{ca}\right\}}$

$\sin\frac{C}{2}=\sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}}$

4 $\cos\frac{A}{2}=\sqrt{\left\{\frac{s(s-a)}{bc}\right\}}$, $\cos\frac{B}{2}=\sqrt{\left\{\frac{s(s-b)}{ca}\right\}}$

$\cos\frac{C}{2}=\sqrt{\left\{\frac{s(s-c)}{ab}\right\}}$

5 $\tan\frac{A}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}$, $\tan\frac{B}{2}=\sqrt{\left\{\frac{(s-c)(s-a)}{s(s-b)}\right\}}$

$\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}$

6 $\cot\frac{A}{2}=\sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)}\right\}}$, $\cot\frac{B}{2}=\sqrt{\left\{\frac{s(s-b)}{(s-c)(s-a)}\right\}}$

$\cot\frac{C}{2}=\sqrt{\left\{\frac{s(s-c)}{(s-a)(s-b)}\right\}}$

7 $\sin A=\frac{2}{bc}\sqrt{\left\{s(s-a)(s-b)(s-c)\right\}}=\frac{2S}{bc}$ with similar expressions for $\sin B$ and $\sin C$

8 $a=b\cos C+c\cos B$, $b=c\cos A+a\cos C$, $c=a\cos B+b\cos A$

9 $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$ (Napier's Analogy)

Thus $\cot A = \cot B = \cot C$, $\therefore A = B = C$

It follows that $\triangle ABC$ is equilateral

44 We have

$c\,(a+b) \cos \frac{1}{2} B = b\,(a+c) \cos \frac{1}{2} C$

$$c\,(a+b) \sqrt{\left\{\frac{s\,(s-b)}{ca}\right\}} = b\,(a+c) \sqrt{\left\{\frac{s\,(s-c)}{ab}\right\}}$$

or $(a+b)\sqrt{[c\,(s-b)]} = (a+c)\sqrt{[b\,(s-c)]}$

Squaring $(a+b)^2\, c\,(s-b) = (a+c)^2\, b\,(s-c)$

or $s\,\{c\,(a+b)^2 - b\,(a+c)^2\} - bc\,\{(a+b)^2 - (a+c)^2\} = 0$

or $s\,\{ca^2 + 2abc + cb^2 - ba^2 - 2abc - bc^2\}$
$\qquad - bc\,(2a+b+c)\,(b-c) = 0$

or $s\,\{bc\,(b-c) - a^2\,(b-c)\} - bc\,(b-c)\,(2a+b+c) = 0$

or $(b-c)\,\{s\,(bc - a^2) - bc\,(2a+b+c)\} = 0$

or $(b-c)\,\{s\,(bc - a^2) - bc\,(2s+a)\} = 0$

or $-(b-c)\,\{s\,(bc+a^2) + abc\} = 0$

Since a, b, c are all positive and so $s\,(bc+a^2) + abc \neq 0$

It follows that $b - c = 0$

Hence $\triangle ABC$ is isosceles

45 We have

$\cos A + 2\cos B + \cos C = 2$

or $\cos A + \cos C = 2\,(1 - \cos B)$

or $2 \cos \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \cdot 2 \sin^2 \frac{B}{2}$

or $\cos \frac{A-C}{2} = 2 \sin \frac{B}{2}$ $\left[\because \cos \frac{A+C}{2} = \sin \frac{B}{2}\right]$

Multiplying both sides by $2 \cos \frac{B}{2}$ we get

$2 \cos \frac{B}{2} \cos \frac{A-C}{2} = 2\left(2 \sin \frac{B}{2} \cos \frac{B}{2}\right)$

or $2 \sin \frac{A+C}{2} \cos \frac{A-C}{2} = 2 \sin B$

or $\sin A + \sin C = 2 \sin B$ or $a + c = 2b$

$\therefore a, b, c$ are in A P

46 (i) $\cot \frac{1}{2} A, \cot \frac{1}{2} B, \cot \frac{1}{2} C$ are in A P

if $2 \cot \frac{1}{2} B = \cot \frac{1}{2} A + \cot \frac{1}{2} C$

i e if $2 \sqrt{\left\{\frac{s\,(s-b)}{(s-c)\,(s-a)}\right\}} = \sqrt{\left\{\frac{s\,(s-a)}{(s-b)\,(s-c)}\right\}}$
$\qquad + \sqrt{\left\{\frac{s\,(s-c)}{(s-a)\,(s-b)}\right\}}$

18 $(b-c)\cot\frac{1}{2}A+(c-a)\cot\frac{1}{2}B+(a-b)\cot\frac{1}{2}C=0$

19 $(a+b+c)(\cos A+\cos B+\cos C)$
$$=2\left(a\cos^2\tfrac{1}{2}A+b\cos^2\tfrac{1}{2}B+c\cos^2\tfrac{1}{2}C\right)$$

20 $\dfrac{a\sin(B-C)}{b^2-c^2}=\dfrac{b\sin(C-A)}{c^2-a^2}=\dfrac{c\sin(A-B)}{a^2-b^2}$

21 $\dfrac{b^2-c^2}{a}\sin 2A+\dfrac{c^2-a^2}{b^2}\sin 2B+\dfrac{a^2-b^2}{c}\sin 2C=0$

22 $a^3\sin(B-C)+b^3\sin(C-A)+c^3\sin(A-B)=0$

23 $\dfrac{b^2-c^2}{\cos B+\cos C}+\dfrac{c^2-a}{\cos C+\cos A}+\dfrac{a^2-b^2}{\cos A+\cos B}=0$

24 $1-\tan\frac{1}{2}A\tan\frac{1}{2}B=2a/(a+b+c)$ **(Roorkee 1973)**

25 $2abc\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C=(a+b+c)\,S$

26 $bc\cos^2\frac{1}{2}A+ca\cos^2\frac{1}{2}B+ab\cos^2\frac{1}{2}C=s^2$

27 $\dfrac{b-c}{a}\cos^2\frac{1}{2}A+\dfrac{c-a}{b}\cos^2\frac{1}{2}B+\dfrac{a-b}{c}\cos^2\frac{1}{2}C=0$

28 $\dfrac{\tan\frac{1}{2}A}{(a-b)(a-c)}+\dfrac{\tan\frac{1}{2}B}{(b-c)(b-a)}+\dfrac{\tan\frac{1}{2}C}{(c-a)(c-b)}=\dfrac{1}{S}$

29 $\dfrac{(a+b+c)^2}{a^2+b^2+c^2}=\dfrac{\cot\frac{1}{2}A+\cot\frac{1}{2}B+\cot\frac{1}{2}C}{\cot A+\cot B+\cot C}$ **(I I T 1977)**

30 $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)=3abc$ **(I I T 1970)**

31 $a^2-2ab\cos(60^\circ+C)=c^2-2bc\cos(60^\circ+A)$

32 $\sin^3 A\cos(B-C)+\sin^3 B\cos(C-A)+\sin^3 C\cos(A-B)$
$$=3\sin A\sin B\sin C$$

23 If in any triangle the angles be to one another as 1 2 3 prove that the corresponding side are as 1 $\sqrt{3}$ 2

34 In a triangle ABC if $\tan\frac{1}{2}A=\frac{5}{6}$ and $\tan\frac{1}{2}B=\frac{20}{37}$ find $\tan C/2$, and prove that in this triangle $a+c=2b$

35 In an isosceles right angled triangle a straight line is drawn from the middle point of one of the equal sides to the opposite angle Show that it divides the angle into parts whose cotangents are 2 and 3

36 The sides of a triangle are in A P and the greatest angle exceeds the least by 90°, prove that the sides are proportional to $\sqrt{7}+1, \sqrt{7}$ and $\sqrt{7}-1$

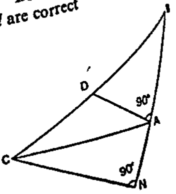
37 If the sides of a triangle are in A P and if its greatest angle exceeds the least angle by α show that the sides are in the ratio $1-x$ 1 $1+x$ where

$$= \frac{2 \sin 60^\circ \cos \frac{A-C}{2}}{\sin 60^\circ} = 2 \cos\left(\frac{A-C}{2}\right)$$

(b) Let $A > B > C$ so that $a > b > c$ i e $a = 10, b = 9, c = ?$ Also as in part (a) $B = 60^\circ$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{1}{2} \qquad c^2 - 10c + (100 - 81) = 0$$

or $c^2 - 10c + 19 = 0$ $\quad c = 5 \pm \sqrt{6}$ Both the values of c are less than 9 and hence both a and d are correct



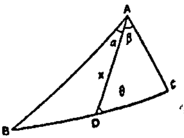
51 We are given $BD = DC$ and $\angle DAB = 90^\circ$
Draw $CN \perp$ to BA produced
Then in $\triangle BCN$, we have
$DA = \frac{1}{2} CN$ and $AB = AN$

$$\tan A = \tan(\pi - \angle CAN)$$

$$= -\tan(\angle CAN) = -\frac{CN}{AN}$$

or $\tan A = -\frac{2AD}{AB} = -2 \tan B$

or $\tan A + 2 \tan B = 0$



52 (i) We are given

$$\frac{m}{n} = \frac{BD}{DC} \qquad (1)$$

or $m\ DC = n\ BD$

Now $\frac{m}{n} = \frac{BD}{DC} = \frac{BD}{AD} \frac{AD}{DC}$

$$= \frac{\sin \alpha}{\sin(\theta - \alpha)} \frac{\sin[\pi - (\theta + \beta)]}{\sin \beta}$$

or $m \sin \beta \sin(\theta - \alpha) = n \sin \alpha \sin(\theta + \beta)$

or $m \sin \beta \sin \theta \cos \alpha - m \sin \beta \cos \theta \sin \alpha$
$= n \sin \alpha \sin \theta \cos \beta + n \sin \alpha \cos \theta \sin \beta$

Dividing by $\sin \alpha \sin \beta \sin \theta$, we get

$m \cot \alpha - m \cot \theta = n \cot \beta + n \cot \theta$

or $m \cot \alpha - n \cot \beta = (m + n) \cot \theta$

Similarly second result can be proved,

(ii) From (1), we have

$$\frac{BD}{m} = \frac{DC}{n} = \frac{BD + DC}{m + n} = \frac{BC}{m + n} = \frac{a}{m + n}$$

48 If a, b, c are in H P, prove that $\sin^2 \frac{1}{2}A$, $\sin^2 \frac{1}{2}B$, $\sin^2 \frac{1}{2}C$ are also in H P

49 If in $\triangle ABC$, $\frac{\sin A}{\sin C}=\frac{\sin (A-B)}{\sin (B-C)}$
prove that a^2, b^2, c^2 are in A P

50 (a) In a triangle the angles A, B, C are in A P show that

$$2 \cos \frac{A-C}{2}=\frac{a+c}{\sqrt{(a^2-ac+c^2)}}$$ (I I T 74)

(b) In a triangle the lenghts of two larger sides are 10 and 9 respectively. If the angles are in A P, then the length of the third side can be
(a) $5-\sqrt{6}$ (b) $3\sqrt{3}$ (c) 5 (d) $5+\sqrt{6}$ (e) None (I I T 87)

51 If the median of $\triangle ABC$ through A is perpendicular to AB prove that $\tan A+2 \tan B=0$ (I I T 66)

52 In a triangle ABC if D be any point of the base BC such that $BD : DC = m : n$ if $\angle BAD=\alpha$, $\angle DAC=\beta$, $\angle CDA=\theta$, and $AD=x$, prove that
(i) $(m+n) \cot \theta=m \cot \alpha-n \cot \beta=n \cot B-m \cot C$
(ii) and $(m+n)^2 x^2=(m+n)(mb^2+nc^2)-mna^2$

53 In $\triangle ABC$ prove that if θ be any angle, then
$$b \cos \theta=c \cos (A-\theta)+a \cos (C+\theta)$$

54 In $\triangle ABC$ prove that
$a^2=(b-c)^2+4bc \sin^2 A/2$ and hence show that
$a=(b-c) \sec \phi$ where $\tan \phi=\frac{2\sqrt{(bc)} \sin A/2}{b-c}$

55 If p and q be perpendiculars from the angular points A and C on any line passing through the vertex C of the $\triangle ABC$ prove that
$$a^2p^2+b^2q^2-2abpq \cos C=a^2b^2 \sin^2 C$$

56 ABC is a triangle and D is the middle point of BC. If AD is perpendicular to AC, prove that
$$\cos A \cos C=\frac{2(c^2-a^2)}{3ac}$$ (I I T 80)

57 If p_1, p_2, p_3 are the altitudes of a triangle from the vertices A, B, C and Δ, the area of the triangle, prove that
$$\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{2ab}{(a+b+c)\Delta} \cos^2 \frac{C}{2}$$ (I I T)

58 If α, β, γ are lengths of the altitudes of a triangle ABC prove that

Now $C=\pi-(\theta+\phi)$

$\cos C=\cos[\pi-(\theta+\phi)]=-\cos(\theta+\phi)$

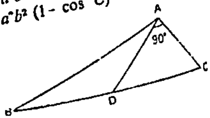
or $\cos C=-\cos\theta\cos\phi+\sin\theta\sin\phi$,

Hence $\cos C=-\sqrt{\left(1-\frac{p^2}{b^2}\right)}\sqrt{\left(1-\frac{q^2}{a^2}\right)}+\frac{p}{b}\frac{q}{a}$

or $\sqrt{\left(1-\frac{p^2}{b^2}\right)\left(1-\frac{q^2}{a^2}\right)}=\frac{pq}{ab}-\cos C$

Squaring $1-\frac{p^2}{b^2}-\frac{q^2}{a^2}+\frac{p^2q^2}{a^2b^2}=\frac{p^2q^2}{a^2b^2}-\frac{2pq}{ab}\cos C+\cos^2 C$

or $a^2p^2+b^2q^2 \quad 2abpq\cos C=a^2b^2(1-\cos C)=a^2b^2\sin^2 C$



56 From $\triangle ADB$,

$\frac{AD}{\sin B}=\frac{BD}{\sin BAD}$

or $AD=\frac{a/2\sin B}{\sin(A-90)}$ (1)

$=-\frac{a\sin B}{2\cos A}$

And from $\triangle ADC$, (2)

$\frac{AD}{DC}=\sin C$ or $AD=DC\sin C=a/2\sin C$

Hence from (1) and (2), we get $-\frac{a\sin B}{2\cos A}=\frac{a\sin C}{2}$ (3)

or $\cos A=-\frac{\sin B}{\sin C}=-\frac{b}{c}$ (4

or $\frac{b^2+c^2-a^2}{2bc}=-\frac{b}{c}$

This gives $b^2=\frac{a^2-c^2}{3}$

Now multiplying (3) by cos C, we get

$\cos A\cos C=-\frac{b}{c}\cos C=-\frac{b}{c}\frac{b}{a/2}=-\frac{2b^2}{ac}=\frac{2(c^2-a^2)}{2ac}$ by

(Note that from $\triangle ADC$, $\cos C=\frac{AC}{DC}=\frac{b}{a/2}$)

57 Since p_1, p_2, p_3 are perpendiculars from the vertices A, B to the opposite sides, we have

$\Delta=\frac{1}{2}ap_1=\frac{1}{2}bp_2=\frac{1}{2}cp_3$

Hence $\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p_3}=\frac{a}{2\Delta}+\frac{b}{2\Delta}-\frac{c}{2\Delta}$

3 Do yourself

4 Do yourself

5 We have

$$\frac{b+c}{a}=\frac{k\sin B+k\sin C}{k\sin A}=\frac{2\sin\frac{B+C}{2}\cos\frac{B-C}{2}}{2\sin\frac{A}{2}\cos\frac{A}{2}}$$

$$=\frac{\cos\frac{A}{2}\cos\frac{B-C}{2}}{\sin\frac{A}{2}\cos\frac{A}{2}}=\frac{\cos\frac{B-C}{2}}{\sin\frac{A}{2}}=\frac{\sin\left(\frac{A}{2}+B\right)}{\sin\frac{A}{2}}$$

$$\left[\because\ \sin\frac{B+C}{2}=\sin\left(\frac{\pi}{2}-\frac{A}{2}\right)=\cos\frac{A}{2}\right]$$

and $\frac{B-C}{2}=\frac{1}{2}[B-(\pi-A-B)]=\frac{1}{2}[A+2B-\pi]$

$$\left[\therefore\ \cos\frac{B-C}{2}=\sin\left(\frac{A}{2}+B\right)\right]$$

6 Do yourself

7 (*a*) $\text{L H S}=\frac{\sin\frac{(A+B)}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}\,k\left\{\sin A\sin^2\frac{B}{2}+\sin B\sin^2\frac{A}{2}\right\}$

$$\left[\because\ \frac{a}{\sin A}=k\right]$$

$$=\cos\frac{C}{2}\,k\left\{2\cos\frac{A}{2}\sin\frac{B}{2}+2\cos\frac{B}{2}\sin\frac{A}{2}\right\}$$

$$=\cos\frac{C}{2}\,\frac{c}{\sin C}\,2\sin\left(\frac{A+B}{2}\right)=c\,\frac{2\cos^2\frac{C}{2}}{2\sin\frac{C}{2}\cos\frac{C}{2}}=c\cot\frac{C}{2}$$

Note: you may also use the values of $\cot\frac{A}{2}$, $\sin\frac{A}{2}$ in terms of s

(*b*) Do yourself

8 $$\frac{1+\cos(A-B)\cos C}{1+\cos(A-C)\cos B}=\frac{1-\cos(A-B)\cos(A+B)}{1-\cos(A-C)\cos(A+C)}$$

$$[\because\ A+B+C=\pi \text{ so that } \cos C=-\cos(A+B)]$$

$$=\frac{1-(\cos^2A-\sin^2B)}{1-(\cos^2A-\sin^2C)}=\frac{\sin^2A+\sin^2B}{\sin^2A+\sin^2C}=\frac{a^2+b^2}{a^2+c^2}$$

or $\quad \sin C \sin \omega \sin A \cos B + \sin C \sin \omega \cos A \sin B$
$= \sin A \sin B \sin C \cos \omega - \sin A \sin B \cos C \sin \omega$

Dividing by $\sin A \sin B \sin C \sin \omega$, we get

$\cot B + \cot A = \cot \omega - \cot C$

or $\cot \omega = \cot A + \cot B + \cot C$

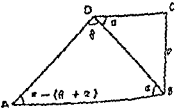
Squaring $\cot^2 \omega = \cot^2 A + \cot^2 B + \cot^2 C + 2 \cot A \cot B$
$+ 2 \cot B \cot C + 2 \cot C \cot A$

or $\operatorname{cosec}^2 \omega - 1 = \operatorname{cosec}^2 A - 1 + \operatorname{cosec}^2 B - 1 + \operatorname{cosec}^2 C - 1 + 2$

[In a $\triangle$ $\cot A \cot B + \cot B \cot C + \cot C \cot A = 1$]

or $\operatorname{cosec}^2 \omega = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$

60 From right angled $\triangle BCD$, we get



$BD = \sqrt{(p^2 + q^2)}$

Let $\angle ABD = \angle BDC = \alpha$

Then $\angle DAB = \pi - (\alpha + \theta)$

and $\tan \alpha = p/q$

Now from ABD we have

$$\frac{AB}{\sin \theta} = \frac{BD}{\sin[\pi - (\theta + \alpha)]} = \frac{BD}{\sin(\theta + \alpha)}$$

$$AB = \frac{BD \sin \theta}{\sin(\theta + \alpha)} = \frac{BD^2 \sin \theta}{BD \sin(\theta + \alpha)}$$

$$= \frac{BD^2 \sin \theta}{BD \sin \theta \cos \alpha + BD \cos \theta \sin \alpha}$$

$$= \frac{(p^2 + q^2) \sin \theta}{q \sin \theta + p \cos \theta}$$

[$BD^2 = p^2 + q^2$, $BD \cos \alpha = q$ and $BD \sin \alpha = p$]

61 Do yourself

62 If each ratio be k then we have

$b + c = 11k$, $c + a = 12k$, $a + b = 13k$

so that $2(a + b + c) = 36k$ or $a + b + c = 18k$

$a = 7k$, $\quad b = 6k$, $\quad c = 5k$

Now $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc} = \dfrac{36 + 25 - 49}{2 \cdot 6 \cdot 5} = \dfrac{12}{60} = \dfrac{1}{5}$

Similarly $\cos B = 19/35$ and $\cos C = 5/7$

$\cos A : \cos B : \cos C = \dfrac{1}{5} : \dfrac{19}{35} : \dfrac{5}{7} = 7 : 19 : 25$

63 Multiplying both sides by 2 we have

$2 \cos A \cos B + 2 \sin A \sin B \sin C = 2 = 1 + 1$
$= (\cos^2 A + \sin^2 A) + (\cos^2 B + \sin^2 B)$ (Note)

$$=\frac{k^2 \sin A \sin (B+C) \sin (B-C)}{\sin B+\sin C}=\frac{k^2 \sin A (\sin^2 B-\sin^2 C)}{\sin B+\sin C}$$

$=k^3 \sin A(\sin B-\sin C)$

L.H S $=k^2$ [$\sin A (\sin B-\sin C)+\sin B (\sin C-\sin A)$ $+\sin C (\sin A-\sin B)]=0$

14 Do yourself

15 $(b^2-c^2) \cot A=(b^2-c^2)\frac{\cos A}{\sin A}=(b^2-c^2)\frac{b^2+c^2-a^2}{2bc\,ka}$

$$=\frac{1}{2k\,abc}[(b^4-c^4)-a^2(b^2-c^2)]$$

$$\text{L H S}=\frac{1}{2k\,abc}[(b^4-c^4)+(c^4-a^4)+(a^4-b^4)]$$

$$-[\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\}]=0$$

16 $a \sin \frac{1}{2} A \sin \frac{1}{2}(B-C)$

$=k \sin A \sin \frac{1}{2} A \sin \frac{1}{2}(B-C)$

$=2k \sin^2 \frac{1}{2} A \cos \frac{1}{2} A \sin \frac{1}{2}(B-C)$, $\quad \sin A=2\sin A/2 \cos A/2$

$=2k \sin^2 \frac{1}{2} A \sin \frac{1}{2}(B+C) \sin \frac{1}{2}(B-C)$

$[\because \cos \frac{1}{2} A=\sin \frac{1}{2}(B+C)]$

$=2k \sin^2 \frac{1}{2} A \{\sin^2 \frac{1}{2} B-\sin^2 \frac{1}{2} C\}$

L H S $=2k [\sin^2 \frac{1}{2} A \{\sin^2 \frac{1}{2} B-\sin^2 \frac{1}{2} C\}$

$+\sin^2 \frac{1}{2} B \{\sin^2 \frac{1}{2} C-\sin^2 \frac{1}{2} A\}+\sin^2 C/2 \{\sin^2 \frac{1}{2} A-\sin^2 \frac{1}{2} B\}]$
$=0$

17 Do yourself

18 $(b-c) \cot \frac{1}{2} A=k (\sin B-\sin C) \cot \frac{1}{2} A$

$$=2k \cos \frac{B+C}{2} \sin \frac{B-C}{2} \cot \frac{1}{2} A$$

$$=2k \sin \tfrac{1}{2} A \sin \frac{B-C}{2} \frac{\cos \frac{1}{2} A}{\sin \frac{1}{2} A}$$

$$=2k \sin \frac{B-C}{2} \sin \frac{B+C}{2}=2k\left(\sin^2 \frac{B}{2}-\sin^2 \frac{C}{2}\right)$$

$$\text{L H S}=2k\left[\left(\sin^2 \frac{B}{2}-\sin^2 \frac{C}{2}\right)+\left(\sin^2 \frac{C}{2}-\sin^2 \frac{A}{2}\right)+\left(\sin^2 \frac{A}{2}-\sin^2 \frac{B}{2}\right)\right]=0$$

19 $(a+b+c)(\cos A+\cos B+\cos C)$

$=a \cos A+b \cos B+c \cos C$

$+(b \cos C+c \cos B)+(c \cos A+a \cos C)+(a \cos B+b \cos A)$

$=a(2\cos^2 A/2-1)+b(2\cos^2 B/2-1)+c(2\cos^2 C/2-1)$

$+(a+b+c)\quad [\because b \cos C+c \cos B=a$ etc$]$

$=2(a \cos^2 A/2+b \cos^2 B/2+c \cos^2 C/2)$

6

Properties of a Triangle, Quadrilateral and Polygon

§ 1 Some definitions and Formulae

(i) **Circum-circle** The circle which passes through the angular points of a $\triangle ABC$ is called circumscribing circle or, more briefly, its *circumcircle* We shall denote its radius by R

(ii) **In-circle** The circle which can be inscribed within the triangle so as to touch each of the sides is called inscribed circle or, more briefly, its in circle We denote its radius by r

(iii) **Escribed circles** The circle which touches BC and the two sides AB and AC produced is called the escribed circle opposite the angle A We denote its radius by r_1 Similarly we denote by r_2 and r_3 the radii of the escribed circles opposite the angles B and C respectively

Students should commit to memory the following formulae

$$R=\frac{a}{2\sin A}=\frac{b}{2\sin B}=\frac{c}{2\sin C} \quad (1)$$

$$R=\frac{abc}{4S} \quad (2)$$

$$r=\frac{S}{s} \quad (3)$$

$$r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2} \quad (4)$$

$$r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \quad (5)$$

$$r_1=\frac{S}{s-a},\ r_2=\frac{S}{s-b},\ r_3=\frac{S}{s-c} \quad (6)$$

$$r_1=s\tan\frac{A}{2},\ r_2=s\tan\frac{B}{2},\ r_3=s\tan\frac{C}{2} \quad (7)$$

$$=\left[\frac{(c-b)(s-b)(s-c)+(a-c)(s-c)(s-a)+(b-a)(s-a)(s-b)}{(a-b)(b-c)(c-a)\sqrt{\{s(s-a)(s-b)(s-c)\}}}\right] \quad (1)$$

Now $(c-b)(s-b)(s-c)+(a-c)(s-c)(s-a)$

$+(b-a)(s-a)(s-b)$

$=(c-b)\{s^2-s(b+c)+bc\}+(a-c)\{s^2-s(c+a)+ca\}$

$+(b-a)\{s^2-s(a+b)+ab\}$

$=s^2(c-b+a-c+b-a)-s(c^2-b^2+a^2-c^2+b^2-a^2)$

$-\{bc(b-c)+ca(c-a)+ab(a-b)\}$

$=(a-b)(b-c)(c-a)$

[Note that on factorising $bc(b-c)+ca(c-a)+ab(a-b)$

$=-(b-c)(c-a)(a-b)$]

Substituting in (1), we get

$$\text{L H S}=\frac{1}{\sqrt{\{s(s-a)(s-b)(s-c)\}}}=\frac{1}{S}$$

29 $\cot\frac{1}{2}A+\cot\frac{1}{2}B+\cot\frac{1}{2}C$

$$=\sqrt{\left\{\frac{s(s-a)}{(s-b)(s-c)}\right\}}+\sqrt{\left\{\frac{s(s-b)}{(s-c)(s-a)}\right\}}+\sqrt{\left\{\frac{s(s-c)}{(s-a)(s-b)}\right\}}$$

$$=\frac{\sqrt{s}[(s-a)+(s-b)+(s-c)]}{\sqrt{\{(s-a)(s-b)(s-c)\}}}$$

$$=\frac{\sqrt{s}\sqrt{s}\,[3s-2s]}{\sqrt{\{s(s-a)(s-b)(s-c)\}}}=\frac{s^2}{S}$$

and $\cot A+\cot B+\cot C=\dfrac{\cos A}{\sin A}+\dfrac{\cos B}{\sin B}+\dfrac{\cos C}{\sin C}$

$$=\frac{(b^2+c^2-a^2)/2bc}{2S/bc}+\frac{(c^2+a^2-b^2)/2ca}{2S/ca}+\frac{(a^2+b^2-c^2)/2ab}{2S/ab}$$

$$=\frac{1}{4S}[b^2+c^2-a^2+c^2+a^2-b^2+a^2+b^2-c^2]$$

$$=\frac{a^2+b^2+c^2}{4S}$$

Hence $\text{R H S}=\dfrac{s^2}{S}\times\dfrac{4S}{a^2+b^2+c^2}=\dfrac{4s^2}{a^2+b^2+c^2}=\dfrac{(a+b+c)^2}{a^2+b^2+c^2}$

30 $a^3\cos(B-C)+b^3\cos(C-A)+c^3\cos(A-B)$

$=k^3\sin^3 A\cos(B-C)+k^3\sin^3 B\cos(C-A)$

$+k^3\sin^3 C\cos(A-B)$

$k^3[\sin^2 A\sin(B+C)\cos(B-C)$

$+\sin^2 B\sin(C+A)\cos(C-A)$

$+\sin^2 C\sin(A+B)\cos(A-B)]$

$=\frac{1}{2}k^3[\sin^2 A(\sin 2B+\sin 2C)+\sin^2 B(\sin 2C+\sin 2A)$

$+\sin^2 C(\sin 2A+\sin 2B)]$

§ 3 **Circum-centre, Centroid and Ortho centre are collinear**

Let O be the circum centre and P the ortho-centre in a $\triangle ABC$. If OD is perpendicular to BC, then D will be its mid point. Let the median AD meet OP in G. Then it is clear that $\triangle OGD$ and $\triangle PGA$ are similar

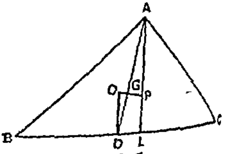


Fig 2

$$\frac{DG}{AG}=\frac{OG}{PG}=\frac{OD}{PA}$$

But $\quad OD=R\cos A$ and $PA=2R\cos A$

Hence $\quad \dfrac{OC}{PA}=\dfrac{R\cos A}{2R\cos A}=\dfrac{1}{2}$

Thus $\quad \dfrac{DG}{AG}=\dfrac{OG}{PD}=\dfrac{OD}{OA}=\dfrac{1}{2}$

If follows that G is the centroid of the $\triangle ABC$ and is situated on the line OP and divides it in the ratio 1 : 2

§ 4 **Bisectors of the angles**

Let AD be the bisector of angle A and suppose AD divides the base BC into two part x and y. Then by geometry,

$$\frac{x}{y}=\frac{AB}{AC}=\frac{c}{b}$$

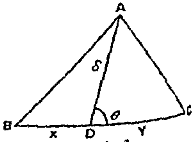


Fig 3

$$\frac{x}{c}=\frac{y}{b}=\frac{x+y}{c+b}=\frac{a}{b+c} \tag{1}$$

Again if δ be the length of AD and θ the angle it makes with BC, then

$$\triangle ABD+\triangle ACD=\triangle ABC$$

or $\quad \dfrac{1}{2}c\delta\sin\dfrac{A}{2}+\dfrac{1}{2}b\delta\sin\dfrac{A}{2}=\dfrac{1}{2}bc\sin A$

or $\quad \delta=\dfrac{bc}{b+c}\dfrac{\sin A}{\sin\dfrac{A}{2}}=\dfrac{2bc}{b+c}\cos\dfrac{A}{2} \qquad (2)$

Also $\quad \theta=\angle DAB+\angle B=\dfrac{A}{2}+B \qquad (3)$

§ 5 **Area of Cyclic Quadrilateral**

Let $ABCD$ be a cyclic quadrilateral whose sides AB, BC, CD and DA are respectively a, b, c and d

$$\frac{a}{\sin 30^\circ}=\frac{b}{\sin 60^\circ}=\frac{c}{\sin 90^\circ}$$

$$a : b : c=\sin 30^\circ : \sin 60^\circ : \sin 90^\circ=\frac{1}{2} : \frac{\sqrt{3}}{2} : 1=1 : \sqrt{3} : 2$$

34 We have

$$\tan\frac{C}{2}=\tan\left(90^\circ-\frac{A+B}{2}\right)=\cot\left(\frac{A}{2}+\frac{B}{2}\right)$$

$$=\frac{\cot\frac{A}{2}\cot\frac{B}{2}-1}{\cot\frac{B}{2}+\cot\frac{A}{2}}=\frac{\frac{6}{5}\cdot\frac{37}{20}-1}{\frac{37}{20}+\frac{6}{5}}$$

$$=\frac{222-100}{5\times 61}=\frac{122}{5\times 61}=\frac{2\times 61}{5\times 61}=\frac{2}{5}$$

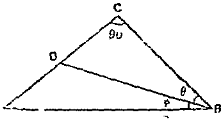
Again $\tan\frac{A}{2}\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\right\}}\sqrt{\left\{\frac{(s-a)(s-b)}{s(s-c)}\right\}}$

Hence $\frac{5}{6}\cdot\frac{2}{5}=\frac{s-b}{s}$

$3s-3b=s$ or $2s=3b$

or $a+b+c=3b$ or $a+c=2b$

35 Let ABC be the triangle, right angled at C and D be the mid point of AC. Join DB.



Since $AC=BC$, we have

$DC=\frac{1}{2}AC=\frac{1}{2}BC$

Also $\angle CAB=\angle CBA=45^\circ$

If $\angle DBC=\theta$ and $\angle DBA=\phi$ then $\cot\theta=\frac{BC}{CD}=\frac{2CD}{CD}=2$

and $\cot\phi=\cot(45^\circ-\theta)=\frac{\cot 45^\circ\cot\theta+1}{\cot\theta-\cot 45^\circ}=\frac{2+1}{2-1}=3$

36 Let A be the greatest and C the least angle of $\triangle ABC$. It is given that a, b, c are in AP, so that

$$a+c=2b \qquad (1)$$

Also $A-C=90^\circ$

From (1), $\sin A+\sin C=2\sin B=2\sin(A+C)$ (2)

$$2\sin\frac{A+C}{2}\cos\frac{A-C}{2}=4\sin\frac{A+C}{2}\cos\frac{A+C}{2}$$

or $\cos\frac{A-C}{2}=2\cos\frac{A+C}{2}=2\sin\frac{B}{2}$

$$=\frac{\{(a^2+b^2)\,cd+ab\,(c^2+d^2)\}}{(ab+cd)}$$

$$=\frac{(ac+bd)\,(ad+bc)}{(ab+cd)}$$

Similarly $\quad BD^2=\frac{(ab+cd)\,(ac+bd)}{(ad+bc)}$

Hence $\quad AC^2\,BD^2=(ac+bd)^2$

or $\quad AC\,BD=ac+bd$

Thus $\quad$ AC BD = AB CD + BC AD

Cor Since circum circle of quadrilateral $ABCD$ is also the circum circle of $\triangle ABC$, the circum radius of $ABCD$

$$=\frac{AC}{2\sin B}=\frac{(ab+cd)}{4\Delta}\,AC$$

where Δ is the area of quadrilateral

$$=\frac{ab+cd}{4\Delta}\sqrt{\left[\frac{(ac+bd)\,(ad+bc)}{(ab+cd)}\right]}$$

$$=\frac{1}{4}\sqrt{\left[\frac{(ab+cd)\,(ac+bd)\,(ad+bc)}{(s-a)\,(s-b)\,(s-c)\,(s-d)}\right]}$$

§ 7 Radii of the inscribed and circumscribing circles of a regular polygon

Let AB, BC and CD be three successive sides of the polygon and let n be the number of its sides let the angles ABC and BCD

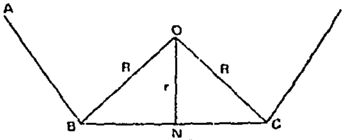


Fig 5

be bisected by the lines BO and CO meeting at O Draw ON perpendicular to BC Then it is clear that O is the centre of both the in circle and the circum circle of the polygon

Also $\quad BN=CN$

Hence $OB=OC=R$ the radius of the circum circle and r the radius of in circle or $c^2=a^2+b^2-ab$ (1)

Also $\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$ if $\frac{a+b+2c}{(a+c)(b+c)}=\frac{3}{a+b+c}$

i e, if $(a+b+2c)(a+b+c)=3(a+c)(b+c)$

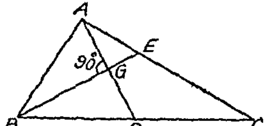
i e if $(a+b)^2+2c^2+3c(a+b)=3(ab+ac+bc+c^2)$

i e if $a^2+b^2+2ab+2c^2+3ca+3cb=3ab+3ac+3bc+3c^2$

i e if $a^2+b^2-ab=c^2$,

which is the same as (1)

(b) We are given, $AC=3$, $BC=4$ and medians AD and BE are perpendicular so that $\angle AGB=90°$. Now $AB^2=BC^2+AC^2-2\,BC\,AC\cos C=16+9-2\cdot4\cdot3\cos C$



or $AB^2=25-24\cos C$ (1)

Again $AD^2=AC^2+DC^2-2\,AC\,DC\cos C$

$=9+4-2\cdot3\cdot2\cos C=13-12\cos C$

$BG=\frac{2}{3}AD=\frac{2}{3}\sqrt{(13-12\cos C)}$ (2)

And $BE^2=BC^2+CE^2-2BC\,CE\cos C$

$=16+\frac{9}{4}-2\cdot4\cdot\frac{3}{2}\cos C$

$=\frac{73}{4}-12\cos C$

so that $AG=\frac{2}{3}BE=\frac{2}{3}\sqrt{\left(\frac{73}{4}-12\cos C\right)}$ (3)

Now since $\angle AGB=90°$, we have

$AB^2=BG^2+AG^2$,

which with the help of (1), (2) and (3) gives

$25-24\cos C=\frac{4}{9}(13-12\cos C)+\frac{4}{9}\left(\frac{73}{4}-12\cos C\right)$

$\frac{40}{3}\cos C=\frac{100}{9}$ $\cos C=\frac{5}{6}$

Area of $\triangle ABC=\frac{1}{2}BC\,AC\sin C$

$=\frac{1}{2}\cdot4\cdot3\sqrt{\left(1-\frac{25}{36}\right)}=\sqrt{(11)}$.

39 (i) Since $\cos A=\frac{\sin B}{2\sin C}$, we have $\frac{b^2+c^2-a^2}{2bc}=\frac{b}{2c}$

or $b^2+c^2-a^2=b^2$ or $c^2=a^2$

Hence $c=a$ and so the $\triangle ABC$ is isosceles

2 If r is the radius of in circle of $\triangle ABC$, prove the following formulae

(i) $r=S/s$, (ii) $r=4R \sin A/2 \sin B/2 \sin C/2$

(iii) $r=(s-a) \tan A/2=(s-b) \tan B/2=(s-c) \tan C/2$ (Roorkee 74)

3 Prove the formulae

(i) $r_1=S/(s-a)$, $r_2=S/(s-b)$, $r_3=S/(s-c)$

(ii) $r_1=s \tan A/2$, $r_2=s \tan B/2$, $r_3=s \tan C/2$

(iii) $r_1=4R \sin A/2 \cos B/2 \cos C/2$

where r_1 r_2 r_3 have their usual meanings

Prove the following

4 $\sqrt{(rr_1r_2r_3)}=S=r^2 \cot A/2 \cot B/2 \cot C/2$

5 (a) $r_1r_2r_3=r^3 \cot^2 A/2 \cot^2 B/2 \cot^2 C/2$

(b) $r_1=r \cot B/2 \cot C/2$

6 (a) $rr_1 \cot A/2=S$

(b) $r_1 \cot A/2=r_2 \cot B/2=r_3 \cot C/2=S$

(c) $r_2r_3=S \cot A/2$

7. (a) $r_2r_3+r_3r_1+r_1r_2=s^2$

(b) $r_1r_2r_3=rs^2$

8 $1/r_1+1/r_2+1/r_3=1/r$

9 $(r_1+r_2) \tan C/2=(r_3-r) \cot C/2=c$

10 $(r_1-r)/a+(r_2-r)/b=c/r_3$

11 $a\,(rr_1+r_2r_3)=b\,(rr_2+r_3r_1)=c\,(rr_3+r_1r_2)$

12 $r_1\,(r_2+r_3)/a=r_2\,(r_3+r_1)/b=r_3\,(r_1+r_2)/c$

13 $(ab-r_1r_2)/r_3=(bc-r_2r_3)/r_1=(ca-r_3r_1)/r_2$.

14 (a) $r_1+r_2+r_3-r=4R$

(b) $r_1+r_2-r_3+r=4R \cos C$

15 (a) $(r_1-r)\,(r_2-r)\,(r_3-r)=4r^2R$

(b) $(r_1-r)\,(r_2+r_3)=a^2$

16 $\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right)=\frac{16R}{r^2(a+b+c)^2}$

17 $\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r^2}=\frac{a^2+b^2+c^2}{S^2}$ (Roorkee 76)

18 $\left(\frac{1}{r_1}+\frac{1}{r_2}\right)\left(\frac{1}{r_2}+\frac{1}{r_3}\right)\left(\frac{1}{r_3}+\frac{1}{r_1}\right)=\frac{64R^3}{a^2b^2c^2}$

19 $\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$

20 $\frac{1}{2Rr}=\frac{1}{bc}+\frac{1}{ca}+\frac{1}{ab}$

and $2b=2k \sin B=2k \sin 75°=2k \sin (45°+30°)$

$$=2k\,\frac{\sqrt{(3+1)}}{2\sqrt{2}}=\frac{\sqrt{3}+1}{\sqrt{2}}\,k \qquad (2)$$

$a+c\sqrt{2}=2b$ by (1) and (2)

41 We have

$a \tan A+b \tan B=(a+b) \tan \frac{1}{2}(A+B)$

$a\,[\tan A-\tan \frac{1}{2}(A+B)]=b\,[\tan \frac{1}{2}(A+B)-\tan B] \qquad (1)$

or $a\left[\dfrac{\sin A \cos \frac{1}{2}(A+B)-\cos A \sin \frac{1}{2}(A+B)}{\cos A \cos \frac{1}{2}(A+B)}\right]$

$=b\left[\dfrac{\sin \frac{1}{2}(A+B) \cos B-\cos \frac{1}{2}(A+B) \sin B}{\cos \frac{1}{2}(A+B) \cos B}\right]$

or $\dfrac{k \sin A \sin \frac{1}{2}(A-B)}{\cos A}=\dfrac{k \sin B \sin \frac{1}{2}(A-B)}{\cos B}$

or $\sin \frac{1}{2}(A-B)\left(\dfrac{\sin A}{\cos A}-\dfrac{\sin B}{\cos B}\right)=0$

or $\sin \frac{1}{2}(A-B)(\tan A-\tan B)=0$

From this equation, we conclude that $A=B$

42 We have proved in the chapter on identities that

$\cos^2 A+\cos^2 B+\cos^2 C=1-2 \cos A \cos B \cos C$

(Q 5 (ii) P 59)

$\cos^2 A+\cos^2 B+\cos^2 C=1$ gives

$1-2 \cos A \cos B \cos C=1$ or $\cos A \cos B \cos C=0$

Hence either $\cos A=0$, $\cos B=0$ or $\cos C=0$

i e either $A=\pi/2$ or $B=\pi/2$ or $C=\pi/2$

It follows that the $\triangle ABC$ is right angled

43 We have

$\cot A+\cot B+\cot C=\sqrt{3}$

Squaring, we get

$\cot^2 A+\cot^2 B+\cot^2 C+2 \cot A \cot B+2 \cot B \cot C$
$+2 \cot C \cot A=3 \qquad (1)$

Since from identities we know that

$\cot A \cot B+\cot B \cot C+\cot C \cot A=1$ (Q 10 (ii) P 60)

we may write (1) as

$\cot^2 A+\cot^2 B+\cot^2 C-\cot A \cot B-\cot B \cot C$
$-\cot C \cot A=0$

or $2 \cot^2 A+2 \cot^2 B+2 \cot^2 C-2 \cot A \cot B$
$-2 \cot B \cot C-2 \cot C \cot A=0$

or $(\cot A-\cot B)^2+(\cot B-\cot C)^2+(\cot C-\cot A)^2=0$

Hence $\cot A-\cot B=0$, $\cot B-\cot C=0$ and $\cot C-\cot A=0$

37 Prove that the distances of the ortho-centre of the ΔABC from its vertex A and side BC are respectively

$$2R \cos A \text{ and } 2R \cos B \cos C$$

38 Prove that the distances of the in-centre of ΔABC from A is

$$4R \sin B/2 \sin C/2$$

39 O is the circumcentre of ΔABC and R_1, R_2 and R_3 are respectively the radii of the circum circles of the $\Delta OBC, OCA$ and OAB Prove that

$$\frac{a}{R_1}+\frac{b}{R_2}+\frac{c}{R_3}=\frac{abc}{R^3}$$

40 I is the incentre of ΔABC and P_1, P_2 and P_3 are respectively the radii of the circum-circles of Δ^s IBC, ICA and IAB Prove that $P_1P_2P_3=2R^2r$

41 (a) If in triangle $(1-r_1/r_2)(1-r_1/r_3)=2$, prove that the triangle is right-angled

(b) In a right angled ΔABC with C as a right angle a perpendicular CD is drawn to AB The radii of the circles inscribed into the triangles ACD and BCD are equal to x and y respectively Find the radius of the circle inscribed into the ΔABC

42 If in a triangle $8R^2=a^2+b^2+c^2$ prove that the triangle is right angled

43 (a) If in a triangle ABC $(a-b)(s-c)=(b-c)(s-a)$, prove that r_1 r_2 r_3 are in A P

(b) The ex radii r_1, r_2 r_3 of ΔABC are in H P Show that its sides a b, c are in A P (I I T 83)

44 In triangle ABC prove that the area of the in circle is to the area of the triangle itself as

$$\pi \quad \cot A/2 \cot B/2 \cot C/2$$

45 In a cyclic quadrilateral $ABCD$ prove that

$$\tan^2 \frac{B}{2}=\frac{(s-a)(s-b)}{(s-c)(s-d)}$$

46 Prove that the sum of the radii of the circles which are respectively inscribed and circumscribed ... polygon of n sides is $\frac{a}{2} \cot \frac{\pi}{2n}$ where a is the side of t...

47 Prove that the area of ... gon in a circle is a mean ... the regular inscribed and ci...

Vikram Singh

i e if $2(s-b)=(s-a)+(s-c)$ i e if $2b=a+c$
i e if a, b, c are in A P

(ii) we have

$$\cos A \cot \tfrac{1}{2}A=(1-2\sin^2 \tfrac{1}{2}A)\frac{\cos \frac{1}{2}A}{\sin \frac{1}{2}A}=\cot \tfrac{1}{2}A-\sin A$$

Now a, b, c are in A P

$\sin A$ $\sin B$, $\sin C$ are also in A P

Also we have proved in part (I) that $\cot A/2$, $\cot B/2$, $\cot C/2$ are in A P Hence their differences are also in A P

47 Do yourself

48 $\frac{1}{b}-\frac{1}{a}=\frac{1}{c}-\frac{1}{b}$

$$\Rightarrow \frac{(\sin A-\sin B)}{2\sin A \sin B}=\frac{\sin B-\sin C}{\sin B \sin C}$$

or
$$\frac{2\sin\frac{A-B}{2}\cos\frac{A+B}{2}}{\sin A/2 \cos A/2}=\frac{2\sin\frac{B-C}{2}\cos\frac{B+C}{2}}{\sin C/2\cos C/2}$$

or
$$\sin\frac{C}{2}\cos\frac{C}{2}\sin\frac{A-B}{2}=\sin^2\frac{A}{2}\cos\frac{A}{2}\sin\frac{B-C}{2}$$

$$\sin^2\frac{C}{2}\sin\frac{A+B}{2}\sin\frac{A-B}{2}=\sin^2\frac{A}{2}\sin\frac{B+C}{2}\sin\frac{B-C}{2}$$

$$\sin^2\frac{C}{2}\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}\right)=\sin^2\frac{A}{2}\left(\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)$$

Divide by $\sin^2 A/2 \sin^2 B/2 \sin^2 C/2$

$$\frac{1}{\sin^2 B/2}-\frac{1}{\sin^2 A/2}=\frac{1}{\sin^2 C/2}-\frac{1}{\sin^2 B/2}$$

$$\frac{1}{\sin^2 A/2}, \frac{1}{\sin^2 B/2}, \frac{1}{\sin^2 C/2} \text{ are in A P}$$

or $\sin^2 A/2$, $\sin^2 B/2$, $\sin^2 C/2$ are in H P

49 Do yourself

50, (a) Since A, B, C are in A P, we have $2B=A+C$

But $A+B+C=180°$

Hence $3B=180°$ or $B=60°$ and $A+C=120°$

Now $b^2=c^2+a^2-2ca \cos B=c^2+a^2-2ca\cos 60°$

$=c^2+a^2-ca$ (1)

Hence $\frac{a+c}{\sqrt{(a^2-ac+c^2)}}=\frac{a+c}{b}$ from (1)

$$=\frac{k(\sin A+\sin C)}{k\sin B}=\frac{2\sin\frac{A+C}{2}\cos\frac{A-C}{2}}{\sin B}$$

(ii) $a=BC=BD+DC=r\cot B/2+r\cot C/2$

or $2R\sin A=r\left(\dfrac{\cos B/2\sin C/2+\cos C/2\sin B/2}{\sin B/2\sin C/2}\right)$ [From 1 (i)]

Or $2R\,2\sin A/2\cos A/2\sin B/2\sin C/2$
$=r\sin((B+C)/2)=r\cos A/2$

$4R\sin A/2\sin B/2\sin C/2=r,$ $(B+C)/2=90^\circ-A/2$

(iii) We know by Geometry that

$BD=BF=\beta$ (say), $CD=CF=\gamma$, $AE=AF=\alpha$

$2\alpha+2\beta+2\gamma=a+b+c=2s$

or $\alpha+\beta+\gamma=s,$ $AF=\alpha=s-(\beta+\gamma)=s-BC=s-a$

or $BD=\beta=s-(\gamma+\alpha)=s-AC=s-b,$

$CD=\gamma=s-(\alpha+\beta)=s-AB=s-c$

Now $\dfrac{ID}{BD}=\tan\dfrac{B}{2},$ $r=\beta\tan\dfrac{B}{2}=(s-b)\tan\dfrac{B}{2}$

Similarly,

$\dfrac{ID}{CD}=\tan\dfrac{C}{2},$ $r=\gamma\tan\dfrac{C}{2}=(s-c)\tan\dfrac{C}{2}$

and $\dfrac{IF}{AF}=\tan\dfrac{A}{2},$ $r=\alpha\tan\dfrac{A}{2}=(s-a)\tan\dfrac{A}{2}$

3 (i) Produce the sides AB, AC of $\triangle ABC$ and let the bisectors of the exterior angles B and C meet in I_1. From I_1 draw perpendiculars to the sides. By Geometry,

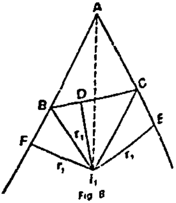


Fig 8

$ID=IE=IF=r_1$

(i) $\triangle ABC=\triangle I_1AF+\triangle I_1AE-\triangle I_1AC$

or $S=\frac{1}{2}r_1c+\frac{1}{2}r_1b-\frac{1}{2}r_1a$

or $S=\frac{1}{2}r_1(b+c-a)$

$=\frac{1}{2}r_1\,2(s-a)$

$r_1=\dfrac{S}{s-a}$ Similarly, $r_2=\dfrac{S}{s-b}$, $r_3=\dfrac{S}{s-c}$

(ii) $AF=AB+BF=AB+BD$, $AE=AC+CE=AC+CD$

$AF+AE=AB+AC+(BD+CD)=a+b+c=2s$

or $2AF=2s$ or $AF=s$ $AF=AE$

$\dfrac{I_1F}{AF}=\tan\dfrac{A}{2}$ or $r_1=s\tan\dfrac{A}{2}$

$$BD=\frac{am}{m+n},\ DC=\frac{an}{m+n} \quad (1)$$

Now $AC^2=AD^2+DC^2-2AD\ DC\cos\theta$

or $b^2=x^2+DC^2-2x\ DC\cos\theta$ (2)

Similarly $c^2=x^2+BD^2-2x\ BD\cos(\pi-\theta)$

or $c^2=x^2+BD^2+2x\ BD\cos\theta$ (3)

Multiplying (2) by m and (3) by n and adding we get

$$mb^2+nc^2=(m+n)\ x^2+m\ DC^2+n\ BD^2$$
$$-2x\cos\theta\ (m\ DC-n\ BD)$$
$$=(m+n)\ x^2+m\ DC^2+n\ BD^2 \quad \text{from (1)}$$

Hence substituting the values of BD and DC, we get

$$mb^2+nc^2=(m+n)\ x^2+m\frac{a^2n^2}{(m+n)^2}+n\frac{a^2m^2}{(m+n)^2}$$
$$=(m+n)\ x^2+\frac{a^2mn}{(m+n)}$$

or $(m+n)\ (mb^2+nc^2)=(m+n)^2\ x^2+a^2\ mn$

or $(m+n)^2\ x^2=(m+n)\ (mb^2+nc^2)-a^2\ mn$

53 R H S $=k\sin C\cos(A-\theta)+k\sin A\cos(C+\theta)$

$=k\ [\sin C\cos A\cos\theta+\sin C\sin A\sin\theta$

$+\sin A\cos C\cos\theta-\sin A\sin C\sin\theta]$

$=k\cos\theta\ (\sin A\cos C+\cos A\sin C)$

$=k\cos\theta\sin(A+C)=k\sin B\cos\theta=b\cos\theta$

54 R H S$=(b-c)^2+4bc\sin^2 A/2=(b-c)^2+2bc\ (1-\cos A)$

$$=(b-c)^2+2bc-2bc\ \frac{b^2+c^2-a^2}{2bc}$$

$$=b^2+c^2-2bc+2bc-b^2-c^2+a^2=a^2 \quad (1)$$

Now squaring

$(b-c)\ \tan\phi=2\sqrt{bc}\sin A/2,$

we get $(b-c)^2\tan^2\phi=4bc\ \sin^2 A/2$ (2)

Putting the value of $4bc\sin^2 A/2$ from (2) in (1), we get

$a^2=(b-c)^2+(b-c)^2\tan^2\phi=(b-c)^2\sec^2\phi$

or $a=(b-c)\sec\phi$

55 We are given

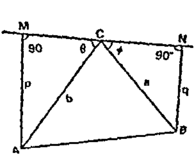
$AM=p,\ BN=q$

Let $\angle\ ACM=\theta$ and $\angle BCN=\phi$

Then $\sin\theta=p/b$ and

$\cos\theta=\sqrt{\left(1-\frac{p^2}{b^2}\right)},\sin\phi=\frac{q}{a}$

and $\cos\phi=\sqrt{\left(1-\frac{q^2}{a^2}\right)}$



$$=\frac{ab}{r_3}\left(1-\cos^2\frac{C}{2}\right)=\frac{ab}{r_3}\sin^2\frac{C}{2}$$

$$=ab\frac{s-c}{S}\quad\frac{(s-a)(s-b)}{ab}=\frac{(s-a)(s-b)(s-c)}{S}$$

Similarly each other term is equal to above

14 (a) L H S $=\left[\frac{S}{s-a}+\frac{S}{s-b}\right]+\left[\frac{S}{s-c}-\frac{S}{s}\right]$

$$=S\frac{[2s-b-a]}{(s-a)(s-b)}+S\frac{[s-s+c]}{s(s-c)}$$

$$=S\,c\;\frac{[s^2-cs+s^2-s(a+b)+ab]}{s(s-a)(s-b)(s-c)}$$

$$=S\,c\;\frac{[2s^2-s(a+b+c)+ab]}{S^2}=\frac{c}{S}[2s^2-s\,2s+ab]$$

$$=\frac{abc}{S}=4\;\frac{abc}{4S}=4R$$

(b) As above L H S

$$S\,c\left[\frac{1}{(s-a)(s-b)}-\frac{1}{s(s-c)}\right]$$

$$=S\,c\left[\frac{[s(s-c)-(s-a)(s-b)]}{S^2}\right]$$

$$=\frac{c}{S}[s(a+b-c)-ab]$$

$$=\frac{c}{2S}\left[(a+b+c)(a+b-c)-2ab\right]$$

$$=\frac{abc}{2S}\left[\frac{(a+b)-c^2-2ab}{ab}\right]$$

$$=\frac{abc}{S}\quad\frac{a^2+b^2\quad c^2}{2ab}=4R\cos C$$

15 L H S $=\left(\frac{S}{s-a}-\frac{S}{s}\right)\left(\frac{S}{s-b}-\frac{S}{s}\right)\left(\frac{S}{s-c}-\frac{S}{s}\right)$

$$=S^3\frac{abc}{s^2\,s\,(s-a)\,(s-b)\,(s-c)}=\frac{S^3\,abc}{s^2\,S^2}$$

$$=\frac{S}{s^2}\,abc=4\,\frac{S^2}{s^2}\left(\frac{abc}{4S}\right)=4r^2\,R$$

16 L H S $=\frac{s-(s-a)}{S}\quad\frac{s-(s-b)}{S}\quad\frac{s-(s-c)}{S}=\frac{abc}{S^3}$

R H S $=16\,\frac{abc}{4S}\quad\frac{s^2}{S^2\,(2s)^2}=\frac{abc}{S^3}=$ L H S

$$=\frac{a+b-c}{2\Delta}=\frac{a+b+c-2c}{2\Delta}=\frac{2s-2c}{2\Delta}$$

$$=\frac{s-c}{\Delta}=\frac{ab}{\Delta s}\cdot\frac{s(s-c)}{ab}$$

$$=\frac{ab}{\Delta s}\cos\tfrac{1}{2}C=\frac{2ab}{(a+b+c)\,\Delta}\cos^2\tfrac{1}{2}C,$$

58 Since α, β, γ are the lengths of altitudes of $\triangle ABC$, we have

$$\tfrac{1}{2}\alpha\, a=\tfrac{1}{2}\beta\, b=\tfrac{1}{2}\gamma\, c=\Delta \qquad (1)$$

Also $\frac{1}{2}ab\sin C=\frac{1}{2}bc\sin A=\frac{1}{2}ca\sin B=\Delta$ (2)

Hence from (1), we have

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{a^2}{4\Delta^2}+\frac{b^2}{4\Delta^2}+\frac{c^2}{4\Delta^2}=\frac{a^2+b^2+c^2}{4\Delta^2} \qquad (3)$$

And $\frac{1}{\Delta}(\cot A+\cot B+\cot C)=\frac{1}{\Delta}\left[\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}\right]$

$$=\frac{b^2+c^2-a^2}{\Delta\, 2bc\sin A}+\frac{c^2+a^2-b^2}{\Delta\, 2ca\sin B}+\frac{a^2+b^2-c^2}{\Delta\, 2ab\sin C}$$

$$=\frac{b^2+c^2-a^2}{4\Delta^2}+\frac{c^2+a^2-b^2}{4\Delta^2}+\frac{a^2+b^2-c^2}{4\Delta^2} \text{ from (2)}$$

$$=\frac{a^2+b^2+c^2}{4\Delta^2} \qquad (4)$$

Hence from (3) and (4), we get

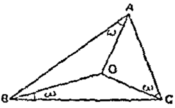
$$\alpha^{-2}+\beta^{-2}+\gamma^{-2}=(\cot A+\cot B+\cot C)/\Delta$$

59 $\angle OCB=C-\omega$ and

$\angle BOC=180^\circ-\omega-(C-\omega)=180^\circ-C$

Similarly $\angle AOB=180^\circ-B$

Now from $\triangle OAB$, we have



$$\frac{OB}{\sin\omega}=\frac{AB}{\sin(180^\circ-B)}=\frac{c}{\sin B}$$

so that $OB=\dfrac{c\sin\omega}{\sin B}$ (1)

Again from $\triangle OBC$, we get

$$\frac{OB}{\sin(C-\omega)}=\frac{BC}{\sin(180^\circ-C)}=\frac{a}{\sin C}$$

$\therefore \quad OB=\dfrac{a\sin(C-\omega)}{\sin C}$ (2)

From (1) and (2) we get

$$\frac{c\sin\omega}{\sin B}=\frac{a\sin(C-\omega)}{\sin C}$$

or $k\sin C\sin\omega\sin C=k\sin A\sin B\sin(C-\omega)$

or $\sin C\sin\omega\sin(A+B)=\sin A\sin B\sin(C-\omega)$

$$=2R\,(1+4\sin A/2\sin B/2\sin C/2)\ \text{(prove)}$$

$$=2R\,(1+r/R)=2\,(R+r)$$ Q 4 (i) P 53

26 (a) Now $(b+c)\tan A/2=2R\,(\sin B+\sin C)\tan A/2$

$$=2R\ 2\sin\frac{B+C}{2}\cos\frac{B-C}{2}\ \frac{\sin A/2}{\cos A/2}$$

$$=2R\ 2\cos\frac{B-C}{2}\cos\frac{B+C}{2}=2R\,(\cos B+\cos C)$$

$$\text{L H S}=4R\,(\cos A+\cos B+\cos C)$$

$$=4R\left(1+\frac{r}{R}\right)=4\,(R+r)\ \text{as in Q 25}$$

(b) $\text{L H S}=\frac{1}{2}\left[1+\cos A+1+\cos B+1+\cos C\right]=\frac{1}{2}\left[3+1+\frac{r}{R}\right]$

$$=2+\frac{r}{2R}$$

27 We know that

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}$$

$$(r+r_1)\tan\frac{B-C}{2}=\left(\frac{S}{s}+\frac{S}{s-a}\right)\ \frac{b-c}{b+c}\ \sqrt{\left\{\frac{s\,(s-a)}{(s-b)\,(s-c)}\right\}}$$

$$=\frac{S\,(s-a+s)}{s\,(s-a)}\ \frac{b-c}{b+c}\ \frac{\sqrt{s}\,\{(s-a)\}}{\sqrt{\{(s-b)\,(s-c)\}}}$$

$$=\frac{S}{\sqrt{\{s\,(s-a)\,(s-b)\,(s-c)\}}}\times(a+b+c-a)\ \frac{b-c}{b+c}$$

$$=\frac{S}{S}\,(b+c)\ \frac{b-c}{b+c}=b-c$$

$$\text{L H S}=b-c+c-a+a-b=0$$

28 $\text{L H S}=(b-c)\ \frac{(s-a)}{S}+(c-a)\ \frac{(s-b)}{S}+(a-b)\ \frac{(s-c)}{S}$

$$=\frac{s}{S}[b-c+c-a+a-b]$$

$$-\frac{1}{S}[a\,(b-c)+b\,(c-a)+c\,(a-b)]=0$$

29 $1+\cos A=2\cos^2\frac{A}{2}=2\ \frac{s\,(s-a)}{bc}$

$$\frac{r_2+r_3}{1+\cos A}=\left(\frac{S}{s-b}+\frac{S}{s-c}\right)\ \frac{bc}{2s\,(s-a)}$$

$$=\frac{S\,(s-c+s-b)}{2s\,(s-a)\,(s-b)\,(s-c)}\ bc=\frac{S\,abc}{2S^2}=\frac{abc}{2S}$$

Similarly each may be proved to be equal to $abc/2S$

or $(\cos^2 A+\cos^2 B-2\cos A\cos B)+(\sin^2 A+\sin^2 B$
$-2\sin A\sin B)+2\sin A\sin B\,(1-\sin C)=0$

or $(\cos A-\cos B)^2+(\sin A-\sin B)^2$
$+2\sin A\sin B\,(1-\sin C)=0$ (1)

Now in a triangle $\sin A>0$, $\sin B>0$ and $1-\sin C\geqslant 0$ and hence result (1) will hold good if

$\cos A-\cos B=0$, $\sin A-\sin B=0$ and $1-\sin C=0$

$A=B$ and $C=90$ $A=B=45°$

$a:b:c=\sin A:\sin B:\sin C=\frac{1}{\sqrt{2}}:\frac{1}{\sqrt{2}}:1$

or $a:b:c=1:1:\sqrt{2}$

64 Putting $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$ etc in the given relation, we get

$$\frac{b^2+c^2-a^2}{2bc}+\frac{c^2+a^2-b^2}{2ca}+\frac{a^2+b^2-c^2}{2ab}-\frac{3}{2}=0$$

or $a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)=3abc$

or $a(b^2+c^2)+b(c^2+a^2)+c(a^2+b^2)=a^3+b^3+c^3+3abc$

Now subtract 6 abc from each side

$a(b-c)^2+b(c-a)^2+c(a-b)^2=a^3+b^3+c^3-3abc$
$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca)$

Multiply both sides by 2

$2a(b-c)^2+2b(c-a)^2+2c(a-b)^2$
$=(a+b+c)[(b-c)^2+(c-a)^2+(a-b)^2]$

or $(b-c)^2(b+c-a)+(c-a)^2(c+a-b)+(a-b)^2(a+b-c)$
$=0$ (1)

In a triangle $b+c-a>0$ etc and hence (1) will hold good if each factor is zero Now $(b-c)^2(b+c-a)=0$ gives $b-c=0$ since $b+c-a\neq 0$ Similarly $c-a=0$ and $a-b=0$ $a=b=c$

Hence the triangle is equilateral

$$P_1 = \frac{2R\sin A}{2\cos A/2} = 2R\sin A/2 \qquad a=2R\sin A$$

$$P_1P_2P_3 = 8R^3\sin A/2\sin B/2\sin C/2$$
$$=2P^2\ 4R\sin A/2\sin B/2\sin C/2 = 2R^2 r,$$

41 (a) We have $\left(1-\frac{s-b}{s-a}\right)\left(1-\frac{s-c}{s-a}\right)=2, \qquad r_1=\frac{s}{s-a}$

or $\frac{b-a}{s-a}\cdot\frac{c-a}{s-a}=2$

or $2(b-a)(c-a)=4(s-a)^2=(2s-2a)^2$

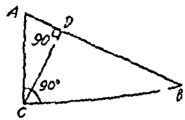
$2(bc-ac-ab+a^2)=(b+c-a)^2$

or $2bc-2ac-2ab+2a^2=b^2+c^2+a^2+2bc-2ca-2ab$

or $a^2=b^2+c^2$

the triangle is right angled

(b) Let x, y and r be radii of the circles inscribed into the Δ's ACD, BCD and ABC respectively Then from similar Δ's ABC and ACD, we get



$$\frac{r}{x}=\frac{AB}{AC}=\frac{c}{a}, \text{ whence } b=\frac{cx}{r}$$

Similarly from similar Δ's ABC and BCD, we get

$$\frac{r}{y}=\frac{AB}{BC}=\frac{c}{b}, \text{ whence } a=\frac{cy}{r}$$

Now $c^2=AB^2=a^2+b^2=\frac{c^2y^2}{r^2}+\frac{c^2x^2}{r^2}=\frac{c^2(x^2+y^2)}{r^2}$

This gives $r=\sqrt{(x^2+y^2)}$

[Note that in similar triangles radii of inscribed circles are proportional to corresponding sides]

42 We know that $a=2R\sin A$ etc

We are given that $8R^2=4R^2(\sin^2 A+\sin^2 B+\sin^2 C)$

or $\sin^2 A+\sin^2 B+\sin^2 C=2$

or $1-\cos^2 A+1-\cos^2 B+\sin^2 C=2$

or $(\cos^2 A-\sin^2 C)+\cos^2 B=0$

or $\cos(A+C)\cos(A-C)+\cos^2 B=0$

or $-\cos B[\cos(A-C)-\cos B]=0$

$[\because \cos B=-\cos(A+C)]$

or $\cos B[\cos(A-C)+\cos(A+C)]=0$

$$r_1 = a\,\frac{\cos\frac{B}{2}\cos\frac{C}{2}}{\cos\frac{A}{2}} = 4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} \qquad (8)$$

$$r_2 = b\,\frac{\cos\frac{C}{2}\cos\frac{A}{2}}{\cos\frac{B}{2}} = 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} \qquad (9)$$

$$r_3 = c\,\frac{\cos\frac{A}{2}\cos\frac{B}{2}}{\cos\frac{C}{2}} = 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2} \qquad (10)$$

§ 2 **Orthocentre** Let AL, BM and CN be the perpendiculars form A, B, C on opposite sides in a $\triangle ABC$ Then from geometry, we know that these perpendiculars are concurrent Their point of intersection P is called the orthocentre of the triangle ABC The $\triangle LMN$ is called the pedal triangle of $\triangle ABC$

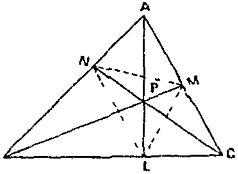


Fig 1

The distance of the Ortho centre from the vertices

From $\triangle APM$, we have

$$AP = AM \sec PAM = AM \sec LAC$$
$$= AM \sec(90^\circ - C) = AM \operatorname{cosec} C$$
$$= AB\cos A \operatorname{cosec} C = \frac{c\cos A}{\sin C}$$
$$= \frac{2R\sin C\cos A}{\sin C} \qquad [\because c = 2R\sin C]$$
$$= 2R\cos A$$

Thus $AP = 2R\cos A$

Similarly $BP = 2R\cos B$ and $CP = 2R\cos C$

Again $PL = BL\tan LBP = BL\tan(90^\circ - C)$

$$= AB\cos B\cot C = c\cos B\cot C$$
$$= 2R\sin C\cos B\,\frac{\cos C}{\sin C} = 2R\cos B\cos C$$

Similarly $PM = 2R\cos C\cos A$

and $PN = 2R\cos A\cos B$

46 We have

$$r + R = \frac{a}{2}\cot\frac{\pi}{n} + \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$$

$$= \frac{a}{2}\left[\frac{\cos(\pi/n)+1}{\sin(\pi/n)}\right] = \frac{a}{2}\,\frac{2\cos^2(\pi/2n)}{2\sin(\pi/2n)\cos(\pi/2n)}$$

$$= \frac{a}{2}\cot\frac{\pi}{2n}$$

47 Let ρ be the radius of the circle If A_1, A_2, A_3 denote the areas of the three polygons (i e an inscribed polygon of $2n$ sides, inscribed polygon of n sides and circumscribed polygon of n sides) then

$$A_1 = n\rho^2\sin\frac{\pi}{n},\ A_2 = \frac{n}{2}\rho^2\sin\frac{2\pi}{n},\ \text{and } A_3 = n\rho^2\tan\frac{\pi}{n},$$

respectively

Now $A_1^2 = n^2\rho^4\sin^2\frac{\pi}{n} = n\rho^2\tan\frac{\pi}{n}\cdot n\rho^2\sin\frac{\pi}{n}\cos\frac{\pi}{n}$

$$= n\rho^2\tan\frac{\pi}{n}\cdot\frac{n}{2}\rho^2\sin\frac{2\pi}{n} = A_3A_2$$

$$A_1 = \sqrt{(A_2A_3)},$$

that is, A_1 is mean proportional between A_2 and A_3.

48 Let p be the perimeter of both the polygons Then each side of first polygon is of length $\frac{p}{n}$ and that of the second polygon is $\frac{p}{2n}$ If A_1 A_2 denote their areas, then by § 8,

$$A_1 = \frac{1}{4}n\,\frac{p^2}{n^2}\cot\frac{\pi}{n} \quad\text{and}\quad A_2 = \frac{1}{4}\,2n\,\frac{p^2}{4n^2}\cot\frac{\pi}{2n}$$

$$\frac{A_1}{A_2} = \frac{2\cot\frac{\pi}{n}}{\cot\left(\frac{\pi}{2n}\right)} = \frac{2\cos\frac{\pi}{n}\sin\frac{\pi}{2n}}{\sin\frac{\pi}{n}\cos\frac{\pi}{2n}} = \frac{2\cos\frac{\pi}{n}\sin\frac{\pi}{2n}}{2\sin\frac{\pi}{2n}\cos\frac{\pi}{2n}\cos\frac{\pi}{2n}}$$

$$= \frac{2\cos\frac{\pi}{n}}{2\cos^2\left(\frac{\pi}{2n}\right)} = \frac{2\cos\frac{\pi}{n}}{1+\cos\left(\frac{\pi}{n}\right)}$$

49 Let A B, C be the centres of the three circles respectively, and ID, IE and IF be the three common tangents meeting at I Join AB BC and CA Since ID, IE and IF are perpendicular to BC, CA and AB respectively and are equal to one

Then the area of quadrilateral

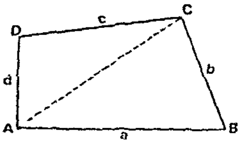


fig 4

$= \text{Area of } \triangle ABC$

$\quad + \text{Area of } \triangle ACD$

$\frac{1}{2} ab \sin B + \frac{1}{2} cd \sin D$

$= \frac{1}{2} ab \sin B$

$\quad + \frac{1}{2} cd \sin (180^\circ - B) \qquad [\because \angle B + \angle D = 180^\circ]$

$= \frac{1}{2} (ab + cd) \sin B \qquad (1)$

But $\quad a^2 + b^2 - 2ab \cos B = AC^2 = c^2 + d^2 - 2cd \cos D$

$\qquad = c^2 + d^2 + 2cd \cos B$

or $\quad a^2 + b^2 - c^2 - d^2 = 2 (ab + cd) \cos B$

Hence $(a^2 + b^2 - c^2 - d^2)^2 = 4 (ab + cd)^2 \cos^2 B$

$\qquad = 4 (ab + cd)^2 (1 - \sin^2 B)$

or $\quad 4 (ab + cd)^2 \sin^2 B = 4 (ab + cd)^2 - (a^2 + b^2 - c^2 - d^2)^2$

$\qquad = (2ab + 2cd + a^2 + b^2 - c^2 - d^2)$

$\qquad \times (2ab + 2cd - a^2 - b^2 + c^2 + d^2)$

$\qquad = [(a+b)^2 - (c-d)^2] [(c+d)^2 - (a-b)^2]$

$\qquad = (a + b + c - d) (a + b - c + d)$

$\qquad \times (c + d + a - b) (c + d - a + b)$

$\qquad = (2s - 2d) (2s - 2c) (2s - 2b) (2s - 2a)$

where $a + b + c + d = 2s$

$(ab + cd) \sin B = 2 \{(s-a)(s-b)(s-c)(s-d)\}^{1/2} \qquad (2)$

Hence from (1) and (2) we get

The area of quadrilateral

$= \sqrt{[(s-a)(s-b)(s-c)(s-d)]}$

Note Remember $\cos B = \frac{a^2 + b^2 - c^2 - d^2}{2 (ab + cd)}$

§ 6 **Ptolemy's Theorem**

If $ABCD$ is a cyclic quadrilateral then

$AC \cdot BD = AB \cdot CD + BC \cdot AD$

Proof We have

$AC^2 = a^2 + b^2 - 2ab \cos B \qquad$ [See fig of § 5]

But $\quad \cos B = \frac{(a^2 + b^2 - c^2 - d^2)}{2 (ab + cd)} \qquad$ [See § 5]

$AC^2 = a^2 + b^2 - \frac{ab (a^2 + b^2 - c^2 - d^2)}{(ab + cd)}$

7

Solution of Triangles

In a triangle the three sides and the three angles are often called the elements of the triangle When three elements are given such that a triangle is formed with them, then the process of calculating the remaining three elements is called the Solution of the Triangles The formulae of the preceding chapter will be used in the solution of triangles and the students are therefore advised to commit them to memory

We shall use the symbols $L \sin A$ for $10+\log \sin A$, $L \tan A$ for $10+\log \tan A$ etc

Problem Set

On Right Angled Triangle

1 Find the acute angles of a right angled triangle whose hypotenuse is four times as long as the perpendicular drawn to it from the opposite angles

2 The length of perpendicular from one angle of a triangle upon the base is 3 c m and the lengths of the sides containing this angle are 4 c m and 5 c m, Find the angles, having given $\log 2 = 3010300$ $\log 3 = 4771213$, $L \sin 36°52' = 9\ 7781186$ diff for $1' = 1684$, and $L \sin 48°\ 35' = 9\ 8750142$, diff for $1' = 1115$

3 In a $\triangle ABC$ $\angle C=90°$ $a=3$ $b=4$ and D is a point on AB so that $\angle BCD=30°$ Find the length CD (I I T 1974)

When Three sides are given

4 The sides of a triangle are x^2+x+1, $2x+1$, and x^2-1, prove that the greatest angle is 120°

5 The sides of a triangle are a b, and $\sqrt{(a^2+ab+b^2)}$ find the greatest angle

6 The sides of a triangle are in the ratio 2 $\sqrt{6}$ $(\sqrt{3}+1)$, find its angles

Now $\angle BOC = \frac{2\pi}{n}$ radians

Hence $\angle BON = \angle CON = \frac{1}{2}\angle BOC = \frac{\pi}{n}$

If a be the length of a side of the polygon, then

$a = BC = 2BN = 2R \sin BON = 2R \sin \frac{\pi}{n}$

or $R = \frac{a}{2} \operatorname{cosec} \frac{\pi}{n}$

Again $a = 2BN = 2ON \tan BON = 2r \tan \frac{\pi}{n}$

$$r = \frac{a}{2} \cot \frac{\pi}{n}$$

§ 8 Area of a regular polygon

Area of the polygon is n times the area of the $\triangle BOC$

Hence the area of the polygon

$$= n \times \frac{1}{2} ON\ BC = n\ ON\ BN$$

$$= n\ BN \cot \frac{\pi}{n}\ BN = n \left(\frac{a}{2}\right)^2 \cot \frac{\pi}{n} \left[\quad BN = \frac{a}{2}\right]$$

$$= \frac{1}{4} na^2 \cot \frac{\pi}{n} \tag{1}$$

Above is an expression for the area in terms of the side

Also the area

$$= n\ ON\ BN = n\ OB \cos \frac{\pi}{n}\ OB \sin \frac{\pi}{n}$$

$$= R^2 \sin \frac{\pi}{n} \cos \frac{\pi}{n} = \frac{1}{2}\ nR^2 \sin \frac{2\pi}{n} \tag{2}$$

(Roorkee 82)

Again the area

$$= n\ ON\ BN = n\ ON\ ON \tan \frac{\pi}{n} = nr^2 \tan \frac{\pi}{n} \tag{3}$$

The formulae (2) and (3) give the area in terms of the radius of the circumscribed and inscribed circles

Problem Set

1 If R is the radius of circum circle of $\triangle ABC$, prove the following formulae

(i) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$ (ii) $R = \frac{abc}{4S}$

2

2

2

8

15 The sides of triangle are 9 and 3, and the difference of the angles opposite to them is 90° Find the base and angles having given
log 2= 3010300 log 3= 4771213,
log 75894=4 8802074, log 75895=4 8802132,
L tan 26° 33 =9 6986847, L tan 26° 34′=9 6990006

16 If $\tan\phi=\frac{a-b}{a+b}\cot\frac{C}{2}$, prove that $c=(a+b)\frac{\sin C/2}{\cos\phi}$ (Roorkee 77)

Let $a=3$ $b=1$, and $C=53°\ 7'\ 48'$, find c without getting A and B, given log 2= 3010300 log 25298=4 4030862, log 25299=4 4031034, L cos 26° 33′ 54″=9 9515452, and L tan 26° 33′ 54″=9 6989700

Two angles and one side given

17 Two angles of a triangle are 41° 13′ 22″ and 71° 19′ 5″ and the side opposite to the first angle is 55, find the side opposite the latter angle, given
log 55=1 7403627, log 79063=4 8979775,
L sin 41° 13′ 22″=9 8188779 L sin 71° 19′ 5″=9·9764927

18 From each of two ships, one kilometer apart, the angle is observed which is subtended by the other ship and a beacon on shore these angles are found to be 52° 25′ 15″ and 75° 9′ 30 respectively Given L sin 75° 9 30″=9 9852635, L sin 52° 25′ 15 =9 8990055, log 1 1197= 0862530 and log 1 2198= 0862886 Find the distance of the beacon from each of the ships

19 In a $\triangle ABC$ $A=38°\ 20'$, $B=45°$, and $b=64$ cm Find c having given log 2= 30103, L sin 83° 20 =9 99705 and log 089896=$\bar{2}$ 95374 (IIT 71)

Two sides and one angle opposite to one of these sides

20 In the ambiguous case, given a b and A, prove that the difference between the two values of c is $2\sqrt{(a^2-b^2\sin^2 A)}$

21 Solve the $\triangle ABC$ when $A=15°$ $a=4$ and $b=4(\sqrt{3}+1)$

22 In the ambiguous case, if a and A are given and c_1 c_2 are the two values of third side, prove that
(i) $c_1+c_2=2b\cos A$ and $c_1c_2=b^2-a^2$
(ii) $c_1^2-2c_1c_2\cos 2A+c_2^2=4a^2\cos^2 A$

21 $S = Rr (\sin A + \sin B + \sin C)$

22 $4R \sin A \sin B \sin C = a \cos A + b \cos B + c \cos C$

23 $S = \{2abc/(a+b+c)\} \cos A/2 \cos B/2 \cos C/2$

24 (a) $S = 4Rr \cos A/2 \cos B/2 \cos C/2$

(b) $s = 4R \cos A/2 \cos B/2 \cos C/2$

25 (a) $\cos A + \cos B + \cos C = 1 + r/R$

(b) $a \cot A + b \cot B + c \cot C = 2(R+r)$

26 (a) $(b+c) \tan A/2 + (c+a) \tan B/2 + (a+b) \tan C/2$
$= 4R (\cos A + \cos B + \cos C) = 4(R+r)$

(b) $\cos^2 A/2 + \cos^2 B/2 + \cos^2 C/2 = 2 + \frac{r}{2R}$

27 $(r+r_1) \tan (B-C)/2 + (r+r_2) \tan (C-A)/2 + (r+r_3) \tan (A-B)/2 = 0$

28 $\frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3} = 0$

29 $(r_2+r_3)/(1+\cos A) = (r_3+r_1)/(1+\cos B) = (r_1+r_2)/(1+\cos C)$

30 $\frac{r_1}{bc} + \frac{r_2}{ca} + \frac{r_3}{ab} = \frac{1}{r} - \frac{1}{2R}$

31 $r^2 + r_1^2 + r_2^2 + r_3^2 = 16R^2 - a^2 - b^2 - c^2$

32 If α, β, γ are the distances of the vertices of a triangle from the corresponding points of contact with the in circle, prove that $$r^2 = \alpha\beta\gamma/(\alpha+\beta+\gamma)$$

33 If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(i) $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = \frac{1}{r}$, (ii) $p_1 p_2 p_3 = \frac{a^2b^2c^2}{8R^3}$

(iii) $\frac{\cos A}{p_1} + \frac{\cos B}{p_2} + \frac{\cos C}{p_3} = \frac{1}{R}$

34 If A, A_1, A_2 and A_3 are respectively the areas of the inscribed and escribed circles prove that
$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}$$

35 If x, y, z are respectively perpendiculars from the circumcentre to the sides of the $\triangle ABC$ prove that
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{4xyz}$$

36 If x, y, z are respectively the distances of the vertices of the $\triangle ABC$ from its ortho-centre prove that
$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{abc}{xyz}$$

Substituting these values of AB and CD in (1), we get

$$AC \sec A = 4AC \sin A$$

or $4 \sin A \cos A = 1$ or $2 \sin 2A = 1$

Thus $\sin 2A = \frac{1}{2}$

$2A = 30°$ or $150°$

That is, $A = 15°$ or $75°$

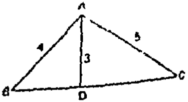
$B = 75°$ or $15°$

2 Let $AD \perp BC$

where $AD = 3$ cm,

$AB = 4$ cm

and $AC = 5$ cm



We first find B We have

$$\sin B = \frac{3}{4} = \frac{3}{2^2}$$

$\therefore$ $L \sin B = 10 + \log 3 - 2 \log 2$

$= 10 + 4771213 - 2 \times 3010300 = 9\ 8750613$

We thus have

$L \sin B = 9\ 8750613$

$L \sin 48°35' = 9\ 8750142$

Diff $= 000471$

1115 is the diff on 60

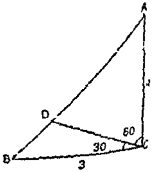
471 is the diff on $\frac{60}{1115} \times 171 = 25$ nearly

Hence $B = 48°\ 35'\ 25''$

Similarly C can be found

Ans $C = 36°\ 52'\ 12''$,

$A = 94°\ 32'\ 23$



3 $AB = \sqrt{(3^2 + 4^2)} = 5$

$$\sin A = \frac{3}{5}, \ \sin B = \frac{4}{5}$$

From $\triangle BCD$, $\frac{BD}{\sin 30°} = \frac{CD}{\sin B}$

$$BD = \frac{\sin 30°}{\sin B} CD = \frac{1}{2} \times \frac{5}{4} CD = \frac{5}{8} CD$$

and from $\triangle ACD$, $\frac{AD}{\sin 60°} = \frac{CD}{\sin A}$

so that $AD = \frac{\sin 60°}{\sin A} CD = \frac{\sqrt{3}}{2} \times \frac{5}{3} CD = \frac{5\sqrt{3}}{6} CD$

But $BD + AD = AB$

$$\frac{5}{8} CD + \frac{5\sqrt{3}}{6} CD = 5$$

48 If the number of sides of two regular polygons having the same perimeter be n and $2n$ respectively, prove that their areas are in the ratio

$$2 \cos (\pi/n) \quad (1+\cos \pi/n)$$

49 Three circles whose radii are a, b, c, touch one another externally, and the tangents at their points of contact meet in a point prove that the distance of this point from either of their points of contact is

$$\left(\frac{abc}{a+b+c}\right)^{1/2}$$

Solutions to Problem Set

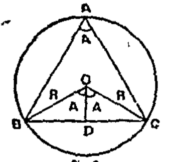


Fig 6

1 (i) Describe a circle with centre O and passing through the vertices of the $\triangle ABC$ Clearly $\triangle BOD = COD$, and we know that the angle subtented at the centre is double the angle subtended on the cirumference i e

$$\angle BOC = 2A$$
$$\angle BOD = \angle COD = A$$

Now if OD is perp to side BC then

$$BD = DC = \frac{a}{2} \text{ and } \frac{BD}{OB} = \sin A$$

or $\frac{a}{2R} = \sin A$, $\qquad \frac{a}{\sin A} = 2R$

Hence by sine formula, $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

(ii) We have proved in (i) that

$$a = 2R \sin A = 2R\ 2 \sin A/2 \cos A/2$$

or $\quad a = 2R \ \frac{2}{bc}\sqrt{[s(s-a)(s-b)(s-c)]} = \frac{4RS}{bc}$,

$$R = \frac{abc}{4S}$$

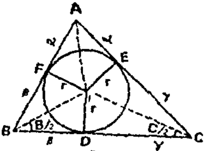


Fig 7

2 (i) The bisectors of the angles of $\triangle ABC$ meet in I Draw ID, IE IF perpendiculars to the sides from I By Geometry,

$$ID = IE = IF = r$$
$$\triangle ABC = \triangle IBC + \triangle ICA + \triangle IAB$$

or $\quad S = \frac{1}{2}ar + \frac{1}{2}br + \frac{1}{2}cr$

or $\quad S = \frac{1}{2}r(a+b+c) = \frac{1}{2}r\, 2s = rs$,

$$r = S/s$$

We thus have

$L \tan C/2 = 10\ 1109244$, $L \tan 52^\circ\ 15' = 10\ 1111004$

$L \tan 52^\circ\ 14' = 10\ 1108395$ $\quad L \tan 52^\circ\ 14' = 10\ 1108359$

Diff $= 0000849$ $\quad$ Diff $= 0002609$

$$\text{Angular diff} = \frac{849 \times 60}{2609} = \frac{50940}{2609} = 19\ 5''$$

Hence $C/2 = 52^\circ\ 14'\ 19\ 5''$, $\quad C = 104^\circ\ 28'\ 39''$

8 Let $a = 32$, $b = 40$, $c = 66$

$$\text{Then } s = \frac{32+40+66}{2} = 69$$

$$\cot\frac{C}{2} = \sqrt{\left\{\frac{s(s-c)}{(s-b)(s-a)}\right\}} = \sqrt{\left\{\frac{69 \times 3}{29 \times 37}\right\}}$$

Hence $L \cot C/2 = 10 + \frac{1}{2}[\log 69 + \log 3 - \log 29 - \log 37]$

$= 10 + \frac{1}{2}[1\ 83885 + 47712 - 1\ 46240 - 1\ 56820]$

$= 10 + \frac{1}{2}[2\ 31597 - 3\ 03060] = 10 - \frac{1}{2} \times 71463$

$= 9\ 64269$

$L \cot C/2 = 9\ 64269$ $\quad L \cot 66^\circ\ 10 = 9\ 64517$

$L \cot 66^\circ\ 20 = 9\ 64175$ $\quad L \cot 66^\circ\ 20 = 9\ 64175$

Diff 00094 $\quad$ Diff $10 = 00342$

$$\text{Angular difference} = \frac{10}{342} \times 94 = 2\ 75' = 2\ 45$$

Since as angle increases, cotangent decreases, we have

$\frac{1}{2}C = 66^\circ\ 20 - 2\ 45 = 66^\circ\ 17'\ 15''$, $\quad C = 132^\circ\ 34'\ 30''$

9 Let $a = 5$, $b = 8$, $c = 11$, $\quad s = \dfrac{5+8+11}{2} = 12$

$$\sin\tfrac{1}{2}C = \sqrt{\left\{\frac{(s-a)(s-b)}{ab}\right\}} = \sqrt{\left\{\frac{(12-5)(12-8)}{5 \times 8}\right\}} = \sqrt{\frac{7}{10}}$$

$L \sin\frac{1}{2}C = 10 + \frac{1}{2}[\log 7 - \log 10] = 10 + \frac{1}{2}[8450980 - 1]$

$= 10 - 0774510 = 9\ 9220490$

Thus we have

$L \sin\frac{1}{2}C = 9\ 9225490$ $\quad L \sin 56^\circ\ 48 = 9\ 9226032$

$L \sin 56^\circ\ 47 = 9\ 9225205$ $\quad L \sin 56^\circ\ 47' = 9\ 9225205$

Diff $= 0000285$ $\quad$ Diff $1 = 0000827$

$$\text{Angular difference } \frac{60 \times 285}{827} = 20\ 6''$$

Hence $C/2 = 56^\circ\ 47'\ 20\ 6''$, $\quad C = 113^\circ\ 34'\ 41''$

10 Since A B C are in A P We have $2B = A + C$

But $A + B + C = 180^\circ$ so that $3B = 180^\circ$ Then $B = 60^\circ$

Let $a = 16$ and $c = 24$

Similarly, $r_2 = s \tan B/2$, and $r_3 = s \tan C/2$

(iii) $a = BD + DC = r_1 \cot \frac{180° - B}{2} + r_1 \cot \frac{180° - C}{2}$

or $a = r_1 (\tan B/2 + \tan C/2)$

or $2R \sin A = r_1 \frac{(\sin B/2 \cos C/2 + \cos B/2 \sin C/2)}{\cos B/2 \cos C/2}$

or $2R\, 2 \sin \frac{A}{2} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} = r_1 \sin \frac{B+C}{2}$

or $4R \sin A/2 \cos B/2 \cos C/2 = r_1$

Similarly, $4R \sin B/2 \cos C/2 \cos A/2 = r_2$,

and $4R \sin C/2 \cos A/2 \cos B/2 = r_3$

4 $\text{L H S} = \sqrt{\left(\frac{S}{s} \frac{S}{s-a} \frac{S}{s-b} \frac{S}{s-c}\right)}$

$= \sqrt{\left(\frac{S^4}{S^2}\right)} = S \quad [\quad S = \sqrt{\{s(s-a)(s-b)(s-c)\}}]$

Again $\text{R H S} = r^2 \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2}$

$= \frac{S^2}{s^2} \sqrt{\left(\frac{s(s-a)}{(s-b)(s-c)} \frac{s(s-b)}{(s-c)(s-a)} \frac{s(s-c)}{(s-a)(s-b)}\right)}$

$= \frac{S^2}{s^2} \sqrt{\left[\frac{s^4}{s(s-a)(s-b)(s-c)}\right]} = \frac{S^2}{s^2} \frac{s^2}{S} = S = \text{L H S}$

5 Proceed as in Q 4

6 $r_1 = s \tan A/2 \quad r_1 \cot A/2 = s$ etc

7 to 9 Do yourself

10 $\text{L H S} = \frac{1}{a}\left[\frac{S}{s-a} - \frac{S}{s}\right] + \frac{1}{b}\left[\frac{S}{s-b} - \frac{S}{s}\right]$

$= \frac{S}{a} \frac{s-s+a}{s(s-a)} + \frac{S}{b} \frac{s-s+b}{s(s-b)}$

$= \frac{S}{s}\left[\frac{1}{s-a} + \frac{1}{s-b}\right] = \frac{S}{s} \frac{2s-a-b}{(s-a)(s-b)}$

$= \frac{Sc}{s(s-a)(s-b)} \frac{s-c}{s-c} = \frac{Sc(s-c)}{S^2}$

$= \frac{c(s-c)}{S} = \frac{c}{r_3}$

11 and 12 Do yourself

13 $\frac{ab - r_1 r_2}{r_3} = \frac{1}{r_3}\left(ab - \frac{S}{s-a} \frac{S}{s-b}\right) = \frac{ab}{r_3}\left\{1 - \frac{s(s-c)}{ab}\right\}$

$[\quad S^2 = s(s-a)(s-b)(s-c)]$

$\frac{A-B}{2}=14^\circ\ 20'\ 40''$ and $\frac{A-B}{2}=90^\circ-26^\circ\ 3'=63^\circ\ 57'$

Adding $A=78^\circ\ 17'\ 40''$ Subtracting $B=42^\circ\ 36'\ 20''$

12 We have

$a^2=b^2+c^2-2bc\cos A$

$=(b^2+c^2)(\cos^2 A/2+\sin^2 A/2)-2bc(\cos^2 A/2-\sin^2 A/2)$

$=(b-c)^2\cos^2 A/2+(b+c)^2\sin^2 A/2$

$=(b+c)^2\sin^2\frac{A}{2}\left[1+\left(\frac{b-c}{b+c}\right)^2\cot^2\frac{A}{2}\right]$

$=(b+c)^2\sin^2(A/2)\,[1+\cot^2\theta]=(b+c)^2\sin^2(A/2)\operatorname{cosec}^2\theta$ (1)

or $a^2=(b+c)^2\sin^2(A/2)\ldots$ (2)

where $\cot\theta=\frac{b-c}{b+c}\cot\frac{A}{2}$

From (2) $\cot\theta=\frac{6-3}{6+3}\cot 18^\circ\ 26'\ 6''$

$L\cot\theta=L\cot 18^\circ\ 26'\ 6''-\log 3=10.47712-.47712=10$

Hence $\log\cot\theta=0$ that is $\cot\theta=1$, $\theta=45^\circ$

Now from (1) we have

$a=(6+3)\sin 18^\circ\ 26'\ 6''\ \operatorname{cosec}45^\circ=3^2\sin 18^\circ\ 26'\ 6''\times\sqrt{2}$

Hence $\log a=2\log 3+\frac{1}{2}\log 2+L\sin 18^\circ\ 26'\ 6''-10$

$=2\times.47712+\frac{1}{2}\times.30103+9.5-10=.60476$

Thus $\log a=.60476$

$\log 4.02=.60423$

Diff $=.00053$

$\log 4.03=.60531$

$\log 4.02=.60423$

Diff for $.01=.00108$

Corresponding difference in $a=\frac{53}{108}\times.01=.0049$

Hence $a=4.02+.0049=4.0249$

13 $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{\frac{5}{2}-2}{\frac{5}{2}+2}\cot 11^\circ\ 10'=\frac{\cot 11^\circ\ 10'}{3^2}$

$L\tan(B-C)/2=L\cot 11^\circ\ 10'-2\log 3$

$=10.60465+2\times.47712$

$=10.70465-.95424=9.75041$

Difference for $10''=9.75043-9.75038=.00005$

And $L\tan\frac{B-C}{2}-L\tan 29^\circ\ 22'\ 20''=9.75041-9.7538=.00003$

Angular difference $=\left(\frac{10}{5}\times 3\right)''=6''$

Hence $\frac{B-C}{2}=29^\circ\ 22'\ 26''$

17 L H S $= \dfrac{(s-a)^2+(s-b)^2+(s-c)^2+s^2}{S^2}$

$= \dfrac{4s^2-2s(a+b+c)+(a^2+b^2+c^2)}{S^2} = \dfrac{a^2+b^2+c^2}{S^2}$

18 L H S $= \dfrac{(s-a)+(s-b)}{S} \cdot \dfrac{(s-b)+(s-c)}{S} \cdot \dfrac{(s-c)+(s-a)}{S}$

$= \dfrac{abc}{S^3}$, $\quad 2s-(a+b)=c$ etc

R H S $= \dfrac{64}{a^2b^2c^2}\left(\dfrac{abc}{4S}\right)^3 = \dfrac{abc}{S^3} =$ L H S

19 $\dfrac{r_1}{(s-b)(s-c)} = \dfrac{S}{(s-a)(s-b)(s-c)} = \dfrac{S}{(S^2/s)} = \dfrac{s}{S} = \dfrac{1}{r}$

Similarly each term is equal to $1/r$ etc

20 R H S $= \dfrac{a+b-c}{abc} = \dfrac{2s}{4SR} = \dfrac{1}{2r\ R}$

21 R H S $= R\ \dfrac{S}{s}\left(\dfrac{a}{2R}+\dfrac{b}{2R}+\dfrac{c}{2R}\right) = \dfrac{S}{s}\ \dfrac{2s}{2} = S =$ L H S

22 R H S $= R\,(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C)$

$= R\,(\sin 2A + \sin 2B + \sin 2C)$

$= R\ 4 \sin A \sin B \sin C$ (why ?)

See Q 1 (i) P 58

23 R H S $= \dfrac{2abc}{2s}\sqrt{\left\{\dfrac{s(s-a)}{bc}\ \dfrac{s(s-b)}{ca}\ \dfrac{s(s-c)}{ab}\right\}}$

$= \sqrt{\{s(s-a)(s-b)(s-c)\}} = S =$ L H S

24 $4Rr = 4\ \dfrac{abc}{4S}\ \dfrac{S}{s} = \dfrac{abc}{s} = \dfrac{2abc}{a+b+c}$

Now proceed as in Q 23 Alternate method is below

We know that

$\sin A+\sin B+\sin C = 4 \cos A/2 \cos B/2 \cos C/2$ [Ex 3 (i) P 59]

R H S $= Rr\,(\sin A+\sin B+\sin C) = (r/2)\,(a+b+c)$

$= r\ s = S$

25 (a) $\cos A+\cos B+\cos C$ (Refer Q 4 (i) P 59)

$= 1+4 \sin A/2 \sin B/2 \sin C/2$ (prove) (1)

$= 1+\dfrac{4R \sin A/2 \sin B/2 \sin C/2}{R} = 1+\dfrac{r}{R}$

(b) L H S $= \left(2R \sin A\ \dfrac{\cos A}{\sin A} + \quad + \quad\right)$

$= 2R\,(\cos A+\cos B+\cos C)$

Again $\cos C=\frac{1-\tan^2 C/2}{1+\tan^2 C/2}=\frac{1-\frac{1}{4}}{1+\frac{1}{4}}=\frac{3}{5}$

$c^2=a^2+b^2-2ab\cos C$

$c^2=9^2+3^2-2\times 9\times 3\times \frac{3}{5}=81+9-\frac{162}{5}=90-\frac{162}{5}$

$=\frac{288}{5}=\frac{3^2\times 2^6}{10}$

$2\log c=2\log 3+6\log 2-\log 10$

$=2\times 4771213+6\times 30103-1$

or $\log c=4771213+90309-5$ or $\log c=8802113$ (1)

We have $\log 7\ 5894=8802074$ (2)

and $\log 7\ 5895=8802132$ (3)

Let $\log(7\ 5894+x)=8802113$

From (1) and (2), the diff for $0001=0000058$

Also from (1) and (3), the diff for $x=0000039$

Hence $x=\frac{39}{58}\times 0001=\frac{0039}{58}=000067$

$a=7\ 589467$

16 We are given $\tan\phi=\frac{a-b}{a+b}\cot\frac{C}{2}$ (1)

To prove $c=(a+b)\frac{\sin C/2}{\cos\phi}$, we use the relation (2)

$c^2=a^2+b^2\quad 2ab\cos C$

We can write (2) as

$c^2=(a^2+b^2)(\cos^2 C/2+\sin^2 C/2)-2ab(\cos^2 C/2-\sin^2 C/2)$

$=(a^2+b^2-2ab)\cos^2 C/2+(a^2+b^2+2ab)\sin^2 C/2$

$=(a-b)^2\cos^2 C/2+(a+b)^2\sin^2 C/2$

$=(a+b)^2\sin^2\frac{C}{2}\left[\left(\frac{a-b}{a+b}\right)^2\cot^2\frac{C}{2}+1\right]$

$=(a+b)^2\sin^2 C/2)\,[\tan^2\phi+1]=(a+b)^2\sin^2(C/2)\sec^2\phi$

$c=(a+b)\sin\frac{C}{2}\Big/\cos\phi$ from (1)

If $a=3$, $b=1$, and $C=53°\ 7'\ 48''$,

We have

$\tan\phi=\frac{3-1}{3+1}\cot 26°\ 33'\ 54''=\frac{1}{2\tan 26°\ 33'\ 54''}$

$L\tan\phi-10=\log 1-\log 2-(L\tan 26°\ 33'\ 54''-10)$

or $L\tan\phi=20-30103-9\ 69897=10$

$10+\log\tan\phi=10$ or $\log\tan\phi=0$

$\tan\phi=1$ so that $\phi=45°$

30 $L H S = \frac{S}{(s-a)\,bc} + \frac{S}{(s-b)\,ca} + \frac{S}{(s-c)\,ab}$

$= \frac{S}{abc}\left\{\frac{a}{s-a} + \left(\frac{b}{s-b}+1\right) + \left(\frac{c}{s-c}+1\right) - 2\right\}$ (Note)

$= \frac{S}{abc}\left\{\frac{a}{s-a} + \frac{s}{s-b} + \frac{s}{s-c} - 2\right\} = \frac{S}{abc}\left\{\frac{a}{s-a} + \frac{s\,(2s-b-c)}{(s-b)\,(s-c)}\right\} - \frac{2S}{abc}$

$= \frac{S}{abc}\left\{\frac{a}{s-a} + \frac{s\,a}{(s-b)\,(s-c)}\right\} - \frac{1}{2}\ \frac{4S}{abc}$

$= \frac{aS}{abc}\left\{\frac{(s-b)\,(s-c)+s\,(s-a)}{(s-a)\,(s-b)\,(s-c)}\right\} - \frac{1}{2}\ \frac{1}{R}$

$= \frac{s\,S}{bc}\left\{\frac{s^2-s\,(a+b+c)+s^2+bc}{s\,(s-a)\,(s-b)\,(s-c)}\right\} - \frac{1}{2R}$

$= \frac{s\,S}{bc}\ \frac{bc}{S^2} - \frac{1}{2R} = \frac{s}{S} - \frac{1}{2R} = \frac{1}{r} - \frac{1}{2R}$

31 $(r_1+r_2+r_3-r)^2 = r_1^2+r_2^2+r_3^2+r^2-2r\,(r_1+r_2+r_3) + 2\,(r_1r_2+r_2r_3+r_3r_1)$

Now from Q 14, $r_1+r_2+r_3-r=4R$

and from Q 7 (a) $r_1r_2+r_2r_3+r_3r_1=s^2$

$(4R)^2 = r_1^2+r_2^2+r_3^2+r^2-2r\,(r_1+r_2+r_3)+2s^2$

$r_1^2+r_2^2+r_3^2+r^2 = 16R^2+2\,(rr_1+rr_2+rr_3)-2s^2$ (1)

Now $2\,(rr_1+rr_2+rr_3)$

$= 2\left[\frac{S^2}{s\,(s-a)} + \frac{S^2}{s\,(s-b)} + \frac{S^2}{s\,(s-c)}\right]$

$= 2S^2\,\frac{(s-b)\,(s-c)+(s-c)\,(s-a)+(s-a)\,(s-b)}{S^2}$

$= 2\,[3s^2-s\,(b+c+c+a+a+b)+bc+ca+ab]$

$= 2\,[3s^2-s\ \ 4s+bc+ca+ab] = -2s^2+2\,(ab+bc+ca)$

Hence from (1)

$r_1^2+r_2^2+r_3^2+r^2 = 16\,R^2-2s^2+2\,(ab+bc+ca)-2s^2$

$= 16R^2-[(a+b+c)^2-2\,(ab+bc+ca)]$

$= 16R^2-(a^2+b^2+c^2)$

32 It is easy to prove that

$2\alpha+2\beta+2\gamma=2s$ and $\alpha=s-a,\ \beta=s-b,\ \gamma=s-c$

$r^2 = \frac{S^2}{s^2} = \frac{s\,(s-a)\,(s-b)\,(s-c)}{s^2} = \frac{\alpha\,\beta\,\gamma}{\alpha+\beta+\gamma}$

$\log c = \frac{1}{2} \log 2 + L \sin 83° 20' - 10$

$= \frac{1}{2} \times 30103 + 9\ 99705 - 10$

$= 1\ 95669 + 9\ 99705 - 10 = 1\ 95374$

We are given $\log\ 089896 = \bar{2}\ 95374$ Hence $c = 89\ 896$ cms

20 We have

$a^2 = b^2 + c^2 - 2bc \cos A$

or $c^2 - 2bc \cos A + b^2 - a^2 = 0$

Let the two values of c be c_1 and c_2 Then

$c_1 + c_2 = 2b \cos A$ and $c_1 c_2 = b^2 - a$

Hence $(c_1 \sim c_2)^2 = (c_1 + c_2)^2 - 4c_1c_2 = 4b^2 \cos^2 A - 4(b^2 - a^2)$

$= 4 [a^2 - b^2 (1 - \cos^2 A)] = 4 [a^2 - b^2 \sin^2 A]$

$c_1 \sim c_2 = 2\sqrt{(a^2 - b^2 \sin^2 A)}$

21 Here $b \sin A = 4(\sqrt{3} + 1) \sin 15°$

$$= 4(\sqrt{3}+1)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) = 2\sqrt{2}$$

$4(\sqrt{3}+1) > 4 > (\sqrt{3}+1) \sin 15°$

that is, $b > a > b \sin A$

Hence in this case, two triangles will be formed Now

$$\sin B = \frac{b \sin A}{a} = \frac{4(\sqrt{3}+1)}{4} \sin 15°$$

$$= \frac{(\sqrt{3}+1)(\sqrt{3}-1)}{2\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$B = 45°$ or $135°$

and so $C = 180° - (15° + 45°)$ or $180° - (15° + 135°)$

that is $C = 120°$ or $30°$

Again since $c = \dfrac{b \sin C}{\sin B}$ we have

$c_1 = 4(\sqrt{3}+1) \sin 120°/\sin 45° = 2\sqrt{6}(\sqrt{3}+1)$

and $c = 4(\sqrt{3}+1) \sin 30°/\sin 135° = 2\sqrt{2}(\sqrt{3}+1)$

22 As in problem 20 the quadratic giving the two values $c_1 c_2$ of c is

$c^2 - 2bc \cos A + (b^2 - a^2) = 0$

Then we have

(i) $c_1 + c_2 = 2b \cos A$ (1) and $c_1 c_2 = b^2 - a^2$ (2)

(ii) From (1) and (2)

$(c_1 + c_2)^2 = 4b^2 \cos^2 A = 4(c_1c_2 + a^2) \cos^2 A$

or $c_1^2 + 2c_1c_2 + c_2^2 - 4c_1c_2 \cos^2 A = 4a^2 \cos^2 A$

or $c_1^2 - 2c_1c_2 (2\cos^2 A - 1) + c_2^2 = 4a^2 \cos^2 A$

or $c_1^2 - 2c_1c_2 \cos 2A + c_2^2 = 4a^2 \cos^2 A$

or $\cos B\, 2\cos A\cos C=0$

$\cos A=0$ or $\cos B=0$ or $\cos C=0$

or $A=\pi/2$ or $B=\pi/2$ or $C=\pi/2$

43 (a) r_1, r_2, r_3 are in A P, if $r_2-r_1=r_3-r_2$

or $\dfrac{S}{s-b}-\dfrac{S}{s-a}=\dfrac{S}{s-c}-\dfrac{S}{s-b}$ or $\dfrac{b-a}{s-a}=\dfrac{c-b}{s-c}$

or $(a-b)(s-c)=(b-c)(s-a)$ which is given,

(b) Since r_1, r_2, r_3 are in H P, it follows that

$\dfrac{s-a}{S}, \dfrac{s-b}{S}, \dfrac{s-c}{S}$ are in A P

Hence $\dfrac{2(s-b)}{S}=\dfrac{s-a}{S}+\dfrac{s-c}{S}$

or $-2b=-a-c$ or $b=\frac{1}{2}(a+c)$

Hence a, b, c are in A P

44 $\dfrac{\text{area of circle}}{\text{area of triangle}}=\dfrac{\pi r^2}{S}=\dfrac{\pi}{S}\dfrac{S^2}{s^2}=\pi\dfrac{S}{s^2}$ (1)

Now $\cot\dfrac{A}{2}\cot\dfrac{B}{2}\cot\dfrac{C}{2}$

$$=\left[\frac{s(s-a)}{(s-b)(s-c)}\frac{s(s-b)}{(s-c)(s-a)}\frac{s(s-c)}{(s-a)(s-b)}\right]^{1/2}$$

$$=\left[\frac{s^3}{(s-a)(s-b)(s-c)}\right]^{1/2}$$

$$=\left[\frac{s^4}{s(s-a)(s-b)(s-c)}\right]^{1/2}=\frac{s^2}{S}$$

Now $\dfrac{\pi}{\cot A/2\cot B/2\cot C/2}=\dfrac{\pi}{s^2/S}=\dfrac{\pi S}{s^2}$ (2)

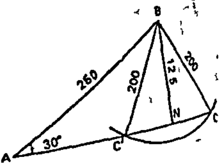
Hence from (1) and (2), we prove the required relation

45 By § 5, we have $\cos B=\dfrac{a^2+b^2-c^2-d^2}{2(ab+cd)}$

$$\tan^2\frac{B}{2}=\frac{1-\cos B}{1+\cos B}=\frac{2(ab+cd)-(a^2+b^2-c^2-d^2)}{2(ab+cd)+(a^2+b^2-c^2-d^2)}$$

$$=\frac{(c+d)^2-(a-b)^2}{(a+b)^2-(c-d)^2}=\frac{(c+d+a-b)(c+d-a+b)}{(a+b+c-d)(a+b-c+d)}$$

$$=\frac{(2s-2b)(2s-2a)}{(2s-2d)(2s-2c)}=\frac{(s-a)(s-b)}{(s-c)(s-d)}$$



and so $c > a > c \sin A$

Hence in this case, there are two $\Delta^s ABC$, and ABC'. Now from ΔABC, we have

$$\frac{\sin C}{c} = \frac{\sin A}{a}$$

or $\sin C = \frac{250 \sin 30°}{200} = \frac{5}{8} = \frac{10}{16} = \frac{10}{2^4}$

$L \sin C = 10 + \log 10 - 4 \log 2 = 10 + 1 - 4 \times 30103$

$= 11 - 1{\cdot}20412 = 9\ 79588$

Since we are given $L \sin 38° 41' = 9\ 7958800$,

we see from (1) that

$\angle BCA = 38° 41' = \angle BC'C$

Hence $\angle ABC = 180° - (30° + 38° 41') = 111° 19'$

$\angle AC'B = 180° - \angle BC'C = 180° - 38° 41' = 141° 19'$

and $\angle ABC' = \angle BC'C - \angle BAC' = 38° 41' - 30° = 8° 41'$

For the smaller value of b (i e AC'), we have

$$\frac{AC'}{\sin ABC'} = \frac{BC'}{\sin BAC'} \text{ or } \frac{AC'}{\sin 8° 41'} = \frac{200}{\sin 30°} = 400$$

Hence $AC = 100 \times 2^2 \sin 8° 41'$

$\log AC' = \log 100 + 2 \log 2 + L \sin 8° 41' - 10$ (1)

$= 2 + 2 \times\ 30103 + 9\ 1789001 - 10$

$= 1\ 7809601$

Since we are given $\log 6\ 038993 = 7809601$, we see from (1) that $AC = 60\ 3893$ meter

27 Here $c < b$ and C is acute and so $c > b \sin C$

Hence there are two triangles in this case

We have

$$\sin B = \frac{b \sin C}{c} = \frac{63 \sin 29° 23' 15''}{36}$$

$= \frac{7}{4} \sin 29° 23' 15 = (7/2^2), \sin 29° 23' 15''$

$L \sin B = \log 7 - 2 \log 2 + L \sin 29° 23' 15'' = 9\ 9338662$

$= 8450980 - 2 \times\ 3010300 + 9\ 6908282 = 9\ 9338222 = 0000755$

Now diff for $1' = 9\ 9338977 - 9\ 9338222 = 00\ 0440$

Also $9\ 338662 - 9{\cdot}9338222 = $

angular difference $= \left(\frac{60}{755} \times 440\right)'' = 35$ approx.

another, it follows that I is the in-centre of the triangle ABC. Now the sides of the $\triangle ABC$ are $b+c$ ($=a'$, say), $c+a$ ($=b'$), and $a+b$ ($=c'$), so that $2s=\Sigma a'=2(a+b+c)$ or $s=a+b+c$. Hence the required distance

$$=IE=ID=IF=\frac{\triangle ABC}{s}$$

$$=\frac{1}{s}\sqrt{\{s(s-a')(s-b')(s-c')\}}$$

$$=\sqrt{\left(\frac{(s-a')(s-b')(s-c')}{s}\right)}$$

$$=\sqrt{\left[\left(\frac{abc}{a+b+c}\right)\right]}$$

$$c=\tfrac{1}{2}\left[b\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}\right.$$
$$\left.\pm\sqrt{\left\{\frac{b^2(m-1)(3+m)}{m}-\frac{4b^2(m-1)(3+m)}{(m+1)^2}\right\}}\right]$$
$$=\frac{b}{2}\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}\left[1\pm\sqrt{\left\{1-\frac{4m}{(m+1)}\right\}}\right]$$
$$\frac{b}{2}\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}\left[1\pm\frac{m-1}{m+1}\right]$$

Hence if c_1 c_2 are two values of c, then

$$c_1=\frac{b}{2}\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}\times\frac{2m}{m+1}=\frac{mb}{m+1}\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}$$

and $c_2=\dfrac{b}{2}\sqrt{\left\{\dfrac{(m-1)(3+m)}{m}\right\}}\times\dfrac{2}{m+1}$

$$=\frac{b}{m+1}\sqrt{\left\{\frac{(m-1)(3+m)}{m}\right\}}$$

Hence $c_1=mc_2$

Thus one value of third side is m times the other

7 The sides of a triangle are 2, 3 and 4, find the greatest angle, having given
log 2 = 3010300, log 3 = 4771213,
L tan 52° 14′ = 10 1108395,
L tan 52°15′ = 10 1111004

8 The sides of a triangle are 32, 40 and 66 meter find the angle opposite the greatest side, having given
log 3 = 47712 log 69 = 1 83885,
log 37 = 1 56820 log 29 = 1 46240,
L cot 66° 10 = 9 64517 L cot 66° 20′ = 9 64175

9 The sides of a triangle are 5, 8 11, find the greatest angle having given
log 7 = 8450980
L sin 56° 47′ = 9 9225205, L sin 56° 48′ = 9 9226032,
(I I T 73)

Two sides and Included Angle

10 The angles of a triangle are in A P and the lengths of the greatest and least sides are 24 and 16 meters respectively Find the length of the third side and the angles given
log 2 = 3010300 log 3 = 4771213, L tan 19° 6 = 9 5394287, diff for 1 = 4084

11 The two sides of a triangle are 540 and 420 meters respectively and include angle of 52° 6 Find the remaining angles, given that
log 2 = 3010300 L tan 26° 3 = 9 6891430
L tan 14° 20 = 9 4074189, L tan 14° 21 = 9 4079453
(Roorkee 76)

12 In a $\triangle ABC$, $b=6$, $c=3$ and $A=36°\ 52\ 12$ Find a having given
log 2 = 30103 log 3 = 47712, L cot 18° 26′ 6 = 10 47712,
L sin 18° 26 6 = 9 5 log 4 02 = 60423 log 4 03 = 60531

13 In a $\triangle ABC$, $b=2\frac{1}{2}$ meter $c=2$ meter and $\angle A=22°\ 20'$ Find the other angles and show that the third side is nearly one meter, given
log 2 = 30103, log 4 = 47712
L cot 11° 10 = 10 70465 L sin 22° 20′ = 9 57971,
L tan 29° 22 20″ = 9 75038, L tan 29° 22′ 30″ = 9 75043
and L sin 49° 27 34 = 9 88079

14 If $a=5$, $b=4$ and cos $(A-B)=31/32$, prove that the third side c will be 6

(a) 1 (b) $\sqrt{3}$

(c) $\frac{\sqrt{3}}{2}$ (d) 2

9. (b) In a right angled triangle, the hypotenuse is four times as long as the perpendicular drawn to it from the opposite vertex. One of the acute angle is

(a) 15° (b) 30°

(c) 45° (d) None of these

(M N R 80)

10 The value of $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}$ is

(i) 0 (ii) $\frac{1}{2}$

(iii) $\frac{1}{8}$ (iv) $-\frac{1}{8}$

11 The value of $\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}$ is

(i) 1/16 (ii) 1/8

(iii) $\frac{1}{2}$ (iv) 1

12 If $A=580°$ then

(a) $\sin\frac{1}{2}A=\frac{1}{2}[\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}]$

(b) $\sin\frac{1}{2}A=-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}$

(c) $\sin\frac{1}{2}A=\frac{1}{2}[-\sqrt{(1+\sin A)}-\sqrt{(1-\sin A)}]$

(d) $\cos\frac{1}{2}A=\sqrt{(1+\sin A)}-\sqrt{(1-\cos A)}$

13 If $2\cos A/2=\sqrt{(1+\cos A)}-\sqrt{(1-\sin A)}$, then

(a) $n\pi+\frac{\pi}{4}<\frac{A}{2}<n\pi+\frac{3\pi}{4}$

(b) $n\pi-\frac{\pi}{4}<\frac{A}{2}<2n\pi-\frac{3\pi}{4}$

(c) $2n\pi-\frac{3\pi}{4}<\frac{A}{2}<2n\pi+\frac{5\pi}{4}$

(d) $2n\pi+\frac{\pi}{2}<\frac{A}{2}<2n\pi+\frac{3\pi}{4}$

14 The general solution of the trigonometrical equation

$$\sin x+\cos x=1$$

is given by

(a) $x=2n\pi$, $n=0, \pm 1, \pm 2$

(b) $x=2n\pi+\pi/2$, $n=0, \pm 1, \pm 2,$

(c) $x=n\pi+(-1)^n\pi/4-\pi/4$, $n=0, \pm 1, \pm 2,$

(d) None of these

(I. I. T. 81)

23 In a $\triangle ABC$, a, c, A are given and $b_2=2b_1$ where b_1, b_2 are two values of the third side then prove that

$$3a=c\sqrt{(1+8\sin^2 A)}$$

24 In the ambiguous case, if two triangles are formed with a, b and A, then prove that the sum of the areas of these triangles is $\frac{1}{2}b^2\sin 2A$

25 In the ambiguous case, if the remaining angles of the triangles formed with a b and A be B_1, C_1 and B_2, C_2, then prove that

$$\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=2\cos A$$

26 Point out whether or not the solutions of the following triangles are ambiguous

Find the smaller value of the third side in the ambiguous case and the other angles in both cases

(*i*) $A=30°$, $c=250$ meter, $a=125$ meter

(*ii*) $A=30°$, $c=250$ meter and $a=200$ meter, given

log 2= 3010300, log 6 03893= 7809601

$L\sin 38°\ 41'=9\ 7958800$ $L\sin 8°, 41'=9\ 1789001$

27 If $b=63$, $c=36$, $c=29°\ 23'\ 15''$, find B

Given log 2= 3010300, log 7= 8450980,

$L\sin 29°\ 23'\ 15=9\ 6908282$,

$L\sin 59°\ 10=9\ 9338222$, $L\sin 59°\ 11'=9\ 9338977$

28 In a $\triangle ABC$, $A=45°$ and c_1, c_2 are the two values of side c in the ambiguous case, show that

$$\cos B_1 C B_2=\frac{2c_1c_2}{c_1^2+c_2^2}$$

29 Let $1<m<3$ In a $\triangle ABC$, if $2b=(m+1)a$, and

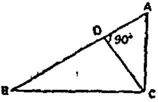
$$\cos A=\frac{1}{2}\sqrt{\left[\frac{(m-1)(m+3)}{m}\right]},$$

prove that there are two values of the third side one of which is m times the other (I I T 76)

Solutions

1 Let ABC by the triangle, C the right angle and CD by the perpendicular from C upon AB, we are given

$$AB=4CD \qquad (1)$$



Now $AB=AC\sec A$

and $CD=AC\sin A$

(C) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$ meters (D) None of these

(I.I.T 83)

25 If $\tan A=(1-\cos B)/\sin B$, then $\tan 2A=\tan B$

(a) True (b) False (I.I.T 83)

26 The larger of $\cos \log \theta$ and $\log \cos \theta$ if $e^{-\pi/2}<\theta<\pi/2$, is

(I.I.T 83)

27 If $\sin \alpha=-\frac{3}{5}$ $\left(\pi<\alpha<\frac{3}{2}\pi\right)$, then the value of $\cos \frac{1}{2}\alpha$ is

(a) $-\frac{1}{\sqrt{10}}$, (b) $\frac{1}{\sqrt{10}}$

(c) $\frac{3}{\sqrt{10}}$ (d) None of these

28 If the sum of the acute angles $\tan^{-1} x$ and $\tan^{-1} \frac{1}{2}$ is 45°, then the value of x is $\frac{1}{3}$

(a) True (b) False

29 If $\sin \alpha=\frac{12}{13}$ $\left(0<\alpha<\frac{1}{2}\pi\right)$ and $\cos \beta=-\frac{3}{5}$ $\left(\pi<\beta<\frac{3}{2}\pi\right)$

Then the value of $\sin(\alpha+\beta)$ is

(a) $-\frac{56}{65}$, (b) $\frac{16}{65}$,

(c) $\frac{56}{65}$, (d) $-\frac{16}{65}$,

30 In a $\triangle ABC$, $\frac{a \cos A+b \cos B+c \cos C}{a+b+c}$

(a) $\frac{r}{R}$, (b) $\frac{R}{r}$,

(c) $\frac{2r}{R}$, (d) $\frac{R}{2r}$

31 For any real number α such that $0<|\alpha|<\pi/2$, the in equality $\sin \alpha<\alpha<\tan \alpha$ or $\cos \alpha<\frac{\sin \alpha}{\alpha}<1$ is fulfilled

(a) True (b) False

32 The inequality $\frac{\tan \alpha}{\alpha}<\frac{\alpha}{\sin \alpha}$ is fulfilled for all real α for which $0<\alpha<\frac{\pi}{2}$

or $\quad \dfrac{15+20\sqrt{3}}{24} CD=5$

or $\quad CD=\dfrac{24}{3+4\sqrt{3}}=\dfrac{24\,(4\sqrt{3}-3)}{48-9}$

Hence $CD=\dfrac{24}{39}(4\sqrt{3}-3)=\dfrac{8}{13}(4\sqrt{3}-3)$

4 Since lengths of sides are positive, we must have

$2x+1>0$, that is, $x>-\frac{1}{2}$ (1)

and $x^2-1>0$, that is, $-1>x>1$ (2)

Both the conditions (1) and (2) will be satisfied if $x>1$

Now it is easy to see that x^2+x+1 is the greatest side For we have

$$(x^2+x+1)-(2x+1)=x\,(x-1)>0$$

and $(x^2+x+1)-(x^2-1)=x+2>0$

Let $a=x^2+x+1$, $b=2x+1$, $c=x^2-1$,

then $\cos A=\dfrac{b^2+c^2-a^2}{2bc}$

$$=\frac{(2x+1)^2+(x^2-1)^2-(x^2+x+1)^2}{2\,(2x+1)\,(x^2-1)}$$

$$=\frac{4x^2+4x+1+x^4-2x^2+1-x^4-x^2-1-2x^3-2x-2x^2}{2\,(2x^3+x^2-2x-1)}$$

$$=\frac{-2x^3-x^2+2x+1}{2\,(2x^3+x^2-2x-1}=-\frac{1}{2}$$

Hence $A=120^\circ$

Note You may use the formula

$$\cos\frac{A}{2}=\sqrt{\left(\frac{s\,(s-a)}{bc}\right)}=\frac{1}{2} \qquad \frac{A}{2}=60^\circ \text{ or } A=120^\circ$$

5 Ans 120°

6 Ans $45^\circ, 60^\circ, 75^\circ$

7 Let $a=2$, $b=3$, $c=4$ Then C is the greatest angle

We have $\quad s=\dfrac{2+3+4}{2}=\dfrac{9}{2}$

Hence $\tan\dfrac{C}{2}=\sqrt{\left\{\dfrac{(s-a)(s-b)}{s(s-c)}\right\}}=\sqrt{\left\{\dfrac{(9/2-2)\,(9/2-3)}{9/2\,(9/2-4)}\right\}}$

$$=\sqrt{\left(\frac{5\times 3}{9\times 1}\right)}=\sqrt{\frac{5}{3}}=\sqrt{\left(\frac{10}{3\times 2}\right)}$$

$L\tan C/2=10+\frac{1}{2}\,[\log 10-\log 3-\log 2]$

$=10+\frac{1}{2}\,[1-4771213-3010300]$

$=10+\frac{1}{2}\times 2218487=10\ 1109244$

(c) $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$, (I I T 79)

(d) None of these

42 The value of $5\cos\theta+3\cos(\theta+\pi/3)+3$ lies between -4 and 4

(a) True (b) False

43 $A=\sin^8\theta+\cos^{14}\theta$ Then for all values of θ,

(a) $A\geqslant 1$, (b) $0<A\leqslant 1$,

(c) $\frac{1}{2}<A\leqslant\frac{3}{2}$ (d) None of these

44 In a $\triangle ABC$, cosec A ($\sin B\cos C+\cos B\sin C$) equals

(i) $\frac{c}{a}$ (ii) $\frac{a}{c}$,

(iii) 1, (iv) None of these

(O U J N T U 80)

45 If $\tan x=\frac{b}{a}$, then $\sqrt{\left(\frac{a+b}{a-b}\right)}+\sqrt{\left(\frac{a-b}{a+b}\right)}$ equals

(i) $\frac{2\sin x}{\sqrt{(\sin 2x)}}$, (ii) $\frac{2\cos x}{\sqrt{(\cos 2x)}}$,

(iii) $\frac{2\cos x}{\sqrt{(\sin 2x)}}$ (iv) $\frac{2\sin x}{\sqrt{(\cos 2x)}}$

46 tan 54° can be expressed as

(i) $\frac{\cos 9^\circ-\sin 9^\circ}{\cos 9^\circ+\sin 9^\circ}$, (ii) $\frac{\sin 9^\circ-\cos 9^\circ}{\sin 9^\circ+\cos 9^\circ}$,

(iii) $\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}$, (iv) $\frac{\sin 9^\circ+\cos 9^\circ}{\sin 9^\circ+\cos 9^\circ}$

47 $\frac{\cos 17^\circ+\sin 17^\circ}{\cos 17^\circ-\sin 17^\circ}$ is equal to

(i) tan 54°, (ii) tan 56°, (iii) tan 62°,

(iv) tan 73°

48 In a triangle ABC, $\frac{a\cos A+b\cos B+c\cos C}{a+b+c}=$

(i) $\frac{r}{R}$ (ii) $\frac{R}{r}$ (iii) $\frac{2r}{R}$ (iv) $\frac{R}{2r}$

49 In a triangle ABC, if $3a=b+c$, then the value of $\cot\frac{B}{2}\cot\frac{C}{2}$ is

(i) 1 (ii) 2 (iii) $\sqrt{3}$ (iv) 3

Now $\tan\frac{C-A}{2}=\frac{c-a}{c+a}\cot\frac{B}{2}=\frac{24-16}{24+16}\cot 30°=\frac{\sqrt{3}}{5}=\frac{2\sqrt{3}}{10}$

Hence $L\tan\frac{C-A}{2}=10+\log 2+\frac{1}{2}\log 3-\log 10$

$=10+\ 3010300+\frac{1}{2}\times\ 4771213-1=9\ 5395906$

Thus we have

$L\tan\frac{C-A}{2}=9\ 5395906$

$L\tan 19°\ 6'=9\ 5394287$

Diff $=$ 0001619

Since 4084 is the diff on 60″

1619 is the diff on $\frac{60}{4084}\times 1619=24''$

Hence $\frac{C-A}{2}=19°\ 6'\ 24$

Also $\frac{C+A}{2}=90°-\frac{B}{2}=90°-30°=60°$

$C=79°\ 6'\ 24$ and $A=40°\ 53'\ 36''$

Again $\cos B=\frac{a^2+c^2-b^2}{2ac}$, $\cos 60°=\frac{16^2+24^2-b^2}{2\times 16\times 24}$

or $\frac{1}{2}\times 2\times 16\times 24=256+576-b^2$

or $b^2=448$ or $b=8\sqrt{7}$

11 Let $a=540$, $b=420$, $C=52°\ 6$

Now $\tan\frac{A-B}{2}=\frac{540-420}{540+420}\cot 26°\ 3$

$=\frac{1}{8\tan 26°\ 3}=\frac{1}{2^3\tan 26°\ 3'}$

Hence $L\tan\frac{A-B}{2}-10=\log 1-3\log 2-[L\tan 26°\ 3'-10]$

$L\tan\frac{A-B}{2}=20-3\times\ 3010300-9\ 6891430$

$=20-\ 9030900-9\ 6891430$

$=20-10\ 5922330=9\ 4077670$

Thus we have

$L\tan\frac{A-B}{2}=9\ 4077670$, $L\tan 14°\ 21'=9\ 4079453$

$L\tan 14°\ 20=9\ 4074189$, $L\tan 14°\ 20=9\ 4074189$

Diff $=$ 0003481 , Diff for 1 $=$ 0005264

Hence angular difference $=\left(\frac{60}{5264}\times 3481\right)''=40''$ approx is 57° therefore 2 radian is an obtuse an

Hence $\tan 1 > 0$ and $\tan 2 < 0$

$\tan 1 > \tan 2$

8 Ans (iii) 9 (a) Ans (c)

(b) If p be the perpendicular then hyp

$x^2+y^2=16p^2$ where x and y are sides

Also $p = x \sin\theta$

and $p = y \cos\theta$ $x=p \sin\theta$ $y=p$ c

Putting in

$x^2+y^2=16p^2$ we get $p^2\left[\frac{1}{\sin^2\theta}+\frac{1}{\cos}\right.$

or $1=16\sin^2\theta\cos^2\theta$

or $1=4(2\sin\theta\cos\theta)^2$ $\frac{1}{4}=\sin^2 2\theta$

or $\sin 2\theta=\frac{1}{2}=\sin 30°$ $\therefore$ $\theta=15°$

10 Ans (iv) See Q 43 (iv) P 24

11 Ans (ii) See Q 43 (viii) P 24

12 Ans (c)

Since $\frac{A}{2}=290°$ we have $225°<\frac{A}{2}<31$

$\sin\frac{A}{2}+\cos\frac{A}{2}$ and $\sin\frac{A}{2}-\cos\frac{A}{2}$

are both negative and so

$$\sin\frac{A}{2}+\cos\frac{A}{2}=-\sqrt{(1+\sin A)}$$

and $\sin\frac{A}{2}-\cos\frac{A}{2}=-\sqrt{(1-\sin A)}$

Adding these $2\sin\frac{A}{2}=-\sqrt{(1-\sin A)}$

13 Ans (d) We know that

$\sin\frac{A}{2}+\cos\frac{A}{2}=\pm\sqrt{(1+\sin A)}$

and $\sin\frac{A}{2}-\cos\frac{A}{2}=\pm\sqrt{(1-\sin A)}$

But we are given

$$2\cos\frac{A}{2}=\sqrt{(1+\sin A)}-\sqrt{(1-}$$

Now (3) will be obtai we take po

and (2) But we know

$\sin\frac{A}{2}+$ cc $-\cos\frac{A}{2}$

Also $\frac{B+C}{2}=90°-\frac{A}{2}=90°-11° 10'=78° 50$

$B=78° 50 +29° 22\ 26 =108° 12' 26''$

and $C=78° 50'-29° 22\ 26 =49° 27' 34$

To find a, we have

$\frac{a}{\sin A}=\frac{c}{\sin C}$ or $a=\frac{c \sin A}{\sin C}=\frac{2 \sin 22° 20}{\sin 49° 27\ 34}$

$\log a=\log 2+(L \sin 22° 20'-10)-(L \sin 49° 27\ 34''-10)$

$= 30103+9\ 57977-9\ 88079$

$=9\ 88080-9\ 88079= 00001$

Since 00001 is very small, we can take it to be nearly 0

Hence log $a=0$ approx and so $a=1$ m nearly

14 We have

$\tan \frac{A-B}{2}=\sqrt{\left[\left(\frac{1-\cos (A-B)}{1+\cos (A-B)}\right)\right]}=\sqrt{\left[\frac{1-\frac{31}{32}}{1+\frac{31}{32}}\right]}=\sqrt{\left(\frac{1}{63}\right)}=\frac{1}{3\sqrt{7}}$

Also $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$ gives

$\frac{1}{3\sqrt{7}}=\frac{5-4}{5+4}\cot\frac{C}{2}$ or $\tan\frac{C}{2}=\frac{\sqrt{7}}{3}$

$\cos C=\frac{1-\tan^2 C/2}{1+\tan^2 C/2}=\frac{1-\frac{7}{9}}{1+\frac{7}{9}}=\frac{1}{8}$

$c^2=a^2+b^2-2ab \cos C=25+16-2\times5\times4\times\frac{1}{8}=36$

Hence $c=6$

15 $a=9$, $b=3$ Then as given, $A-B=90°$

We have $\tan\frac{A-B}{2}=\frac{a-b}{a+b}\cot\frac{C}{2}$

or $\tan 45°=\frac{9-3}{9+3}\cot\frac{C}{2}$ or $\tan C/2=\frac{1}{2}$

$L \tan C/2=10-\log 2=10-3010300=9\ 6989700$

Now difference for $1=9\ 6990006-9\ 69869847= 0002853$

Also $L \tan C/2-L \tan 26° 33'=9\ 6989700-9\ 6986847$

$= 0002853$

Hence angular diff $\equiv \frac{60}{3159}\times 2853=54$

Hence $C/2=26° 33\ 54''$ or $C=53° 7' 48$

Now $A/2+B/2=90°-C/2=90°-26° 33' 54 =63° 26\ 6''$

And $\frac{A-B}{2}=45°$ $A=108° 26' 6''$ and $B=18° 26' 6''$

and $\sin\phi=\frac{1}{\sqrt{(10)}}$ gives $\tan\phi=\frac{1}{3}$,

so that $\sin\phi=\frac{2\tan\phi}{1+\tan^2\phi}=\frac{2/3}{1+1/9}=\frac{3}{5}$ and so $\cos 2\phi=\frac{4}{5}$

Hence $\sin(\theta+2\phi)=\frac{1}{5\sqrt{2}}\cdot\frac{4}{5}+\frac{7}{5\sqrt{2}}\cdot\frac{3}{5}=\frac{25}{25\sqrt{2}}=\frac{1}{\sqrt{2}}$

$\theta+2\phi=45°$

18 Ans (iv) We have $\sin x+\cos x=2$

or $\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\sqrt{2}$

or $\sin(x+\pi/4)=\sqrt{2}>1$

Since the sine of an angle cannot be greater than 1, the given equation has no solution

19 Ans (a)

Since $\sec^2\theta\geqslant 1$, we have

$$\frac{4xy}{(x+y)^2}\geqslant 1 \text{ or } 4xy\geqslant(x+y)^2$$

or $0\geqslant(x-y)^2$ or $(x-y)^2\leqslant 0$

Since x, y are real, $(x-y)^2$ cannot be negative Hence the only possibility is $(x-y)^2=0$, i e $x=y$

20 Ans $-2/\sqrt{5}$

21 Ans (b)

[Hint Proceed as in problem 18]

22 Ans $\sqrt{\left(\frac{x^2+1}{x^2+2}\right)}$

23 Ans 0

Let $\tan^{-1}\sqrt{\left\{\frac{a(a+b+c)}{bc}\right\}}=A$, $\tan^{-1}\sqrt{\left\{\frac{b(a+b+c)}{ca}\right\}}=B$, $\tan^{-1}\sqrt{\left\{\frac{c(a+b+c)}{ab}\right\}}=C$

Then $\theta=A+B+C$

$$\tan\theta=\frac{\tan A+\tan B+\tan C-\tan A\tan B\tan C}{1-\tan A\tan B-\tan B\tan C-\tan C\tan A}$$

Now $\tan A+\tan B+\tan C$

$$=\sqrt{(a+b+c)}\left\{\sqrt{\left(\frac{a}{bc}\right)}+\sqrt{\left(\frac{b}{ca}\right)}+\sqrt{\left(\frac{c}{ab}\right)}\right\}$$

$$=\sqrt{(a+b+c)}\left\{\frac{a+b+c}{\sqrt{(abc)}}\right\}$$

17 Let $A=41° 13' 22$, $B=71° 19' 5''$, and $a=55$

We then have

$$\frac{\sin A}{a}=\frac{\sin B}{b} \quad \text{or} \quad b=\frac{a \sin B}{\sin A}$$

$$\log b=\log a+L \sin B-L \sin A$$
$$=\log 55+L \sin 71° 19' 5''-L \sin 41° 13' 22$$
$$=1\ 7403627+9\ 9764927-9\ 8188779=1\ 8979775$$

We are given

log 79063=4 8979775 Hence $b=79\ 063$

18 The third angle of the triangle formed by the ships and the beacon

$=180°-(52° 25' 15''+75° 9' 30)=52° 25' 15$

Hence the triangle is isosceles, so that if a and b be the required distances respectively, we have $b=1$ kilometer

$$\text{Also } \frac{a}{\sin 75° 9' 30}=\frac{b}{\sin 52° 25' 15}=\frac{1}{\sin 52° 25' 15'}$$

$$a=\frac{\sin 75° 9' 30''}{\sin 52° 25' 15''}$$

Hence $\log a=L \sin 75° 9' 30-L \sin 52° 25' 15''$
$$=9\ 9852635-9\ 8990055=\ 0862580$$

We have

log 1 2197= 0862530 (1)

log 1 2198= 0862886 (2)

Let log (1 2197+x)= 0862580 (3)

From (1) and (2), we have

the diff for 0001= 0000356

and from (1) and (3)

diff for x= 0000050

$$\text{Hence } x=\frac{50}{356} \times\ 0001=\frac{005}{356}=\ 000014 \text{ approx}$$

$$a=1\ 219714$$

Hence the required distance=1 219714 kilometer

19 We are given

$A=38° 20'$, $B=45°$ and $b=64$

$$C=180°-(38° 20+45°)=180°-83° 20$$
$$\sin C=\sin [180°-83° 20]=\sin 83° 20$$

$$\text{Now } c=\frac{b \sin C}{\sin B}=\frac{64\times \sin 83° 20}{\sin 45°}$$

or $c=64\sqrt{2} \sin 83° 20=(2)^{13/2} \sin 83° 20$

[illegible]

$\sin^2\theta - \cos^4\theta = 1 - \cos^2\theta - \cos^4\theta = -\left(\cos^2\theta + \frac{1}{2}\right)^2 + \frac{5}{4}$ [illegible]

[illegible] $\leq t \leq 1$

[illegible]

[illegible] we have $-1 \leq \cos\theta \leq 1$ [illegible]

[illegible] Since $-\pi/2 < -1$ and $1 < \pi/2$ [illegible]

$-\frac{\pi}{2} < x < \frac{\pi}{2}$

Now we know that for all the values of x for which

$-\frac{\pi}{2} < x < \frac{\pi}{2}$, we have $\cos x > 0$ or $\cos(\cos\theta) > 0$

[illegible]

The inequality is equivalent to

$\cos(\sin\theta) - \cos\left(\frac{\pi}{2} - \cos\theta\right) > 0$

$2\sin\left(\frac{\pi}{4} + \frac{\sin\theta - \cos\theta}{2}\right)\sin\left(\frac{\pi}{4} - \frac{\sin\theta + \cos\theta}{2}\right) > 0$

[illegible] show that the factors in the [illegible]

[illegible]

[illegible] $\sin\theta - \cos\theta = \sqrt{2}\sin\left(\theta - \frac{\pi}{4}\right)$

[illegible]

23 Here the quadratic for third side b is given by

$b^2-2bc\cos A+(c^2-a^2)=0$

$b_1+b_2=2c\cos A$ (1)

and $b_1b_2=c^2-a^2$ (2)

Also it is given that $b=2b_1$ (3)

Hence from (1) and (3) $3b_1=2c\cos A$ (4)

and from (2) and (3), $2b_1^2=c^2-a^2$ (5)

Finally from (4) and (5), we have

$$2\cdot\frac{4c^2\cos^2 A}{9}=c^2-a^2$$

or $8c^2(1-\sin^2 A)=9c^2-9a^2$ or $9a^2=c^2(1+8\sin^2 A)$

Hence $3a=c\sqrt{(1+8\sin^2 A)}$

24 *Sum of the area of the two triangles*

$=\frac{1}{2}ab\sin C_1+\frac{1}{2}ab\sin C_2=\frac{1}{2}ab(\sin C_1+\sin C_2)$ (1)

Now from problem 22, we have

$c_1+c_2=2b\cos A$

or $k(\sin C_1+\sin C_2)=2k\sin B\cos A$

or $\sin C_1+\sin C_2=2\sin B\cos A$

Hence from (1)

sum of the areas of the two triangles

$=\frac{1}{2}ab\cdot 2\sin B\cos A$

$=b^2\left(\frac{a\sin B}{b}\right)\cos A=b^2\sin A\cos A$

$=\frac{1}{2}b^2\cdot 2\sin A\cos A=\frac{1}{2}b^2\sin 2A$

25 c_1, c_2 are two values of c, we have

$c_1+c_2=2b\cos A$ (problem 22) (1)

Also B_1, B_2 are supplementary angles, that is

$B_2=180°-B_1$ so that $\sin B_2=\sin B_1$

Hence $\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_1}=\frac{\sin C_1+\sin C_2}{\sin B_1}$

$=\frac{k\,c_1+k\,c_2}{kb}=\frac{c_1+c_2}{b}=2\cos A$ from (1)

26 (i) Here $A=30°$, $c=250\ m$, $a=125\ m$

We have, $c\sin A=250\sin 30°=125=a$

Hence the triangle ABC is right angled at C so that $\angle C=90°$ and $\angle B=90°-30°=60°$

(ii) Here $A=30°$, $c=250\ m$, $a=200\ m$

In this case, $c\sin A=250\sin 30°=125\ m$

Again

$\sin^2\theta+\cos^4\theta=1-\cos^2\theta+\cos^4\theta=(\cos^2\theta-\frac{1}{2})^2+\frac{3}{4}\geqslant\frac{3}{4}$

Hence $\frac{3}{4}\leqslant A\leqslant 1$

38 Ans (b)

For all θ, we have $-1\leqslant\cos\theta\leqslant 1$ Put $x=\cos\theta$ to get $-1\leqslant x\leqslant 1$ Since $-\pi/2<-1$ and $1<\pi/2$, it follows that

$-\frac{\pi}{2}<x<\frac{\pi}{2}$

Now we know that for all the values of x for which

$-\frac{\pi}{2}<x<\frac{\pi}{2}$, we have $\cos x>0$ or $\cos(\cos\theta)>0$

39 Ans (a)

The inequality is equivalent to

$$\cos(\sin\theta)-\cos\left(\frac{\pi}{2}-\cos\theta\right)>0$$

or $2\sin\left(\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\sin\theta+\cos\theta}{2}\right)>$

We will show that the factors in the left hand member positive

Since $|\sin\theta-\cos\theta|=\sqrt{2}\sin\left(\theta-\frac{\pi}{4}\right)|\leqslant\sqrt{2}<\frac{1}{2}\pi$

It follows that

$$-\frac{\pi}{2}<\sin\theta-\cos\theta<\frac{\pi}{2}\text{ or }-\frac{\pi}{4}<\frac{\sin\theta-\cos\theta}{2}<\frac{\pi}{4}$$

or $0<\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}<\frac{\pi}{2}$

Consequently

$$\sin\left(\frac{\pi}{4}+\frac{\sin\theta-\cos\theta}{2}\right)>0\text{ for all }\theta$$

Similarly

$$\sin\left(\frac{\pi}{4}-\frac{\sin\theta+\cos\theta}{2}\right)>0$$

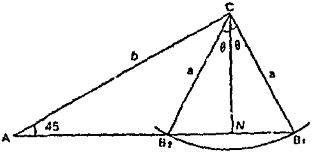
Hence the given inequality holds true

40 Ans (b) 41 Ans (a)

42 Ans (b) we have

Hence if B_1, B_2 are two values of B, we have
$B_1=59^\circ\ 10'\ 35''$ and $B_2=180^\circ-59^\circ\ 10\ 35''=120\ 49\ 25''$

28 Let the two triangles formed be AB_1C and AB_2C Draw $CN \perp AB_1$ Then since $\triangle B_1CB_2$ is isosceles, we have $\angle B_1CN=\angle B_2CN=\theta$, say



Since $\angle A=45^\circ$, we have $CN=b \sin 45^\circ=b/\sqrt{2}$

Now $a^2=b^2+c^2-2bc \cos A=b^2+c^2-2bc \cos 45^\circ=b^2+c^2-\sqrt{2}bc$

or $c^2-\sqrt{2}bc+b^2-a^2=0$

If c_1 c_2 are the two values of c, then

$c_1+c_2=\sqrt{2}b$ and $c_1c_2=b^2-a^2$ (1)

$c_1^2+c_2^2=(c_1+c_2)^2-2c_1c_2=2b^2-2\ (b^2-a\ )=2a^2$ (2)

Now $\cos B_1CB_2=\cos 2\theta=2\ \cos^2\theta-1$

$=2\left(\frac{C\backslash}{a}\right)^2-1=2\ \frac{b^2}{2a^2}-1$ ($CN=b/\sqrt{2}$)

$\frac{b^2-a^2}{a^2}=\frac{c_1c_2}{\frac{1}{2}(c_1^2+c_2^2)}=\frac{2c_1c_2}{c_1^2+c_2^2}$

29 We are given $2b=(m+1)\ a$ (1)

Since $1<m<3$, we have $0<\frac{(m-1)(m-3)}{m}<4$

and so $\cos A=\frac{1}{2}\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}$ lies between 0 and 1

Thus the given value of $\cos A$ is meaningful

Now $\frac{b^2+c^2-a^2}{2bc}=\cos A=\frac{1}{2}\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}$,

so that $b^2+c^2-a=bc\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}$

or $b^2+c^2-\frac{4b^2}{(m+1)^2}=bc\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}$ from (1)

or $c^2-cb\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}+b^2\frac{(m+1)^2-4}{(m+1)^2}=0$

or $c^2-cb\sqrt{\left\{\frac{(m-1)\ (3+m)}{m}\right\}}+\frac{(m-1)\ (3+m)}{(m+1)^2}\ b^2=0$

48 Ans (i)

$$\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$$

$$=\frac{2R\sin A\cos A+2R\sin B\cos B+2R\sin C\cos C}{2R\sin A+2R\sin B+2R\sin C}$$

$$=\frac{\sin 2A+\sin 2B+\sin 2C}{2(\sin A+\sin B+\sin C)}=\frac{1}{2}\frac{4\sin A\sin B\sin C}{4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}$$

$$=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{r}{R}$$

Q 3 (iii) P 58

49 Ans (ii),

$$\cot\frac{B}{2}\cot\frac{C}{2}=\sqrt{\left[\left(\frac{s(s-b)}{(s-c)(s-a)}\right)\frac{s(s-c)}{(s-a)(s-b)}\right]}$$

$$=\frac{s}{s-a}=\frac{2s}{2s-2a}$$

$$=\frac{a+b+c}{b+c-a}=\frac{a+3a}{3a-a}=2$$

8
Objective Questions.

Problem Set

1 An arc of a circle of length 11 cm subtends an angle of 30° at the centre of the circle Then the radius of the circle is

(i) (121/21) cm (ii) 21 cm
(iii) 22 cm (iv) 42 cm [take $\pi = 22/7$]

2 The value of $\cos 10° - \sin 10°$ is

(i) positive (ii) negative
(iii) 0 (iv) 1

3 The value of $\cos 1° \cos 2° \cos 3° \quad \cos 179°$ is

(i) $\frac{1}{\sqrt{2}}$ (ii) 0
(iii) 1 (iv) None of these

4 The value of $\tan 1° \tan 2° \tan 3° \quad \tan 89°$ is

(i) ∞ (ii) 0
(iii) 1 (iv) $\frac{1}{2}$

5 The maximum value of $24 \sin \theta + 7 \cos \theta$ is

(i) 1 (ii) 24
(iii) 25 (iv) 7

6 Which of the following is correct

(a) $\sin 1° > \sin 1$ (b) $\sin 1° < \sin 1$
(c) $\sin 1° = \sin 1$ (c) $\sin 1° = \frac{\pi}{180} \sin 1$

7 Which is greater $\tan 1$ or $\tan 2$

8 If $\sin \alpha = \sin \beta$, then the angles α and β are related by

(i) $\alpha = 2n\pi + (-1)^n \beta$ (ii) $\alpha = n\pi \pm \beta$
(iii) $\beta = n\pi + (-1)^n \alpha$ (iv) $\beta = (2n+1)\pi + \alpha$

where n is any integer

9 (a) The value of $\frac{1 - \tan^2 15°}{1 + \tan^2 15°}$ is

15 Suppose that $\sin^n x \sin 3x=\sum_{m=0}^{n} c_m \cos mx$ is an identity in x, where $c_0, c_1, \ldots, c_n$ are constants and $c_n \neq 0$. Then the value of n is (I I T 81)

16 The maximum distance of a point on the graph of the function $\sqrt{3} \sin x+\cos x$ from x axis is

(i) 4 (ii) 2

(iii) 1 (iv) $\sqrt{3}$

17 If θ and ϕ are angles in the first quadrant, such that $\tan \theta=1/7$ and $\sin \phi=1/\sqrt{(10)}$, then

(a) $\theta+2\phi=90°$ (b) $\theta+2\phi=30°$

(c) $\theta+2\phi=75°$ (d) $\theta+2\phi=45°$

18 The equation $\sin x+\cos x=2$ has

(i) only one solution (ii) two solutions

(iii) infinite number of solutions

(iv) no solution

19 For real x and y the equation $\sec^2 \theta=\frac{4xy}{(x+y)^2}$ is possible only when $x=y$

(a) True (b) False

20 If the angle α is in the third quadrant and $\tan \alpha=2$, then $\sin \beta=$

21 The equation $a \sin x+b \cos x=c$ where $|c| > \sqrt{(a^2+b^2)}$ has

(a) One solution (b) No solution.

(c) Infinite number of solutions.

22 $\sin \cot^{-1} \cos \tan^{-1} x=$

23 Let a, b, c be positive real numbers

Let $\theta=\tan^{-1}\sqrt{\left[\frac{a(a+b+c)}{bc}\right]}+\tan^{-1}\sqrt{\left[\frac{b(a+b+c)}{ca}\right]}+\tan^{-1}\sqrt{\left[\frac{c(a+b+c)}{ab}\right]}$

Then $\tan \theta$ equals (I I T 81)

24 From the top of a light house 60 meters high with its base at the sea level, the angle of depression of a boat is 15°. The distance of the boat from the foot of the light house is

(A) $\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)$ 60 meters, (B) $\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right)$ 60 meters

7 **Properties of geometrical figures**

(i) **Equilateral triangle** All sides equal

(ii) **Isosceles triangle** Two sides equal

(iii) **Rohmbus** All sides equal and no angle a right angle but diagonals are at right angles and unequal

(iv) **Square** All sides equal and each angle is a right angle The diagonals are also equal

(v) **Parallelogram** Opposite sides parallel and equal, diagonals bisect each other

(vi) **Rectangle** Opposite sides equal and each angle is a right angle Diagonals equal

8 **Coordinates of standard points**

(i) **Centroid of a triangle** The point is the intersection of the medians (i e line joining a vertex to the mid point of the opposite side) This point divides each median in the ratio 2 1

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$$

(ii) **Circum centre of a triangle** This is a point which is equidistant from the three vertices of the triangle It is also the point of intersection of right bisectors of the sides of the triangle (i e the lines through the mid point of a side and perpendicular to it) It is the centre of the circle that passes through the vertices of the triangle

(iii) **Incentre of a triangle** This is the centre of the circle which touches the sides of a given triangle It is the point of intersection of the internal bisectors of the angles of the triangle Its coordinates are given by the formula

$$x=\frac{ax_1+bx_2+cx_3}{a+b+c},\ y=\frac{ay_1+by_2+cy_3}{a+b+c}$$

where a, b, c are the lengths of the sides of the triangle

(iv) **Orthocentre of a triangle** This point is the intersection of the altitudes, (i e the lines through the vertices and perpendicular to opposite sides)

9 **Straight line (First degree) Equations**

(i) $Ax+By+C=0$ General form

(ii) $x=0$ y axis

(iii) $y=0$ x axis

(iv) $x=a$ Parallel to y axis

(a) True (b) False

33 The value of $\tan 9° - \tan 27° - \tan 63° - \tan 81°$ is

(a) 2, (b) 3,

(c) 4, (d) none of these

34 Given that $\pi < \alpha < \frac{3}{2}\pi$ Then the expression

$\sqrt{(4 \sin^4 \alpha + \sin^2 2\alpha)} + 4 \cos^2 (\pi/4 - \frac{1}{2}\alpha)$ is equal to

(a) 2, (b) $2+4 \sin \alpha$,

(c) $2-4 \sin \alpha$, (d) none

35 Given that $\frac{\pi}{2} < \alpha < \pi$ Then the expression

$$\sqrt{\left(\frac{1-\sin \alpha}{1+\sin \alpha}\right)}+\sqrt{\frac{(1+\sin \alpha)}{(1-\sin \alpha)}}=$$

(a) $\frac{1}{\cos \alpha}$ (b) $-\frac{2}{\cos \alpha}$,

(c) $\frac{2}{\cos \alpha}$, (d) none of these

36 If $\sin A = \sin B$, then we always have $\sin 2A = \sin 2B$

(a) True (b) False

37 Given $A = \sin^2 \theta + \cos^4 \theta$, then for all real values of θ

(a) $1 \leqslant A \leqslant 2$, (b) $\frac{3}{4} \leqslant A \leqslant 1$,

(c) $\frac{13}{16} \leqslant A \leqslant 1$, (d) $\frac{3}{4} \leqslant A \leqslant \frac{13}{16}$ (I I T 80)

38 The inequality $\cos \theta \leqslant 0$ holds for all real θ

(a) True (b) False

39 For all θ in $[0, \frac{1}{2}\pi]$, we have $\cos (\sin \theta) \geqslant \sin (\cos \theta)$

40 If $\tan \theta = -\frac{4}{3}$, then $\sin \theta$ is

(a) $-\frac{4}{5}$ but not $\frac{4}{5}$, (b) $-\frac{4}{5}$ or $\frac{4}{5}$

(c) $\frac{4}{5}$ but not $-\frac{4}{5}$ (d) None of these. (I I T 79)

41 If $\alpha+\beta+\gamma=2\pi$, then

(a) $\tan \frac{\alpha}{2}+\tan \frac{\beta}{2}+\tan \frac{\gamma}{2}=\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2}$,

(b) $\tan \frac{\alpha}{2} \tan \frac{\beta}{2}+\tan \frac{\beta}{2} \tan \frac{\gamma}{2}+\tan \frac{\gamma}{2} \tan \frac{\alpha}{2}=1$

12 Three points collinear

Find the equation of the line joining any two points by [2(x)] and show that the coordinates of the third point satisfies it (see also 6)

or If the three points be A, B, C then show that

slope of AB = slope of BC [9 (x)]

13 Angle between two given lines

$$\tan\theta = \frac{m_1 \sim m_2}{1 + m_1 m_2}$$

where m_1 and m_2 are the slopes of the given lines, $m_1 \sim m_2$ means the difference between m_1 and m_2 i e it stands for both $m_2 - m_1$ if $m_2 > m_1$ and for $m_1 - m_2$ if $m_1 > m_2$. If α is the angle between two given lines then $\pi - \alpha$ is also the angle between them and $\tan(\pi - \alpha) = -\tan\alpha$

14 Condition for two given lines to be parallel

In this case $\theta = 0$ or 180°, $\tan\theta = 0$, hence from 13

$$m_1 \sim m_2 = 0 \quad \text{or} \quad m_1 = m_2 \text{ i e their slopes are equal}$$

or

$$-\frac{A_1}{B_1} = -\frac{A_2}{B_2} \qquad \frac{A_1}{A_2} = \frac{B_1}{B_2}$$

i e coefficients of x and y are proportional in their equations

15 Methods to write a line parallel to a given line

Change only the constant term in the given equation, the terms of x and y remaining unchanged. The changed constant will be found by an additional given condition

For example, a line parallel to

$$2x - 3y - 5 = 0$$

is

$$2x - 3y + k = 0$$

16 Condition for two lines to be perpendicular

In this case, $\theta = 90^\circ$ and $\tan\theta = \tan 90^\circ = \infty$, hence from 13

$$m_1 m_2 = -1 \quad \text{or} \quad m_2 = -1/m_1$$

or Product of their slopes $= -1$

i e slope of one is negative reciprocal of the slope of the other

or $\left(-\frac{A_1}{B_1}\right)\left(-\frac{A_2}{B_2}\right) = -1$ or $A_1A_2 + B_1B_2 = 0$

i e Product of the coeffs of x + Product of the coeff of y = 0

17 Rule to write a line perpendicular to a given line

Interchange the coefficients of x and y in the given equation and change the sign in between them. Also change the

Solutions

1 Ans (ii)

If radius is a cm, then

$a\ \frac{\pi}{6}=11$ or $a=\frac{11\times 6}{\pi}=\frac{11\times 6\times 7}{22}=21$ cm

[Formula $s=a\theta$, that is, length of arc of a circle is equal to the product of the length of its radius and the magnitude of the angle in radians subtended by the arc at the centre of the circle]

2 Ans (i)

We know that if $0\leqslant\theta<45°$ then $\cos\theta>\sin\theta$

Hence $\cos 10°>\sin 10°$, that is $\cos 10°-\sin 10°>0$

3 Ans (ii)

Since $\cos 90°=0$, we have

$\cos 1°\cos 2°\cos 3° \ldots \cos 90° \ldots \cos 179°=0$

4 Ans (iii)

We have

$\tan 1°\tan 2°\tan 3° \ldots \tan 44°\tan 45°\tan 46° \ldots \tan 88°\tan 89°$

$=(\tan 1°\tan 89°)(\tan 2°\tan 88°) \ldots (\tan 44°\tan 46°)\tan 45°$

$=(\tan 1°\cot 1°)(\tan 2°\cot 2°) \ldots (\tan 44°\cot 44°)\tan 45°$

$=1$ [$\because \tan\theta\cot\theta=1$ for all θ and $\tan 45°=1$]

5 Ans (iii)

We have

$$24\sin\theta+7\cos\theta=25\left(\frac{24}{25}\sin\theta+\frac{7}{25}\cos\theta\right)$$

$$=25\,[\cos\alpha\sin\theta+\sin\theta\cos\alpha]$$

where $\cos\alpha=\frac{24}{25}$ so that $\sin\alpha=\frac{7}{25}$

$$=25\sin(\theta+\alpha)$$

Since the value of $\sin(\theta+\alpha)$ cannot exceed 1, the maximum value of

$24\sin\theta+7\cos\theta$ is 25

[Note we multiply and divide the given expression by $\sqrt{(24^2+7^2)}$ i e by 25]

6 Ans (b)

Since 1 radian$=57°$ approximately and $\sin 57°>\sin 1°$, we conclude that $\sin 1°<\sin 1$

7 Ans tan 1

Sin 1 radian is an acute angle and its apporximate value which will be the requird distance between the given parallel lines.

Note In all the questions to follow we shall give reference to the above formulae For the proofs of the above, students are advised to look to the reference books as mentioned Students are further advised to work out the problems independently and in case of difficulty look to the solutions given after the exercise

Problem Set (A)

Distance between two points

1 The vertices A B, C of a triangle are (2, 1), (5, 2) and (3, 4) respectively Find the coordinates of the circumcentre and also the radius of the circum circle

2 (a) Prove that the points (0, −1) (2, 1), (0, 3) and (−2, 1) are the vertices of a square

(b) Show that the points A (−4, −1), B (−2, −4) C (4, 0), D (2, 3) are the vertices of rectangle (Roorkee 73)

3 (a) Show that the points $(2a, 4a)$, $(2a, 6a)$, $(2a+\sqrt{3}a, 5a)$ are the vertices of an equilateral triangle

(b) The straight lines $x+y=0$, $3x+y-4=0$, $x+3y-4=0$ form a triangle which is

(i) isosceles (ii) equilateral

(iii) right angled (iv) none of these (IIT 83)

4 Show that the points (12, 8), (−2, 6) and (6, 0) are the vetices of right angled triangle

5 Find the coordinates of the incentre and centroid of the triangle whose vertices are (−36, 7) (20 7) (0, −8)

6 If the point $P(x, y)$ be equidistant from the points

A $(a+b, b-a)$ and B $(a-b, a+b)$

then prove that $bx=ay$

7 If G be the centroid of a triangle ABC and O be any other point then prove that

(i) $AB^2+BC^2+CA^2=3\,(GA^2+GB^2+GC^2)$

(ii) $OA^2+OB^2+OC^2=GA^2+GB^2+GC^2+3GO^2$

8 Prove analytically that in a right angled triangle the mid point of the hypotenuse is equidistant from the three angular points

9 Prove analytically that the line joining the mid points of two sides of a triangle is half the third side

are both positive when $\pi/4 < A/2 < 3\pi/4$ or in general $2n\pi+\pi/4 < A/2 < 2n\pi+3\pi/4$

14 Ans (c) We have $\sin x+\cos x=1$

or $\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x=\frac{1}{\sqrt{2}}$

$\sin x\cos\frac{\pi}{4}+\cos x\sin\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ or $\sin\left(x+\frac{\pi}{4}\right)=\sin\frac{\pi}{4}$

Solution is

$$x+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{4} \quad\text{or}\quad x=n\pi+(-1)^n\frac{\pi}{4}-\frac{\pi}{4}$$

15 Ans $n=6$ we have

$\sin^3 x\sin 3x=\frac{1}{4}(3\sin x-\sin 3x)\sin 3x$

$=\frac{3}{4}\sin x\sin 3x-\frac{1}{4}\sin^2 3x$

$=\frac{3}{8}\ 2\sin x\sin 3x+\frac{1}{8}\ 2\sin^2 3x$

$=\frac{3}{8}(\cos 2x-\cos 4x)-\frac{1}{8}(1-\cos 6x)$

$=-\frac{1}{8}+\frac{3}{8}\cos 2x-\frac{3}{8}\cos 4x+\frac{1}{8}\cos 6x$

Hence we have the identity

$$\sum_{m=0}^{n} c_m\cos mx=-\frac{1}{8}+\frac{3}{8}\cos 2x-\frac{3}{8}\cos 4x+\frac{1}{8}\cos 6x$$

or $c_0\ c_1\cos x+c_2\cos 2x+\quad c_n\cos nx$

$=-\frac{1}{8}+\frac{3}{8}\cos 2x-\frac{3}{8}\cos 4x+\frac{1}{8}\cos 6x$

Comparing, we see that $n=6$

[Observe that $c_1=c_3=c_5=0$ and $c_0=-\frac{1}{8}$, $c_2=\frac{3}{8}$, $c_4=-\frac{3}{8}$, $c_6=\frac{1}{8}$]

16 Ans (a) We have

$y=\sqrt{3}\sin x+\cos x=2\left(\frac{\sqrt{3}}{2}\sin x+\frac{1}{2}\cos x\right)$

$=2\sin\left(x+\frac{\pi}{6}\right)$

max value of $y=2$

17 Ans (d) We have

$\sin(\theta+2\phi)=\sin\theta\cos 2\phi+\cos\theta\sin 2\phi$

Now $\tan\theta=\frac{1}{7}$ gives $\sin\theta=\frac{1}{5\sqrt{2}}$ and $\cos\theta=\frac{7}{5\sqrt{2}}$

5 Determine the sides

$$BC=a=25,\ CA=b=39,\ AB=c=56$$

where A, B, C are the vertices as given. Then the incentre by the formula (8), (iii) Page 262 is given by

$$x=\frac{\Sigma ax_1}{\Sigma a},\ y=\frac{\Sigma ay_1}{\Sigma a} \text{ is } (-1, 0)$$

Centroid by 8 (i) i.e. $x=\frac{\Sigma x_1}{3},\ y=\frac{\Sigma y_1}{3}$ is $\left(-\frac{16}{3}, 2\right)$

6 $\qquad PA=PB, \qquad PA^2=PB^2$

or $[x-(a+b)]^2+[y-(b-a)]^2=[x-(a-b)]^2+[y-(a+b)]^2$

or $[(x-a)-b]^2-[(x-a)+b]^2=[(y-b)-a]^2-[(y-b)+a]^2$

or $\qquad -4(x-a)\,b=-4(y-b)\,a$

$\qquad [\quad (L-M)^2-(L+M)^2=-4LM]$

or $\qquad bx-ab=ay-ab$

or $\qquad bx=ay$

7. For the sake of convenience let us choose G as origin and the points A, B, C as (x_1, y_1), (x_2, y_2) and (x_3, y_3) respectively and the other point O be taken as (h, k). Coordinates of G are

$$\left(\frac{\Sigma x_1}{3}, \frac{\Sigma y_1}{3}\right)$$

But we have chosen G as origin

$$\frac{\Sigma x_1}{3}=0,\ \frac{\Sigma y_1}{3}=0$$

or $x_1+x_2+x_3=0$ and $y_1+y_2+y_3=0$

(i) $\quad AB^2+BC^2+CA^2=(x_1-x_2)^2+(y_1-y_2)^2+\ldots+$

$\qquad =2(\Sigma x_1^2+\Sigma y_1^2)-2(\Sigma x_1x_2+\Sigma y_1y_2)$

$=3(\Sigma x_1^2+\Sigma y_1^2)-\Sigma x_1^2-\Sigma y_1^2-2\Sigma x_1x_2-2\Sigma y_1y_2$

$\qquad$[By adding and subtracting $\Sigma x_1^2+\Sigma y_1^2$]

$=3(GA^2+GB^2+GC^2)-(x_1+x_2+x_3)^2-(y_1+y_2+y_3)^2$

$=3(GA^2+GB^2+GC^2) \quad [\quad \Sigma x_1=0,\ \Sigma y_1=0$ by (2)]

and $GA^2=(x_1-0)^2+(y_1-0)^2=x_1^2+y_1^2$ etc]

(ii) $OA^2+OB^2+OC^2=(h-x_1)^2+(k-y_1)^2+\ldots+$

$\qquad (x_1^2+y_1^2)+(x_2^2+y_2^2)+(x_3^2+y_3^2)+(h^2+k^2)+(h^2+k^2)$

$\qquad +(h^2+k^2)-2h(x_1+x_2+x_3)-2k(y_1+y_2+y_3)$

$=GA^2+GB^2+GC^2+GO^2+GO^2+GO^2-0-0$ by (1)

$\quad GA^2+GB^2+GC^2+3GO^2$

$\quad [\because \quad GO^2=(h-0)^2+(k-0)^2=h^2+k^2]$

$$=\frac{(a+b+c)^{3/2}}{\sqrt{(abc)}}$$

and $\tan A \tan B \tan C=\sqrt{\left[\frac{a(a+b+c)\,b(a+b+c)\,c(a+b+c)}{bc\;ca\;ab}\right]}$

$$=\frac{(a+b+c)^{3/2}}{\sqrt{(abc)}}$$

Hence $\tan A+\tan B+\tan C=\tan A \tan B \tan C$
and so $\tan \theta=0$

24 Ans (B)

25 Ans (a)

Note that $\frac{1-\cos B}{\sin B}=\frac{2\sin^2 B/2}{2\sin B/2\cos B/2}=\tan\frac{B}{2}$
so that $\tan A=\tan B/2$ which implies $A=n\pi+B/2$

$$\Rightarrow 2A=2n\pi+B$$

$$\Rightarrow \tan 2A=\tan(2n\pi+B)=\tan B$$

26 Ans $\cos\log\theta$.

For $e^{-\pi/2}<\theta<\frac{\pi}{2}$, we have
$\cos\log\theta>0$ and $\log\cos\theta<0$ It follows that
$\cos\log\theta>\log\cos\theta$

27 Ans (a)

Since $-<\alpha<\frac{3}{2}\pi$, we have $\frac{\pi}{2}<\frac{\alpha}{2}<\frac{3}{4}\pi$

and so $\cos\alpha<0$ and $\cos\frac{\alpha}{2}<0$

Hence,

$\sin\alpha=-\frac{3}{4}$ gives $\cos\alpha=-\frac{4}{5}$ But $2\cos^2\frac{\alpha}{2}-1=\cos\alpha=-\frac{4}{5}$

which gives $\cos^2\frac{\alpha}{2}=\frac{1}{10}$ and $\cos\frac{\alpha}{2}=-\frac{1}{\sqrt{10}}$

$$\left[\quad \frac{-}{2}<\frac{\alpha}{2}<\frac{3}{2}\pi\right]$$

28 Ans (a)

29 Ans (a)

$\sin\alpha=\frac{12}{13}\left(0<\alpha<\frac{1}{2}-\right)$ we have $\cos\alpha=\frac{5}{12}$

and $\cos\beta=-\frac{3}{5}\left(\pi<\beta<\frac{3}{2}\pi\right)$ gives $\sin\beta=-\frac{4}{5}$

Hence $\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$

$$=\frac{12}{13}\left(-\frac{3}{5}\right)+\frac{5}{13}\left(-\frac{4}{5}\right)=-\frac{56}{65}$$

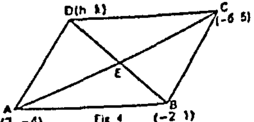


Fig 4

Then E is also the mid point of diagonal BD If D be the point (h, k) then

$$\left(\frac{h-2}{2}, \frac{k+1}{2}\right) \text{ is the mid point } E \text{ i e } \left(-\frac{3}{2}, \frac{1}{2}\right)$$

$$\frac{h-2}{2}=-\frac{3}{2} \text{ and } \frac{k+1}{2}=\frac{1}{2} \qquad h=-1, k=0$$

Hence the fourth vertex is the point $(-1, 0)$

13 Since the distance of origin from the given points is each equal to c as $\cos^2 \alpha+\sin^2 \alpha=1$, hence the three points will form an isosceles triangle with base as the given points and vertex as origin Now from geometry we know that in the case of an isosceles triangle the perpendicular from the vertex to the base bisects it Hence proved

14 Suppose that the coordinates of the points are (x, y), then by the given condition (1)

$\sqrt{[(x-ae)^2+y^2]}+\sqrt{[(x+ae)^2+y^2]}=2a$ (2)

Now $[(x-ae)^2+y^2]-[(x+ae)^2+y^2]=-4aex$

$[\quad (a-b)^2-(a+b)^2=-4ab\ ]$

On dividing (2) by 1, we get (3)

$\sqrt{[(x-ae)^2+y^2]}-\sqrt{[(x+ae)^2+y^2]}=-2ex$

$$\left[\quad \frac{L-M}{\sqrt{L}+\sqrt{M}}=\sqrt{L}-\sqrt{M}\quad\right]$$

Adding (1) and (3) we get

$2\sqrt{[(x-ae)^2+y^2]}=2\ (a-ex)$ Square

$x^2-2aex+a^2e^2+y^2=a^2-2aex+e^2x^2$

or $\quad x^2(1-e^2)+y^2=a^2(1-e^2)$

or $\quad \frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$

Problem Set (B)

Area of a triangle and collinear points

1 The coordinates of two points A and B are $(3, 4)$ and $(5, -2)$ respectively Find the coordinates of any point P if

$$A=5\cos\theta-3\cos\left(\theta+\frac{\pi}{3}\right)+3$$

$$=5\cos\theta+3\cos\theta\cos\frac{\pi}{3}-3\sin\theta\sin\frac{\pi}{3}+3$$

$$=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3=7\left(\frac{13}{14}\cos\theta-\frac{3\sqrt{3}}{14}\sin\theta\right)+3$$

$$=7[\cos\alpha\cos\theta-\sin\alpha\sin\theta]+3$$

where $\cos\alpha=\frac{13}{14}$ and $\sin\alpha=\frac{3\sqrt{3}}{14}$

$$=7\cos(\theta+\alpha)+3$$

Since $-1\leqslant\cos(\theta+\alpha)\leqslant 1$, we have

$$-4\leqslant A\leqslant 10$$

43 Ans (b)

It is clear that $\sin^8\theta+\cos^{14}\theta\geqslant 0$. But the equality $\sin^8\theta+\cos^{14}\theta=0$ can hold only when $\sin\theta=0$ and $\cos\theta=0$ simultaneously which is obviously impossible. Hence we must have $\sin^8\theta+\cos^{14}\theta>0$ (1)

Since $\sin^2\theta\leqslant 1$ and $\cos^2\theta\leqslant 1$, we have $\sin^8\theta\leqslant\sin^2\theta$ and $\cos^{14}\theta\leqslant\cos^2\theta$ and so (2)

$$\sin^8\theta+\cos^{14}\theta\leqslant\sin^2\theta+\cos^2\theta=1$$

From (1) and (2), we have

$$0<\sin^8\theta+\cos^{14}\theta\leqslant 1$$

44 Ans (iii)

45 Ans (ii) We have

$$\sqrt{\left(\frac{a+b}{a-b}\right)}+\sqrt{\left(\frac{a-b}{a+b}\right)}=\frac{a+b+a-b}{\sqrt{(a^2-b^2)}}=\frac{2a}{\sqrt{(a^2-b^2)}}$$

$$\frac{2}{\sqrt{\{1-(b/a)^2\}}}=\frac{2}{\sqrt{(1-\tan^2 x)}}=\frac{2\cos x}{\sqrt{(\cos^2 x-\sin^2 x)}}$$

$$=\frac{2\cos x}{\sqrt{(\cos 2x)}}$$

46 Ans (iii) For we have

$$\tan 54^\circ=\tan(45^\circ+9^\circ)=\frac{\tan 45^\circ+\tan 9^\circ}{1-\tan 45^\circ\tan 9^\circ}=\frac{1+\tan 9^\circ}{1-\tan 9^\circ}$$

$$=\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}$$

47 Ans (iii) For we have

$$\frac{\cos 17^\circ+\sin 17^\circ}{\cos 17^\circ-\sin 17^\circ}=\frac{1+\tan 17^\circ}{1-\tan 17^\circ}=\tan(45^\circ+17^\circ)=\tan 62^\circ$$

11 For what value of λ are the points $(\lambda, 2-2\lambda)$, $(-k+1, 2k)$, $(-4-k, 6-2\lambda)$ collinear

12 Show that the points A (2, 5), B (4, 6) and R (8, 8) are collinear

13 Prove that the quadrilateral whose vertices are A (2 5) B (4, −1), C (9, 1) and D (3, 7) is a parallelogram and find its area Find the coordinates of a point E in AC such that it divides AC in the ratio 2 1 Prove that D, E and F, the mid point of BC, are collinear

14 Show that the straight lines

$$7x-2y+10=0$$
$$7x+2y-10=0$$
and
$$y+2=0$$
(I I T 7

form an isosceles triangle and find its area

15 The points (0, 8/3) (1, 3) and (82, 30) are vertices of—
(a) Obtuse angled triangle (b) Acute angled triangle
(c) right angled triangle (d) Isosceles triangle (I I T 8
(e) none

Solutions to Problem Set (B)

1 Let the point P be $(x \; y)$
As $PA=PB$ $\quad PA^2=PB^2 \quad x-3y-1=0$ (
$\triangle PAB=10$ (Ref 5 261) $\quad 3x+y-23=0$ (
Solving (1) and (2) we get $x=7$, $y=2$ Point P is (7,

2 Let the vertex be $(x_r \; y_r)$ $r=1$ 2, 3 where both x_r and y_r a integers Hence its area $=\frac{1}{2}\,\Sigma x_1\,(y_2-y_3)=$ rational numb Also if a be its side then $a^2=(x_1-x_2)^2+(y_1-y_2)^2=a$ positi integer

But the area of an equilateral triangle

$$=\tfrac{1}{2}\,bc\,\sin A=\tfrac{1}{2}a\,a\,\sin 60^\circ=\frac{\sqrt{3}}{4}\,a^2$$

$$a=b=c$$

and $\angle A=\angle B=\angle C=60^\circ$

Area $=\dfrac{\sqrt{3}}{4}\,a^2$, which is irrational since a^2 is a positive integer

But we found earlier that area is a rational number Thus two statements are contradictory Therefore if the vertices are integers then that triangle cannot be an equilateral triangle

3 By Rule 3 the coordinates of point A which divides the join of P (−5, 1) and Q (3, 5) in the ratio λ 1 is

$A=5\cos\theta-3\cos\left(\theta+\frac{\pi}{3}\right)+3$

$=5\cos\theta+3\cos\theta\cos\frac{\pi}{3}-3\sin\theta\sin\frac{\pi}{3}+3$

$=\frac{13}{2}\cos\theta-\frac{3\sqrt{3}}{2}\sin\theta+3=7\left(\frac{13}{14}\cos\theta-\frac{3\sqrt{3}}{14}\sin\theta\right)+3$

$=7[\cos\alpha\cos\theta-\sin\alpha\sin\theta]+3$

where $\cos\alpha=\frac{13}{14}$ and $\sin\alpha=\frac{3\sqrt{3}}{14}$

$=7\cos(\theta+\alpha)+3$

Since $-1\leqslant\cos(\theta+\alpha)\leqslant 1$, we have

$-4\leqslant A\leqslant 10$

43 Ans (b)

It is clear that $\sin^8\theta+\cos^{14}\theta\geqslant 0$ But the equality $\sin^8\theta+\cos^{14}\theta=0$ can hold only when $\sin\theta=0$ and $\cos\theta=0$ simultaneously which is obviously impossible Hence we must have $\sin^8\theta+\cos^{14}\theta>0$ (1)

Since $\sin^2\theta\leqslant 1$ and $\cos^2\theta\leqslant 1$, we have $\sin^8\theta\leqslant\sin^2\theta$ and $\cos^{14}\theta\leqslant\cos^2\theta$ and so (2)

$\sin^8\theta+\cos^{14}\theta\leqslant\sin^2\theta+\cos^2\theta=1$

From (1) and (2), we have

$0<\sin^8\theta+\cos^{14}\theta\leqslant 1$

44 Ans (iii)

45 Ans (ii) We have

$\sqrt{\left(\frac{a+b}{a-b}\right)}+\sqrt{\left(\frac{a-b}{a+b}\right)}=\frac{a+b+a-b}{\sqrt{(a^2-b^2)}}=\frac{2a}{\sqrt{(a^2-b^2)}}$

$\frac{2}{\sqrt{\{1-(b/a)^2\}}}=\frac{2}{\sqrt{(1-\tan^2 x)}}=\frac{2\cos x}{\sqrt{(\cos^2 x-\sin^2 x)}}$

$=\frac{2\cos x}{\sqrt{(\cos 2x)}}$

46 Ans (iii) For we have

$\tan 54^\circ=\tan(45^\circ+9^\circ)=\frac{\tan 45^\circ+\tan 9^\circ}{1-\tan 45^\circ\tan 9^\circ}=\frac{1+\tan 9^\circ}{1-\tan 9^\circ}$

$=\frac{\cos 9^\circ+\sin 9^\circ}{\cos 9^\circ-\sin 9^\circ}$

47 Ans (iii) For we have

$\frac{\cos 17^\circ+\sin 17^\circ}{\cos 17^\circ-\sin 17^\circ}=\frac{1+\tan 17^\circ}{1-\tan 17^\circ}=\tan(45^\circ+17^\circ)=\tan 62^\circ$

7 (a) Let the third vertex be (p, q) which lies on $y=x+3$ ...(1)

$$q=p+3$$

$\Delta=\frac{1}{2}\{2(-2-q)+3(q-1)+p(1+2)\}=\pm 5$ given

$-4-2q+3q-3+3p=\pm 10$ or $q+3p-7=\pm 10$

+ive sign $q=17-3p=p+3$ by (1), $14=4p$

or $p=7/2$ and $q=13/2$

−ive sign $q=-3-3p=p+$ by (1), $4p=-6$

or $p=-3/2, q=3/2$

Third vertex is $(7/2, 13/2)$ or $(-3/2, 3/2)$

(b) $(1, -1)$ or $(-2, -10)$

8 $2\,\Delta DBC=\Delta ABC$ $\quad 2(14x-7)=\frac{49}{2}$ $\quad x=\frac{77}{56}=\frac{11}{8}$

9 $$\Delta=\frac{1}{2}\begin{vmatrix} a & b+c & 1 \\ b & c+a & 1 \\ c & a+b & 1 \end{vmatrix}=\frac{1}{2}\begin{vmatrix} a & a+b+c & 1 \\ b & a+b+c & 1 \\ c & a+b+c & 1 \end{vmatrix} \text{ by } C_2+C_1$$

$$=\frac{a+b+c}{2}\begin{vmatrix} a & 1 & 1 \\ b & 1 & 1 \\ c & 1 & 1 \end{vmatrix}=0 \quad \text{Two columns identical}$$

Since $\Delta=0$, the points are collinear

Note You may use

$$\Delta=\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}=0$$

10 Proceed as in Q 9

11 $$\Delta=\frac{1}{2}\begin{vmatrix} k & 2-2k & 1 \\ -k+1 & 2k & 1 \\ -4-k & 6-2k & 1 \end{vmatrix}=0 \text{ Apply } R_2-R_1 \text{ and } R_3-R_1$$

$$\Delta=\frac{1}{2}\begin{vmatrix} k & 2-2k & 1 \\ -2k+1 & 4k-2 & 0 \\ -4-2k & 4 & 0 \end{vmatrix}=0 \text{ or } \begin{vmatrix} -2k+1 & 4k-2 \\ -4-2k & 4 \end{vmatrix}=0$$

or $4(1-2k)-(-4-2k)(4k-2)=0$

or $(1-2k)+(k+2)(2k-1)=0$ or $1-2k+(2k^2+3k-2)=$

or $2k^2+k-1=0$ or $(k+1)(2k-1)=0$ $\quad k=\frac{1}{2}, -1$

12 Do yourself

CO-ORDINATE GEOMETRY

(*b*) Find the area of the triangle formed by *y* axis, the straight line *L* passing through the points (1, 1) and (2, 0) and the straight line perpendicular to the line *L* and passing through the point ($\frac{1}{2}$, 0) and show that it is $\frac{25}{16}$ sq units (I I T 72)

(*c*) The line $2x+3y=12$ meets the x axis at A and y axis at B. The line through (5, 5) perpendicular to AB meets the x axis and the line AB at C, D, E respectively. If O is the origin of coordinates, find the area of figure $OCEB$ (I I T 76)

10 Prove that the line $y-x+2=0$ divides the join of points $(3,-1)$ and $(8, 9)$ in the ratio 2 : 3

11 (*a*) Prove that perpendicular drawn from the point (4,1) on the join of $(2, -1)$ and $(6, 5)$ divides it in the ratio 5 : 8

(*b*) Find the ratio in which the join of $(-5, 1)$ and $(1, -3)$ divides the st. line passing through (3,4) and (7, 8)

12 If the straight line through the point P (3, 4) makes an angle $\pi/6$ with the x-axis and meets the line $12x+5y+10=0$ at Q find the length of PQ

13 The line joining two points A (2, 0), B (3, 1) is rotated about A in anti clock wise direction through an angle of 15°. Find the equation of the line in the new position. If B goes to C in the new position, what will be the coordinates of C?

14 Find the equation of the straight line which passes through (4, 5) and is (a) parallel (b) perpendicular to the straight line $3x-2y+5=0$

15 The vertices of a triangle OBC are O (0, 0), B $(-3,-1)$, $C(-1, -3)$. Find the equation of the line parallel to BC and intersecting the sides OB and OC whose perpendicular distance from the point (0, 0) is $\frac{1}{2}$ (I I T 1976)

16 (*a*) Prove that the equation to the straight line passing through the point $(a\cos^3\theta, a\sin^3\theta)$ and perpendicular to the line $x\sec\theta+y\csc\theta=a$ is $x\cos\theta-y\sin\theta=a\cos 2\theta$

(*b*) One side of a square makes an angle α with x axis and one vertex of the square is at the origin. Prove that the equations of its diagonals are

$$y(\cos\alpha-\sin\alpha)=x(\sin\alpha+\cos\alpha)$$

1
The Straight Line

(First Degree)

Reference Books

1 Coordinate Geometry by Dr H C Gupta

2 Coordinate Geometry by Prof M L Khanna

Important formulae

1 **Distance between two points**

$$\sqrt{[\{(x_2-x_1)^2+(y_2-y_1)^2\}]}$$

$=\sqrt{\{(\text{Diff of abscissas})^2+(\text{Diff of ordinates})^2\}}$

Distance from the origin $=\sqrt{(x_1^2+y_1^2)}$

2 Point which divides the join of two given points in a given ratio m_1 m_2 (Internally and Externally

$$x=\frac{m_1x_2+m_2x_1}{m_1+m_2},\ y=\frac{m_1y_2+m_2y_1}{m_1+m_2} \quad \text{(Internally)}$$

$$x=\frac{m_1x_2-m_2x_1}{m_1-m_2},\ y=\frac{m_1y_2-m_2y_1}{m_1-m_2} \quad \text{(Externally)}$$

Coordinates of any point on the join of (x_1, y_1) and (x_2, y_2)

$$\left(\frac{\lambda x_2+x_1}{\lambda+1}, \frac{\lambda y_2+y_1}{\lambda+1}\right)$$

This point divides the given line in the ratio λ 1

4 **Mid point** $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

5 **Area of a triangle**

$$\tfrac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$$

or $$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

Note that if one vertex (x_3, y_3) is at the origin (0, 0), then

Area $=\frac{1}{2}(x_1y_2-x_2y_1)$ $\quad x_3=0\ y_3=0$

6 **Condition for the three points to be collinear**

If the area of the triangle be zero then the three points will be collinear

(b) The points (1, 3) and (5, 1) are two opposite vertices of a rectangle The other two vertices lie on the line $y=2x+c$ Find c and the remaining vertices (I I T 1981)

25 (a) Two consecutive sides of a parallelogram are $4x+5y=0$ and $7x+2y=0$ If the equation to one diagonal is $11x+7y=9$, find the equation of other diagonal ((I.I T 1970)

(b) Show that the diagonal of the parallelogram whos sides are $u=p$, $u=q$, $v=r$, $v=s$ where $u=ax+by+c=0$ and $v=a'x+b'y+c'=0$ which passes through the points of intersection of $u=p, v=r$ and $u=q$, $v=s$ is given by

$$\begin{vmatrix} u & v & 1 \\ p & r & 1 \\ q & s & 1 \end{vmatrix}=0$$

26 The extremities of the diagonal of a square are the points (1, 5) and (8,8) Find the equation to its sides and the coordinates of the other vertices

27 (a) Find the equations to the straight lines passing through the point (4, 5) and equally inclined to the lines

$$3x=4y+7 \text{ and } 5y=12x+6$$

(b) A ray of light is sent along the line $x-2y+5=0$, upon reaching the line $3x-2y+7=0$, the ray is reflected from it Find the equation of the line containing the reflected ray

(c) From a point $(-2, 3)$ a ray of light is sent at an angle α ($\tan\alpha=3$) to the axis of x Upon reaching the x axis the ray is reflected from it Find the equation of the straight line which contains the reflected ray

28 Find the equation of the lines joining the origin to the points of trisection of the portion of the line $3x+y=12$ intercepted between the axes

29 Find the coordinates of the foot the perpendicular drawn from the point (2, 3) to the line $y=3x+4$,

30 A line $4x+y=1$ through the point A (2 - 7) meets the line BC whose equation is $3x-4y+1=0$ at the point B Find the equation to the line AC so that $AB=AC$ (I I T 1971)

(v) $y=b$ Parallel to x-axis

(vi) $y=mx+c$ Line which cuts off an intercept c on y-axis and makes an angle θ with the +ive direction (anticlockwise) of x-axis and $\tan\theta=m$ is called its slope or gradient

(vii) $y=mx$ Any line through the origin

(viii) $\frac{x}{a}+\frac{y}{b}=1$ Intercept form here a and b are the intercepts on the axis of x and y respectively

(ix) $y-y_1=m(x-x_1)$ Equation of a line through a given point (x_1, y_1) and having slope m

(x) $y-y_1=\frac{y_2-y_1}{x_2-x_1}(x-x_1)$ Equation of a line passing through two given points (x_1, y_1) and (x_2, y_2)

Its $\text{slope}=m=\frac{y_2-y_1}{x_2-x_1}=\frac{\text{Diff of ordinates}}{\text{Diff of abscissas}}$

(xi) $x\cos\alpha+y\sin\alpha=p$ Equation of a line on which the length of perpendicular from origin is p and α is the angle which this perpendicular makes with the +ive direction of x axis

(xii) $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$ This is another form of the equation (ix)

i e $y-y_1=m(x-x_1)$ Any point on this line is

$(x_1+r\cos\theta, y_1+r\sin\theta)$

Its distance from the given point

(x_1, y_1) is $\sqrt{\{(x_1+r\cos\theta-x_1)^2+(y_1+r\sin\theta-y_1)^2\}}$

$=\sqrt{\{r^2(\cos^2\theta+\sin^2\theta)\}}=r$

10 **Slope and intercepts on axes by the line $Ax+By+C=0$**

$\text{Slope}=m=\frac{-A}{B}=-\frac{\text{coeff of } x}{\text{coeff of } y}$

Intercepts : Put $y=0$ in the given equation and find

$$x=\frac{-C}{A}$$

and then put $y=0$ and find

$$y=\frac{-C}{B}$$

which are respectively the intercepts on the axis of x and y.

18 **Point of intersection of two given lines**

Solve their equations for x and y by the method of cross multiplication or by any other method

Since the intercepts are +ive, $a=3$, here $b=7-a=4$

Hence its equation is $\frac{x}{3}+\frac{y}{4}=1$ or $4x+3y=12$

7 $\alpha=30°$ Line is $x\cos 30°+y\sin 30°=p$ or $x\frac{\sqrt{3}}{2}+y\frac{1}{2}=p$

$a=$ Intercept on x axis, $y=0$, $a=\frac{2p}{\sqrt{3}}=OA$

$b=$ Intercept on y axis, $x=0$, $b=2p=OB$

Area of right angled $\triangle OBA=\frac{1}{2}ab=\frac{1}{2}\frac{2p}{\sqrt{3}}\ 2p=\frac{50}{\sqrt{3}}$ given.

$p^2=25 \qquad p=\pm 5$

Hence the lines are $x\sqrt{3}+y=\pm 10$

8 The line through (a, b) and (c, d) is

$$y-b=\frac{d-b}{c-a}(x-a), \text{ by (9 (x) p 263} \qquad (1)$$

The three points will be collinear if $(a-c, b-d)$ lies on (1)

$$\therefore \quad (b-d)-b=\frac{d-b}{c-a}(a-c-a) \text{ or } -d(c-a)=-c(d-b), \qquad (2)$$

$$ad=bc$$

The line (1) passes through (0, 0) if $-b(c-a)=-a(d-b)$

or $\qquad -bc+ab=-ad+ab$ or $ad=bc$

which is true by (2)

9 (a) Any line L perpendicular to $5x-y=1$ is $x+5y=k$ (9 17), Its intercepts on the axes are $a=k$ (as $y=0$), $b=k/5$ (as $x=0$) $\triangle=\frac{1}{2}ab=5$,

$$\frac{k^2}{10}=5 \text{ or } k^2=50, \qquad k=\pm 5\sqrt{2}$$

Hence the line L is $\quad x+5y=\pm 5\sqrt{2}$,

(c) $O(0, 0)$, $B(0, 4)$ $\quad C(5/3, 0)$, $E(3, 2)$

Area of $OCEB=\triangle_1+\triangle_2$

10 Any point on the line joining (3, −1) and (8, 9) dividing it in the ratio $\lambda : 1$ is $\left(\frac{8\lambda+3}{\lambda+1}, \frac{9\lambda-1}{\lambda+1}\right)$ If it lies on $y-x+2=0$, then

$(9\lambda-1)-(8\lambda+3)+2(\lambda+1)=0$, $3\lambda-2=0$, $\lambda=2/3$

Hence the required ratio is 2 : 3

11 (a) The line joining given points (2, −1) and (6, 5) is $\qquad$ (1)

$3x-2y=8 \qquad$ (9 (X) P.263)

constant term The value of the new constant is to be found by an additional given condition

For example, a line perpendicular to $2x-3y+5=0$ is

$$3x+2y+k=0$$

18 The general equation of a line through the intersection of two given lines $P=0$ and $Q=0$ is $P+\lambda Q=0$

The value of λ is to be found by an additional given condition

19 *Length of perpendicular from a given point (x_1, y_1) to a* given line $ax+by+c=0$ is $\dfrac{ax_1+by_1+c}{\sqrt{(a^2+b^2)}}$

i e Substitute the co-ordinates of the points and divide by square root of the sum of the squares of the coefficients of x and y

20 Equation of bisectors of given lines are

$$\frac{a_1x+b_1y+c_1}{\sqrt{(a_1^2+b_1^2)}}=\pm\frac{a_2x+b_2y+c_2}{\sqrt{(a_2^2+b_2^2)}}$$

Any point on the bisectors is equidistant from the given lines

21 Method to prove three lines to be concurrent

1st Method Find the point of intersection of any two lines and show that it satisfies the third also

2nd Method If $P=0$, $Q=0$, and $R=0$ be the equations of the given lines and if $lP+mQ+nR=0$ takes the form

$$0x+0y+0=0$$

where l, m and n are any three constants to be found by inspection then the three given lines are concurrent

3rd Method. If the given lines be

$$a_1x+b_1y+c_1=0,\quad a_2x+b_2y+c_2=0$$

ahd $$a_3x+b_3y+c_3=0$$

then they are concurrent if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}=0$$

22 Distance between parallel lines

First Method Find the distance of each from the origin and retain their signs If these be p_1 and p_2 respectively, the requird distance between parallel lines is $p_1 \sim p_2$

Second Method Choose a point on any of the lines (Put $x=0$ and find the value of y or put $y=0$ and find the valve of x) Now find the perpendicular distance of this point from the other

Hence any line parallel to BC will have its slope $=-1$

Hence its equation is

$$y=-x+c \quad \text{or} \quad x+y-c=0$$

Its distance from origin is 1/2 $\quad \dfrac{-c}{\sqrt{(1+1)}}=\pm\frac{1}{2}$

$$c=\pm\frac{\sqrt{2}}{2}$$

Required equation is $x+y\pm\dfrac{\sqrt{2}}{2}=0$

Now the lines OB and OC are in 3rd quadrant This line meets both OB and OC and hence it will also be in 3rd quadrant so that the intercepts on the axes will be $-$ve Therefore we should choose $+$sign out of $\pm$

Hence the required line

$$\text{is } x+y+\frac{\sqrt{2}}{2}=0 \quad \text{or} \quad 2x+2y+\sqrt{2}=0$$

Note You should draw a figure,

16 (a) Slope of given line $=-\dfrac{\sec\theta}{\operatorname{cosec}\theta}=-\dfrac{\sin\theta}{\cos\theta}$ Rule 10 P 263.

Slope of a line perpendicular to it will be $\dfrac{\cos\theta}{\sin\theta}$ Rule 16 P 264.

The line passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and hence its equation is $(y-a\sin^3\theta)=\dfrac{\cos\theta}{\sin\theta}(x-a\cos^3\theta)$

or $\quad x\cos\theta-y\sin\theta=a(\cos^4\theta-\sin^4\theta)=a\cos 2\theta$

$\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)$
$=(\cos 2\theta)\ 1=\cos 2\theta$

(b) Let the side OA make an angle α with x axis and since the side of the square $OABC$ is a, therefore co ordinates of point A are $(a\cos\alpha, a\sin\alpha)$ Now the diagonal OB will make an angle of $45°+\alpha$ with axis and pass through origin O Hence its equation is

$$y=\tan(45°+\alpha)\ x \quad \text{or} \quad y=\frac{1+\tan\alpha}{1-\tan\alpha}x$$

or $y(\cos\alpha-\sin\alpha)-x(\cos\alpha+\sin\alpha)=0$

10 In any triangle prove that

$$AB^2+AC^2=2\,(AO^2+OC^2)$$

where O is mid point of BC

11 If O be the origin and if coordinates of any two points Q_1 and Q_2 be (x_1, y_1) and (x_2, y_2) respectively, prove that

$$OQ_1\ OQ_2 \cos Q_1OQ_2 = x_1x_2 + y_1y_2$$ (I I T 61)

12 The extremities of the diagonal of a parallelogram are the points $(3, -4)$ and $(-6, 5)$ Third vertex is the point $(-2, 1)$ Find the coordinates of the fourth vertex

13 Prove that perpendicular from origin to the line joining the points

$$(c\cos\alpha, c\sin\alpha) \quad \text{and} \quad (c\cos\beta, c\sin\beta)$$

bisects it as well

14 A point moves such that the sum of its distances from two fixed points $(ae, 0)$ and $(-ae, 0)$ is always $2a$ prove that the equation of the locus is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2(1-e^2)}=1$$ M N R 81)

Solutions to Problem Set (A)

1 Let $P(x, y)$ be the co ordinates of the centre of the circum-circle

$PA^2=PB^2$ and $PB^2=PC^2$ Result I, page 261

$3x+y=12$ and $x+3y=10$

Solving these two we get the centre as $(13/4\ 9/4)$ Having found P we can now say that

$$PA^2=r^2=\frac{50}{16} \qquad r=\frac{5}{4}\sqrt{2}$$

2 (*a*) Show that

$AB=BC=CD=DA$ and $AC^2=AB^2+BC^2$ i e $\angle B=\frac{\pi}{2}$

(*b*) Do yourself

3 (*a*) Show that

$AB=BC=CA=2a$

(*b*) Solving the equations in pairs we get the points $A(1, 1)$, $B(2, -2)$ $C(-2, 2)$

Clearly $AB=AC=\sqrt{10}$ Hence Δ is isosceles

4 Determine the lengths of sides and show that sum of the squares of two is equal to square of the third

Hence any line parallel to BC will have its slope $=-1$

Hence its equation is

$$y=-x+c \quad \text{or} \quad x+y-c=0$$

Its distance from origin is 1/2 $\quad \frac{-c}{\sqrt{(1+1)}}=\pm\frac{1}{2}$

$$c=\pm\frac{\sqrt{2}}{2}$$

Required equation is $x+y\pm\frac{\sqrt{2}}{2}=0$

Now the lines OB and OC are in 3rd quadrant This line meets both OB and OC and hence it will also be in 3rd quadrant so that the intercepts on the axes will be $-$ve Therefore we should choose $-$sign out of $\pm$

Hence the required line

$$\text{is } x+y+\frac{\sqrt{2}}{2}=0 \quad \text{or} \quad 2x+2y-\sqrt{2}=0$$

Note You should draw a figure.

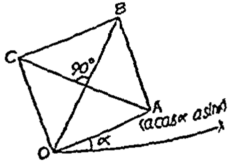
16 (a) Slope of given line $=-\frac{\sec\theta}{\operatorname{cosec}\theta}=-\frac{\sin\theta}{\cos\theta}$ Rule 10 P 263

Slope of a line perpendicular to it will be $\frac{\cos\theta}{\sin\theta}$ Rule 16 P 264

The line passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and hence its equation is $(y-a\sin^3\theta)=\frac{\cos\theta}{\sin\theta}(x-a\cos^3\theta)$

or $\quad x\cos\theta-y\sin\theta=a(\cos^4\theta-\sin^4\theta)=a\cos 2\theta$

$\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)$
$=(\cos 2\theta)\quad 1=\cos 2\theta$



(b) Let the side OA make an angle α with x-axis and since the side of the square $OABC$ is a therefore co-ordinates of point A are $(a\cos\alpha, a\sin\alpha)$ Now the diagonal OB will make an angle of $45°+\alpha$ with axis and pass through origin O Hence its equation is

$$y=\tan(45°+\alpha)\ x \quad \text{or} \quad y=\frac{1+\tan\alpha}{1-\tan\alpha}x$$

or $y(\cos\alpha-\sin\alpha)-x(\cos\alpha+\sin\alpha)=0$

8 Choose the vertices of the right angled triangle as $O(0, 0)$, $A(a, 0)$, $B(0, b)$

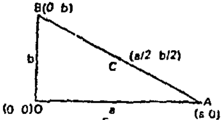


Fig 1

Since the triangle is right angled

$$AB^2 = a^2 + b^2$$

or $AB = \sqrt{(a^2+b^2)}$

If C be the mid point of AB then $CA = CB = \frac{1}{2}AB = \frac{1}{2}\sqrt{(a^2+b^2)}$

Also $CO^2 = (a/2-0)^2 + (b/2-0)^2 = \frac{1}{4}(a^2+b^2)$ so that

$CO = \frac{1}{2}\sqrt{(a^2+b^2)}$

Hence $CA = CB = CO$ Proved

9 Let the base $BC = 2a$ be chosen along x-axis and its mid point be taken as origin so that the coordinates of the point B are $(-a, 0)$ and C are $(a, 0)$

Let A be the point (h, k)

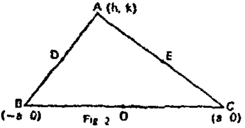


Fig 2

The mid point of AB is $D\left(\frac{h-a}{2}, \frac{k}{2}\right)$ and of AC is $E\left(\frac{h+a}{2}, \frac{k}{2}\right)$

$$DE^2 = \left(\frac{h-a}{2} - \frac{h+a}{2}\right)^2 + \left(\frac{k}{2} - \frac{k}{2}\right)^2 = a^2 + 0 = a^2$$

$$DE = a = \tfrac{1}{2}(2a) = \tfrac{1}{2}BC$$

10, Refer figure Q 9 and prove

11 From triangle OQ_1Q_2 by applying cosine formula we get

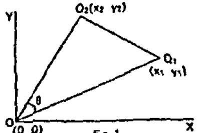


Fig 3

$$Q_1Q_2^2 = OQ_1^2 + OQ_2^2 - 2OQ_1\,OQ_2 \cos Q_1OQ_2$$

or $(x_1-x_2)^2 + (y_1-y_2)^2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2OQ_1OQ_2 \cos\theta$$

or $x_1^2 + x_2^2 - 2x_1x_2 + y_1^2 + y_2^2 - 2y_1y_2$

$$= x_1^2 + y_1^2 + x_2^2 + y_2^2 - 2OQ_1\,OQ_2 \cos\theta$$

or $x_1x_2 + y_1y_2 = OQ_1\,OQ_2 \cos Q_1OQ_2$

12 We know that in a parallelogram the diagonals bisect each other Mid point E of diagonal AC is

$$\left(\frac{3-6}{2}, \frac{-4+5}{2}\right) \text{ or } \left(-\frac{3}{2}, \frac{1}{2}\right)$$

Hence any line parallel to BC will have its slope $=-1$
Hence its equation is

$$y=-x+c \quad \text{or} \quad x+y-c=0$$

Its distance from origin is 1/2 $\qquad \frac{-c}{\sqrt{(1+1)}}=\pm\frac{1}{2}$

$$c=\pm\frac{\sqrt{2}}{2}$$

Required equation is $x+y\pm\frac{\sqrt{2}}{2}=0$

Now the lines OB and OC are in 3rd quadrant. This line meets both OB and OC and hence it will also be in 3rd quadrant so that the intercepts on the axes will be $-$ive. Therefore we should choose $+$sign out of $\pm$
Hence the required line

$$\text{is } x+y+\frac{\sqrt{2}}{2}=0 \quad \text{or} \quad 2x+2y+\sqrt{2}=0$$

Note You should draw a figure,

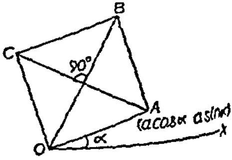
16 (a) Slope of given line $=-\frac{\sec\theta}{\operatorname{cosec}\theta}=-\frac{\sin\theta}{\cos\theta}$ Rule 10 P 263,

Slope of a line perpendicular to it will be $\frac{\cos\theta}{\sin\theta}$
Rule 16 P 264

The line passes through the point $(a\cos^3\theta, a\sin^3\theta)$ and hence its equation is $(y-a\sin^3\theta)=\frac{\cos\theta}{\sin\theta}(x-a\cos^3\theta)$

or $\quad x\cos\theta-y\sin\theta=a(\cos^4\theta-\sin^4\theta)=a\cos 2\theta$

$\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)$
$=(\cos 2\theta)\ 1=\cos 2\theta$



(b) Let the side OA make an angle α with x axis and since the side of the square $OABC$ is a, therefore co-ordinates of point A are $(a\cos\alpha, a\sin\alpha)$. Now the diagonal OB will make an angle of $45°+\alpha$ with axis and pass through origin O. Hence its equation is

$$y=\tan(45°+\alpha)\ x \quad \text{or} \quad y=\frac{1+\tan\alpha}{1-\tan\alpha}x$$

or $y(\cos\alpha-\sin\alpha)-x(\cos\alpha+\sin\alpha)=0$

$$PA = PB \text{ and } \triangle PAB = 10$$

2 If the vertices of a triangle have integral coordinates, prove that the triangle cannot be equilateral (I I T 75)

3 The point A divides the join of P $(-5, 1)$ and Q $(3, 5)$ in the ratio k 1 Find the two values of k for which the area of $\triangle ABC$ where B is $(1, 5)$ and C $(7, -2)$ is equal to 2 units (I I T 67)

4 The coordinates of three points O, A, B are $(0, 0)$, $(0\ 4)$ and $(6, 0)$ respectively A point P moves so that area of $\triangle POA$ is always twice the area of $\triangle POB$ Find the equation to both parts of the locus of P (I.I T 64)

5 Prove that the area of a triangle is four times the area of a triangle formed by joining the mid point of its sides

6 The equations to the sides of a triangle are

$$y = m_1 x + c_1, \quad y = m_2 x + c_2 \text{ and } x = 0$$

prove that its area is

$$\frac{1}{2}\,\frac{(c_1 - c_2)^2}{m_1 - m_2}$$

7 (*a*) The area of a triangle is 5 Two of its vertices are $(2, 1)$ and $(3, -2)$ The third vertex lies on $y = x + 3$ Find the third vertex (I I T 78)

(*b*) The area of a triangle is 3/2 sq units Two of its vertices are the points A $(2, -3)$ and B $(3, -2)$ the centroid of the triangle lies on the line $3x - y - 8 = 0$ Find the third vertex C

8 (*a*) If the coordinates of points A B, C D are $(6, 3)$, $(-3, 5)$, $(4, -2)$ and $(x, 3x)$ respectively and if $\dfrac{\triangle DBC}{\triangle ABC} = \frac{1}{2}$ then prove that the value of x is $\dfrac{11}{8}$.

(*b*) The coordinates of A B, C are $(6, 3)$ $(-3, 5)$ and $(4, -2)$ respectively and P is any point (x, y) Show that the ratio of the areas of the triangles PBC and ABC is

$$\left|\frac{x + y - 2}{7}\right|$$ (I I T 83)

9 Prove that the points $(a\ b + c)$, $(b, c + a)$, $(c, a + b)$ are collinear

10 Prove that the points $(a, 0)$, $(0, b)$ and $(1, 1)$ will be collinear if $\dfrac{1}{a} + \dfrac{1}{b} = 1$

Solving (1) and (2), we get the point B as $\left(\frac{-1}{2}, \frac{5}{2}\right)$ i e (x_1, y_1)

If the point D be $(x_2 \ \ y_2)$ then mid points of AC and BD are same

$$\frac{3+1}{2}=\frac{x_1+x_2}{2} \quad \text{or} \quad x_2=4-x_1=4+\frac{1}{2}=\frac{9}{2}$$

$$\frac{4-1}{2}=\frac{y_1+y_2}{2} \quad \text{or} \quad y_2=3-y_1=3-\frac{5}{2}=\frac{1}{2}$$

Point D is (9/2, 1/2)

Note We could directly find the point D by writing the equations of AD and CD as above whose slopes are known and then solving them

(b) Hint The sides of the square through given vertex will be inclined at $\pm 45°$ to the diagonal

Ans $23x-7y=9,\ 7x+23y=53$

(c) $x-2y+3=0,\ 2x+y-4=0$ and diagonal is $3x-y-1=0$

24 AB is $4x+7y+5=0$ and the coordinates $(-3, 1)$ of the point A satisfies it

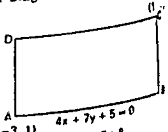


Fig 9

AD is $\perp$ to AB and passes through $(-3, 1)$

AD is $7x-4y+k=0$

where $7(-3)-4(1)+k=0$

$k=25$

Equation of AD is $7x-4y+25=0$

DC is parallel to AB and passes through C (1, 1)

DC is $4x+7y+k=0$ where $4 \cdot 1+7 \cdot 1+k=0$, $\quad k=-11$

Eq to DC is $\quad 4x+7y-11=0$

CB is parallel to AD and passes through (1, 1)

Eq to CB is

$7x-4y+k'=0$ where $7 \cdot 1-4 \cdot 1+k'=0 \quad k'=-3$

CB is $\quad 7x-4y-3=0$

(b) Let A (1, 3) and C be (5 1) whose mid point (3, 2) will be on the line joining other vertices as the diagonals bisect $\quad 2=2 \cdot 3+c \qquad c=-4$ (1)

Hence the other diagonal is $y=2x-4$

$$\left(\frac{3k-5}{k+1}, \frac{5k+1}{k+1}\right)$$

Also B is (1, 5) and C is (7, −2)

Area of $\triangle ABC=\frac{1}{2}\{x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)\}$

$$=\pm 2 \text{ (given)}$$

or $\frac{1}{2}\left\{\frac{3k-5}{k+1}(5+2)+1\left(-2-\frac{5k+1}{k+1}\right)+7\left(\frac{5k+1}{k+1}-5\right)\right\}$

$$=\pm 2$$

or $\quad 21k-35-7k-3-28=\pm 4(k+1)$

$14k-66=4k+4 \quad$ or $\quad 14k-66=-4k-4$

$10k=70 \quad$ or $\quad 18k=62 \qquad k=7$ or $31/9$.

4 The three given points are origin, one on x axis the other on y axis. Let P be (x, y)

We know that area of a triangle $=\frac{1}{2}$ base $\times$ height

$\triangle POA=2\,\triangle POB$

$\frac{1}{2}\,4x=\pm 2\,\frac{1}{2}\,6y$

or $2x=\pm 6y$ or $x=\pm 3y$

Hence the required locus of both the parts is

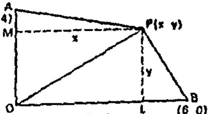


Fig 5

$$(x-3y)(x+3y)=0$$

5 Refer figure Q 9 P 269

Area of $\triangle ABC=\frac{1}{2}$ base $\times$ height $=\frac{1}{2}(2a)k=ak$

Area of $\triangle ODE$ where O, D and E are the mid points of the sides and O is (0, 0), D is

$$\left(\frac{h-a}{2}, \frac{k}{2}\right), E \text{ is } \left(\frac{h+a}{2}, \frac{k}{2}\right)$$

Hence by the formula $\Delta=\frac{1}{2}(x_1y_2-x_2y_1)$ as one vertex is origin (Ref 5 Page 261)

$$\Delta\; ODE=\frac{1}{2}\left\{\frac{h-a}{2}\;\frac{k}{2}-\frac{h+a}{2}\;\frac{k}{2}\right\}=\frac{1}{8}(-2ak)$$

$$=-\frac{1}{4}ak=\frac{1}{4}ak$$

6 Solving in pairs the coordinates of the vertices are

$$\left(\frac{c_1-c_2}{m_2-m_1}, \frac{m_2c_1-m_1c_2}{m_2-m_1}\right) \ (0, c_1) \text{ and } (0, c_2)$$

Area $=\frac{1}{2}x_1(y_2-y_3) \qquad x_2=0 \;\; x_3=0$

$$=\frac{1}{2}\,\frac{c_1-c_2}{m_2-m_1}\,(c_1-c_2)=\frac{1}{2}\,\frac{(c_1-c_2)^2}{m_2-m_1}$$

13 $AB=CD=\sqrt{(72)}$
$BC=AD=\sqrt{(29)}$
Area $=2\ \triangle\ ABC=42$
E is $(16/3,\ 7/3)$,
F is $(13/2,\ 0)$
Now show as usual that D, E, F are collinear

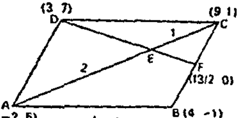


Fig 6

14 Solving in pairs the vertices are
$A\ (0\ \ 5)\ \ B\ (-2,\ -2),\ C\ (2,\ -2)$
Clearly $AB=AC=\sqrt{(5_5)}$ Hence isosceles and area $=14$ units

15 If the given points be A, B and C then
$m_1=$ slope of $AB=\dfrac{3-8/3}{1-0}=1/3$
$m_2=$ slope of $BC=\dfrac{30-3}{82-1}=\dfrac{27}{81}=1/3$
Hence the points are collinear E is correct

Problem Set (C)

The straight line

Find the equation of the following lines

1 Which makes an angle α with x-axis and cuts an intercept of length a on it
2 Whose slope is 3 and bisects the join of the points $(-2,\ 5)$ and $(3,\ 4)$
3 The portion of which intercepted between the axes is divided by the point $(-^5,\ 4)$ in the ratio 1 2
4 Which passes through the points $(3,\ 3)$ and $(7\ \ 6)$ and find the length of the intercept cut off by the axes
5 Whose intercept by the axes is bisected at the point (x_1, y_1)
6 Which passes through the point $(-3,\ 8)$ and cuts off $+$ ive intercepts on the axes whose sum is 7
7 On which the perpendicular from origin makes an angle of 30° with x-axis and which forms a triangle of area $50/\sqrt{3}$ with the axes
8 Prove that the points $(a,\ b)$, $(c,\ d)$ and $(a-c,\ b-d)$ are collinear if $ad=bc$ Also show that the straight line passing through these points passes through origin.
9 (a) straight line L is perpendicular to the line $5x-y=1$ The area of the triangle formed by the line L and coordinate axes is 5 Find the equation of the line
(I I T 1980)

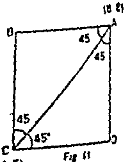


Fig 11

26 Slope of $AC=\frac{8-5}{8-1}=\frac{3}{7}$

The lines AB and AD pass through (8, 8) and are inclined at an angle of $\pm 45°$ to AC whose slope is 3/7 Hence proceeding as in Ex 18 their equations are

AB $5x-2y-24=0$

AD $2x+5y-56=0$

CD is parallel to AB and passes through (1, 5),

CD is $5x-2y+5=0$

CB is parallel to AD and passes through (1, 5),

CB is $2x+5y-27=0$

Solving AB and CB we get the point B as

$$\frac{x}{54+120}=\frac{y}{-48+135}=\frac{1}{25+4} \quad \therefore \; B \text{ is } \left(\frac{174}{29}, \frac{87}{29}\right)$$

Solving AD and CD we get the point D as

$$\frac{x}{112-25}=\frac{y}{10+280}=\frac{1}{25+4}, \quad \therefore \; D \text{ is } \left(\frac{87}{29}, \frac{380}{29}\right)$$

27 (a) The slopes of the given lines are 3/4 and 12/5 respectively. If m be the slope of the required line then since it passes through (4, 5) its equation is (1)

$$y-5=m(x-4)$$

By the given condition we have

$$\tan\theta=\frac{m-3/4}{1+m\,3/4}=-\frac{m-\frac{12}{5}}{1+m\frac{12}{5}}$$

or $(4m-3)(12m+5)=-(5m-12)(3m+4)$

or $48m^2 \; 16m \; 15=-15m^2+16m+48$

or $63m^2-32m-63=0$ or $63m^2-81m+49m \; 63=0$

or $(7m-9)(9m+7)=0$ $\quad m=9/7$ or $-7/9$

On putting the values of m in (1) the required lines are

$$9x-7y=1 \quad \text{and} \quad 7x+9y=73 \text{ respectively}$$

(b) Students of Physics know that the incident ray and refracted ray are equally inclined to the normal to the surface Here incident ray is along the line $x-2y+5=0$ (slope $m_1=1/2$) Surface is given by the line $3x-2y+7=0$ (slope $=3/2$) and the incident ray meets it at P whose co-ordinates are easily found to be $(-1\;2)$, Normal to the surface (i e perpendicular) will have slope $-2/3=m$ If the slope of reflected ray be m_2 then it passes through $P(-1\;2)$ and hence its equation is and $y(\sin\alpha+\cos\alpha)+x(\cos\alpha-\sin\alpha)=a$ where a is the length of the side of the square

17 Find the equation to the straight line which divides the join of the points (2, 3) and (−5, 8) in the ratio 3 4 and is also perpendicular to it

18 Find the equation to the straight lines passing through the point (3, −2) and inclined at 60° to the line $\sqrt{3}x+y=1$ **(I I T 1974)**

19 A vertex of an equilateral triangle is (2,3) and the equation of the opposite side is $x+y=2$ Find the equation of the other sides of the triangle **(I I T 1975)**

20 Show that the point (3, −5) lies between the parallel lines $2x+3y=7$ and $2x+3y+12=0$ and find the equation of lines through (3, −5) cutting the above lines at an angle of 45° **(Roorkee 1958)**

21 Find the equations to the straight lines passing through the point (2 3) and inclined at an angle of 45° to the line

$$3x+y-5=0$$

22 Find the equation to the sides of an isosceles right angled triangle the equation of whose hypotenuse is $3x+4y=4$ and the opposite vertex is the point (2, 2)

23 (*a*) The opposite angular points of a square are (3, 4) and (1, −1) Find the co-ordinates of the other two vertices **(Roorkee 1985)**

(*b*) The diagonal of a square lies along the line $8x-15y=0$ and one vertex of the square is (1, 2) Find the equations to the sides of the square passing through this vertex

(*c*) (1, 2), (3, 8) are a pair of opposite vertices of a square Find the equation of the sides and the diagonal of the square passing through (1, 2) **(Roorkee 1981)**

24 (*a*) One side of a rectangle lies along the line $4x+7y+5=0$ Two of its vertices are (−3, 1) and (1, 1) Find the equations of the other three sides **(I I T**

28 On putting $y=0$ and then $x=0$ we get the coordinates of the points where the line meets the axes as

$A\ (4, 0),\ B\ (0, 12)$

The points of trisection are the points which divide AB in the ratio 1 2 and 2 1

C is $(8/3, 4)$ $D\ (4/3, 8)$ and O is $(0, 0)$

OC is $(y\ \ 0)=\frac{4\ \ 0}{8/3\ \ 0}(x-0)$ or $2y=3x$ (1)

Similarly OD is $y=6x$

29 The given line is $3x-y+4=0$ (1)

Any line perpendicular to it and passing through (2, 3) is $x+3y+k=0$ where $2+3\ 3+k=0$ or $k=-11$

i e $x+3y-11=0$ (2)

Solving (1) and (2) as in Q 26 the coordinates of foot of perpendicular are

$$\left(-\frac{1}{10}\ \ \frac{37}{10}\right)$$

30 AB is $4x+y=1$ Its slope is -4

BC is $3x\ \ 4y+1=0$ Its slope is $3/4$

They meet at B and if α be the angle between them, then

$$\tan\alpha=\frac{-4-3/4}{1-4\ 3/4}=\frac{19}{8}$$ (1)

Since $AB=AC$ therefore $\triangle ABC$ is isosceles triangle whose vertex A is $(2\ \ 7)$ and base angle is α Also slope of base BC is $3/4$

Any line through $(2, -7)$ is

$$y+7=m\ (x-2)$$ (2)

$$\tan(\pm\alpha)=\frac{m-3/4}{1+m\ 3/4}=\frac{4m-3}{4+3m}$$

or $\pm\frac{19}{8}=\frac{4m-3}{4+3m}$ by (1)

or $19\ (4+3m)=\pm 8\ (4m-3)$ or $76+57m=\pm(32m-24)$

Taking +ive sign we get $m=-4$

Taking −ive sign we get $m=-52/89$

The value of $m=-4$ corresponds to slope of given line AB

The other value of $m=-52/89$ corresponds to the other AC

Equation of AC from (2) is

31 Two sides of an isosceles triangle are given by the equation $7x-y+3=0$ and $x+y-3=0$ and its third side passes through *the point (1, −10). Determine the equation of the third side*
(I I T 1984)

32. The equations of perpendicular bisectors of the sides AB and AC of a triangle ABC are $x-y+5=0$ and $x+2y=0$ respectively. If the point A is $(1, -2)$, find the equation of the line BC
(I I T 1986)

Solutions to Problem set (C)

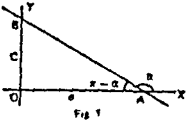


Fig 1

1 $m=\tan\alpha$, $c=a\tan(\pi-\alpha)$

$=-a\tan\alpha$

$y=mx+c$ $\quad y=x\tan\alpha-a\tan\alpha$

or $x-y\cot\alpha=a$

2 Mid point is $(1/2, 9/2)$, $m=3$

$y-y_1=m(x-x_1)$

or $y-9/2=3(x-\frac{1}{2})$

or $3x-y+3=0$

3 $\frac{x}{a}+\frac{y}{b}=1$ $\quad A(a, 0), B(0, b)$

Now $(-5, 4)$ divides AB in the ratio 1 : 2

$$-5=\frac{1\cdot 0+2a}{2+1},\ 4=\frac{1\cdot b+2\cdot 0}{2+1} \qquad a=-\frac{15}{2}, b=12$$

Hence the line is $5y-8x=60$

4 Slope $=\frac{y_2-y_1}{x_2-x_1}=\frac{6-3}{7-3}=\frac{3}{4}$

$y-3=\frac{3}{4}(x-3)$ or $3x-4y+3=0$

Intercept on x axis put $y=0$ $\quad a=-1$

Intercept on y-axis put $x=0$ $\quad b=3/4$

Length of the portion intercepted between the axes

$=\sqrt{(a^2+b^2)}=5/4$

5 $A(a, 0)$, $B(0, b)$, mid point of AB is $(a/2, b/2)=(x_1, y_1)$ given

$x_1=a/2$ or $a=2x_1$, $y_1=b/2$, $\quad b=2y_1$,

Line is $\frac{x}{a}+\frac{y}{b}=1$ or $\frac{x}{2x_1}+\frac{y}{2y_1}=1$ or $xy_1+yx_1=2x_1y_1$

6 Here $a+b=7$ and $-3/a+8/b=1$ as $x/a+y/b=1$ passes through $(-3, 8)$

$-3b+8a=ab$ but $b=7-a$,

$-3(7-a)+8a=a(7-a)$ or $a^2+4a-21=0$,

$(a+7)(a-3)=0$, $\quad a=3, -7$

(iii) Is parallel to the line

$$3x-4y+5=0$$

2 Find the value of λ such that the straight line

$$(2x+3y+4)+\lambda(6x-y+12)=0$$

(i) Is parallel to y-axis

(ii) Is perpendicular to the line

$$7x+5y=4$$

3 Show that the straight lines given by

(i) $x(a+2b)+y(a+3b)=a+b$
for different values of a and b pass through a fixed point

(ii) $(2+\lambda)x+(1+\lambda)y=5+7\lambda$
for different values of λ pass through a fixed point (I I T 1971)

(iii) The set of lines
$ax+by+c=0$ where $3a+2b+4c=0$ (I I T 1982)
is concurrent at the point

(iv) The straight line $5x+4y=0$ passes through the point of intersection of the straight lines
$x+2y-10=0$ and $2x+y+5=0$ (I I T 1983)
State true or false

(v) If $u=a_1x+b_1y+c_1=0$ and $v=a_2x+b_2y+c_2=0$ and
$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$ prove that the curve $u+kv=0$ is nothing but any of the given straight lines $u=0$ or $v=0$ (M N R 87)

4 (a) Find the equations of the lines through the point of intersection of the lines
$x-3y+1=0$ and $2x+5y-9=0$
and whose distance from the origin is $\sqrt{5}$

(b) Given the pencil of lines $a(2x+y+4)+b(x-2y-3)=0$ Prove that among the lines of the pencil there exists only one line whose distance from the point $(2, -3)$ is $\sqrt{10}$ Write the equation of this line

5 Find the equation of the straight line joining the point (a, b) to the point of intersection of the lines

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{y}{a}=1$$

6 A variable straight line drawn through the point of intersection of the straight lines

Any line perpendicular to it is $2x+3y=k$. If it passes through (4, 1), then $k=11$. Hence the line is

$$2x+3y=11 \qquad (2)$$

Now prove as in Q 10 that line (2) divides the join of given points in the ratio 5 : 8

(*b*) Equation of Ist line is $2x+3y+7=0$ and it divides the second line externally in the ratio 5 : 9

12 Any line through P (3, 4) making an angle $\pi/6$ with x axis is

$$\frac{x-3}{\cos 30^\circ}=\frac{y-4}{\sin 30^\circ}=r$$

where r represents the distance of any point on this line from the given point P (3, 4)

Any point Q on it is $\left(r\frac{\sqrt{3}}{2}+3,\ r\frac{1}{2}+4\right)$ and Q lies on

$$12x+5y+10=0$$

$$12\left(r\frac{\sqrt{3}}{2}+3\right)+5\left(r\cdot\tfrac{1}{2}+4\right)+10=0$$

$$(12\sqrt{3}+5)\,r+132=0 \qquad r=\frac{-132}{12\sqrt{3}+5}=\frac{132}{12\sqrt{3}+5}$$

13 A is (2, 0), B is (3, 1)

Also $AB=\sqrt{2}$. Slope of

$$AB=\frac{1-0}{3-2}=1=\tan\theta \qquad \theta=45^\circ$$

Now AB is rotated through 15° in anticlockwise direction and hence it makes an angle of 60° with x-axis. Since it passes through (2, 0) its equation in new position is

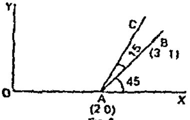


Fig 8

$$\frac{x-2}{\cos 60^\circ}=\frac{y-0}{\sin 60^\circ}=r=AC=AB=\sqrt{2}$$

Hence the coordinates of C are $2+\sqrt{2}\cos 60^\circ,\ 0+\sqrt{2}\sin 60^\circ$

or $\left(\frac{4+\sqrt{2}}{2},\ \frac{\sqrt{6}}{2}\right)$

14 Refer Rule 15, 16 page 264

$$3x-2y=2,\ 2x+3y=23$$

15 B (−3, −1) and C is (−1, −3)

$$\text{Slope of } BC=\frac{-3-(-1)}{-1-(-3)}=-1$$

15 The equation of the base of an equilateral triangle is $x+y=2$ and the vertex is $(2, -1)$ Find the length of the side of the triangle (I I T 1973, 83)

16 (a) Find the coordinates of the points on the line $x+5y=13$ at a distance of 2 units from the line
$$12x-5y+26=0$$
(b) A point moves so that square of its distance from the point $(3, -2)$ is numerically equal to its distance from the line $5x-12y=13$ Find the equation of its locus (Roorkee 1974)

17 Find all the points on the line $x+y=4$ that lie at a unit distance from the line $4x+3y=10$ (I I T 1976)

18 A straight line makes an intercept on the y axis twice as along as that on x axis and is at a unit distance from the origin Determine its equation

19 Find the equation of the line joining the point $(3, 5)$ to the point of intersection of the lines
$$4x+y-1=0 \quad \text{and} \quad 7x-3y-35=0$$
and prove that the line is equidistant from the origin and the point $(8, 34)$ (Roorkee 1944)

20 If p and p' be the lengths of perpendiculars from origin to the lines
$$x \sec \theta - y \operatorname{cosec} \theta = a$$
and
$$x \cos \theta - y \sin \theta = a \cos 2\theta$$
respectively, then prove that
$$4p^2+p'^2=a^2$$

21 Prove that the product of the perpendiculars from the points $[\pm\sqrt{(a^2-b^2)}, 0]$ to the line
$$\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \text{ is } b^2$$

22 If p is the length of perpendicular from origin on the line which cuts an intercept a on x axis and b from y axis, prove that
$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$$

23 Find the direction in which a straight line must be dra[wn] through the point $(1, 2)$ so that its point of intersection w[ith] the line $x+y=4$ may be at a distance $\sqrt{6}/3$ from this poin[t]

The other diagonal AC will be perpendicular to (I) and pass through the point A and hence its equation is
$y(\cos\alpha+\sin\alpha)+x(\cos\alpha-\sin\alpha)=k$ where $k=a$

17 Slope of the line is

$$\frac{8-3}{-5-2}=\frac{5}{-7}$$

and hence slope of a line perpendicular to it will be 7/5
Also the coordinates of a point dividing the join of

$(2\ \ 3)$ and $(-5\ \ 8)$ in the ratio $3\ \ 4$ is $\left(1, \frac{36}{7}\right)$

Rule 2 P 261

Hence the line is $y-36/7=(7/5)(x-1)$ or $49x-35y+229=0$

18 Any line through $(3, -2)$ is $y+2=m(x-3)$ (1)
Slope of $\sqrt{3}x+y=1$ is $-\sqrt{3}$ **Rule 10 P 263**
Angle between the two lines is 60°

$$\tan(\pm 60°)=\frac{m\ \ (-\sqrt{3})}{1+m(-\sqrt{3})} \text{ or } \pm\sqrt{3}(1-m\sqrt{3})=m+\sqrt{3}$$

+ive sign, $\sqrt{3}-3m=m+\sqrt{3}$ or $4m=0$ $m=0$,
− ive sign, $-3+3m=m+\sqrt{3}$ or $2m=2\sqrt{3}$ $m=\sqrt{3}$,
Putting in (1) the required lines are
$y+2=0$ and $\sqrt{3}x-y=2+3\sqrt{3}$

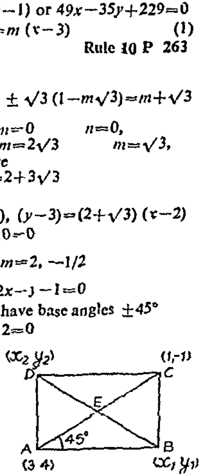
19 Proceed as in Ex 18
$(y-3)=(2-\sqrt{3})(x-2)$, $(y-3)=(2+\sqrt{3})(x-2)$

20 Ans $x-5y-28=0$ or $5x+y-10=0$

21 $\tan(\pm 45°)=\frac{m-(-3)}{1+m(-3)}$, $m=2, -1/2$
$x+2y-8=0$ and $2x-y-1=0$

22 Isosceles right angled triangle will have base angles $\pm 45°$
$7x+y=16$, $x-7y+12=0$

23 (*a*) Slope of AC is 5/2 and slopes of two lines inclined at an angle of $\pm 45°$ are found as above to be 3/7 and −7/3 Hence AB is $y-4=(3/7)(x-3)$
or $3x\ \ 7y+19=0$



Also CB is (1)

$$y+1=-\frac{7}{3}(x-1)$$

or $7x+3y-4=0$ (2)

(g) The area enclosed within the curve (I I T 1981)
$|x|+|y|=1$ is

28 Show that the four lines

$$ax \pm by \pm c=0$$

enclose a rhombus whose area is $\frac{2c^2}{ab}$

29 (a) Find the equation of the straight lines passing through $(-2, -7)$ and having an intercept of length 3 between the straight lines (I I T 77)

$$4x+3y=12,\ 4x+3y=3$$

(b) Find the equation of the straight line through the point $(-\frac{5}{3}, 4)$ given that its segment intercepted by the lines $x+2y+1=0$ and $x+2y-1=0$ is of length $2/\sqrt{5}$

(c) P is the point $(-1, 2)$, a variable line through P cuts the co ordinate axes in A and B respectively Q is a point on AB such PA, PQ and PB are in H P Show that the locus of Q is the line $y=2x$

(d) A and B are two fixed points whose co-ordinates respectively are $(3, 2)$ and $(5, 1)$ ABP is an equilateral triangle on AB situated on the side opposite to that of the origin Find the co ordinates of P and those of the orthocentre of triangle ABP

(e) A line is such that its segment between the straight lines $5x-y-4=0$ and $3x+4y-4=0$ is bisected at the point $(1, 5)$, obtain its equation (Roorkee 88)

30 If $x \cos \alpha+y \sin \alpha=p$ where $p=-\frac{\sin^2 \alpha}{\cos \alpha}$ be a straight line, prove that perpendiculars p_1, p_2 and p_3 on this line from the points $(m^2, 2m)$, $(mm', m+m')$ and $(m'^2, 2m')$ respectively are in geometrical progression

Solutions to Problems Set (D)

1 Any line through the intersection of given lines is $P+\lambda Q=0$

or $\quad 2x-3y+\lambda(4x-5y-2)=0 \quad$ (1)

or $\quad x(2+4\lambda)-y(3+5\lambda)-2\lambda=0$

Its slope $=\frac{2+4\lambda}{3+5\lambda}$

(i) Since it passes through the point $(2, 1)$

$\therefore \quad 2(2+4\lambda)-1(3+5\lambda)-2\lambda=0 \quad \lambda=-1$

The other diagonal AC will be perpendicular to (I) and pass through the point A and hence its equation is
$y\ (\cos\alpha+\sin\alpha)+x\ (\cos\alpha-\sin\alpha)=k$ where $k=a$

17 Slope of the line is

$$\frac{8-3}{-5-2}=\frac{5}{-7}.$$

and hence slope of a line perpendicular to it will be 7/5

Also the coordinates of a point dividing the join of

(2, 3) and (−5, 8) in the ratio 3 4 is $\left(1, \frac{36}{7}\right)$

Rule 2 P 261

Hence the line is $y-36/7=(7/5)\ (x-1)$ or $49x-35y+229=0$

18 Any line through (3, −2) is $y+2=m\ (x-3)$ (1)

Slope of $\sqrt{3}x+y=1$ is $-\sqrt{3}$ **Rule 10 P 263**

Angle between the two lines is 60°

$$\tan\ (\pm 60°)=\frac{m-(-\sqrt{3})}{1+m\ (-\sqrt{3})} \text{ or } \pm\sqrt{3}\ (1-m\sqrt{3})=m+\sqrt{3}$$

+ive sign, $\sqrt{3}-3m=m+\sqrt{3}$ or $4m=0$ $m=0$,
− ive sign, $-3+3m=m+\sqrt{3}$ or $2m=2\sqrt{3}$ $m=\sqrt{3}$,
Putting in (1) the required lines are
$y+2=0$ and $\sqrt{3}x-y=2+3\sqrt{3}$

19 Proceed as in Ex 18

$(y-3)=(2-\sqrt{3})\ (x-2)$, $(y-3)=(2+\sqrt{3})\ (x-2)$

20 Ans $x-5y-28=0$ or $5x+y-10=0$

21 $\tan\ (\pm 45°)=\dfrac{m-(-3)}{1+m(-3)}$, $\therefore$ $m=2, -1/2$

$x+2y-8=0$ and $2x-y-1=0$

22 Isosceles right angled triangle will have base angles ±45°

$7x+y=16$, $x-7y+12=0$

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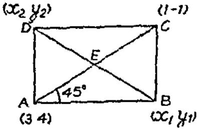

Also CB is (1)

$$y+1=-\frac{7}{3}\ (x-1)$$

or $7x+3y-4=0$ (2)

or $\quad x(1+2\lambda)+y(5\lambda-3)+(1-9\lambda)=0$ (I)

Its distance from origin is given to be $\sqrt{5}$

$\therefore \quad \dfrac{1-9\lambda}{[(1+2\lambda)^2+(5\lambda-3)^2]^{1/2}}=\sqrt{5}$, Square

$(1-9\lambda)^2=5(1+4\lambda^2+4\lambda+25\lambda^2-30\lambda+9)$

or $\quad 64\lambda^2-112\lambda+49=0$ or $(\lambda-7)^2=0 \quad \lambda=7/8$

Putting for λ in (I) and simplifying we get the required line as $\quad 2x+y-5=0$

(b) Put $b/a=\lambda$ then $(\lambda-1)^2=0, \quad \lambda=1,\ 3x-y+1=0$

5 The required line by $P+\lambda Q=0$ is

$$\left(\frac{x}{a}+\frac{y}{b}-1\right)+\lambda\left(\frac{x}{b}+\frac{y}{a}-1\right)=0$$

or $\quad x\left(\dfrac{1}{a}+\dfrac{\lambda}{b}\right)+y\left(\dfrac{1}{b}+\dfrac{\lambda}{a}\right)-(1+\lambda)=0$ (I)

Since it passes through the point (a, b) and hence we have

$$(1+1-1)+\lambda\left(\frac{a}{b}+\frac{b}{a}-1\right)=0 \qquad \lambda=-\frac{ab}{a^2+b^2-ab}$$

Putting for λ in (I) the required line is

$$x\left(\frac{1}{a}-\frac{a}{a^2+b^2-ab}\right)+y\left(\frac{1}{b}-\frac{b}{a^2+b^2-ab}\right)-\left(1-\frac{ab}{a^2+b^2-ab}\right)=0$$

or $\quad x\dfrac{b}{a}(b-a)+y\dfrac{a}{b}(a-b)-(a-b)^2=0$

or $\quad a^2y-b^2x=ab(a-b)$

Alternative Method

Solving the given equations we get the point

$$\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$$

Hence we have to find the line joining the points

$$(a, b), \left(\frac{ab}{a+b}, \frac{ab}{a+b}\right)$$

whose equation by 9 (X) is

$$y-b=\frac{\dfrac{ab}{a+b}-b}{\dfrac{ab}{a+b}-a}(x-a)$$

The other diagonal AC will be perpendicular to (I) and pass through the point A and hence its equation is
$y(\cos\alpha+\sin\alpha)+x(\cos\alpha-\sin\alpha)=k$ where $k=a$

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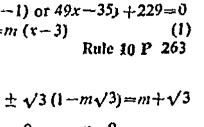


Also CB is (1)

$$y+1=-\frac{7}{3}(x-1)$$

or $7x+3y-4=0$ (2)

8 The first two lines meet at (0, 0) and the last two at and hence the diagonal is

$$y-0=(3/2)(x-0) \quad \text{or} \quad 3x-2y=0$$

or use $P-\lambda Q=0$ and find λ by making it pass throu

9 The first two lines meet at the point (5, 5) whic satisfies the equation $4x+5y-45=0$ of the third line the three lines are concurrent

10 The first and third lines meet at $(1, -1)$ and puttin equation of second line we get $a=5$

11 $\lambda=-45$ (5 5) is point of intersection

12 (a) The lines will be concurrent if

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0$$

or $a(bc-a^2)-b(b^2-ac)+c(ab-c^2)=0$

or $a^3+b^3+c^3-3abc=0$

or $(a+b+c)(a^2+b^2+c^2-ab-bc-ca)=0$

or $a+b+c=0,$

(b) As above

$$\begin{vmatrix} a & 2 & 1 \\ b & 3 & 1 \\ c & 4 & 1 \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} a+c-2b & 0 & 0 \\ b & 3 & 1 \\ c & 4 & 1 \end{vmatrix} \text{ by } R_1+R_3-R_2$$

or $(a+c-2b)(3-4)=0$ $a+c-2b=0$

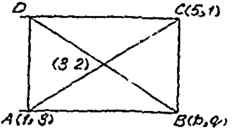
or $a+c=2b$ a, b, c are in A P

13 (a) The lines will be concurrent if

$$\begin{vmatrix} p_1 & q_1 & -1 \\ p_2 & q_2 & -1 \\ p_3 & q_3 & -1 \end{vmatrix}=0$$

or $$\frac{1}{2}\begin{vmatrix} p_1 & q_1 & 1 \\ p_2 & q_2 & 1 \\ p_3 & q_3 & 1 \end{vmatrix}=0$$

or



Now let (p, q) be the vertex B, then

$$BC \perp BA \qquad \frac{q-1}{p-5}\times\frac{q-3}{p-1}=-1$$

or $\quad p^2+q^2-6p-4q+8=0 \qquad (2)$

But (p, q) lies on (1) $\quad q=2p-4 \qquad (3)$

From (2) and (3) eliminating q, we get

$$p^2-6p+8=0 \quad \text{or} \quad (p-4)(p-2)=0$$

$p=4$ and hence $q=4$

$p=2$ and hence $q=0$

Hence the other vertices are (4, 4) and (2, 0)

(a) AB is $4x+5y=0$
AD is $7x+2y=0$
Both of them intersect at (0 0) A Then diagonal $11x+7y=9$ does not pass through A i e (0 0) and hence it cannot be the equation of AC Therefore it represents diagonal BD

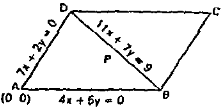


Fig 10

Solving the diagonal with AB and AD respectively we get the points B and D as $(5/3, -4/3)$, $(-2/3, 7/3)$
The mid point P of diagonal BD is $(1/2, 1/2)$
Since the diagonal of a parallelogram bisect each other therefore diagonal AC will pass through A and P i e (0 0) and $(\frac{1}{2}, \frac{1}{2})$
Its equation is clearly $y=x$ or $x-y=0$

(b) The given determinant when expanded gives an equation of first degree in x and y therefore it represents a straight line Further it is satisfied by $u=p$ and $v=r$ [two rows identical] Hence this line passes through the vertex given by $u=p$ and $v=r$ Similarly it passes through the vertex given by $u=q$ and $v=s$ which is opposite vertex and as such it is the equation of the diagonal

$$\frac{|-31|}{\sqrt{(145)}}=\frac{31}{\sqrt{(145)}}$$

Its distance from (8, 34) is

$$\frac{96-34-31}{\sqrt{(145)}}=\frac{31}{\sqrt{(145)}}$$

20 $p=\frac{-a}{\sqrt{(\sec^2\theta+\operatorname{cosec}^2\theta)}}$ $\qquad 4p^2=\frac{4a^2\sin^2\theta\cos^2\theta}{\sin^2\theta+\cos^2\theta}$

or $4p^2=a^2(2\sin\theta\cos\theta)^2=a^2\sin^2 2\theta$

$$p'=\frac{-a\cos 2\theta}{\sqrt{(\cos^2\theta+\sin^2\theta)}} \qquad p'^2=a^2\cos^2 2\theta$$

$$4p^2+p'^2=a^2(\sin^2 2\theta+\cos^2 2\theta)=a^2$$

21 Equation of the given line is

$$x(b\cos\theta)+y(a\sin\theta)-ab=0$$

$$p_1=\frac{\sqrt{(a^2-b^2)}(b\cos\theta)-ab}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}$$

$$p_2=\frac{-\sqrt{(a^2-b^2)}(b\cos\theta)-a}{\sqrt{(b^2\cos^2\theta+a^2\sin^2\theta)}}$$

$$p_1p_2=-\frac{(a^2-b^2)b^2\cos^2\theta-a^2b^2}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$(L+M)(L-M)=L^2-M^2$$

$$=b^2\frac{[a-a^2\cos^2\theta+b^2\cos^2\theta]}{b^2\cos^2\theta+a^2\sin^2\theta}$$

$$=b^2\frac{[a^2\sin^2\theta+b^2\cos^2\theta]}{b^2\cos^2\theta+a^2\sin^2\theta}=b^2$$

22 $p=\frac{1}{\sqrt{(1/a^2+1/b^2)}}$ $\qquad \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}$

23 Let the line through (1, 2) make an angle θ with x axis so that the equation is

$$\frac{x-1}{\cos\theta}=\frac{y-2}{\sin\theta}=r$$

where r is the distance of any point on the line from the point (1 2)

Any point on it is $(r\cos\theta+1, r\sin\theta+2)$

If it lies on $x+y=4$ Then

$$(r\cos\theta+1)+(r\sin\theta+2)=4$$

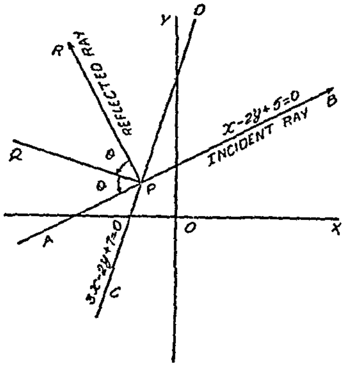
or $r(\cos\theta+\sin\theta)=1$ But $r=\frac{1}{3}\sqrt{6}$ given

$\frac{1}{3}\sqrt{6}(\cos\theta+\sin\theta)=1$

or $\cos\theta+\sin\theta=\frac{3}{\sqrt{6}}$

$$y-2=m_2(x+1) \quad \text{...(1)}$$

Now normal ($m=-2/3$) is equally inclined to incident ray ($m_1=1/2$) and reflected ray m_2



$$\frac{m_1-(-2/3)}{1+m_1(-2/3)}=-\frac{m_2-(-2/3)}{1+m_2(-2/3)} \quad \text{(Note)}$$

or $\frac{1/2+2/3}{1-1/2\,2/3}=\frac{3m_2+2}{2m_2-3}$ or $\frac{7}{4}=\frac{3m_2+2}{2m_2-3}$

$(14-12)\,m_2=8+21$ or $m_2=29/2$

Putting in (1) the reflected ray is, $y-2=(29/2)(x+1)$

or $29x-2y+33=0$

(c) Proceed as above and draw the line Incident ray $y-3=3(x+2)$ $\tan\alpha=3$ Surface is x axis i e $y=0$ point P is $(-3\ 0)$ Normal to surface is y axis Incident ray makes an angle $90°-\alpha$ with normal y-axis and so will the reflected ray but in opposite sense Hence it will make an angle $-\alpha$ with x axis where $\tan(-\alpha)=-\tan\alpha=-3$ Hence its equation is $y-0=-3(x+3)$

or $y+3x+9=0$

$\frac{p_1}{AD}=\sin\theta, \frac{p_2}{AB}=\sin\theta$

$$\text{Area}=\frac{p_2}{\sin\theta}\quad\frac{p_1}{\sin\theta}\quad\sin\theta=\frac{p_1p_2}{\sin\theta}$$

Distance between parallel sides

$3x-2y-a=0$ and $3x-2y-2a=0$ is

$$\frac{-a}{\sqrt{13}}-\frac{(-2a)}{\sqrt{13}}=\frac{a}{\sqrt{13}}$$

Distance between parallel sides

$2x-3y+a=0$ and $2x-3y+3a=0$

$$=\frac{a}{\sqrt{(13)}}-\frac{3a}{\sqrt{(13)}}=\frac{-2a}{\sqrt{(13)}}=\frac{2a}{\sqrt{(13)}}$$

Also θ is the angle between the lines whose slopes are 3/2 and 2/3

$$\tan\theta=\frac{\frac{3}{2}-\frac{2}{3}}{1+\frac{3}{2}\cdot\frac{2}{3}}=\frac{5}{12}\qquad\sin\theta=\frac{5}{13}$$

$$\text{Area}=\frac{p_1p_2}{\sin\theta}=\frac{a}{\sqrt{(13)}}\cdot\frac{2a}{\sqrt{(13)}}\cdot\frac{13}{5}=\frac{2a^2}{5}\qquad\text{Proved}$$

23

(b) Use $\text{Area}=\frac{p_1p_2}{\sin\theta}=p_1p_2\operatorname{cosec}\theta$ by part (a)

Here $p_1=p-q,\ p_2=r-s$ $\qquad\theta=\alpha-\beta$ (A)

and $\tan\theta=\tan(\alpha-\beta)$

(c) Here $p_1=\frac{d_1-c_1}{\sqrt{(a_1^2+b_1^2)}},\ p_2=\frac{d_2-c_2}{\sqrt{(a_2^2+b_2^2)}}$

and $\tan\theta=\frac{-\frac{a_1}{b_1}+\frac{a_2}{b_2}}{1+\frac{a_1a_2}{b_1b_2}}=\frac{a_2b_1-a_1b_2}{(a_1a_2+b_1b_2)}$

$$\operatorname{cosec}^2\theta=1+\cot^2\theta=1+\frac{(a_1a_2+b_1b_2)^2}{(a_2b_1-a_1b_2)^2}$$

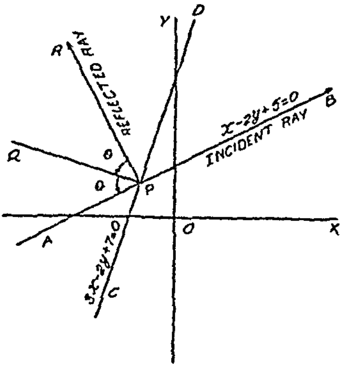
$$=\frac{(a_2b_1-a_1b_2)^2+(a_1a_2+b_1b_2)^2}{(a_2b_1-a_1b_2)^2}$$

$$=\frac{a_1^2(a_2^2+b_2^2)+b_1^2(a_2^2+b_2^2)}{(a_2b_1-a_1b_2)^2}\qquad\text{(B)}$$

$$=\frac{(a_1^2+b_1^2)(a_2^2+b_2^2)}{(a_1b_2-a_2b_1)^2}$$

$$y-2=m_2(x+1) \qquad \ldots (1)$$

Now normal $(m=-2/3)$ is equally inclined to incident ray $(m_1=1/2)$ and reflected ray m_2



$$\frac{m_1-(-2/3)}{1+m_1(-2/3)}=-\frac{m_2-(-2/3)}{1+m_2(-2/3)} \qquad \text{(Note)}$$

or $\dfrac{1/2+2/3}{1-1/2 \cdot 2/3}=\dfrac{3m_2+2}{2m_2-3}$ or $\dfrac{7}{4}=\dfrac{3m_2+2}{2m_2-3}$

$(14-12)\,m_2=8+21$ or $m_2=29/2$

Putting in (1) the reflected ray is, $y-2=(29/2)(x+1)$

or $29x-2y+33=0$

(c) Proceed as above and draw the line Incident ray $y-3=3(x+2)$, $\tan\alpha=3$ Surface is x axis i.e. $y=0$ point P is $(-3, 0)$ Normal to surface is y axis Incident ray makes an angle $90°-\alpha$ with normal y axis and so will the reflected ray but in opposite sense Hence it will make an angle $-\alpha$ with x axis where $\tan(-\alpha)=-\tan\alpha=-3$ Hence its equation is $y-0=-3(x+3)$

or $y+3x+9=0$

29 (a) The given parallel lines are

$$4x+3y=3 \quad (1)$$
$$4x+3y=12 \quad (2)$$

Any line through the point $(-2, -7)$ is

$$\frac{x+2}{\cos\theta}=\frac{y+7}{\sin\theta}=r \quad (3)$$

This line makes an intercept of length 3 between parallel lines (1) and (2)

Hence we shall choose two values of r say r and $r+3$ and the two corresponding points obtained will lie on lines (1) and (2)

$r\cos\theta-2,\ r\sin\theta-7$ will lie on $4x+3y=3$

$(r+3)\cos\theta-2,\ (r+3)\sin\theta-7$ will lie on $4x+3y=12$

$$4(r\cos\theta-2)+3(r\sin\theta-7)=3 \quad (4)$$
$$4[(r+3)\cos\theta-2]+3[(r+3)\sin\theta-7]=12 \quad (5)$$

Subtracting (4) from (5) we get

$12\cos\theta+9\sin\theta=9$ or $12\cos\theta=9(1-\sin\theta)$

$144(1-\sin^2\theta)=81(1-2\sin\theta+\sin^2\theta)$

$16-16\sin^2\theta=9-18\sin\theta+9\sin^2\theta$

or $25\sin^2\theta-18\sin\theta-7=0$

or $(\sin\theta-1)(25\sin\theta+7)=0$

$$\sin\theta=1, \frac{-7}{25} \quad \therefore \quad \cos\theta=0, \frac{24}{25}$$

Hence from (3) the lines are

$$\frac{x+2}{24/25}=\frac{y+7}{-7/25} \quad \text{or} \quad 7x+24y+182=0$$

and $\frac{x+2}{0}=\frac{y+7}{1}$ or $\frac{y+7}{x+2}=\infty$, $\quad x+2=0$

Hence the two lines are

$7x+24y+182=0$ and $x+2=0$

(b) The given lines are parallel and distance between them is $2/\sqrt{5}$ which is perpendicular distance. Hence the required line is one through the point $(-5, 4)$ and perpendicular to $x+2y+1=0$

Its equation is $\quad 2x-y+\lambda=0$

where $\quad -10-4+\lambda=0$

or $\lambda=14 \quad 2x-y+14=0$

(c) Any line through $P(-1, 2)$ is

$$\frac{x+1}{\cos\theta}=\frac{y-2}{\sin\theta}=\frac{r_1}{A}=\frac{r_2}{Q}=\frac{r_3}{B}$$

$$y+7=\frac{-52}{89}(x-2)$$

or $\quad 89y+623=-52x+104$

or $\quad 52x+89y+519=0$

31 Any line through $(1, -10)$ is $y+10=m(x-1)$ Since it makes equal angles say θ, with the given lines, we have

$$\tan\theta=\frac{m-7}{1+7m}=-\frac{m-(-1)}{1+m(-1)} \qquad \textbf{(Note)}$$

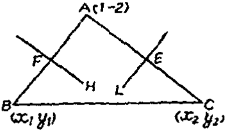
This gives $m=\frac{1}{3}$ or -3

Hence the equations of third side is

$y+10=\frac{1}{3}(x-1)$ or $y+10=-3(x-1)$

i e $\quad x-3y-31=0$ or $3x+y+7=0$

32 Let the coordinates of B and C be (x_1, y_1) and (x_2, y_2)



Perpendicular bisector of AB is $x-y+5=0$ (1)

Mid point $F\left(\frac{x_1+1}{2}, \frac{y_1-2}{2}\right)$ of AB lies on (1)

$\therefore \quad x_1-y_1+13=0 \qquad (2)$

Also AB is perpendicular to MF *i e* $m_1m_2=-1$

or $\frac{y_1+2}{x_1-1}\times 1=-1$ or $x_1+y_1+1=0 \qquad (3)$

Solving (2) and (3), we get $x_1=-7, y_1=6$

B is $(-7, 6)$ Similarly C is $\left(\frac{11}{5}, \frac{2}{5}\right)$

Equation of BC is $y-6=\frac{2/5-6}{11/5+7}(x+7)$

or $23(y-6)=14(x+7)$ or $14x+23y-40=0$

Problem Set (D)

Lines through the intersection of given lines, concurrent lines and perpendicular distance of a point from a line

1 Find the equations to the straight lines that pass through the intersection of the lines

$$2x-3y=0 \text{ and } 4x-5y=2 \text{ and}$$

(*i*) Passes through the point $(2\ 1)$

(*ii*) Is perpendicular to the line

$$x+2y+1=0$$

Point P $\left(4+\frac{1}{2}\sqrt{3}, \frac{3}{2}+\sqrt{3}\right)$

Point P_2 $\left(5+\frac{1}{6}\sqrt{3}, \frac{3}{2}+\frac{1}{2}\sqrt{3}\right)$

29 (e) Any line through mid point (1, 5) of intercept AB may be taken as $\frac{x-1}{\cos\theta}=\frac{y-5}{\sin\theta}=r, -r$ for A and B respectively

Point A $(r\cos\theta+1, r\sin\theta+5)$ lies on Ist line and point B $(-r\cos\theta+1, -r\sin\theta+5)$ lies on 2nd line

$r(5\cos\theta-\sin\theta)=4$ and $r(3\cos\theta+4\sin\theta)=19$

Eliminating r on dividing the two we get

$$\frac{5\cos\theta-\sin\theta}{3\cos\theta+4\sin\theta}=\frac{4}{19} \quad \text{or} \quad 83\cos\theta=35\sin\theta$$

$$\frac{\cos\theta}{35}=\frac{\sin\theta}{83} \quad \text{Hence the required line is}$$

$$\frac{x-1}{35}=\frac{y-5}{83} \quad \text{or} \quad 83x-35y+92=0$$

30 $$p_1=\frac{m^2\cos\alpha+2m\sin\alpha-\left(-\frac{\sin^2\alpha}{\cos\alpha}\right)}{\sqrt{(\cos^2\alpha+\sin^2\alpha)}}$$

$$p_1=\left(\frac{m\cos\alpha+\sin\alpha}{\sqrt{(\cos\alpha)}}\right)^2$$

Similarly $$p_3=\left(\frac{m'\cos\alpha+\sin\alpha}{\sqrt{(\cos\alpha)}}\right)^2$$

Also $$p_2=\frac{mm'\cos\alpha+(m'+m)\sin\alpha-\left(-\frac{\sin^2\alpha}{\cos\alpha}\right)}{\sqrt{(\cos^2\alpha+\sin^2\alpha)}}$$

or $$p_2=\frac{1}{\cos\alpha}\left[mm'\cos^2\alpha+(m'+m)\sin\alpha\cos\alpha+\sin^2\alpha\right]$$

or $$p_2=\frac{(m\cos\alpha+\sin\alpha)}{\sqrt{(\cos\alpha)}}\cdot\frac{m'\cos\alpha+\sin\alpha}{\sqrt{(\cos\alpha)}}$$

or $p_2=\sqrt{(p_1 p_3)}$ or $p_2^2=p_1 p_3$

Hence p_1, p_2, p_3 are in G P

Problem Set (E)

Orthocentre circumcentre centroid bisectors and locus

1 The sides of a triangle are

$$y=x,\ y=2x \text{ and } y=3x+4$$

(a) Find the equation of the medians and hence the coordinates of its centroid

$$\frac{x}{a}+\frac{y}{b}=1 \text{ and } \frac{x}{b}+\frac{y}{a}=1$$

meets the coordinate axes in A and B Show that the locus of the mid point of AB is the curve

$2xy(a+b)=ab(x+y)$ (I I T 1977)

7 Two parallel straight lines inclined at an angle of 135° to the axis of x meet the x axis at the points A, B and y axis at the points C and D respectively Find the equation of the locus of point of intersection of AD and BC

8 Find the equation of the diagonal through the origin of the quadrilateral formed by

$x=0$ $y=0, x+y=1$ and $6x+y=3$ (I I T 1973)

9 Show that the lines

$3x-4y+5=0, 7x-8y+5=0$

and $4x+5y=45$ are concurrent

10 For what value of a will the lines

$3x+4y+1=0, ax+2y-3=0$

and $2x-y-3=0$ are concurrent ?

11 For what value of k will the following lines be concurrent

$3x-4y+5=0,\ 7x-8y+5=0,\ 4x+5y+k=0$

12 (a) Prove that the lines

$ax+by+c=0$ $bx+cy+a=0$

and $cx+ay+b=0$ are concurrent

if $a^3+b^3+c^3=3abc$

or if $a+b+c=0$

(b) The lines $ax+2y+1=0$, $bx+3y+1=0$ and $cx+4y+1=0$ are concurrent show that a, b, c are in A P

13 (a) If the lines

$p_1x+q_1y=1,\ p_2x+q_2y=1\ \ p_3x+q_3y=1$

are concurrent then prove that the points (p_1, q_1), $(p_2\ q_2)$, and $(p_3\ q_3)$ are collinear

(b) Find the condition of concurrency of the lines

$y=m_1x+c_1,\ y=m_2x+c_2,\ y=m_3x+c_3$

14 Prove that the lines

$(p-q)\,x+(q-r)\,y+(r-p)=0$

$(q-r)\,x+(r-p)\,y+(p-q)=0$

$(r-p)\,x+(p-q)\,y+(q-r)=0$ are concurrent

8 Find the equation of the line which bisects the obtuse angle between the lines

$$x-2y+4=0 \text{ and } 4x-3y+2=0 \quad \text{(IIT 79)}$$

9, (a) Find the coordinates of the incentre of the triangle equations of whose sides are

$$x+1=0,\ 4y+5=0,\ 5x+12y=27$$

(b) How many circles can be drawn each touching all the three lines $x+y=1$, $y=x+1$, $7x-y$ 6 ? Find the centre and radius of one of them (IIT 77)

10 (a) The sides of a triangle are

$$4x+3y+7=0,\ 5x+12y=27$$

and $$3x+4y+8=0$$

Find the equations of the internal bisectors of the angles and show that they are concurrent

(b) Write the equation of the internal bisector of the angle between the lines $2x-3y-5=0$, $6x-4y+7=0$ which is the supplement of the angle containing the point (2, −1)

11 Find the locus of the mid point of the portion of the line

$$x \cos \alpha + y \sin \alpha = p$$

which is intercepted between the axes (MNR 85)

12 (a) A variable line through the point (6/5, 6/5) cuts the co ordinate axes in the points A and B If the point P divides AB internally in the ratio 2 1, show that the equation to the locus of P is

$$5xy=2(2x+y)$$

(b) A straight line passing through the point (1, 1) is terminated by the axes of coordinates Show that the locus of the mid point of the line has equation $2xy=x+y$

13 (a) A line drawn through the origin intersects the lines $2x+y-2=0$ and $x-2y+2=0$ in A and B Show that locus of the mid point of AB is $2x^2 \quad 3xy-2y^2+x+3y=0$

(b) A point moves so that square of its distance from the point (3 −2) is numerically equal to its distance from the line $5x-12y=13$ Find the equation to its locus (Roorkee 76)

14 (a) If the line

$$\frac{x}{a}+\frac{y}{b}=1$$

24 A line is drawn through P (3, 4) inclined to x axis at an angle $3\pi/4$ Find its equation and also the coordinates of two points on it on opposite sides of P at a distance $\sqrt{2}$ from it

25 (a) Find the equations of two lines each parallel to

$$3x+4y+7=0$$

one of which passes through (1, 1) and the other through $(-2, -1)$ Also find the perpendicular distance between them

(b) Prove that the distance between $4x+3y=11$ and $8x+6y=15$ is $\frac{7}{10}$ (M N R 87)

26 Prove that the lines

$$2x+3y=19 \quad \text{and} \quad 2x+3y=-7$$

are equidistant from the line $2x+3y=6$

27 (a) Show that the area of the parallelogram formed by the lines

$3y-2x=a$ $2y-3x+a=0$, $2x-3y+3a=0$ and $3x-2y=2a$ is $2a^2/5$

(b) Show that the area of the parallelogram whose sides are $x\cos\alpha+y\sin\alpha=p$, $x\cos\alpha+y\sin\alpha=q$, $x\cos\beta+y\sin\beta=r$, $x\cos\beta+y\sin\beta=s$ is

$$\pm(p-q)(r-s)\operatorname{cosec}(\alpha-\beta)$$

(c) Prove that the area of the parallelogram, the equations of whose sides are $a_1x+b_1y+c_1=0$, $a_1x+b_1y+d_1=0$ and $a_2x+b_2y+c_2=0$ and $a_2x+b_2y+d_2=0$ is

$$\frac{(d_1-c_1)(d_2-c_2)}{a_1b_2-a_2b_1}$$

(d) Prove that the area of the parallelogram formed by the lines $4y-3x-a=0$, $3y-4x+a=0$, $4y-3x-3a=0$ and $3y-4x+2a=0$ is $\frac{2}{7}a^2$

(e) Prove that the four straight lines

$$\frac{x}{a}+\frac{y}{b}=1, \frac{x}{b}+\frac{y}{a}=1, \frac{x}{a}+\frac{y}{b}=2 \text{ and } \frac{x}{b}+\frac{y}{a}=2$$

form a rhombus Find its area

(f) Prove that the diagonals of the parallelogram formed by the four straight lines

$\sqrt{3}x+y=0$, $\sqrt{3}y+x=0$, $\sqrt{3}x+y=1$ and $\sqrt{3}y+x=1$ are perpendicular to each other

20 O is a fixed point and AP and BQ are two fixed parallel straight lines BOA is perpendicular to both and $\angle POQ$ is a rt angle Prove that the locus of the foot of the perpendicular from O on PQ is a circle whose diameter is AB

21 The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $y^2-8xy-9x^2=0$ Determine the locus of its vertex (Roorkee 85)

22 Find the co ordinates of the feet of the perpendiculars let fall from the point (5, 0) upon the sides of the triangle formed by joining the three points (4, 3), (−4, 3) and (0, − 5), Prove also that the points so determined lie on a straight line

23 The vertices of a triangle are $A\ (p, p \tan \alpha)$, $B\ (q, q \tan \beta)$, $C\ (r, r \tan \gamma)$ If circumcentre O of triangle ABC is at the origin and $H\ (\bar{x}, \bar{y})$ be its orthocentre, then show that

$$\frac{\bar{x}}{\bar{y}}=\frac{\cos \alpha+\cos \beta+\cos \gamma}{\sin \alpha+\sin \beta+\sin \gamma}$$

24 (a) The line $lx+my+n=0$ bisects the angle between a pair of straight lines of which one is $px+qy+r=0$ Show that the equation to the other line is

$$(px+qy+r)\ (l^2+m^2)-2\ (lp+mq)\ (lx+my+n)=0,$$

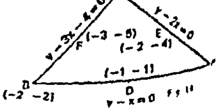
(b) Lines $L_1=ax+by+c=0$ and $L_2=lx+my+n=0$ intersect at the point P and makes an angle θ with each other Find the the equation of a line L different from L_2 which passes through P and makes the same angle θ with L_1 (I I T 84)

25 If P, Q, R are three collinear points on BC, CA and AB respectively of a triangle ABC then prove that $BP\ CQ\ AR+PC\ QA\ RB=0$

Solution to Problems Set (E)

1 Solving the given equations in pairs the coordinates of the vertices of the triangle are $A\ (-4, -8)$, $B\ (-2, -2)$, $C\ (0, 0)$

(a) The mid points of the sides of the triangle are



Putting the value of λ in (1) the required line is

$$x-y-1=0$$

(*ii*) Since it is perpendicular to the line $x+2y+1=0$ whose slope is $-\frac{1}{2}$ hence using the condition $m_1m_2=-1$ for perpendicular lines we get

$$-\frac{1}{2}\left(\frac{2+4\lambda}{3+5\lambda}\right)=-1 \qquad \lambda=-\frac{2}{3}$$

Putting the value of λ in (*1*) the required line is $2x-y=4$.

(*iii*) Since it is parallel to the line $3x-4y+5=0$ whose slope is 3/4, hence using the condition $m_1=m_2$ for parallel lines

we get $\quad \dfrac{2+4\lambda}{3+5\lambda}=\dfrac{3}{4} \qquad \lambda=1$

Putting the value of λ in (1), the required line is

$$3x-4y-1=0$$

(*iv*) True The given line satisfies $(-20/3, 25/3)$ which is the point of intersection of other two given lines

2 Slope of the given line $=-\dfrac{2+6\lambda}{3-\lambda}$

(*i*) If it is parallel to y-axis i e $\perp$ to x-axis then its slope $=\tan 90°=\infty$ and hence $3-\lambda=0 \quad \lambda=3$

(*ii*) If it is perpendicular to $7x+5y=4$ whose slope is $-7/5$

then as in last question $\left(-\dfrac{7}{5}\right)\left(-\dfrac{2+6\lambda}{3-\lambda}\right)=-1$

or $14+42\lambda=-15+5\lambda$ or $29=-37\lambda \qquad \lambda=-29/37$

3 The given equation can be put in the form

(*i*) $a(x+y-1)+b\,(2x+3y-1)=0$

or $(x+y-1)+\lambda\,(2x+3y-1)=0$ where $\lambda=b/a$

Above is of the form $P+\lambda Q=0$ and as such it represents a line through the intersection of the lines $P=0$ and $Q=0$

i e $\quad x+y-1=0$ and $2x+3y-1=0$

Solving the above we get the point $(2, -1)$ which is fixed,

(*ii*) Proceed as above

(*iii*) Dividing by 4 we get

$$\frac{3}{4}a+\frac{1}{2}b+c=0$$

Hence the set of lines pass through the point (3/4, 1/2)

4 (*a*) The required line by $P+\lambda Q=0$ is

$$(x-3y+1)+\lambda\,(2x+5y-9)=0$$

where $-4-8+k=0$ or $k=12$ or $x+y+12=0$ (1)
Similarly the equations of altitudes through the vertices B and C are respectively

$$2y+x+6=0 \quad (2)$$

and $$3y+x=0 \quad (3)$$

Solving any two of (1), (2) and (3) we get the point $(-18, 6)$ which satisfies the third also. Hence the altitudes of a triangle are concurrent and the point of concurrency is called the *orthocentre* of the triangle.

(*d*) Area of the triangle. Since one vertex is at the origin the area of the triangle is $\frac{1}{2}(x_1y_2-x_2y_1)$

$$=\frac{1}{2}[-4(-2)-(2)(-8)]=-4=4 \text{ sq units}$$

2 Putting $x=0$ in $3x+2y=24$, we get the point $A(0, 12)$
Putting $y=0$ in $3x+2y=24$, we get the point $B(8, 0)$
Mid point of AB is $(4, 6)$
Right bisector of AB is $2x-3y+k=0$
where $2\cdot4-3\cdot6+k=0$, $k=10$ (1)
Hence its equation is $2x-3y+10=0$
Any line parallel to x axis is $y=c$. If it passes through $(0, -1)$, then $-1=c$ and hence its equation is (2)

$$y=-1$$

It meets (1) at $C(-13/2, -1)$
Hence we know the three points $A(0, 12)$, $B(8, 0)$, $C(-13/2, -1)$. Its area as usual is 91 sq units (Rule 5 P 26?)

3 (*a*) The three vertices are $A(7, -1)$, $B(-2, 8)$, $C(1, 2)$
Altitudes through A and B are $x-2y=9$ and $x=-11$ respectively
Solving these we get the orthocentre as $(-11, -10)$

(*b*) $(-1, -2)$ orthocentre

(*c*) Let the given points be A, B and C respectively

Slope of BC by $\frac{y_2-y_1}{x_2-x_1}$ is $1/t_2$
Hence the line through A perpendicular to BC is (1)
$y-a(t_1+t_2)=-t_3(x-at_1t_2)$
Similarly the line through $B \perp$ to CA is (2)
$y-a(t_2+t_3)=-t_1(x-a\,t_2t_3)$
Subtracting we get $x=-a$ and
hence $y=a\,(t_1+t_2+t_3+t_1t_2t_3)$

(*d*) Proceed as above

or $\quad y-b=\dfrac{b^2}{a^2}(x-a)$

or $\quad a^2y-b^2x=ab\,(a-b)$

6 The required line by $P+\lambda Q=0$ is (1) of Q 5

Putting $y=0$, we get the point A as $\left(\dfrac{ab\,(1+\lambda)}{b+a\lambda},\ 0\right)$

Putting $x=0$, we get the point B as $\left(0,\ \dfrac{ab\,(1+\lambda)}{a+b\lambda}\right)$

If (h, k) be the mid point of AB, then

$$2h=\frac{ab\,(1+\lambda)}{b+a\lambda},\quad 2k=\frac{ab\,(1+\lambda)}{a+b\lambda}$$

In order to find the locus of (h, k), we have to eliminate the variable λ

$$\frac{1}{2h}+\frac{1}{2k}=\frac{(b+a\lambda)+(a+b\lambda)}{ab(1+\lambda)}=\frac{(a+b)\,(1+\lambda)}{ab\,(1+\lambda)}=\frac{a+b}{ab}$$

$$\frac{h+k}{2hk}=\frac{a+b}{ab}$$

The variable λ has been eliminated and hence the locus of mid point (h, k) is obtained by generalising (h, k) and is

$$2xy\,(a+b)=ab\,(x+y)$$

7 The line makes an angle of $135°$ with x axis and hence its slope is $\tan 135°$ *i e* -1

Let the two parallel lines with slope -1 be

$x+y=a$ and $x+y=b$

where a and b are parameters

Putting $y=0$, we get the points $A\,(a, 0)$, $B\,(b, 0)$,

Putting $x=0$, we get points $C\,(0, a)$, $D\,(0, b)$

Equation to AD joining $(a, 0)$ and $(0, b)$ is $\dfrac{x}{a}+\dfrac{y}{b}=1$

Equation to BC joining $(b, 0)$ and $(0, a)$ is $\dfrac{x}{b}+\dfrac{y}{a}=1$

In order to find the locus of the point of intersection of these lines we have to eliminate the parameters a and b Subtracting we get

$$x\left(\frac{1}{a}-\frac{1}{b}\right)+y\left(\frac{1}{b}-\frac{1}{a}\right)=0 \quad \text{or} \quad x-y=0$$

The bisectors are $21x+77y-101=0$ and $11x-3y+9=0$. Let θ be the angle between one of the lines say $3x-4y+7=0$ and one of the bisectors say $11x-3y+9=0$. Their slopes are $m_1=3/4$, $m_2=11/3$

$$\tan\theta=\frac{\frac{3}{4}-\frac{11}{3}}{1+\frac{3}{4}\cdot\frac{11}{3}}=\frac{-34}{45}$$

and it is numerically less than 1

Hence $\theta < 45°$ or $2\theta < 90°$. Therefore this is the bisector of the acute angle between the lines and hence the other bisects the obtuse angle

8 The two bisectors are

$$(4\ \ \sqrt{5})\,x-(3-2\sqrt{5})\,y+(2-4\sqrt{5})=0 \qquad (1)$$
$$(-4-\sqrt{5})\,x+y\,(3+2\sqrt{5})-(2+4\sqrt{5})=0 \qquad (2)$$
$$m_1=\tfrac{1}{2},\ m_2=\frac{4-\sqrt{5}}{3-2\sqrt{5}},$$

where m_1 is the gradient of one of the given lines and m_2 the gradient of one of the bisectors

$$\tan\theta=\frac{\frac{1}{2}-\frac{4-\sqrt{5}}{3-2\sqrt{5}}}{1+\frac{1}{2}\,\frac{4-\sqrt{5}}{3-2\sqrt{5}}}=\frac{-5}{10-5\sqrt{5}}=\frac{-1}{2-\sqrt{5}}$$

$$=-\frac{2+\sqrt{5}}{4-5}=2+\sqrt{5}>1$$

$\theta > 45°$ or $2\theta > 90°$. Hence (1) bisects the obtuse angle between the given lines

9 (a) Write the equations of the sides in such a form that the constant term in each is +ive

BC is $x+1$, AB is $-3x-4y+5=0$, AC is $-5x-12y+27=0$

Trace the lines by putting $x=0$ and find y and then $y=0$ and find x. Thus you will find the points where the given lines cut the axes of x and y. We see from the figure that origin lies within all the three angles. Hence finding the bisec-

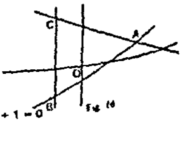


Fig 14

i.e., area of a triangle formed by the points (p_1, q_1), (p_2, q_2) and (p_3, q_3) is zero and as such the points are collinear

(b) Required condition is $\Sigma m_1(c_2-c_3)=0$

14 The lines will be concurrent if

$$\begin{vmatrix} p-q & q-r & r-p \\ q-r & r-p & p-q \\ r-p & p-q & q-r \end{vmatrix}=0$$

Adding column no 2 and 3 to column no 1, we get

$$\begin{vmatrix} 0 & q-r & r-p \\ 0 & r-p & p-q \\ 0 & p-q & q-r \end{vmatrix}=0 \quad \text{Hence proved}$$

15 Let p=perp from vertex $(2, -1)$ to base $x+y=2$ $\quad p=1/\sqrt{2}$

Also $p=a\sin 60°=\frac{a\sqrt{3}}{2}$ $\quad \frac{1}{\sqrt{(2)}}=\frac{a\sqrt{3}}{2}$, $\quad a=\sqrt{\left(\frac{2}{3}\right)}$

16 (a) Let the point be (p, q) so that $p+5q=13$ (1)

Its distance from the line $12x-5y+26=0$ is

$\frac{12p-5q+26}{\sqrt{(144+25)}}=\pm 2$ given

or $\quad 12p-5q+26=\pm 26$

+ive sign $\quad 12p-5q=0$ (2)

−ive sign $\quad 12p-5q=-52$ (3)

Solving (1) and (2) we get the point (p, q) as $(1, 12/5)$

Solving (1) and (3) we get the point (p, q) as $(-3, 16/5)$

(b) Apply $\quad d=\pm p$

$13(x^2+y^2-6x+4y+13)=\pm(5x-12y-13)$

17 Ans $(3, 1), (-7, 11)$

18 If the intercept on x axis be a then on y-axis is $2a$ so that its equation is

$$\frac{x}{a}+\frac{y}{2a}=1 \quad \text{or} \quad 2x+y=2a$$

By given condition

$$\frac{2a}{\sqrt{(4+1)}}=\pm 1 \qquad a=\pm\tfrac{1}{2}\sqrt{5}$$

Hence the lines are $\quad 2x+y=\pm\sqrt{5}$

19 By $P+\lambda Q=0$ the line is $12x-y-31=0$

Its distance from $(0, 0)$ is

4

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8

If the c... ...of $\angle$ E C be taking
... ...bers of EC C4 (5
... ...oppos... to $\angle$ E C) the ...
— Hence ... ...the ...
the results of substituting the co-ordinates
respectively in the left hand members ...
sign (here positive) Then the equations
respectively are

x ... $-1-0$ x ... $+1=0$ and $-7x-$...

Let P be any point on the internal bisector of ...
it is evident from the figure that P lies on the ...
of AB as C and of AC as B Hence the coordinates
make both $-7x+y+6=0$ and x ... $y+1=0$ of the
sign (positive) Therefore the equation of the
bisector of angle BAC must be

$$\frac{7x-y-6}{\sqrt{(49+1)}}=\frac{x-y+1}{\sqrt{2}} \text{ i.e. } -7x+y+6=5x\ 5y$$

or $12x-6y-1=0$

Similarly the internal bisector of the angle ABC is

$$\frac{-7x+y+6}{\sqrt{(49+1)}}=+\frac{x+y-1}{\sqrt{2}} \text{ i.e. } -7x+y+6=5x+5y\quad 5$$

or $12x+4y-11=0$

Solving (1) and (2), the in centre is found to be $(\frac{2}{12}\ 1)$

10 (a) Prove as in Q 9

(b) Substituting the co ordinates of the point $(2\ -1)$ in the left hand side of the given equations the results are 2 and 23 i.e. both +ive Hence the bisector of the angle in which $(2\ -1)$ lies is obtained by taking +sign out of $\pm$ Therefore the equation of the bisector of the supplement of that angle is obtained by taking $-$ive sign Hence the required equation is

$$\frac{2x-3y-5}{\sqrt{(4+9)}}=-\frac{6x-4y+7}{\sqrt{(36+16)}}$$

or $10x-10y-3=0$

11 Putting $y=0$ and then $x=0$ in the given equation we get the points

$$A\left(\frac{p}{\cos\alpha}\ 0\right)\ B\left(0\ \frac{p}{\sin\alpha}\right)$$

Divide both sides by $\sqrt{2}$

$$\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta=\frac{3}{\sqrt{6}\sqrt{2}}=\frac{\sqrt{3}}{2}$$

or $\cos(\theta-45°)=\cos 30°$ $\qquad\theta-45°=\pm 30°$

$\theta=75°$ or $15°$

24 Line is $\dfrac{x-3}{\cos 3\pi/4}=\dfrac{y-4}{\sin 3\pi/4}=r$, say

$$\frac{x-3}{-1/\sqrt{2}}=\frac{y-4}{1/\sqrt{2}}=r$$

Its equation is $\qquad x-3=-(y-4)$

or $\qquad x+y-7=0$

Any point on it is

$$\left(\frac{-r}{\sqrt{2}}+3, \frac{r}{\sqrt{2}}+4\right)$$

Since the two points are at a distance $\sqrt{2}$ from P on opposite sides

Hence choosing $r=\pm\sqrt{2}$, we get the points as

$$\left(-\sqrt{2}\,\frac{1}{\sqrt{2}}+3, \sqrt{2}\,\frac{1}{\sqrt{2}}+4\right)$$

and $$\left(\sqrt{2}\,\frac{1}{\sqrt{2}}+3, -\sqrt{2}\,\frac{1}{\sqrt{2}}+4\right)$$

or $\quad$ (2, 5) and (4, 3)

25 The required lines are

$$3x+4y-7=0 \text{ and } 3x+4y-10=0$$

$$p_1=\frac{-7}{5}, p_2=\frac{10}{5}$$

d=distance between parallel lines,

$$=p_1\sim p_2=\frac{-7}{5}-\frac{10}{5}=\frac{-17}{5}=\frac{17}{5}$$

26 All the three lines are parallel Any point on the third line on putting $x=0$ is (0 2) Its distance from the other two lines are $\dfrac{0+6-19}{\sqrt{13}}=-\sqrt{13}$ and $\dfrac{0+6+7}{\sqrt{13}}=\sqrt{13}$

Above shows that these lines are equidistant from the third line and on opposite sides of it

27 (a) Area of $\|^{m}=2$ Area of $\triangle ABD=2\,\frac{1}{2}AB\,AD\,\sin\theta$

But if p_1 and p_2 be the distances between the parallel sides then from the figure

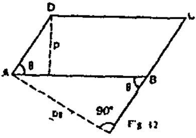


Fig 12

$$h\left(\frac{k}{h}+2\right)\left(\frac{2k}{h}-1\right)=3\frac{k}{h}+1$$

or $(k+2h)(2k-h)=3k+h$

$2k^2+3hk-2h^2-3k-h=0$ Hence the locus is

$2y^2+3xy-2x^2-3y-x=0$ or $2x^2-3xy-2y^2+x+3y=0$

(b) $13(x^2+y^2)-83x+64y+182=0$

14 (a) The given line is $\frac{x}{a}+\frac{y}{b}=1$ (1)

Any line through origin and perpendicular to (1) is

$\frac{x}{b}-\frac{y}{a}=0$ (2)

Now foot of perpendicular is the point of intersection of lines (1) and (2) In order to find its locus we have to eliminate the variables a and b Squaring and adding (1) and (2), we get

$$x^2\left(\frac{1}{a^2}+\frac{1}{b^2}\right)+y^2\left(\frac{1}{b^2}+\frac{1}{a^2}\right)=1$$

or $x^2+y^2=c^2$ $\quad\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c^2}$ (given)

(b) Any line perpendicular to AB i e $bx+ay=ab$ is

$ax-by=\lambda$ (λ parameter)

$A(a, 0)$ $B(0, b)$

$P\left(\frac{\lambda}{a}, 0\right)$ $Q\left(0, -\frac{\lambda}{b}\right)$

Lines AQ and BP by intercepts form are

$\frac{x}{a}+\frac{y}{-\lambda/b}=1$ or $\lambda x-aby=a\lambda$ (1)

$\frac{x}{\lambda/a}+\frac{y}{b}=1$ or $abx+\lambda y=b\lambda$ (2)

In order to find the locus of the point of intersection we have to eliminate λ between (1) and (2)

$$\lambda=\frac{aby}{x-a}=-\frac{abx}{y-b} \text{ from (1) and (2)}$$

$$x(x-a)+y(y-b)=0$$

or $x^2+y^2-ax-by=0$ is the required locus (1)

15 (a) $y-k=m(x-h)$ where m is parameter

Equation of the through (0, 0) perpendicular to (1) is

$$y=-\frac{1}{m}x$$ (2)

Area $=p_1 p \operatorname{cosec}\theta=$ as given by relations A and B

(*d*) Do yourself

(*e*) Here $p_1=p_2=\frac{1}{(1/a^2+1/b^2)^{1/2}}=\frac{ab}{\sqrt{(a^2+b^2)}}=p$, say

Hence the lines enclose a rhombus

Area is $p_1 p_2 \operatorname{cosec}\theta=p^2 \operatorname{cosec}\theta=\frac{a^2b^2}{(a^2+b^2)} \operatorname{cosec}\theta$ (1)

Now θ is the angle between adjacent sides whose slopes are

$-\frac{b}{a}$ and $-\frac{a}{b}$ $\tan\theta=\frac{-b/a+a/b}{1+1}=\frac{a^2-b^2}{2ab}$

$$\operatorname{cosec}^2\theta=1+\cot^2\theta=1+\frac{4a^2b^2}{(a^2-b^2)^2}=\left[\frac{a^2+b^2}{(a^2-b^2)}\right]^2$$

$\operatorname{cosec}\theta=\frac{a^2+b^2}{a^2-b^2}$ Hence from (1) area $=\frac{a^2b^2}{a^2-b^2}$

(*f*) Here $p_1=p_2$ Rhombus and hence diagonals are perpendicular

(*g*) The given lines are $\pm x\pm y=1$ *i e*

$x+y=1$, $x-y=1$, $x+y=-1$, $x-y=-1$

$p_1=p_2=\frac{2}{\sqrt{2}}=\sqrt{2}$ Also $\theta=90°$

Area $p_1p_2 \operatorname{cosec}\theta=p^2 \cdot 1=2$

28 The four sides of the rhombus are

$ax+by+c=0$ (1)

$ax+by-c=0$ (2)

$ax-by+c=0$ (3)

$ax-by-c=0$ (4)

Solving (1) with (3) and (4) we get the vertices $(-c/a\ 0)$ $(c/a\ 0)$

Solving (2) with (3) and (4) we get the vertices as $(0, c/b)$ $(0, -c/b)$

Above shows that one diagonal is of length $2c/a$ and is along x axis whereas the other is along y axis and is of length $2c/b$. Since the diagonals being along the axes, they are perpendicular and hence a rohmbus

The area is $\frac{1}{2}(d_1 d_2)=\frac{1}{2}\left(\frac{2c}{a}\ \frac{2c}{b}\right)=\frac{2c^2}{ab}$

Note You may do it as in Q 27

Slope of line (1) is $-\dfrac{a+b}{a-b}=m_1$

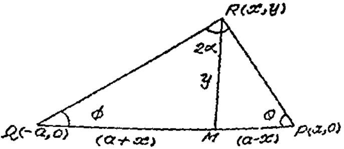
Slope of bisector $x-y=0$ is $1=m_2$

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}=\frac{-\dfrac{a+b}{a-b}-1}{1-1\cdot\dfrac{a+b}{a-b}}=\frac{-2a}{-2b}=\frac{a}{b}$$

$$\alpha=\tan^{-1}\frac{a}{b}$$

or vertical angle is $2\alpha=2\tan^{-1}\dfrac{a}{b}$

18 Let QP be chosen along x axis and its length $=2a$ and mid point as origin so that P is $(a, 0)$ and Q is $(-a, 0)$
Let R be (x, y) Let $\angle RPQ=\theta$ and $\angle RQP=\phi$



$\theta-\phi=2\alpha$ given and RM be perpendicular on x axis

$$\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\tan 2\alpha \qquad (1)$$

But $\tan\theta=\dfrac{RM}{MP}=\dfrac{y}{a-x}$ $\tan\phi=\dfrac{RM}{MQ}=\dfrac{y}{a+x}$

Putting in (1) we get

$$\frac{\dfrac{y}{a-x}-\dfrac{y}{a+x}}{1+\dfrac{y}{a+x}\cdot\dfrac{y}{a-x}}=\tan 2\alpha \quad\text{or}\quad 2xy\cot 2\alpha=a^2-x^2+y^2$$

or $y^2-x^2-2xy\cot 2\alpha-a^2=0$ is the required locus

19 Here $AB=c$ Let $\angle BAO=\theta$ Then $\angle QPA=\theta$, where Q is the foot of the perpendicular from P on AB To find the locus of Q (h, k) we have

$OA=AB\cos\theta=c\cos\theta$ $AP=OB=AB\sin\theta=c\sin\theta$

$PQ=AP\cos\theta=c\sin\theta\cos\theta$

where r_1, r_2, r_3 are respectively the distances of the points A, Q and B from the given point $P(-1, 2)$

Point A is $(r_1 \cos\theta - 1, r_1 \sin\theta + 2)$ where $r_1 \sin\theta + 2 = 0$ (1)

as the point A lies on x axis

Point B is $(r_3 \cos\theta - 1, r_3 \sin\theta + 2)$ where $r_3 \cos\theta - 1 = 0$ (2)

as the point B lies on y-axis

Let point Q be (h, k) $\quad h = r_2 \cos\theta - 1$

and $\quad k = r_2 \sin\theta + 2$ (3)

We have to find the locus of the point Q

Also it is given that r_1, r_2, r_3 are in H P

$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_3} = -\frac{\sin\theta}{2} + \cos\theta \text{ by (1) and (2)}$$

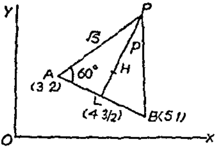
$$\text{or } \frac{2}{r_2} = -\frac{1}{2}\frac{(k-2)}{r_2} + \frac{(h+1)}{r_2} \text{ by (3)}$$

$$\text{or } 2 = -\frac{k}{2} + 1 + h + 1 \text{ or } k = 2h \text{ or } y = 2x$$

(*d*) Equation of AB is $\quad x + 2y = 7$

and its length $a = \sqrt{5}$ and mid point of AB is the point $L(4, 3/2)$. If p be the vertex of the equilateral triangle then its perpendicular distance p from AB is $a \sin 60°$ or $p = \sqrt{5}$

$$\frac{\sqrt{3}}{2} = \frac{1}{2}\sqrt{15}$$



Also distance of orthocentre from AB is $\frac{1}{3}p$

Now both H and P lie on a line perpendicular to AB whose slope will be 2 and passing through $L(4, 3/2)$

$$\tan\theta = 2 \quad \text{or} \quad \sin\theta = \frac{2}{\sqrt{5}} \text{ and } \cos\theta = \frac{1}{\sqrt{5}}$$

Hence H and P lie on

$$\frac{x-4}{\cos\theta} = \frac{y-3/2}{\sin\theta} = p \text{ for } P \text{ and} = \frac{1}{3}p \text{ for } H$$

$x = 4 + p\cos\theta, \quad y = 3/2 + p \sin\theta$ for P

$x = 4 + (3/p)\cos\theta \quad y = 3/2 + (3/p)\sin\theta$ for H

putting the values of p, $\cos\theta$ and $\sin\theta$ in the above we get

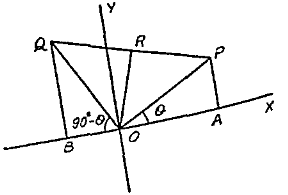
$c^{4/3}(x^{2/3}+y^{2/3})=c^2$ or $x^{2/3}+y^{2/3}=c^{2/3}$

20 We take O as origin and x axis along BOA. Let the co-ordinates of A and B be $(a, 0)$ and $(-b, 0)$. Let the fixed parallel lines AP and BQ be taken parallel to y axis so that BOA is perpendicular to both.

If $\angle POA=\theta$ then as $\angle POQ=90°$

we have $\angle QOB=90°-\theta$

Coordinates of P are $(a, a\tan\theta)$ and those of Q are $(-b, b\cot\theta)$



Equation to line PQ is

$$y-a\tan\theta=\frac{b\cot\theta-a\tan\theta}{-(b+a)}(x-a)$$

or $x(b\cot\theta-a\tan\theta)+y(a+b)=ab(\cot\theta+\tan\theta)$

or changing into $\sin\theta$ and $\cos\theta$ it becomes

$x(b\cos^2\theta-a\sin^2\theta)+y(a+b)\sin\theta\cos\theta=ab$ (1)

Multiplying by 2 it becomes

$x(2b\cos^2\theta-2a\sin^2\theta)+y(a+b)\sin 2\theta=2ab$ (2)

Also equation to perpendicular OR on PQ is

$x(a+b)\sin 2\theta-y(2b\cos^2\theta-2a\sin^2\theta)=0$

But $2b\cos^2\theta-2a\sin^2\theta$

$=b(1+\cos 2\theta)-a(1-\cos 2\theta)$

$=(a+b)\cos 2\theta-(a-b)$

Hence the two lines (1) and (2) can be re written as

$x(a+b)\cos 2\theta+y(a+b)\sin 2\theta=2ab+(a-b)x$ (3)

$x(a+b)\sin 2\theta-y(a+b)\cos 2\theta=-(a-b)y$ (4)

In order to find the locus of point of intersection R we have to eliminate θ for which we square and add the equations (3) and (4)

$x^2(a+b)^2+y^2(a+b)^2=4a^2b^2+4ab(a-b)x+$

(b) Find the equation of the right bisectors of the sides and hence the coordinates of its circumcentre

(c) Find the equations of the perpendiculars drawn from the vertices to the opposite sides and hence the coordinates of its orthocentre

(d) The area of the triangle

2 The line $3x+2y=24$ meets the y axis at and the x axis at B The perpendicular bisector of AB meets the line through $(0, -1)$ parallel to x-axis at C Find the area of the triangle ABC (I I T 76)

3 Find the orthocentre of the triangle with sides

(a) $x+y=6$, $2x+y=4$ and $x+2y=5$

(b) $4x-7y+10=0$, $x+y=5$ $7x+4y=15$ (I I T 76)

(c) The vertices of a triangle are $[at_1t_2\ a(t_1+t_2)]$ $[at_2t_3\ a(t_2+t_3)]$, $[at_3t_1, a(t_3+t_1)]$ Find the coordinates of its orthocentre (I I T 83)

(d) Prove that the orthocentre of the triangle formed by the three lines $y=m_1x+a/m_1$, $y=m_2x+a/m_2$, $y=m_3x+a/m_3$ is

$$\left\{-a\quad a\left(\frac{1}{m_1}+\frac{1}{m_2}+\frac{1}{m_3}+\frac{1}{m_1m_2m_3}\right)\right\}$$

(e) Prove that the coordinates of the orthocentre of the triangle whose vertices are $(0, 0)$, $(2, -1)$ and $(1, 3)$ is $(-4/7, -1/7)$ (I I T 74)

4 (a) Two vertices of a triangle are $(5, -1)$ and $(-2, 3)$ If the orthocentre of the triangle is the origin, find the coordinates of the third point (I I T 79)

(b) Two vertices of a triangle are $(4, -3)$ and $(-2, 5)$ If the orthocentre of the triangle is at $(1\ 2)$, prove that the third vertex is $(33\ 26)$ (Roorkee 87)

5 Find the coordinates of those points on the line $3x+2y=5$ which are equidistant from the lines

$$4x+3y=7 \text{ and } 2y-5=0$$

6 Prove that the lengths of the perpendiculars from any point on the line $7x-9y+10=0$ to the line $3x+4y=5$ and $12x+5y=7$ are equal

7 Find the equation of the bisector of the acute angle between the lines

$$3x-4y+7=0 \text{ and } 12y+5y-2=0$$ (I I T 75)

$x_1y_2 - x_2y_1 = -\frac{1}{41}\left[h(9k-40h) - k(40k+9h)\right]$

$= \frac{40}{41}\left[h^2+k^2\right]$

Putting the above values in (1) we get

$f\frac{(40k+40h)}{41} - g\frac{(50k-50h)}{41} + \frac{40}{41}(h^2+k^2)=0$

Hence the locus of the vertex (h, k) is

$(x^2+y^2)+(4g+5f)x+(4f-5g)y=0$

Above equation represents a circle

22 Let the points be denoted by A, B and C respectively The equations of the sides are found to be as AB, $y=3$, BC, $2x+y=-5$ and CA, $2x-y=5$ Equations of lines perpendicular to above and passing through $(5, 0)$ are respectively $x=5$ $x-2y=5$ and $x+2y=5$

Solving these pairs of perpendicular lines the three feet of perpendiculars are given as $P(5, 3)$ $Q(-1, -3)$ and $R(3\ 1)$ It can be easily shown that area of $\Delta PQR=0$ so that these points are collinear We may also say that slope of PQ=slope of $PR=1$ so that these points are collinear

23 Let $x^2+y^2=R^2$ be the equation of the circle circumscribing the triangle ABC as its centre O is $(0\ 0)$

The three vertices lie on it

$p^2+p^2\tan^2\alpha=R^2$ or $p\sec\alpha=R$ $p=R\cos\alpha$

Hence the point $A(p\ \ p\tan\alpha)$ is $(R\cos\alpha\ \ R\sin\alpha)$

Similarly B and C are $(R\cos\beta\ \ R\sin\beta)$

and $(R\cos\gamma, R\sin\gamma)$

If G be the centroid of triangle ABC then,

$G=\left(\frac{R\Sigma\cos\alpha}{3}\ \ \frac{R\Sigma\sin\alpha}{3}\right)$ $O=(0, 0)$

and $H=(\bar{x}\ \ \bar{y})$

From geometry we know that O G, H are collinear

Hence slope of OG = slope of OH

$$\frac{\bar{y}-0}{\bar{x}-0}=\frac{\Sigma\frac{R\sin\alpha}{3}-0}{\Sigma\frac{R\cos\alpha}{3}-0}$$

$$\frac{\bar{x}}{\bar{y}}=\frac{\cos\alpha+\cos\beta+\cos\gamma}{\sin\alpha+\sin\beta+\sin\gamma}$$

moves in such a way that

$$\frac{1}{a^2}+\frac{1}{b}=\frac{1}{c^2}$$

where c is a constant Prove that the foot of perpendicular from the origin on the straight line describes the circle $x^2+y^2=c^2$

(b) The straight line

$$\frac{x}{a}+\frac{y}{b}=1$$

cuts the axes in A and B and a line perpendicular to AB cuts the axes in P and Q Find the locus of the point of intersection of AQ and BP

15 (a) A straight line passes through a fixed point (h, k) Find the locus of feet of the perpendiculars on it drawn from the origin

(b) Find the projection of the point $(1, 0)$ on the line joining the points $P(-1, 2)$ and $Q(5, 4)$

16 (a) A line APB of constant length meets the x axis at A and y axis at B If $AP=b$, $PB=a$ and the line slides with its extremities on the coordinate axes, show that equation of the locus of the point P is

$$\frac{x^2}{a}+\frac{y^2}{b^2}=1$$

(b) A straight line segment of length l moves with it ends on two mutually perpendicular lines Find the locus of the point which divides the line in the ratio 1 2

(I I T 1978)

17 Prove that the lines

$$(a+b)\,x+(a-b)\,y-2ab=0 \quad (1)$$
$$(a-b)\,x+(a+b)\,y-2ab=0 \quad (2)$$

and $x+y=0$ (3)

form an isosceles triangle whose vertical angle is $2\tan^{-1} a/b$

18 Two points P and Q are given R is a variable point on one side of the line PQ such that $\angle RPQ-\angle RQP$ is a positive constant 2α Find the locus of the point R **(Roorkee 82)**

19 The ends A B of a straight line segment of constant length c slide upon the fixed rectangular axes OX OY respectively If the rectangle $OAPB$ be completed then show that the locus of the foot of the perpendicular drawn from P to AB is $x^{2/3}+y^{2/3}=c^{2/3}$ **(I I T 83)**

co-ordinates of the point P such that $PQRS$ is a parallelogram are

(b) If $P=(1, 0)$, $Q=(-1, 0)$ and $R=(2, 0)$ are three given points, then the locus of a point S satisfying the relation $SQ^2+SR^2=2SP^2=$ is

(a) a st line ∥ to x axis, (b) circle through origin,

(c) circle with centre at the origin, (IIT) 88

(d) a st line ∥ to y axis

3 L, M, N are the points (p,q), $(3, 2)$, $(7,5)$ If LM is perpendicular to LN then

$$p^2+q^2-10p-7q+31=0$$

(a) True, (b) False

4 If A and B are the points $(-3, 4)$ and $(2, 1)$ Then the co ordinates of point C on AB produced such that $AC=2BC$ are

(i) (2 4) (ii) (3, 7), (iii) (7, −2), $(-\frac{1}{2}, \frac{4}{11})$,

5 P and Q are points on the line joining A $(-2, 5)$ and $B(3, 1)$ such that $AP=PQ=QB$ Then the mid point of PQ is

(i) $(\frac{1}{2}, 3)$, (ii) $(-\frac{1}{2}, 4)$, (iii) (2, 3), (iv) (1, 4)

6 If the lines $3y+4x=1$, $y=x+5$ and $5y+bx=3$ are concurrent then the value of b is

(i) 1, (ii) 3, (iii) 6, (iv) 0

7 (a) If the points $(-2, -5)$, $(2, -2)$, $(8, a)$ are collinear, then the value of a is

(b) The distance between the lines $3x+4y=9$ and $6x+8y=15$ is

(i) 3/2 (ii) 3/10 (iii) 6 (iv) None of these (M N R 82)

(c) Let the vertices of a triangle be (0, 0), (3, 0) and (0, 4) Its orthocentre is

(a) (0, 0) (b) (1, 4/3) (c) (3/2, 2)

(d) None of these

8 The gradient of the line joining the points on the curve $y=x^2+2x$ whose abscissas are 1 and 3 is

9 If the lines $y=4-3x$, $ay=x+10$ and $2y+bx+9=0$ form three sides of a rectangle and the fourth side passes through $(1, -2)$ then its equation is

$D(-1, -1), E(-2, -4), F(-3, -5)$

Equation to median AD joining $(-4, -8)$ and $(-1, -1)$

$$y+8=\frac{-1-(-8)}{-1-(-4)}(x+4) \text{ or } 3(y+8)=7(x+4)$$

or $\quad 7x-3y-4=0 \quad (1)$

Similarly the equations of medians BE and CF are respectively $\quad x+2=0 \quad (2)$

and $\quad 5x-3y=0 \quad (3)$

Note If $y-y_1=m(x-x_1)$

or $\frac{y-y_1}{x-x_1}=m=\infty$, then $x-x_1=0$

Solving any two of (1), (2) and (3) we get the point of intersection as $(-2, 10/3)$ which clearly satisfies the third Hence the three medians are concurrent and the point of concurrency is called centroid of the triangle

We can however find the coordinates of the centroid by using the formula

$$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right) \text{ i e } \left(-2, \frac{-10}{3}\right)$$

(*b*) **Right Bisector** Any line which passes through the middle points of any side and is perpendicular to it is called its right bisector

Any line $\perp$ to BC $y-x=0$ is $x+y+k=0$ Since is passes through mid point $D(-1, -1)$,

$\therefore \quad -1-1+k=0$ or $k=2$

Right bisector of BC is $x+y+2=0 \quad (1)$

Similarly the right bisectors of sides CA and AB are respectively $\quad 2y+x+10=4 \quad (2)$

$3y+x+18=0 \quad (3)$

Solving any two of (1), (2) and (3) we get the point $(-6, -8)$ which satsifies the third also Hence the right bisectors of the sides of a triangle are concurrent and the point of concurrency is called the circumcentre of the triangle which is the centre of a circle that passes through the vertices of the triangle

(*c*) **Orthocentre** Any line through the vertex A $(-4 \ -8)$ and perpendicular to the opposite side BC, $y-x=0$ is called the altitude through A Hence its equation is

$$x+y+k=0$$

(b) Ans (d) $2x+3=0$

3 Ans (a)

4 Ans (iii)

5 Ans (i)

From the given condition, it is clear that P divides AB in the ratio 1 2 and Q divides AB in the ratio 2 1 Hence $P\,(-1/3,\ 11/3)$ and $Q\,(4/3,\ 7/3)$

Mid-point of PQ is $(1/2, 3)$ (i) is correct

6 Ans (iii)

7 (a) Ans $2\frac{1}{2}$ (b) (ii) (c) (a) i e $(0, 0)$

The three altitudes are $x=0$, $y=0$ and $\frac{x}{4}-\frac{y}{3}=0$ which meet at $(0, 0)$

8 Ans 6

9 The two points are $(1, 3)$ and $(3, 15)$

The lines may be re written as

$AB,\ 3x+y-4=0$, $BC,\ x-ay+10=0$ $CD,\ bx+2y+9=0$

DA passes through $(1, -2)$

The figure being rectangle i e $AB \perp BC$ $a=3$

AB is parallel to CD $b=6$

Now DA will be parallel to BC and as such its equation is

$$x-3y+\lambda=0 \qquad a=3$$

It passes through $(1, -2)$ $\lambda=-7$

Hence fourth side is $x-3y-7=0$

10 Ans (ii)

11 Ans (b)

Solving the equations

$$3x+4y+6=0,\ 4x+7y+8=0$$

we get their point of intersection as $(-2, 0)$ Since the co-ordinates of this point satisfy the equation

$$\sqrt{2}x+\sqrt{3}y+2\sqrt{2}=0$$

it follows that the three lines are concurrent.

12 Solve any two and put in 3rd,

$$\Delta=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}=0,$$

4 Let the third vertex A be (p, q). O is orthocentre $(0, 0)$ which is intersection of altitudes through A and B

Slope of $BC = \frac{3-(-1)}{-2-5} = -4/7$

Slope of AD = slope of $AO = q/p$

But $m_1 m_2 = -1$

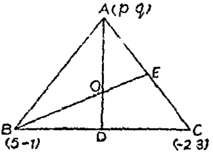
$$\left(-\frac{4}{7}\right)\frac{q}{p} = -1$$

$$7p = 4q \qquad (1)$$

Slope of $CA = \frac{q-3}{p+2}$

Slope of BE = Slope of BO

$$= \frac{-1}{5}$$



But $m_1 m_2 = -1$ $\qquad \left(\frac{q-3}{p+2}\right)\left(\frac{-1}{5}\right) = -1$

or $\qquad 5p - q + 13 = 0 \qquad (2)$

Solving (1) and (2) we get $p = -4$, $q = -7$. Hence the third vertex A is the point $(-4, -7)$

5 Any point which is equidistant from the given lines will lie on the bisector of the lines whose equations are

$$\frac{4x+3y-7}{\sqrt{(16+9)}} = \pm\frac{2y-5}{\sqrt{(4+0)}}$$

or $\qquad 8x - 4y + 11 = 0$ and $8x + 16y - 39 = 0$

If the coordinates of the required point be (α, β) then it will lie on the bisectors

$$8\alpha + 4\beta + 11 = 0 \qquad (1)$$

and $$8\alpha + 16\beta - 39 = 0 \qquad (2)$$

Since it lies on the line $3x + 2y = 5$

$$3\alpha - 2\beta = 5 \qquad (3)$$

Solving (1) and (3) and solving (2) and (3) we get the coordinates of the required points as

$$\left(\frac{-1}{14}, \frac{73}{28}\right) \text{ and } \left(\frac{1}{16}, \frac{77}{32}\right)$$

6 The first line be shown to be one of the bisectors of the other two lines as the distance of any point on the bisector from the given lines are equal

7 The bisectors of given lines are given by

$$\frac{3x-4y+7}{\sqrt{(25)}} = \pm\frac{12x+5y-2}{\sqrt{(169)}}$$

Equation of a Circle

(*a*) Centre (h k), radius a

$$(x-h)^2+(y-k)^2=a^2 \quad (1)$$

$x^2+y^2=a^2$, Centre (0, 0), radius a

(b) Centre (h k) passes through origin Here

$r=\sqrt{[(h-0)^2+(k-0)^2]}=\sqrt{(h^2+k^2)}$

$$(x-h)^2+(y-k)^2=h^2+k^2 \quad (3)$$

$x^2+y^2-2hx-2ky=0$

(c) Centre (h k) and touches the axis of x From the figure it is clear that radius will be k

$(x-h)^2+(y-k)^2=k^2$,

or $x^2+y^2-2hx-2ky+h^2=0$ (4)

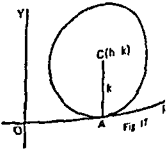


Fig 17

i e its intersection with x axis *i e* $y=0$, is

$$x^2-2hx+h^2=0$$

or $(x-h)^2=0$,

i e, it gives two equal values of x as x axis is a tangent

(*d*) Centre (h k) and touching the axis of y

$$(x-h)^2+(y-k)^2=h^2 \quad (5)$$

or $x^2+y^2-2hx-2ky+k^2=0$

(*e*) Which touches both the axes In this case the centre of the circle will be $(h\ h)$ and radius h

$$(x-h)^2+(y-h)^2=h^2 \quad (6)$$

or $x^2+y^2-2hx-2hy+h^2=0$

But since the centre could be in any of the four quadrants its co-ordinates can be taken as $(\pm h\ \pm h)$ and radius h (7)

$x^2+y^2\pm 2hx\pm 2hy+h^2=0$

(*f*) General equation of a circle

$x^2+y^2+2gx+2fy+c=0$

The coefficients of x^2 and y^2 are both unity and there is no term of xy

$(x^2+2gx)+(y^2+2fy)+$

tors we shall choose +ive sign out of ± sign for internal bisectors

Bisector of $\angle A$

$$\frac{-5x-12y+27}{\sqrt{(25+144)}}=+\frac{-3x+4y+5}{\sqrt{(9+16)}}$$

or $14x-112y+70=0$ or $x-8y+5=0$ (1)

Bisector of $\angle B$

$$\frac{-3x+4y+5}{\sqrt{(9+16)}}=+\frac{x+1}{1}$$

or $y-2x=0$ (2)

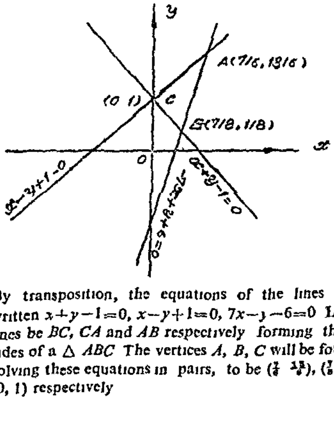
Bisector of $\angle C$

$$\frac{-5x-12y+27}{\sqrt{(25+144)}}=+\frac{x+1}{1}$$

or $9x+6y-7=0$ (3)

Solving (1) and (2) the coordinates of incentre are found to be $(\frac{1}{3}, \frac{2}{3})$ which clearly satisfies the third also

Remark *An alternative method of finding the in centre in given in part (b) below*



(*b*) By transposition, the equations of the lines may be written $x+y-1=0$, $x-y+1=0$, $7x-y-6=0$ Let these lines be *BC*, *CA* and *AB* respectively forming the three sides of a $\triangle$ *ABC* The vertices *A*, *B*, *C* will be found, on solving these equations in pairs, to be $(\frac{7}{6}, \frac{13}{6})$, $(\frac{7}{8}, \frac{1}{8})$ and (0, 1) respectively

(j) Circle through three given points. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

The coordinates of the three points will satisfy it and thus we will have three equations in g, f and c and solving them we shall find their values.

Note. In case the circle circumscribes a triangle the equations of whose sides are given then solving them in pairs we shall get the three vertices of the triangle.

(k) Parametric equations of a circle

$$x^2+y^2=a^2$$

$$x=a\cos\theta,\ y=a\sin\theta$$

$$(x-h)^2+(y-k)^2=r^2$$

$$x-h=r\cos\theta,\ y-k=r\sin\theta \qquad (14)$$

$$x=h+r\cos\theta,\ y=k+r\sin\theta$$

(l) Conditions for the two circles to touch. In case the two circles touch then from the above figure it is clear that

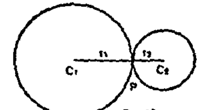


Fig 19

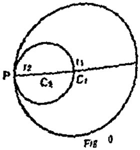


Fig 20

The distance between their centres

= Sum of the radii (Externally) (15)

= Difference of the radii (Internally)

Also the point of contact divides the join of centres in the ratio of the radii internally (for touching externally) and externally (for touching internally).

Note. See another method in part m.

(m) Equation of the common chord of two circles

$$S_1=0 \text{ and } S_2=0$$

Ensure that the coefficients of x^2 and y^2 in both the equations are each unity, then the equation of the common chord is given by

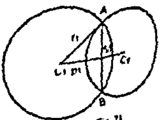


Fig 21

$$S_1-S_2=0 \qquad (16)$$

If (h, k) be the mid point of AB then

$$2h=\frac{p}{\cos\alpha}+0 \qquad 2k=0+\frac{p}{\sin\alpha}$$

or $\cos\alpha=\frac{p}{2h}$ and $\sin\alpha=\frac{p}{2k}$

Eliminating the variable α we get $\cos^2\alpha+\sin^2\alpha=1$

$\frac{p^2}{4h^2}+\frac{p^2}{4k^2}=1$ Generalising (h, k) the required locus is

$$\frac{p^2}{4}\left(\frac{1}{x^2}+\frac{1}{y^2}\right)=1 \quad \text{or} \quad p^2(x^2+y^2)=4x^2y^2$$

12 (a) Let the line be $x/a+y/b=1$ and since it passes through the point

$$\left(\frac{6}{5}, \frac{6}{5}\right) \qquad \frac{1}{a}+\frac{1}{b}=\frac{5}{6} \qquad (1)$$

It meets the axes at $A(a, 0)$, and $B(0\ b)$

Let $(h\ k)$ be the point which divides AB in the ratio 2 1 then

$$h=\frac{2\ 0+1\ a}{3}=\frac{a}{3},\ k=\frac{2\ b+1\ 0}{3}=\frac{2b}{3}$$

$a=3h$, $b=\frac{3k}{2}$ Putting in (1), we get

$\frac{1}{3h}+\frac{2}{3k}=\frac{5}{6}$ Generalising $(h\ k)$ the locus is

$$\frac{y+2x}{3xy}=\frac{5}{6} \quad \text{or} \quad 2(y+2x)=5xy$$

(b) Proceed as in part (a)

13 Any line through origin is $y=mx$

It meets the line $2x+y\ 2=0$ in $A\left(\frac{2}{m+2}, \frac{2m}{m+2}\right)$

It meets the line $x-2y+2=0$ in $B\left(\frac{2}{2m-1}\ \frac{2m}{2m-1}\right)$

If $(h\ k)$ be the mid point of AB then

$$2h=\frac{2}{m+2}+\frac{2}{2m-1} \quad \text{or} \quad h=\frac{3m+1}{(m+2)(2m-1)} \qquad (1)$$

$$2k=\frac{2m}{m+2}+\frac{2m}{2m-1} \quad \text{or} \quad k=\frac{m(3m+1)}{(m+2)(2m-1)} \qquad (2)$$

In order to find the locus we have to eliminate the variable m

Dividing (1) and (2) we get $\frac{k}{h}=m$ Putting in (1) we get (*j*) Circle through th

the circle be x^2+y^2

The coordinates of the t will have three equations in find their values

Note In case the circle of whose sides ar shall get the thre

(*k*) Parametric equatio

x^2

$x=a\cos\theta,\ y=a$

$(x-h)^2+(y-k)^2=r$

$x-h=r\cos\theta\quad y-k=$

$x=h+r\cos\theta,\ y=k+$

(*l*) Conditions for the c circles touch then from the

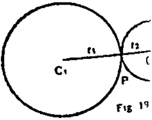


Fig 19

The distance between t =Sum of the r =Difference of

Also the point of con ratio of the radii internally (for touching internally)

Note See another m

(*m*) Equation of the c of two circles

$S_1=0$ and S_2

Ensure that the co and y^2 in both the equa unity, then the equation on chord is given by

$S_1-S_2=$

Foot of $\perp$ is intersection of (1) and (2) and its locus is obtained by eliminating m

$$y-k=-\frac{x}{y}(x-h) \quad \text{or} \quad x^2+y^2-hx-ky=0$$

(b) Line PQ is $x-3y+7=0$ and line through (1, 0) perpendicular to it is $3x+y-3=0$. Solving we get the foot of perpendicular as $\left(\frac{1}{5}, \frac{12}{5}\right)$ which is the projection of (1, 0) on the given line

16 Let the extremities of the line on the axes be $A(p, 0)$, $B(0, q)$. P is a point such that $AP=b$ and $PB=a$

$$AB=b+a=\sqrt{[(p-0)^2+(0-q)^2]}=\sqrt{(p^2+q^2)}$$

$$p^2+q^2=(a+b)^2 \qquad (1)$$

If (x, y) be the coordinates of P then it divides AB in the ratio $b : a$

$$x=\frac{b\cdot 0+a\cdot p}{a+b}=\frac{ap}{a+b}, \quad y=\frac{b\cdot q+a\cdot 0}{a+b}=\frac{bq}{a+b}$$

We have to eliminate p and q

$$\frac{x}{a}=\frac{p}{a+b}, \quad \frac{y}{b}=\frac{q}{a+b}$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{p^2+q^2}{(a+b)^2}=1 \text{ by (1)}$$

Hence the locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

(b) Proceed as above $p^2+q^2=l^2$

$$x=\frac{2p}{3} \quad y=\frac{q}{3} \qquad \frac{x^2}{4}+\frac{y^2}{1}=\frac{l^2}{9}$$

17 We know that is a triangle is isosceles then the bisector of the vertical angle is perpendicular to the base

The bisector of the angle given by lines (1) and (2) is

$$\frac{(a+b)x+(a-b)y-2ab}{\sqrt{[(a+b)^2+(a-b)^2]}}=\pm\frac{(a-b)x+(a+b)y-2ab}{\sqrt{[(a-b)^2+(a+b)^2]}}$$

or $x+y-2b=0$ and $x-y=0$

Clearly the bisector $x-y=0$ is perpendicular to the third side $x+y=0$. Hence the triangle is isosceles

If 2α be the vertical angle then α is the angle between the bisector $x-y=0$ and any of the lines (1) and (2)

8 Which passes through the points (0, 5) and (6, 1) and whose centre lies on the line $12x+5y=25$

9 (a) Which passes through the points (1, −2) and (4, −3) and whose centre lies on the line $3x+4y=7$ (IIT 74)

(b) A circle has radius 3 units and its centre lies on the line $y=x-1$. Find the equation of the circle if it passes through (7, 3) (Roorkee 88)

(c) Find the equation of a circle which passes through the point (2, 0) and whose centre is the limit of the point of intersection of the line $3x+5y=1$, $(2+c)x+5c^2y=1$ as c tends to 1

10 (a) Which passes through the point (2, 3) and touches the line $2x-3y-13=0$ at the point (2, −3)

(b) Prove that the equation of the circle which passes through the point (−2, 1) and is tangent to the line $3x-2y-6=0$ at the point (4, 3) is

$$7(x^2+y^2)+4x-82y+55=0$$

(c) Find the equation of the circle passing through the point A (1, 2), B (3, 4) and tangent to the straight line $3x+y-3=0$ (Roorkee 73)

11 Which passes through the point (1, −2) and (3, −4) and touches the axis of x

12 Which touches both the axes and the line $x=c$

13 Which touches the axes of coordinates at their positive sides and also touches the line $3x+2y=6$

14 Whose diameter is the line joining the points (−4, 3) and (12, −1). Find also the intercept made by it on y axis (IIT 71)

15 Whose diameter is the line joining the points (0, −1) and (2, 3). Find also the intercept made by it on the axis of x

16 Which passes through the points (3, 0), (1, 6) and (4, −1).

17 Which passes through the vertices of the triangle formed by the line

$$x+y+1=0,\ 3x+y-5=0 \text{ and } 2x+y-5=0$$

18 (a) Which passes through the points (9, 1), (7, 9), (−2, 12) and hence show that the four points (9, 1), (7, 9), (−2, 12) and (6, 10) lie on a circle

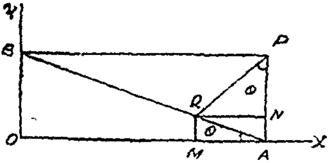
(b) For what value of c are the points (2, 0), (0, 1), (4, 5) and (0, c) concyclic (MNR 82)

$$h = OM = OA - AM = OA - QN = c\cos\theta - PQ\sin\theta$$
$$= c\cos\theta - c\sin^2\theta\cos\theta$$

and $k = QM = AN = AP - PN = c\sin\theta - PQ\cos\theta$
$$= c\sin\theta - c\sin\theta\cos^2\theta$$

Thus $h = c\cos\theta - c\cos\theta\sin^2\theta = c\cos\theta\,(1-\sin^2\theta)$
$$= c\cos^3\theta \qquad (1)$$

and $k = c\sin\theta - c\sin\theta\cos^2\theta = c\sin\theta\,(1-\cos^2\theta)$
$$= c\sin^3\theta \qquad (2)$$



$$\cos\theta = \left(\frac{h}{c}\right)^{1/3} \text{ and } \sin\theta = \left(\frac{k}{c}\right)^{1/3}$$

Squaring and adding we get

$$1 = \cos^2\theta + \sin^2\theta = \left(\frac{h}{c}\right)^{2/3} + \left(\frac{k}{c}\right)^{2/3} = \frac{h^{2/3}}{c^{2/3}} + \frac{k^{2/3}}{c^{2/3}}$$

The locus of Q is

$$x^{2/3} + y^{2/3} = c^{2/3}$$

Alternative Let the coordinates of A and B be $(a, 0)$ and $(0, b)$ on the axes so that its equation is $\quad x/a + y/b = 1$

or $\quad bx + ay - ab = 0 \qquad (1)$

By given condition $\quad AB = c \quad$ or $\quad a^2 + b^2 = c^2 \qquad (2)$

Again if P be the vertex of rectangle then point P is (a, b)

Line through P and perpendicular to AB is

$$(y-b) = \frac{a}{b}(x-a) \quad \text{or} \quad ax - by - (a^2 - b^2) = 0 \qquad (3)$$

Both these lines meet at the point Q whose locus we are to find, the variables being a, b which are connected by

$$a^2 + b^2 = c^2$$

Solving (1) and (3) by the method of cross multiplication

$$\frac{x}{-a^3} = \frac{y}{-b^3} = \frac{1}{-(a^2+b^2)} = -\frac{1}{c^2} \text{ by (2)}$$

or $\quad a = (c^2 x)^{1/3}, \quad b = (c^2 y)^{1/3}\quad$ putting in (2) we get

Show that if the two tangents are mutually perpendicular, the locus of their point of intersection is a circle concentric with the given circles

28 (a) Show that the locus of a point such that the ratio of its distances from two given points is constant, is a circle. Hence show that the circle cannot pass through the given points (IIT 70)

(b) Given the base of a triangle and the ratio of the lengths of the other two unequal sides, prove that the vertex lies on a fixed circle (IIT 73)

29 A point moves such that sum of the squares of its distances from the sides of a square of side unity is equal to 9. Show that the locus is a circle whose centre coincides with the centre of the square. Find also its radius (IIT 76)

30 If a straight line through $C(-\sqrt{8}, \sqrt{8})$ making an angle of $135°$ with the x axis cuts the circle $x=5\cos\theta$, $y=5\sin\theta$ in points A and B find the length of segment AB

31 Find the parametric equation of the circle

$$x^2+y^2+x+\sqrt{3}y=0$$

32 Find the equation of the circle whose radius is 3 and which touches externally the circle $x^2+y^2-2x-4y-20=0$ at the point (5 5)

33 Find the equation of the circle whose radius is 3 and which touches internally the circle $x^2+y^2-4x-6y-12=0$ at the point $(-1\ -1)$

34 (a) Prove that the circles

$x^2+y^2+2x+2y+1=0$ and $x^2+y^2-4x-6y-3=0$

touch each other and find the coordinates of the point of contact

(b) The circles $(x-1)^2+(y-2)^2=16$ and $(x+4)^2+(y+3)^2=1$

(a) Touch, (b) Coaxial (c) Intersect (d) None

35 Show that the circles

$x^2+y^2-10x+4y-20=0$ and $x^2+y^2+14x-6y+22=0$

touch each other. Find the coordinates of the point of contact and the equation of the common tangent at the point of contact

36 (a) A circle of radius 2 lies in the first quadrant and touches both the axes of coordinates. Find the equation of the

$(a-b)^2(x^2+y^2)$

or $(x^2+y^2)[(a+b)^2-(a-b)^2]=4ab[ab+(a-b)x]$

or $x^2+y^2=ab+(a-b)x$

or $x^2-(a-b)x-ab+y^2=0$

or $(x-a)(x+b)+y^2=0$

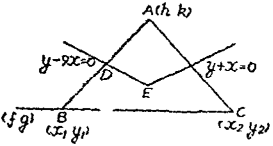
Above represents the equation of a circle on AB as diameter

21 Let the base be BC which passes through a fixed point $L(f, g)$. The three points L, B, C, are collinear

$$\begin{vmatrix} f & g & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

or $f(y_1-y_2)-g(x_1-x_2)+(x_1y_2-x_2y_1)=0$ (1)

Let the vertex A be (h, k) whose locus we are to find



Lines $y^2-8xy-9x^2=0$ or $(y-9x)(y+x)=0$ are the right bisectors of sides

AB is perpendicular to $y-9x=0$ $\quad \therefore \dfrac{k-y_1}{h-x_1}\times 9=-1$

$x_1+9y_1=h+9k$ (2)

Mid point $D\left(\dfrac{h+x_1}{2}, \dfrac{k+y_1}{2}\right)$ of AB lies on $y-9x=0$

$(k+y_1)-9(h+x_1)=0$

or $9x_1-y_1=k-9h$ (3)

Solving (2) and (3) for x_1 and y_1 we get

$x_1=\dfrac{9k-40h}{41},\ y_1=\dfrac{40k+9h}{41}$

Proceeding exactly as above we get the values of x_2, y_2 as

$x_2=-k$ and $y_2=-h$

$x_1-x_2=\dfrac{50k-40h}{41}\quad y_1-y_2=\dfrac{40k+50h}{41}$ and (*b*) The equation of the circle passing through (1, 1) and the points of intersection of the circle $x^2+y^2+13x-3y=0$ and $2x^2+2y^2+4x-7y-25=0$ is $4x^2+4y^2+30x-13y-25=0$ (I I T 83)

45 Find the equation of the circles passing through the intersection of the circles $x^2+y^2-4=0$ and $x^2+y^2-2x-4y+4=0$ and touching the line $x+2y=0$

46 The circle $x^2+y^2+2gx+2fy+c=0$ bisects the circumference of the circle $x^2+y^2+2g'x+2f'y+c'=0$ Prove that

$$2g'(g-g')+2f'(f-f')=c-c'$$

47 (*a*) Prove that the length of the common chord of the circles $(x-a)^2+(y-b)^2=c^2$ and $(x-b)^2+(y-a)^2=c^2$ is $\sqrt{[4c^2-2(a-b)^2]}$ Hence find the condition that the two circles may touch each other

(*b*) If the circle c_1 $x^2+y^2=16$ intersects another circle c_2 of radius 5 in such a manner that the common chord is of maximum length and has a slope equal to 3/4, then the coordinates of the centre of c_2 are (I I T 88)

48 Prove that the length of the common chord of the circles $x^2+y^2+ax+by+c=0$ and $x^2+y^2+bx+ay+c=0$ is

$$\sqrt{[\tfrac{1}{2}(a+b)^2-4c]}$$

49 Find the equation and length of the common chord of the circles $x^2+y^2+2x+3y+1=0$ and $x^2+y^2+4x+3y+2=0$

50 The points of intersection of the line $4x-3y-10=0$ and the circle $x^2+y^2-2x-4y-20=0$ are and (I I T 83)

51 Prove that the length of the common chord of the two circles $x^2+y^2+2hx+a^2=0$ and $x^2+y^2-2ky-a^2=0$ is

$$2\sqrt{\left(\frac{(h^2-a^2)(k^2+a^2)}{h^2+k^2}\right)}$$

52 A circle of diameter 13*m* with centre *O* coinciding with the origin of coordinate axes has diameter *AB* on the *x* axis If the length of the chord *AC* be 5*m* find the following

(*a*) Equation of pair of lines *BC*

(*b*) The area of the smaller portion bounded between the circle and the chord *AC* (Roorkee 83)

53 The abscissas of the two points *A* and *B* are the roots of the equation $x^2+2ax-b^2=0$ and their ordinates are the roots of

24 (*a*) $lx+my+n=0$ is a bisector and let (α, β) be any point on it so that $l\alpha+m\beta+n=0$ (1)

The other line will pass through the intersection of given line and given bisector and hence by $P+\lambda Q=0$ its equation is

$$(px+qy+r)+\lambda\,(lx+my+n)=0 \quad (2)$$

Also $$px+qy+r=0 \quad (3)$$

If (α, β) be a point on the bisector then its perpendicular distance from the lines (2) and (3) is same

$$\frac{(p\alpha+q\beta+r)+\lambda\,(l\alpha+m\beta+n)}{\sqrt{[(p+l\lambda)^2+(q+m\lambda)^2]}}=\frac{p\alpha+q\beta+r}{\sqrt{(p^2+q^2)}}$$

Putting $l\alpha+m\beta+n=0$ by (1) in the above we get

$$(p+l\lambda)^2+(q+m\lambda)^2=p^2+q^2$$

or $$2\lambda\,(pl+qm)+\lambda^2\,(l^2+m^2)=0$$

$$\lambda=-2\frac{(pl+qm)}{l^2+m^2}$$

Putting for λ in (2) the required line is

$$(l^2+m^2)\,px+qy+r)-2\,(lp+mq)\,(lx-my+n)=0$$

(*b*) Hint Same as Q 24 (a) in a different form

Ans $(lx+my+n)\,(a^2+b^2)-2\,(al+bm)\,(ax+by+c)=0$

25 Do yourself

Objective Questions (Straight Line)

1 (*A*) The triangle joining the points A (2 7), B (4, -1), C (-2, 6) is

(*i*) equilateral (*ii*) right-angled (*iii*) isosceles

(*B*) The equation of the straight line which passes through the point (1, -2) and cuts off equal intercepts from the axes will be

(*a*) $x+y=1$ (*b*) $x-y=1$ (*c*) $x+y+1=0$

(*d*) $x-y-2=0$ (M N R 78)

(*C*) The equation of the straight line which is perpendicular to $y=x$ and passes through (3, 2) will be given by

(*a*) $x-y=5$ (*b*) $x+y=5$ (*c*) $x+y=1$

(*d*) $x-y=1$ (M N R 79)

(*D*) The equation of the line passing through (1, 2) and perpendicular to $x+y+1=0$ is

(*i*) $y-x+1=0$ (*ii*) $y-x-1=0$ (*iii*) $y-x+2=0$

(*iv*) $y-x-2=0$ (M N R 81)

2 (*a*) Q R, S are the points (-2 -1), (0, 3), (4, 0) Then the

$$x^2+y^2+2gx+2fy+c=0$$

It touches y axis *i.e.* $x=0$ at $(0, 3)$

$$y^2+2fy+c=(y-3)^2=y^2-6y+9 \quad \text{Eq 11 Page 331}$$

Comparing $f=-3$, $c=9$

Intercept on x axis is

$$2\sqrt{(g^2-c)}=8 \quad g^2-9=16$$

or $g^2=16+9=25$ or $g=\pm 5$

Hence the required equation is

$$x^2+y^2\pm 10x-6y+9=0$$

5 As in Q 4

$$x^2+2gx+c=(x-3)^2=x^2-6x+9$$

$$g=-3 \quad c=9$$

Also $2\sqrt{(f^2-c)}=8$

$$f^2-9=16 \quad \text{or} \quad f=\pm 5$$

$$x^2+y^2-6x\pm 10y+9=0$$

$$y^2+2fy+c=(y-\sqrt{3})^2=y^2-2\sqrt{3}y+3$$

6 Touches axis of y *i.e.* $x=0$ at $(0, \sqrt{3})$

$$y^2+2fy+c=(y-\sqrt{3})^2=y^2-2\sqrt{3}y+3$$

$$f=-\sqrt{3} \quad c=3$$

It intersects x axis *i.e.* $y=0$ at $(-1, 0)$ and $(-3, 0)$

$$x^2+2gx+c=(x+1)(x+3)=x^2+4x+3$$

$$g=2 \text{ and } c=3$$

Hence the circle is

$$x^2+y^2+4x-2\sqrt{3}y+3=0$$

7 Since it passes through origin, $c=0$ and its equation

$$x^2+y^2+2gx+2fy=0$$

Intercept on x axis

$$2\sqrt{(g^2-c)}=a \qquad g=\pm a/2, \qquad c=0$$

Intercept on y axis

$$2\sqrt{(f^2-c)}=b \qquad f=\pm b/2, \qquad c=0$$

Hence the circle is

$$x^2+y^2\pm ax\pm by=0$$

8 Let the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Its centre $(-g, -f)$ lies on

$$12x+5y=25$$

$$-12g-5f=25$$

Again it passes through the points $(0, 5)$ and $(6, 1)$

$$10f+c=0$$

10 The line $3x+4y-24=0$ cuts the x axis at A and y-axis at B. Then the in-centre of the triangle OAB where O is the origin is

(i) (1, 2), (ii) (2, 2) (iii) (12, 12), (iv) (2, 12)

11 The three lines $3x+4y+6=0$, $\sqrt{2}x+\sqrt{3}y+2\sqrt{2}=0$ and $4x+7y+8=0$ are

(a) Sides of a triangle, (b) Concurrent, (c) Parallel, (d) None of these

12 The lines $ax+by+c=0$, $bx+cy+a=0$, $cx+ay+b=0$ are concurrent only when

(i) $a+b+c=0$ (ii) $a^2+b^2+c^2=2abc$
(iii) $a^3+b^3+c^3=3abc$ (iv) $a+b+c=abc$

13 The line $(p+2q)x+(p-3q)y=p-q$ for different values of p and q passes through the point

(i) (3/2, 5/2) (ii) (2/5, 2/5) (iii) (3/5, 3/5) (iv) (2/5 3/5)

14 The equation
$\sqrt{\{(x-2)^2+y^2\}}+\sqrt{\{(x+2)^2+y^2\}}=4$ represents

(i) circle (ii) pair of lines (iii) a parabola (iv) an ellipse *(M N R 1982)*

15 All points lying inside the triangle formed by the points (1, 3) (5, 0) and (−1, 2) satisfy

(a) $3x+2y\geqslant0$ (b) $2x+y-13\geqslant0$ (c) $2x-3y-12\leqslant0$
(d) $-2x+y\geqslant0$ (e) none of these (I I T 86)

Solutions

1 (A) Ans (ii)

(B) Line cutting equal intercepts on axes is of the form $x+y=c$ or $x-y=c$. It passes though $(1, -2)$
$x+y+1=0$ or $x-y-3=0$. Hence C is correct

(C) (b) is the correct answer

(D) (ii),

2 (a) Ans (2 −4)

Let (h, k) be the point P. Now mid points of the diagonals PR and QS are the same

Hence $\frac{h+0}{2}=\frac{-2+4}{2}$ and $\frac{k+3}{2}=\frac{-1+0}{2}$

or $h=2$ and $k=-4$

The point P is $(2, -4)$

$$x^2+y^2+2gx+2fy+c=0$$

It touches y-axis i e $x=0$ at (0, 3)

$$y^2+2fy+c=(y-3)^2=y^2-6y+9$$ Eq 11 Page 331

Comparing $f=-3, c=9$

Intercept on x-axis is

$$2\sqrt{(g^2-c)}=8 \qquad g^2-9=16$$

or $g^2=16+9=25$ or $g=\pm 5$

Hence the required equation is

$$x^2+y^2\pm 10x-6y+9=0$$

5 As in Q 4

$$x^2+2gx+c=(x-3)^2=x^2-6x+9$$

$$g=-3, c=9$$

Also $2\sqrt{(f^2-c)}=8$

$$f^2-9=16 \text{ or } f=\pm 5$$

$$x^2+y^2-6x\pm 10y+9=0$$

$$y^2+2fy+c=(y-\sqrt{3})^2=y^2-2\sqrt{3}y+3$$

6 Touches axis of y i e $x=0$ at $(0, \sqrt{3})$

$$y^2+2fy+c=(y-\sqrt{3})^2=y^2-2\sqrt{3}y+3$$

$$f=-\sqrt{3} \quad c=3$$

It intersects x-axis i e $y=0$ at (−1 0) and (−3, 0)

$$x^2+2gx+c=(x+1)(x+3)=x^2+4x+3$$

$$g=2 \text{ and } c=3$$

Hence the circle is

$$x^2+y^2+4x-2\sqrt{3}y+3=0$$

7 Since it passes through origin $c=0$ and its equation is

$$x^2+y^2+2gx+2fy=0$$

Intercept on x axis

$$2\sqrt{(g^2-c)}=a \qquad g=\pm a/2 \qquad c=0$$

Intercept on y axis

$$2\sqrt{(f^2-c)}=b \qquad f=\pm b/2, \qquad c=0$$

Hence the circle is

$$x^2+y^2\pm ax\pm by=0$$

8 Let the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Its centre $(-g, -f)$ lies on

$$12x+5y=25$$

$$-12g-5f=25$$ ...(1)

Again it passes through the points (0, 5) and (6, 1)

$$25+10f+c=0$$ ...(2)

or $3abc-a^3-b^3-c^3=0$ (iii) is correct

13 $(x+y-1)+q/p\,(2x-3y+1)=0$

i e $P+\lambda Q=0$ which passes through the intersection of $P=0$ and $Q=0$ on solving the point $(2/5,\ 3/5)$ (iv) is correct

14 Ans (ii) We have $\sqrt{\{(x+2)^2+y^2\}}=4-\sqrt{\{(x-2)^2+y^2\}}$

Squaring $(x+2)^2+y^2=16+(x-2)^2+y^2-8\sqrt{\{(x-2)^2+y^2\}}$

or $8\,(x-2)=-8\sqrt{\{(x-2)^2+y^2\}}$

Squaring, $(x-2)^2=(x-2)^2+y^2=0$ or $y^2=0$

Thus the equation represents a pair of co incident lines

Note Actually the given equation represents the line segment $y=0$, $-2\leqslant x\leqslant 2$, since other points on $y=0$ do not satisfy the given equation

15 See solution in the end

10 (a) Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

It passes through the points (2, 3) and (2, −3)

$$4g+6f+c+13=0 \qquad (1)$$

and $4g-6f+c+13=0$ (2)

Subtracting we get $12f=0$ $\quad f=0$

Again $2x-3y-13=0$ is tangent at (2, −3) and its slope is 2/3 so that slope of normal is −3/2

Also slope of normal CP where C is $(-g, -f)$ is

$$\frac{-f+3}{-g-2}=-\frac{3}{2} \quad \text{Put } f=0 \quad \frac{3}{-g-2}=-\frac{3}{2}$$

or $-g-2=-2$ $\quad g=0$

Putting $g=0$ $f=0$ in (1) we get $c=-13$

Required circle is $x^2+y^2-13=0$

(b) Do yourself

(c) Let the circle be $\quad x^2+y^2+2gx+2fy+c=0$ (1)

It passes through $\quad$ (1, 2) and (3, 4)

$$5+2g+4f+c=0 \qquad (2)$$

$$25+6g+8f+c=0 \qquad (3)$$

Subtracting $20+4g+4f=0$ or $g+f+5=0$

Again $3x+y-3=0$ is a tangent

$$\frac{-3g-f-3}{\sqrt{(10)}}=\sqrt{(g^2+f^2-c)}$$

or $(3g+f+3)^2=10\,(g^2+f^2+5+2g+4f)$

or $(2g-5+3)^2=10\,[g^2+(g+5)^2+5+2g-4g-20]$, by (

$(2g-2)^2=10\,[2g^2+8g+10]$

or $4\,[g^2-2g+1]=20\,[g^2+4g+5]$

or $4g^2+22g+24=0$ or $2g^2+11g+12=0$

$(g+4)\,(2g+3)=0$

$g=-4, \quad -3/2,$

$f=-1, \quad -7/2 \quad$ by (3)

and $c=7, \quad 12 \quad$ by (1)

The circles are $\quad x^2+y^2-8x-2y+7=0$

and $\quad x^2+y^2-3x-7y+12=0$

11 $\quad x^2+y^2+2gx+2fy+c=0$

It passes through (1, −2) and (3, −4)

$$5+2g-4f+c=0 \text{ and } 25+6g-8f+c=0$$

Eliminating, f we get

$$15+2g-c=0$$

Add g^2+f^2 in both sides

$$(x^2+2gx+g^2)+(y^2+2fy+f^2)=g^2+f^2-c$$

or $$(x+g)^2+(y+f)^2=[\sqrt{(g^2+f^2-c)}]^2$$

Centre is $(-g, -f)$ and radius is

$$\sqrt{(g^2+f^2-c)} \qquad (8)$$

i e, ($\frac{1}{2}$ coeff of x with sign changed,

$\frac{1}{2}$ coeff of y with sign changed)

Note In case the coefficients of x^2 and y^2 be each λ instead of unity then divide by λ first before finding the coordinates of the centre

(g) Lengths of intercepts on the axes made by the circle

$$x^2+y^2+2gx+2fy+c=0$$

Let it cut the axis of x i e $y=0$ in points $(x_1, 0)$, and $(x_2, 0)$

x_1 x_2 are the roots of $x^2+2gx+c=0$

$x_1+x_2=-2g$ $x_1x_2=c$,

Intercept $=x_2-x_1=[(x_2+x_1)^2-4x_1x_2]^{1/2}=\sqrt{(4g^2-4c)}$

or Intercept $=2\sqrt{(g^2-c)}$ (9)

Similarly intercept on y axis is

$$y_2-y_1=2\sqrt{(f^2-c)} \qquad (10)$$

Note In case the circle touches the axis of x at $(x_1, 0)$ then it will intersect the x axis i e $y=0$ in two coincident points

$$x^2+2gx+c=(x-x_1)^2 \qquad (11)$$

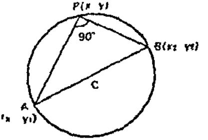
Similarly if it touches the y axis at $(0, y_1)$ then

$$y^2+2fy+c=(y-y_1)^2 \qquad (12)$$

(h) Condition for a given line to be a tangent In this case the perpendicular from the centre should be equal to radius

(i) Circle whose diameter is the line joining two points $A(x_1, y_1)$ and $B(x_2, y_2)$

Angle in a semi circle is a right angle



If $P(x, y)$ be any point on it then PA is perpendicular to PB

$$m_1m_2=-1$$

or $$\frac{y-y_1}{x-x_1}\cdot\frac{y-y_2}{x-x_2}=-1$$

or $$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0 \qquad (13)$$

15 $\quad x(x-2)+(y+1)(y-3)=0$

or $\quad x^2+y^2-2x-2y-3=0,\ g=-1, f=-1, c=-3$

Intercept on x axis

$$=2\sqrt{(g^2-c)}=2\sqrt{(1+3)}=4 \qquad \text{Rule P 331}$$

16 $\quad x^2+y^2+2gx+2fy+c=0$

It passes through the points (3, 0), (1, −6) and (4, −1)

$$6g+0f+9+c=0 \qquad (1)$$
$$2g-12f+37+c=0 \qquad (2)$$
$$8g-2f+17+c=0 \qquad (3)$$

Subtracting (2) from both (1) and (3) thereby eliminating c

$$4g+12f-28=0 \quad \text{or} \quad g+3f-7=0 \qquad (4)$$
$$6g+10f-20=0 \quad \text{or} \quad 3g+5f-10=0 \qquad (5)$$

Solving (4) and (5) we get

$$f=\frac{11}{4},\ g=\frac{-5}{4}$$

and hence from (1), $c=\dfrac{-3}{2}$

Putting for g f and c the required circle is

$$2x^2+2y^2-5x+11y-3=0$$

17 Solving the sides of the triangle in pairs the required vertices are (3, −4), (0, 5) and (6, −7). Now proceeding as in Q 16 the circle is

$$x^2+y^2-30x-10y+25=0$$

18 (a) As in Q 16, $g=0, f=-3, c=-76$

$x^2+y^2-6y-76=0$ and it passes through (6, 10) also

(b) $3(x^2+y^2)-13x-17y+14=0$ It passes through $(0, c)$

$3c^2-17c+14=0 \qquad c=1,\ 14/3$

$c=14/3$ as $c=1$ is already there for point (0, 1)

(c) First find the vertices

19 The lines $y=x$ and $y=-x$ are at right angles to each other and the circle passes through origin

Also $\quad OA=OB=b$

and $\quad \angle AOB=\pi/2$

and hence AB is a diameter of length

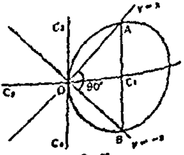


Fig 27

$$\sqrt{(b^2+b^2)}=b\sqrt{2}$$

$$r=\frac{b\sqrt{2}}{2}=\frac{b}{\sqrt{2}}=OC_1$$

Also its length

$$=2AM=2\sqrt{(r_1^2-p_1^2)}=2\sqrt{(r_2^2-p_2^2)}$$

where p_1 and p_2 are the perpendicular distances of the centres from the common chord

Note Conditions for $S_1=0$ and $S=0$ to touch

In this case the common chord $S_1-S=0$ will become common tangent and hence perpendicular from centre of either circle should be equal to corresponding radius

(*n*) **Equation of a circle through the intersection of given circle S=0 and given line P=0 is**

$$S+\lambda P=0 \qquad (17)$$

where λ is found by an additional given condition

(*o*) **Equation of a circle passing through the intersection of** ***given circles $S_1=0$ and $S_2=0$ (in standard form)***

$$S_1+\lambda S_2=0 \quad (\lambda\neq-1)$$

The value of λ is found by an additional given condition

Problem Set (A)

Equation of a circle

Find the equation of a circle in the following cases

1 Whose centre is the point (1 −3) and touches the line

$$2x-y-4=0$$

2 Whose centre is the point (−3 1) and passes through the point $(4, 1+2\sqrt{6})$ Find also the length of the intercept made by the circle on y axis

3 Whose centre is in first quadrant and radius being 4 given that it touches the x axis and the line $4x-3y=0$

4 Which touches the axis of y at (0 3) and cuts an intercept of 8 units on the axis of x (I I T 1972)

5 Which touches the axis of x at (3 0) and makes an intercept of 8 units on the y axis

6 (*a*) Which touches the axis of y at $(0\ \sqrt{3})$ and cuts the axis of x in the points (−1 0) and (−3 0)

(*b*) Find the equation to the circles which pass through the point (1 1), (2 2) and whose radius is 1 and show that there are two such circles

$$(x-2)^2+(y-1)^2=1 \text{ and } (x-1)^2+(y-2)^2=1$$

7 Which passes through origin and cuts off intercepts a and b respectively from the axes

$\sqrt{(34-k)}<5$ or $34-k<25$, $9<k$, or $k>9$.

Similarly it does not touch or intersects y-axis, therefore,

$$r<3$$

or

$$\sqrt{(34-k)}<3 \text{ or } 34-k<9$$

or

$$25<k \quad \text{or} \quad k>25$$

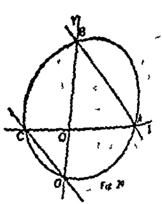
Hence the conditions are $k>9$, $k>25$, $k<29$

or we can say that $k>25$, $k<29$

or

$$25<k<29$$

24 (a) Let the given lines be AB and CD respectively



$AB=a_1x+b_1y+c_1=0$,

$CD=a_2x+b_2y+c_2=0$

On putting first $y=0$ then $x=0$ the intercepts are

$$OA=\frac{-c_1}{a_1},\ OB=\frac{-c_1}{b_1}$$

$$OC=\frac{-c_2}{a_2}\ \ OD=\frac{-c_2}{b_2}$$

Since the points A, B, C, D are concyclic we know from geometry that

$$OA\ OC=OB\ OD$$

$$\left(\frac{-c_1}{a_1}\right)\left(\frac{-c_2}{a_2}\right)=\left(\frac{-c_1}{b_1}\right)\left(\frac{-c_2}{b_2}\right)$$

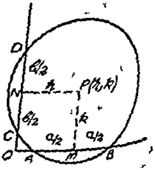
or

$$a_1a_2=b_1b_2,$$

Proved.

(b) Ans (A) as in part (a) $a_1a_2=b_1b_2=18$

25 Let the two rods be AB (along x axis) and CD along y axis so that the four points A, B, C, D are on a circle whose centre is (h, k) say. Perpendicular from centre on either rod will bisect it. Again from geometry we know that OAB and OCD are two secants to the circle



$$OA\ OB=OC\ OD$$

or $(h-a/2)(h+a/2)=(k-b/2)(k+b/2)$

or $h^2-k^2=\frac{a^2}{4}-\frac{b^2}{4}$ or $4(x^2-y^2)=a^2-b^2$

is the required locus.

(c) Show that a cyclic quadrilateral is formed by the lines

$$5x+3y=9,\ x=3y\ \ 2x=y \text{ and } x+4y+2=0$$

taken in order Find the equation of the circumcircle

(Roorkee 80)

19 Which passes through the origin and cuts off chords of length b from lines $y=x$ and $y=-x$

20 On which the coordinates of any point are

$$(2+4\cos\theta\ \ -1+4\sin\theta)$$

θ being the parameter

21 Show that if two tangents are mutually perpendicular the locus of their point of intersection is a circle concentric with the given circles

22 Find whether the point $\left(1\ \ \frac{\sqrt{(11)}+6}{6}\right)$ lies inside or outside the circle whose equation is

$$3x^2+3y^2-5x-6y+4=0$$

Find its centre and radius

23 The circle

$$x^2+y^2-6x-10y+k=0$$

does not touch or intersect the coordinate axes and the point (1, 4) is inside the circle Find the range of the value of k

24 (a) If the lines

$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0$$

cut the coordinate axes in concyclic points, prove

$$a_1a_2=b_1b_2$$

(b) The lines $2x+3y+19=0$ and $9x+6y-17=0$ cut the coordinate axes in concyclic points

(a) True (b) False (IIT 88)

25 Two rods of lengths a and b slide along the axes which are rectangular in such a manner that their ends are concyclic Prove that the locus of the centre of circle passing through these ends is the curve

$$4(x^2-y^2)=a^2-b^2$$

26 Prove that the locus of the point of intersection of the lines

$$x\cos\alpha+y\sin\alpha=a \text{ and } x\sin\alpha-y\cos\alpha=b$$

is a circle whatever α may be

Another form of Q 26 is given in Q 27

27 A tangent is drawn to each of the circles

$$x^2+y^2=a^2,\ x^2+y^2=b^2$$

$$2a^2 - 2a^2 \frac{1+\lambda^2}{1-\lambda^2} = 0 \quad \text{or} \quad -4a^2\lambda^2 = 0$$

Above is not possible as $a \neq 0, \lambda \neq 0$

Therfore the point A does not lie on the circle

Now putting $(-a, 0)$ we get

$$2a^2 + 2a^2 \frac{1+\lambda^2}{1-\lambda^2} = 0 \quad \text{or} \quad 4a^2 = 0$$

Above is also not possible as $a \neq 0$

Hence B also does not lie on the circle (1)

(b) Same as above

29 Let $P(x, y)$ be any point then by the given conditions the sum of the squares of its distances from the four sides of the square is 9

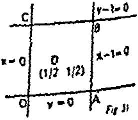


Fig 31

$$x^2 + y^2 + (x-1)^2 + (y-1)^2 = 9$$

$$2(x^2+y^2) - 2x - 2y - 7 = 0$$

or $x^2 + y^2 - x - y - 7/2 = 0$ is a circle whose centre is $(\frac{1}{2}, \frac{1}{2})$ which coincides with the centre of the square (1)

30 Eliminating θ the circle is $x^2+y^2=25$

The line through $[-\sqrt{(8)}, \sqrt{(8)}]$ and having slope $\tan 135° = -1$ is

$$y - \sqrt{8} = -1\,(x+\sqrt{8}) \quad \text{or} \quad y = -x$$

The line $y+x$ passes through the centre $(0, 0)$ of the circle $x^2+y^2=25$ Hence it is a diameter, say AB, whose length is

$$2r = 10$$

31 $x^2+y^2+x+\sqrt{3}y=0$

$g = 1/2 \quad f = \sqrt{3}/2 \quad c = 0$

Centre is $(-g, -f)$ or $(-1/2, -\sqrt{3}/2)$

and $r = \sqrt{(g^2+f^2-c)} = \sqrt{(1/4+3/4)} = 1$

Hence its equation can be written as

$$(x+1/2)^2 + (y+\sqrt{3}/2)^2 = 1$$

Comparing with $\cos^2\theta + \sin^2\theta = 1$ we get

$x+1/2 = \cos\theta \quad y+\sqrt{3}/2 = \sin\theta$

$$x = -1/2 + \cos\theta, \; y = -\sqrt{3}/2 + \sin\theta$$

as the parametric equations

Note In case radius be λ then we shall first divide by λ^2 before comparing with $\cos^2\theta + \sin^2\theta = 1$

(c) Show that a cyclic quadrilateral is formed by the lines
$$5x+3y=9 \quad x=3y,\ 2x=y \text{ and } x+4y+2=0$$
taken in order. Find the equation of the circumcircle
(Roorkee 80)

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Find its centre and radius

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24 (a) If the lines
$$a_1x+b_1y+c_1=0 \text{ and } a_2x+b_2y+c_2=0$$
cut the coordinate axes in concyclic points prove
$$a_1a_2=b_1b_2$$
(b) The lines $2x+3y+19=0$ and $9x+6y-17=0$ cut the coordinate axes in concyclic points
(a) True (b) False **(IIT 88)**

25 Two rods of lengths a and b slide along the axes which are rectangular in such a manner that their ends are concyclic. Prove that the locus of the centre of circle passing through these ends is the curve
$$4(x^2-y^2)=a^2-b^2$$

26 Prove that the locus of the point of intersection of the lines
$$x\cos\alpha+y\sin\alpha=a \text{ and } x\sin\alpha-y\cos\alpha=b$$
is a circle whatever α may be
Another form of Q 26 is given in Q 27

27 A tangent is drawn to each of the circles
$$x^2+y^2=a^2,\ x^2+y^2=b^2$$

35 $AB = r_1 + r_2$ and hence touch externally and point of contact is $\left(\frac{-19}{13}, \frac{9}{13}\right)$ and A is $(5, -2)$ and B is $(-7, 3)$, $r_1=7$, $r_2=6$

Slope of AB is $\frac{5}{-12}$ and hence common tangent will be a line through P and perpendicular to AB so that its slope is $12/5$

Its equation is $y-9/13=(12/5)(x+19/13)$

or $$12x-5y+21=0$$

Note The common tangent is also given by $S_1-S_2=0$ when S_1 and S_2 are the equations of the circles in standard form

36 (a) A circle of radius $r_1=2$, touching both the axes in first quadrant will have its centre A as $(2, 2)$. Also B is $(6, 5)$ given and if its radius be r_2, then since the two circles touch externally therefore $AB=r_1+r_2$

or $\sqrt{[(2-6)^2+(2-5)^2]}=2+r_2$

or $5=2+r_2$ $\quad r_2=3$

Hence its equation is $(x-6)^2+(y-5)^2=3^2$

or $$x^2+y^2-12x-10y+52=0$$

(b) Let the circle be $(x-h)^2+(y-k)^2=r^2$ (1)

$$x^2+y^2=a^2 \quad (2)$$

$$(x-2a)^2+y^2=(2a)^2$$

(1) and (2) touch externally $\quad \sqrt{(h^2+k^2)}=r+a$

(1) and (3) touch externally $\quad \sqrt{[(h-2a)^2+k^2]}=r+2a$

$$\sqrt{\{(h-2a)^2+k^2\}}-\sqrt{(h^2+k^2)}=a$$

Now proceeding as in Q P 271, the locus of centre is

$$12x^2-4y^2-24ax+9a^2=0$$

37 A is $(-a, -2a)$ $r_1=\sqrt{(a^2+4a^2+3a^2)}=2\sqrt{2}a$,

B is $(4a, 3a)$ $r_2=\sqrt{(16a^2+9a^2-7a^2)}=3\sqrt{2}a$

$$AB=\sqrt{[(5a)^2+(5a)^2]}=5\sqrt{2}a=r_1+r_2$$

Hence the two circles touch externally

The point of contact as in Q 32 will divide AB in the ratio $r_1 : r_2$ i.e. in the ratio $2:3$. If (h, k) be the point of contact then

$$h=\frac{2(4a)+3(-a)}{2+3}=a, \quad k=\frac{2(3a)+3(-2a)}{2+3}=0$$

Hence the point of contact is $(a, 0)$

38 A is $(-a, 0)$, $r_1=\sqrt{(a^2-c)}$

B is $(0, -b)$ $r_2=\sqrt{(b^2-c)}$

$AB=r_1 \pm r_2$ if the two circles touch ... true for external and circle with centre at (6, 5) and touching the above externally

(*b*) Determine the locus of the centres of the circles, which touch the two circles

$x^2+y^2=a^2$ and $x^2+y^2=4ax$ and are external to both

(Roorkee 81)

37 Show that the circles

$x^2+y^2+2ax+4ay-3a^2=0$ and $x^2+y^2-8ax-6ay+7a^2=0$

touch and determine the point of contact

38 Prove that the circles

$x^2+y^2+2ax+c=0$ and $x^2+y^2+2by+c=0$

will touch one another if $1/a^2+1/b^2=1/c$

39 If the line $x\cos\alpha+y\sin\alpha=p$ cuts the circle $x^2+y^2=a^2$ in M and N, then show that the circle whose diameter is MN is $x^2+y^2-a^2=2p(x\cos\alpha+y\sin\alpha-p)$ **(Roorkee 67)**

40 $y=mx$ is a chord of the circle of radius a through the origin and whose diameter is along the axis of x. Find the equation of the circle whose diameter is the chord. Hence find the locus of its centre for all values of m

41 The equation of two circles are

$(x-a)^2+y^2=a^2$ and $x^2+(y-b)^2=b^2$

Prove that the equation of the circle whose diameter is their common chord is

$(a^2+b^2)(x^2+y^2)=2ab(bx+ay)$

42 (*a*) Find the equation of the circles passing through the points of intersection of the circles

$x^2+y^2-2x-4y-4=0$ and $x^2+y^2-10x-12y+40=0$

and whose radius is 4

(*b*) Find the equation of the circle passing through the points of intersection of the circles

$x^2+y^2-6x+2y+4=0$, $x^2+y^2+2x-4y-6=0$

and with its centre on the line $y=x$ **(Roorkee 74)**

43 Find the equations of circles which touch $2x-y+3=0$ and pass through the points of intersection of the line $x+2y-1=0$ and the circle $x^2+y^2-2x+1=0$

44 (*a*) Two circles $x^2+y^2=6$ and $x^2+y^2-6x+8=0$ are given. Find the equation of the circle through their points of intersection and the point (1, 1) **(I I T 80)**

or

$$x^2+y^2-x(2a+m\lambda)+\lambda y=0$$

Centre is $\left(\frac{2a+m\lambda}{2}, \frac{-\lambda}{2}\right)$

Since the chord $y=mx$ is a diameter of the required therefore its centre lies on it

$$-\frac{\lambda}{2}=m\,\frac{2a+m\lambda}{2} \quad \text{or} \quad \lambda(1+m^2)=-2am \qquad \lambda=\frac{-2am}{1+m^2}$$

Putting for λ the required circle is

$$(1+m^2)(x^2+y^2-2ax)-2am(y-mx)=0$$

$$(1+m^2)(x^2+y^2)-2a(x+my)=0$$

If (h, k) be the co-ordinates of its centre then

$$h=\frac{a}{1+m^2}, \quad k=\frac{am}{1+m^2}$$

In order to find its locus we have to eliminate m. Dividing

$\frac{k}{h}=m$ and putting in $h(1+m^2)=a$, we get $h\left(1+\frac{k^2}{h^2}\right)=a$

$h^2+k^2=ah$. Generalising the required locus is $x^2+y^2-ax=0$ which is a circle

41 The equation of the common chord by $S-S'=0$ is $ax-by=0$

The required circle by $S+\lambda P=0$ is

$$(x^2+y^2-2ax)+\lambda(ax-by)=0$$

Its centre is $\left(\frac{2a-\lambda a}{2}, \frac{\lambda b}{2}\right)$ and since the common chord $ax-by=0$ is a diameter it will lie on it

$$a\,\frac{2a-\lambda a}{2}-b\,\frac{\lambda b}{2}=0$$

$$2a^2=\lambda(a^2+b^2) \quad \text{or} \quad \lambda=\frac{2a^2}{a^2+b^2}$$

Hence the required circle is

$$(a^2+b^2)(x^2+y^2-2ax)+2a^2(ax-by)=0$$

or

$$(x^2+y^2)(a^2+b^2)=2ab(bx+ay)$$

(a) Any circle through the intersection of given circles is

$$S_1+\lambda S_2=0$$

or $(x^2+y^2-2x-4y-4)+\lambda(x^2+y^2-10x-12y+40)=0$

or $(x^2+y^2)-2\frac{(1+5\lambda)}{1+\lambda}x-2\frac{(2+6\lambda)}{1+\lambda}y+\frac{40\lambda-4}{1+\lambda}=0 \quad (1)$

$r=\sqrt{(g^2+f^2-c)}=4$ given

$$16=\frac{(1+5\lambda)^2}{(1+\lambda)^2}+\frac{(2+6\lambda)^2}{(1+\lambda)^2}-\frac{40\lambda-4}{1+\lambda}$$

the equation $x^2+2px-q=0$ Find the equation and the radius of the circle with AB as diameter **(I I T 1984)**

55 Obtain the equation of the straight lines passing through the point A (2 0) and making an angle 45° with the tangent at A to the circle $(x+2)^2+(y-3)^2=25$ Find the equation of the circles each of radius 3 whose centres are on these straight lines at a distance of $5\sqrt{2}$ units from A **(Roorkee 87)**

Solutions to Problem Set (A)

1 Since it touches $2x-y-4=0$, hence perpendicular from centre $(1, -3)$ is equal to radius

Thus $r=\dfrac{2\cdot1-(-3)-4}{\sqrt{(4+1)}}=\dfrac{1}{\sqrt{5}}$,

Equation is

$$(x-1)^2+(y+3)^2=1/5$$

2 P is $(4, 1+2\sqrt{6})$ and C is $(-3\ 1)$

$$r^2=(4+3)^2+(2\sqrt{6})^2=49+24=73$$

Hence its equation is

$$(x+3)^2+(y-1)^2=73$$

or $x^2+y^2+6x-2y-63=0$

$$g=3, f=-1\ \ c=-63$$

Intercept on y axis is

$$2\sqrt{(f^2-c)}=2\sqrt{(1+63)}=16$$ **Rule (g) P 331**

3 Its radius is 4 and it touches x axis and hence its centre is $(h, 4)$ Its equation is

$$(x-h)^2+(y-4)^2=4^2$$ **Rule (c) P, 330**

It touches the line $4x-3y=0$ $\perp$ from centre $(h, 4)$ is equal to radius 4

$$\frac{4h-3\cdot4}{\pm\sqrt{(4^2+3^2)}}=4$$

or $4h-12=\pm20$

or $4h=12\pm20$

or $4h=32$ or -8

$h=8$ or -2

Since h is to be +ive by given condition therefore $h=8$

Hence the circle is

$$(x-8)^2+(y-4)^2=4^2$$

or $x^2+y^2-16x-8y+64=0$

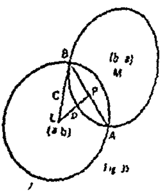
4 Let the circle be

47 (a) By $S_1 - S_2 = 0$ the equation of common chord AB is $x - y = 0$

$LP = p =$ perpendicular from centre (a, b) on this chord,

$$p = \frac{a-b}{\sqrt{2}}$$

Also $LB = r = c$



$$BP^2 = c^2 - p^2 = c^2 - \frac{(a-b)^2}{2}$$

$$AB = 2BP = 2\sqrt{\left(c^2 - \frac{(a-b)^2}{2}\right)} = \sqrt{[4c^2 - 2(a-b)^2]}$$

If the two circles touch each other then length of common chord is zero and hence

$$4c^2 - 2(a-b)^2 = 0 \quad \text{or} \quad (a-b)^2 = 2c^2,$$

is the required condition

(b) $C_1: x^2+y^2=16$, $C_2: (x-h)^2+(y-k)^2=25$

Common chord by $S_1 - S_2 = 0$ is

$2hx + 2ky = (h^2 + k^2 - 9)$ Its slope $= -h/k = 3/4$ given I

If p be the length of $\perp$ on it from the centre $(0, 0)$ of C_1 of radius 4 then $p = \dfrac{h^2+k^2-9}{\sqrt{(4h^2+4k^2)}}$ Also the length of the chord is

$2\sqrt{r^2-p^2} = 2\sqrt{4-p^2}$ The chord will be of maximum length if $p = 0$ or $h^2+k^2- \ =0$ or $h^2 + \frac{16}{9}h^2 = 9$ by (I)

or $25h^2 = 81$ $\quad h = \pm\frac{9}{5}$ $\quad k = \mp\frac{12}{5}$ by I

Centres are $\left(\frac{9}{5}, -\frac{12}{5}\right)$ and $\left(-\frac{9}{5}, \frac{12}{5}\right)$

48 Here also common chord is $x - y = 0$

Centre is

$$\left(\frac{-a}{2}, \frac{-b}{2}\right) \text{ and } r = \tfrac{1}{2}\sqrt{(a^2+b^2-4c)}$$

$$p = \frac{\frac{-a}{2} - \left(\frac{-b}{2}\right)}{\sqrt{2}} = \frac{b-a}{2\sqrt{2}}$$

$$BP^2 = r^2 - p^2 = \frac{a^2+b^2-4c}{4} - \frac{(b-a)^2}{8} = \frac{1}{8}\left[(a+b)^2 - 8c\right]$$

$$37+12g+2f+c=0 \quad \text{... (3)}$$

Subtracting (2) from (3),

$$12g-8f=-12 \quad \text{...(4)}$$

Adding (1) and (4)

$$-13f=13 \qquad f=-1$$

$$g=-5/3 \text{ and } c=-15$$

Hence the circle is

$$x^2+y^2-\tfrac{10}{3}x-2y-15=0$$

9 $$x^2+y^2+2gx+2fy+c=0$$

Centre $(-g, -f)$ lies on $3x+4y=7$

$$-3g-4f=7 \quad (1)$$

Passes through $(1, -2)$ and $(4, -3)$

$$5+2g-4f+c=0 \text{ and } 25+8g-6f+c=0$$

Subtracting we get

$$20+6g-2f=0 \quad (2)$$

Solving (1) and (2)

$$g=-\frac{47}{15}, f=\frac{9}{15}$$

and hence $c=\frac{55}{15}$

Putting for g f and c the required equation is

$$15x^2+15y^2-94x+18y+55=0$$

(b) If (h, k) be the centre then $k=h-1$ as it lies on $y=x-1$

Equation is $(x-h)^2+(y-h+1)^2=9$ as $r=3$

It passes through the point (7,3) $h^2-11h+28=0$ $h=4,7$

Ans $x^2+y^2-8x-6y+16=0$, $x^2+y^2-14x-12y+76=0$

(c) Eliminating y we get

$$(3c^2-c-2)\,x=c^2-1 \qquad x=\frac{(c-1)(c+1)}{(c-1)\ (3c+2)}$$

limit when $c\to 1$ gives $x=\frac{2}{5}$ and hence $y=-\frac{1}{25}$

Centre $(-g\ -f)$ is $\left(\frac{2}{5}, \frac{-1}{25}\right)$

Circle is $x^2+y^2-\frac{4}{5}x+\frac{2}{25}y+k=0$

It passes through (2, 0) $k=-\frac{12}{5}$

Hence the required circle is $25(x^2+y^2)-20x+2y-60=0$

$$\theta = 22^\circ\ 36' = 22\frac{36}{60}\ \text{degree} = \frac{113}{5}\ \text{degrees}$$

$$= \frac{113}{5} \times \frac{\pi}{180}\ \text{radians} = \frac{113}{5} \times \frac{22}{7\times 180} = 4\ \text{nearly}$$

Area of sector $OAIC = \theta\ (\text{radius})^2 = 4 \times 6.5 \times 6.5 = 169$

and area of $\triangle OAC = \frac{1}{2}\ OA.CN = \frac{1}{2} \times 6.5 \times BC \sin\theta$

$$= \frac{1}{2} \times 6.5 \times 12 \times \frac{5}{13} = 15$$

Hence the required area $ALCA = 16.9 - 15 = 1.9$ sq m approx

53 Let α, β and γ, δ be the roots of the first and second of the given equations. Then

$$\alpha+\beta = -2a,\ \alpha\beta = -b^2$$
$$\gamma+\delta = -2p,\ \gamma\delta = -q^2$$

Now coordinates of A and B are (α, γ) and (β, δ) respectively

The equations of the circle on AB as diameter is

$$(x-\alpha)(x-\beta)+(y-\gamma)(y-\delta)=0$$

or $\quad x^2+y^2-(\alpha+\beta)x-(\gamma+\delta)y+\alpha\beta+\gamma\delta=0$

or $\quad x^2+y^2-2ax+2py-b^2-q^2=0$

Its radius $=\sqrt{(a^2+p^2+b^2+q^2)}$

54 The equation of the circle is $x^2+y^2+4x-6y-12=0$ and equation of tangent at the point $A\ (2, 0)$ is $x.2+y.0+2(x+2)-3(y+0)-12=0$ or $4x-3y-8=0$

Now the lines through $A\ (2, 0)$ inclined at an angle of 45° to line (1) are found as in Q 21 P 277 to be

$$y-0=-7(x-2) \quad \text{and} \quad y-0=\frac{1}{7}(x-2)$$

or $\quad \frac{x-2}{-1}=\frac{y}{7}=r \quad \text{and} \quad \frac{x-2}{7}=\frac{y}{1}=r$

writing them in the form $\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$

or $\quad \frac{x-2}{\frac{-1}{\sqrt{(50)}}}=\frac{y-0}{\frac{7}{\sqrt{(50)}}}=\pm 5\sqrt{2}$ and $\frac{x-2}{\frac{7}{\sqrt{(50)}}}=\frac{y}{\frac{1}{\sqrt{(50)}}}=\pm 5\sqrt{2}$

The centres of the required circles are given to be at a distance of $5\sqrt{2}$ from $A\ (2, 0)$

$x-2=-1, +1, 7, -7 \qquad x=1, 3, 9, -5$

$y-0=7, -7, 1, -1 \qquad y=7, -7, 1, -1$

$$37+12g+2f+c=0 \qquad \text{.. (3)}$$

Subtracting (2) from (3),

$$12g-8f=-12 \qquad \text{.. (4)}$$

Adding (1) and (4),

$$-13f=13 \qquad f=-1$$

$$g=-5/3 \text{ and } c=-15$$

Hence the circle is

$$x^2+y^2-\tfrac{10}{3}x-2y-15=0$$

9 $$x^2+y^2+2gx+2fy+c=0$$

Centre $(-g, -f)$ lies on $3x+4y=7$

$$-3g-4f=7 \qquad (1)$$

Passes through $(1, -2)$ and $(4, -3)$

$$5+2g-4f+c=0 \text{ and } 25+8g-6f+c=0$$

Subtracting we get

$$20+6g-2f=0 \qquad (2)$$

Solving (1) and (2)

$$g=-\frac{47}{15}, f=\frac{9}{15}$$

and hence $c=\frac{55}{15}$

Putting for g f and c the required equation is

$$15x+15y^2-94x+18y+55=0$$

(b) If (h, k) be the centre then $k=h-1$ as it lies on $y=x-1$

Equation is $(x-h)^2+(y-h+1)^2=9$ as $r=3$

It passes through the point (7,3) $h^2-11h+28=0$ $h=4,7$

Ans $x^2+y^2-8x-6y+16=0$, $x^2+y^2-14x-12y+76=0$

(c) Eliminating y we get

$(3c^2-c-2)x=c^2-1$ $\qquad x=\frac{(c-1)(c+1)}{(c-1)(3c+2)}$

limit when $c\to1$ gives $x=\frac{2}{5}$ and hence $y=-\frac{1}{25}$

Centre $(-g \quad f)$ is $\left(\frac{2}{5}, \frac{-1}{25}\right)$

Circle is $x^2+y^2-\frac{4}{5}x+\frac{2}{25}y+k=0$

It passes through (2, 0) $\qquad k=-\frac{12}{5}$

Hence the required circle is $25(x^2+y^2)-20x+2y-60=0$

$$px_1+qy_1=a^2 \text{ and } hx_1+ky_1=a^2 \quad (A)$$

Now consider the equation $xx_1+yy_1=a^2$

Above is an equation of first degree in x, y and as such it represents a straight line and by virtue of relations in A it passes through $A(p, q)$ and $B(h, k)$ (4)

Hence the equation $xx_1+yy_1=a^2$ represents the equation of the line AB, and is called the chord of contact of the point (x_1, y_1)

Note The equation of the chord of contact of a point (x_1, y_1) (outside the circle) is of the same form as the equation of tangent to the circle at the point (x_1, y_1)

(e) Polar of a given point (x_1, y_1) w r t circle $x^2+y^2=a^2$

Definition *If through any fixed point $P(x_1, y_1)$ chords of the circle be drawn then the locus of the points of intersection of tangents at the extremities of these chords is called the polar of the point P and the point P is called the pole*

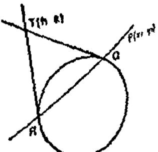


Fig 13

Equation of polar

Let QR be the chord passing through the point $P(x_1, y_1)$ tangents at the extremities of which intersect at the point $T(h, k)$, whose locus we are to find

Now QR is the chord of contact of the point (h, k) and hence its equation is

$$hx+ky=a^2$$

But QR passes through $P(x_1, y_1)$ (A)

$$hx_1+ky_1=a^2$$

Hence from (A) the locus of the point (h, k) is (5)

$$xx_1+yy_1=a^2$$

Above is the equation of the polar of the point (x_1, y_1)

Note The form of the equation of the polar of any point (x_1, y_1) is the same as the equation of tangent at the point (x_1, y_1) or the chord of contact of tangents drawn from the point (x_1, y_1)

Again it touches the axis of x so that intercept on x-axis is zero

$2\sqrt{(g^2-c)}=0$ or $g^2=c$

Putting in (1) we get

$15+2g-g^2=0$ or $g^2-2g-15=0$

or $(g-5)(g+3)=0$

$g=5, -3$ $c=25, 9$ and hence $f=10, 2$

Hence the values of g, f, c and 5, 10, 25 or $-3, 2, 9$,

Therefore the two circles are

$$x^2+y^2+10x+20y+25=0$$

and $x^2+y^2-6x+4y+9=0$

12 A circle which touches both the axes has its centre at (h, h) and radius h Again it touches the line $x-c=0$

$\therefore$ perpendicular from centre is equal to radius

$$\frac{h-c}{\pm\sqrt{1}}=h \quad \text{or} \quad h-c=\pm h$$

or $2h=c$ $h=c/2$

Hence its centre is $(c/2, c/2)$ and radius $c/2$ But the centre can be in fourth quadrant also under given condition Hence its centre is $(c/2, \pm c/2)$ Therefore its equation is

$$(x-c/2)^2+(y\pm c/2)^2=(c/2)^2$$

or $x^2+y^2-cx\pm cy+c^2/4=0$

13 Centre (h, h) and radius h as in Q 12 as it touches coordinate axes on +ive sides Again it touches the line

$$3x+2y-6=0$$

$$\frac{3h+2h-6}{\pm\sqrt{(13)}}=h \quad \text{or} \quad (5h-6)^2=13h^2$$

or $12h^2-60h+36=0$ or $h^2-5h+3=0$

$$h=\frac{5\pm\sqrt{(13)}}{2}$$

Circle $(x-h)^2+(y-h)^2=h^2$,

where $h=\dfrac{5\pm\sqrt{(13)}}{2}$

14 The required circle by rule (*i*) page 331 is

$$(x+4)(x-12)+(y-3)(y+1)=0$$

or $x^2+y^2-8x-2y-51=0$

$g=-4, f=-1, c=-51$

Intercept on y axis

$2\sqrt{(f^2-c)}=2\sqrt{(1+51)}=2\sqrt{(52)}=4\sqrt{(13)}$ Rule (g) P 331 the centre of the other. Hence from Δ PC_1C_2 right angled at P we have

$$C_1C_2^2 = C_1P^2 + C_2P^2$$

or $$(g_1 - g_2)^2 + (f_1 - f_2)^2 = (g_1^2 + f_1^2 - c_1) + (g_2^2 + f_2^2 - c_2)$$

or $$2g_1g_2 + 2f_1f_2 = c_1 + c_2$$

(i) Radical axis

Definition *The radical axis of two circles $S_1=0$ and $S_2=0$ is the locus of a point which moves so that the lengths of the tangents drawn from it to the two circles are equal*

If $P(h, k)$ be the point then by definition $PT^2 = PT'^2$

or $$h^2+k^2+2g_1h+2f_1k+c_1 = h^2+k^2+2g_2h+2f_2k+c_2 \text{ by (g)}$$

$$2h(g_1-g_2)+2k(f_1-f_2)+c_1-c_2=0$$

Hence the locus of the point (h, k) is (9)

$$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0 \text{ or } S_1-S_2=0$$

In case the circles touch, then the radical axis $S_1-S_2=0$ becomes the common tangent and in case they intersect then it becomes the equation of their common chord

Slope of radical axis $= -\dfrac{2(g_1-g_2)}{2(f_1-f_2)} = m_1$

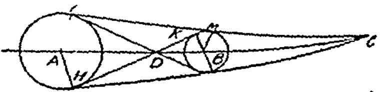
Slope of C_1C_2 is $\dfrac{f_1-f_2}{(g_1-g_2)} = m_2$

Clearly $m_1m_2 = -1$

Hence radical axis is perpendicular to the line joining the centres of the two circles

(l) Common tangents to two circles

The following propositions can be easily proved (i) *The direct common tangents to two circles meet on the line of centres and divide it externally in the ratio of the radii* (This follows from the similar Δ's CBM and CAL in the following diagram)



(ii) *The transverse common tangents also meet on the line of centres and divide it internally in the ratio of radii* (This follows from the similarity of the Δs BKD and AHD)

Clearly C_1 is centre on x axis with coordinates $(b/\sqrt{2}, 0)$
Similarly the other centres could be

$C_2\,(-b/\sqrt{2},\ 0)$ or $C_3\,(0\ \ b/\sqrt{2})$ or $C_4\,(0,\ -b/\sqrt{2})$

Hence the circles are

$$\left(x \pm \frac{b}{\sqrt{2}}\right)^2 + y^2 = \left(\frac{b}{\sqrt{2}}\right)^2 \quad \text{or} \quad x^2+y^2 \pm \sqrt{2}bx = 0$$

or $\quad (x-0)^2 + \left(y \pm \frac{b}{\sqrt{2}}\right) = \left(\frac{b}{\sqrt{2}}\right)^2$

or $\quad x^2+y^2 \pm \sqrt{2}by = 0$

20 $\quad x = 2+4\cos\theta,\ y = -1+4\sin\theta$

Clearly $\quad (x-2)^2+(y+1)^2 = 16\,(\cos^2\theta+\sin^2\theta) = 16$

Above represents a circle with centre $(2, -1)$ and radius 4

21 See Q 26 P 347

22 Dividing by 3 we get

$$x^2+y^2-5x/3-2y+4/3=0$$

Centre is $\quad (-g, -f)$ *i e* $(5/6, 1)$

$$r^2 = g^2+f^2-c = \frac{25}{36}+1-\frac{4}{3} = \frac{13}{36} \qquad r = \frac{1}{6}\sqrt{(13)}$$

The point P

$$\left(1, \frac{\sqrt{(11)}+6}{6}\right)$$

lies outside on or inside the circle if the distance between the point and centre is greater than, equal to or less than the radius,

$$CP = \left(\frac{5}{6}-1\right)^2 + \left(\frac{\sqrt{(11)}}{6}+1-1\right)^2 = \frac{1}{36}+\frac{11}{36} = \frac{12}{36} < \frac{13}{36} = r^2$$

Hence the point P lies inside the circle

23 $\quad x^2+y^2-6x-10y+k=0$

Centre C (3, 5) Point P (1, 4)

$r^2 = 9+25-k = 34-k$

Point P lies inside the circle therefore

$CP < r^2$

or $\quad (3-1)^2+(5-4)^2 < r^2$

or $\quad 4+1 < 34-k$

or $\quad -29+k < 0$ *i e* $-$ive

$k < 29$

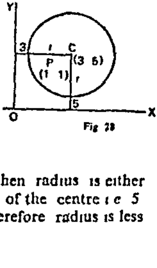


Fig 23

If it touches the x axis or intersects then radius is either equal to or greater than y-coordinate of the centre *i e* 5
Since it neither touches nor intersects therefore radius is less than 5

8 Find the equation of the circle which passes through the point (1, 1) and which touches the circle

$$x^2+y^2+4x-6y-3=0$$

at the point (2, 3) on it,

9 Tangents are drawn to the circle $x^2+y^2=25$ from the point (13, 0) Prove that the angle between them is $2\tan^{-1} 5/12$ and their equations are

$$12y+5x+65=0 \text{ and } 12y-5x-65=0$$

10 Prove that the tangent to the circle $x^2+y^2=5$ at the point (1, −2) also touches the circle

$$x^2+y^2-8x+6y+20=0$$

and find its point of contact (I I T 1975)

11 What is the equation of the family of circles tangent to

$$3x-y=6$$

at P (1, −3)? Select the member or members of the family with radius equal to $2\sqrt{(10)}$

12 Find the locus of the point of intersection of tangents to the circle

$$x=a\cos\theta,\ y=a\sin\theta$$

at the point whose parametric angles differ (i) by $\pi/3$, (ii) by $\pi/2$

13 Find the equation of tangents to the circle $x^2+y^2=a^2$ which makes with axes a triangle of area a^2

14 The extremities of a diameter of a circle are (1 2) and (3, 4) Find its equation Also determine the equations of tangents which are parallel to this diameter

15 Two parallel tangents to a given circle are cut by a third tangent in the points R and Q Show that the lines from R and Q to the centre of the circle are mutally perpendicular

16 If the line $lx+my=1$ be a tangent to the circle $x^2+y^2=a^2$, prove that the point (l m) lies on a circle (M.N R 1978)

17 (a) Find all the common tangents to the circles

$$x^2+y^2-2x-6y+9=0$$

and

$$x^2+y^2+6x-2y+1=0,$$

(b) A circle I of radius 5m, is having its centre A at the origin of the coordinate axes Two circles II and III with centres at B and C and of radii 3 and 4 m respectively touch the circle I and also touch the x axis to the

Alternative Method

Let the centre of the circle be (h, λ), its equation is

$$x^2+\ \ -2hx-2ky+\lambda=0$$

$AB=a=$ Interc on x-axis $=2\sqrt{(g^2-c)}=2\sqrt{(h^2-\lambda)}$

$CD=b=$ Interc on y axis $=2\sqrt{(f^2-c)}=2\sqrt{(b^2-\lambda)}$

$a^2=4\ (h^2$ $-4\ (k^2-\lambda)$

Eliminating the able λ we get on subtracting

$$(a^2-b^2)=4\ (h^2\ \ k\ )$$

Hence locus of centre is $4\ (x^2-y^2)=a^2-b^2$

Therefore the correct answer is *(d)*

26 In order to find the locus of the point of intersection we have to eliminate the variable α between the lines for which we square and add

$$x^2\ (\sin\ \ \alpha+\cos^2\alpha)+y^2\ (\sin^2\alpha-\cos^2\alpha)=a^2+b^2$$

or $x^2+y^2=a^2+b^2$ which represents a circle

27 Any tangent to first circle is $x\cos\alpha+y\sin\alpha=a$ as its distance from centre (0 0) is equal to radius a Any tangent to $x^2+y^2=b^2$ but perpendicular to above is obtained by replacing α by $\alpha-90°$ and its equation is

$$x\cos(\alpha-90°)+y\sin(\alpha-90°)=b$$

or $x\cos(90°-\alpha)\ \ y\sin(90°-\alpha)=b$

or $x\sin\alpha-y\cos\alpha=b$

Locus of the point of intersection of these tangents as in Q 26 is

$$x^2+y^2=a^2+b^2$$

which is a circle concentric with the given circles

28 (a) Let the given points be chosen along x axis and the tance between them be $2a$ and their mid point as origin Hence their co ordinates are

$$A\ (a, 0)\qquad B\ (-a\ \ 0)$$

Let P be any point (x, y) such that $\frac{PA}{PB}=k$

$$PA^2=k^2PB^2$$

or $(x-a)^2+y^2=k^2\ [(x+a)^2+y^2]$

or $(x^2+y^2+a^2)\ (1-k^2)-2ax\ (1+k^2)=0$

or $x^2+y^2+a^2-2ax\ \frac{1+k^2}{1-k^2}=0,\ k\neq 1$ (1)

Above equation represents a circle

If it passes through the point $A\ (a, 0)$ then

(4, 3) to the circle $x^2+y^2=9$ and the line joining their point of contact is $7\frac{17}{25}$ units (I I T 87)

26 If the pole of any line with respect to the circle $x^2+y^2=c^2$ lies on the circle $x^2+y^2=9c^2$, then the line will be a tangent to the circle $x^2+y^2=c^2/9$

27 Prove that the ratio of the distances of any two points from the centre of a circle is the same as the distance of each from the polar of the other

28 If

$$4l^2-5m^2+6l+1=0$$

Prove that the line

$$lx+my+1=0$$

touches a fixed circle

29 (a) A triangle has two of its sides along the axes, its third side touches the circle

$$x^2+y^2-2ax-2ay+a^2=0$$

Prove that the locus of the circum-centre of the triangle is

$$a^2-2a\,(x+y)+2xy=0$$

(b) The circle $x^2+y^2-4x-4y+4=0$ is inscribed in a triangle which has two of the sides along the coordinate axes The locus of the circumcentre of the triangle is (I I T 87)

$$x+y+k\sqrt{(x^2+y^2)}=0 \quad \text{find } k$$

30 The centre of circle S lies on the line $2x-2y+9=0$ and S cuts at right angles the circle $x^2+y^2=4$, show that S passes through each of two fixed points and find their coordinates.

31 If the two circle

$$x^2+y^2+2gx+2fy=0$$

and

$$x^2+y^2+2g'x+2f'y=0$$

touch each other then

(a) $f=f'$ and $g=g'$, (b) $ff'=gg'$,
(c) $f/g=f'/g'$ (d) None

32 Let A be the centre of the circle $x^2+y^2-2x-4y-20=0$ Suppose that the tangent at the point B (1, 7) and D (4, −2) on the circle meet at the point C Find the area of the quadrilateral $ABCD$ (I I T 81)

33 The equations of three circles are given

$$x^2+y^2=1,\ x^2+y^2+8x+15=0,\ x^2+y^2+10y+24=0$$

Determine the coordinates of the point such that the tangents drawn from it to three circles are equal in length (Roorkee 82)

32 $x^2+y^2-2x-4y-20=0$

Centre A (1, 2) and radius$=5$

Point of contact P is (5, 5) If $B(h, k)$ be the centre of the required circle of radius 5 then from the figure P is mid point of AB

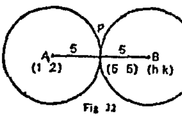


Fig 32

$$5=\frac{h+1}{2} \text{ and } 5=\frac{k+2}{2}, \qquad h=9, k=8$$

Hence the required circle is

$$(x-9)^2+(y-8)^2=5$$

or $\quad x^2+y^2-18x-16y+120=0$

33 $x^2+y^2-4x-6y-12=0$

Centre A is (2, 3) and radius$=5$ $=PA$, $B(h\ k)$ is the centre of the required circle of radius $BP=3$ which touches the given circle internally at $P(-1, -1)$

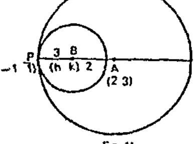


Fig 33

$$BA=PA-PB=5-3=2$$

Thus B divides PA in the ratio 3 2

$$h=\frac{3\ 2+2(-1)}{3+2}=\frac{4}{5},\ k=\frac{3\ 3+2(-1)}{3+2}=\frac{7}{5}$$

Hence the required circle is

$(x-4/5)^2+(y-7/5)^2=3^2$ or $5x^2+5y^2-8x-14y-32=0$

34 (*a*) Centre A is $(-1, -1)$ and radius$=1$

Centre B is (2, 3) and radius$=4$

Distance between the centres

$=AB=\sqrt{(3^2+4^2)}=5=r_1+r_2=4+1$

Hence the two circles touch each other externally If $P(h\ k)$ be the point of contact then it will divide $A(-1, -1)$ $B(2\ 3)$ in the ratio of the radii 1 4

$$h=\frac{1\ 2+4(-1)}{1+4}=\frac{-2}{5} \quad k=\frac{1\ 3+4(-1)}{1+4}=\frac{-1}{5}$$

Hence the point of contact is $(-2/5, -1/5)$

(*b*) The two circle, with centres c_1 and c_2 and radii r_1 and r_2 will

(1) Intersect if $c_1 c_2 < r_1+r_2$

(2) Not Intersect if $c_1 c_2 > r_1+r_2$

(3) Touch if $c_1 c_2=r_1+r_2$ (Externally)$=r_1-r_2$ (Internally)

Here $c_1 c_2=5\sqrt{2}, \neq 4+1, \neq 4-1$ They do not touch

$c_1 c_2=5\sqrt{2} > 4+1$ They do not Intersect

Hence *d* is correct answer

47 The radical axes of the circles

$$x^2+y^2+2gx+2fy+c=0$$

and $2x^2+2y^2+3x+8y+2c=0$ touches the circle $x^2+y^2+2x+2y+1=0$, show that either $g=\frac{3}{4}$ or $f=2$

48 The line $Ax+By+C=0$ cuts the circle $x^2+y^2+ax+by+c=0$ in P and Q. The line $A'x+B'y+C'=0$ cuts the circle $x^2+y^2+a'x+b'y+c'=0$ in R and S. If P, Q, R, S are concyclic then show that

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix}=0$$

(Roorkee 86)

49 If from any point on the circle $x^2+y^2+2gx+2fy+c=0$ tangents are drawn to the circle $x^2+y^2+2gx+2fy+c\sin^2\alpha+(g^2+f^2)\cos^2\alpha=0$, show that angle between the tangents is 2α

50 Distance from the origin to the centres of the three circles $x^2+y^2-2\lambda x=c^2$ (where c is constant and λ is variable) are in G P Prove that the lengths of tangents drawn from any point on the circle $x^2+y^2=c^2$ to the three circles are also in G P (M N R 85)

51 Tangents PQ, PR are drawn to the circle $x^2+y^2=a^2$ from a given point P Find the equation of the circum-circle of the triangle PQR

52 If two curves whose equations are

$$ax^2+2hxy+by^2+2gx+2fy+c=0$$

and $a'x^2+2h'xy+b'y^2+2g'x+2f'y+c'=0$ intersect in four concyclic points prove that

$$\frac{a-b}{h}=\frac{a'-b'}{h'}$$

53 If the four points of intersection of the lines $(2x-y+1)(x-3y+3)=0$ with the axes lies on a circle, then find its centre

54 (a) If the equation of the circles whose radii are r and R be respectively $S=0$ and $S'=0$, then prove that the circles

$$\frac{S}{r}\pm\frac{S'}{R}=0$$ will cut orthogonally

$-$ve for internal $\qquad \sqrt{(a^2+b^2)}=\sqrt{(a^2-c)}+\sqrt{(b^2-c)}$

Square $\quad a^2+b^2=a^2-c+b^2-c+2\sqrt{[(a^2-c)(b^2-c)]}$

or $\quad c=\sqrt{[(a^2-c)(b^2-c)]}$ or $c^2=a^2b^2-c(a^2+b^2)+c^2$

or $$\frac{1}{c}=\frac{a^2+b^2}{a^2b^2} \quad \text{or} \quad \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{c}$$

Alternative If the two circle touch then their common chord $S_1-S_2=0$ i e $ax-by=0$ should be a tangent to either Hence perpendicular from centre $(-a, 0)$ is equal to radius $\sqrt{(a^2-c)}$

$$\frac{-a^2}{\sqrt{(a^2+b^2)}}=\sqrt{(a^2-c)} \quad \text{Square}$$

$$a^4=(a^2-c)(a^2+b^2)=a^4+a^2b^2-c(a^2+b^2)$$

or $$\frac{1}{c}=\frac{a^2+b^2}{a^2b^2} \quad \text{or} \quad \frac{1}{c}=\frac{1}{a^2}+\frac{1}{b^2}$$

39 Equation of the circle through the intersection of circle $S=0$ and line MN i e $P=0$ is given by $S+\lambda P=0$

$$(x^2+y^2-a^2)+\lambda(x\cos\alpha+y\sin\alpha-p)=0 \qquad (1)$$

or $\quad x^2+y^2+x(\lambda\cos\alpha)+y(\lambda\sin\alpha)-(a^2+\lambda p)=0$

Centre is $\left(\frac{-\lambda}{2}\cos\alpha, \frac{-\lambda}{2}\sin\alpha\right)$

Since the line MN is a diameter of the requird circle and hence its centre will lie on $P=0$

$$\frac{-\lambda}{2}\cos\alpha \cos\alpha-\frac{\lambda}{2}\sin\alpha \sin\alpha-p=0$$

$$\frac{\lambda}{2} 1+p=0 \qquad \lambda=-2p$$

Hence the required circle from (1) is

$$x^2+y^2-a^2-2p(x\cos\alpha+y\sin\alpha-p)=0$$

40 Since the diameter is along x axis and the chord $y=mx$ passes through the origin therefore centre is $(a, 0)$ where a is the given radius of the circle Hence its equation is $S=(x-a)^2+y^2=a^2$

or $\quad x^2+y^2-2ax=0$

P is $y=mx$ (m is variable) Circle through the intersection of $S=0$ and $P=0$ is

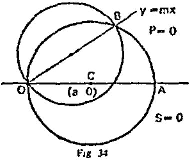


Fig 34

$$S+\lambda P=0 \quad \text{or} \quad (x^2+y^2-2ax)+\lambda(y-mx)=0$$

Solution to Problem Set (B)

1 Perpendicular from the centre (a, b) to the tangent $mx-y+$
should be equal to radius r

$$\frac{ma-b+c}{\sqrt{(m^2+1)}}=r \quad \text{Square}$$

$$m^2a^2+(c-b)^2+2ma\,(c-b)=r^2\,(m^2+1)$$

or $$m\,(a^2-r^2)+2ma\,(c-b)+(c-b)=r^2$$

2 Centre is $\left(\frac{5}{2}, \frac{-5}{2}\right)$, $r=\sqrt{\left(\frac{25}{4}+\frac{25}{4}-0\right)}=\frac{5}{2}\sqrt{2}$

$$7y-x-5=0$$

If it is a tangent then $\perp$ from centre should be equal to r

$$p=\frac{7\,(-5/2)-5/2-5}{\sqrt{(49+1)}}=\frac{-50}{2\sqrt{50}}=\frac{5\sqrt{2}}{2}=r$$

Parallel tangent Any line parallel to

$$7y-x-5=0 \text{ is } 7y-x+\lambda=0$$

Condition of tangency

$$\frac{7\,(-5/2)-5/2+\lambda}{\sqrt{(50)}}=\frac{5\sqrt{2}}{2}$$

$$\lambda-20=\pm 5\sqrt{2}\;5\sqrt{2}/2=\pm 25$$

$$\lambda=-5 \quad \text{or} \quad +45$$

Hence the other parallel tangent is

$7y-x+45=0$ as $7y-x-5=0$
is already a tangent

Point of contact of $7y-x-5=0$ slope $1/7$

Any line through the centre $(5/2, -5/2)$ and perpendi
tangent is

$$y+5/2=-7\,(x-5/2)$$

$$y+7x-15=0$$

Solving above with $7y-x-5=0$ we get $x=2$ $y=1$
the point of contact is $(2, 1)$

2nd Method Let (p, q) be the point of contact of the ta

$$7y-x-5=0$$

Tangent at (p, q) to

$$x^2+y^2-5x+5y=0 \text{ is}$$

$$xp+yq-\tfrac{5}{2}(x+p)+\tfrac{5}{2}(y+q)=0$$

$$(2p-5)\,x+(2q+5)\,y+(5q-5p)=0$$

Comparing (1) and (2) we get

$$\frac{2p-5}{-1}=\frac{2q+5}{7}=\frac{5q-5p}{-5}$$

$$16(1+2\lambda+\lambda^2)=1+10\lambda+25\lambda^2+4+24\lambda$$
$$+36\lambda^2-40\lambda^2-40\lambda+4+4\lambda$$

or $\quad 16+32\lambda+16\lambda^2=21\lambda^2-2\lambda+9$

or $\quad 5\lambda^2-34\lambda-7=0 \quad (\lambda-7)(5\lambda+1)=0 \quad \lambda=7, -1/5$

Putting the values of λ in (1) the required circles are

$$2x^2+2y^2-18x-22y+69=0 \text{ and } x^2+y^2-2y-15=0$$

(b) $S+\lambda P=0$, $\lambda=4/3$, $7(x^2+y^2)-10x-10y-12=0$

43 The required circle by $S+\lambda P=0$ is

$$x^2+y^2-2x+1+\lambda(x+2y-1)=0$$
$$x^2+y^2-x(2-\lambda)+2\lambda y+(1-\lambda)=0$$

Centre $(-g, -f)$ is $\left(\frac{2-\lambda}{2}, -\lambda\right)$

$$r=\sqrt{(g^2+f^2-c)}=\sqrt{\left[\frac{(2-\lambda)^2}{4}+\lambda^2-(1-\lambda)\right]}=\tfrac{1}{2}\sqrt{(5\lambda^2)}=\frac{\lambda}{2}\sqrt{5}$$

Since the circle touches the line $2x-y+3=0$ therefore perpendicular from centre is equal to radius

$$\frac{2\,\frac{2-\lambda}{2}-(-\lambda)+3}{\pm\sqrt{5}}=\frac{\lambda}{2}\sqrt{5} \quad \text{or} \quad 5=\pm\frac{\lambda}{2}\,5 \qquad \lambda=\pm 2$$

Putting the values of λ in (1) the required circles are

$$x^2+y^2+4y-1=0, \quad x^2+y^2-4x-4y+3=0$$

44 (a) $S+\lambda S'=0$ and pass through (1, 1)

$$x^2+y^2-3x+1=0$$

(b) Proceed as above $\lambda=\frac{1}{2}$ etc

45 $$S+\lambda S'=0$$

Centre $\left(\frac{1}{1+\lambda}, \frac{2}{1+\lambda}\right)$ and $r=\frac{1}{1+\lambda}\sqrt{(1+4\lambda^2)}$

Applying the condition of tangency, we get

$\lambda=\pm 1$

For $\lambda=1 \qquad x^2+y^2-x-2y=0$

Note that $\lambda=-1$ gives $x+2y-4=0$

which is the common chord of the two circles

46 Their common chord is

$$2x(g-g')+2y(f-f')+c-c'=0$$

If the first circle bisects the circumference of the second circle then their common chord must pass through the centre $(-g', -f')$ of the second circle Hence the required condition is $\quad -2g'(g-g')-2f'(f-f')+c-c'=0,$

or $\quad 2g'(g-g')+2f'(f-f')=c-c'$

8. The given circle is

$S \equiv x^2+y^2+4x-6y-3=0$

$B(2, 3)$ is a point on the circle tangent at which is

$x\,2+y\,3+2(x+2)$

$-3(y+3)-3=0$

or $\quad 4x-8=0$

or $\quad P\equiv \quad x-2=0$

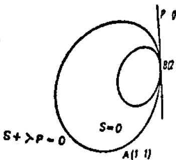


Fig 36

The required circle is a circle through the intersection of $S=0$ and $P=0$. Hence its equation is given by

or $\quad S+\lambda P=0$

$x^2+y^2+4x-6y-3+\lambda(x-2)=0 \qquad (1)$

Since it passes through the point $(1, 1)$ we have

$-3-\lambda=0 \qquad \lambda=-3$

Putting the value of λ in (1) the required circle is

$x^2+y^2+x-6y+3=0$

9. Proceed as in Q 4

10. Tangent at $(1, -2)$ to $x^2+y^2=5$ is

$x\,1+y(-2)=5 \quad$ or $\quad x-2y-5=0 \qquad (1)$

Centre of the other circle is $(4, -3)$ and radius is

$\sqrt{(16+9-20)}=\sqrt{5}$

If (1) is a tangent to the second circle, then perpendicular from centre should be equal to radius

$$\frac{4-2(-3)-5}{\sqrt{(1+4)}}=\frac{10-5}{\sqrt{5}}=\sqrt{5}=\text{radius}$$ Hence it touches the other circle as well. The point of contact be found as in Q 2 to be $(3, -1)$

11. Tangent to the circle at $(1, -3)$ is $3x-y=6$. Let the centre be (h, k) and it will lie on the line through $(1, -3)$ and perpendicular to tangent. Its equation is

$x+3y+\lambda=0 \quad$ where $\quad 1-9+\lambda=0 \qquad \lambda=8$

$x+3y+8=0$

Since centre (h, k) lies on it

$h+3k+8=0 \quad$ or $\quad h=-3k-8$

Also if r be the radius then $(1, -3)$ being a point on the circle

$(h-1)^2+(k+3)^2=r^2 \qquad (1)$

or $\quad (-3k-8-1)^2+(k+3)^2=r^2$

$$AB=2BP=2\,\frac{1}{2\sqrt{2}}\sqrt{[(a+b)^2-8c]}=\sqrt{[\tfrac{1}{2}(a+b)^2-4c]}$$

49 Ans $2x+1=0,\ 2\sqrt{2}$

50 Solve the two equations, the points are (4, 2) and (−2, −6)

51 *Do yourself*

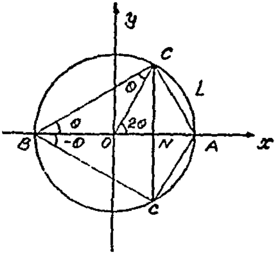
52 Centre O of the circle is the origin and x axis is along the diameter BA, where $BA=13$ m so that co-ordinates of A and B are (6 5, 0) and (−6 5, 0) We are given chord $AC=5$ m To find the equation of pair of lines BC and area of the portion $ALCA$

Since angle in a semi-circle is a right angle, we have

$$\angle ACB=90^\circ$$

Hence $BC=\sqrt{(AB^2-AC^2)}=\sqrt{(13^2-5^2)}=12$

If $\angle ABC=\theta$, then $\angle OCB=\theta$ and $\angle AOC=2\theta$



Also, $\tan\theta=\frac{AC}{BC}=\frac{5}{12}$, $\sin\theta=\frac{5}{13}$ and $\cos\theta=\frac{12}{13}$

Thus the lines BC pass through B (−6 5, 0) and have gradients $\frac{5}{12}$ and $-\frac{5}{12}$ Hence equation of the lines is given by

$$y-0=\pm\frac{5}{12}(x+6\ 5)$$

or $$12y=\pm(5x+32\ 5) \qquad (1)$$

To find the required area, we have

$$\tan\theta=\frac{5}{12}=\ 416=\ 42 \text{ so that from the tables}$$

Area of the triangle formed by the tangent and the axes will be $\frac{1}{2}hk=a^2$ given

$$\pm\ \frac{1}{2}\ \frac{a}{m}\sqrt{(1+m^2)}\ \ a\sqrt{(1+m^2)}=a^2$$

or $\qquad (1+m^2)=\pm 2m \quad$ or $\quad 1+m^2\pm 2m=0$

or $\qquad (1\pm m)^2=0 \qquad m=\pm 1$

Hence the required tangents are

$$y=\pm x\pm a\sqrt{2}$$

14 The equation of the circle is

$$(x-1)(x-3)+(y-2)(y-4)=0$$

or $$x^2+y^2-4x-6y+11=0$$

Centre is (2, 3) $r=\sqrt{(4+9-11)}=\sqrt{2}$

Slope of the diameter joining (1, 2) and (3, 4) is $(4-2)/(3-1)=1$

Hence any line parallel to this diameter is $y=x+c$

or $$x-y+c=0$$

If it is a tangent, then perpendicular from centre (2, 3) is equal to radius $\sqrt{2}$

$$\frac{2-3+c}{\pm\sqrt{2}}=\sqrt{2} \quad \text{or} \quad c-1=\pm 2 \qquad c=3,\ -1$$

Hence the tangents are $y=x+3$, and $y=x-1$

15 Let the circle be $x^2+y^2=a^2$ and two parallel tangents be drawn at $A\ (0, a)$ and at $B\ (0\ -a)$ whose equations are $y=a$ and $y=-a$ Let $P\ (h, k)$ be any point on the circle tangent at which is

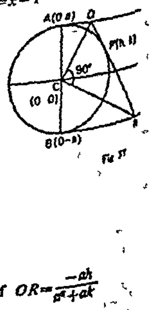


Fig 37

$$hx+ky=a^2$$

where $h^2+k^2=a^2 \qquad (1)$

Solving with $y=a$ we get

$$Q\left(\frac{a^2-ak}{h},\ a\right) \text{ and } O \text{ is } (0, 0)$$

Solving with $y=-a$ we get

$$R\left(\frac{a^2+ak}{h},\ -a\right)$$

m_1=slope of $OQ=\frac{ah}{a^2-ak}$, m_2=slope of $OR=\frac{-ah}{a^2+ak}$

$$m_1m_2=\frac{ah}{a\,(a-k)}\ \frac{-ah}{a\,(a+k)}=\frac{-h^2}{a^2-k^2}=\frac{-h^2}{h^2}=-1 \text{ by (1)}$$

Hence the lines OQ and OR are mutually perpendicular.

Hence the centres of the circles are $(1, 7)$ $(3, -7)$ $(9, 1)$ and $(-5, -1)$

Since radius of each is 3 their equations are

$(x-1)^2+(y-7)^2=3^2$, $(x-3)^2+(y+7)^2=3^2$,

$(x-9)^2+(y-1)^2=3^2$ and $(x+5)^2+(y+1)^2=3^2$

§ 2 Tangent, Normal, Chord of Contact, Polar, Chord with a given middle point, length of tangent, Radical axis, orthogonal intersection of two circles

(a) Tangent of the circle

$$x^2+y^2+2gx+2fy+c=0$$

at any point (x_1, y_1) is

$$xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0 \qquad (1)$$

Particular case $x^2+y^2=a^2$

$$xx_1+yy_1=a^2 \qquad (2)$$

Rule Write $x^2=x\,x_1$, $y^2=y\,y_1$, $2gx=g(x+x_1)$, $2fy=f(y+y_1)$

(b) Normal It will be a line passing through the point $(x_1\ y_1)$ and perpendicular to tangent found above It also passes through the centre of the circle

(c) *Tangent in terms of slope m*

Let the circle be $x^2+y^2=a^2$

Any line whose slope is m is $y=mx+c$

or $mx-y+c=0$

If it is a tangent then perpendicular from the centre $(0, 0)$ is equal to the radius a,

$$\frac{c}{\pm\sqrt{(1+m^2)}}=a \quad \text{or} \quad c=\pm a\sqrt{(1+m^2)}$$

Hence the line $y=mx\pm a\sqrt{(1+m^2)}$ (3)

is always a tangent to the circle Corresponding to a given line there will be two tangents parallel to it

(d) Chord of contact of tangents drawn from a point (x_1, y_1)

Let the points of contact of tangents from any points $T(x_1, y_1)$ be $A(p, q)$, $B(h, k)$ on the circle

$x^2+y^2=a^2$

$T_A=px+qy=a^2$

$T_B=hx+ky=a^2$

Both these tangents pass through $T(x_1\ y_1)$

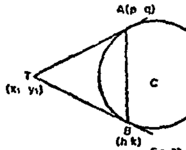
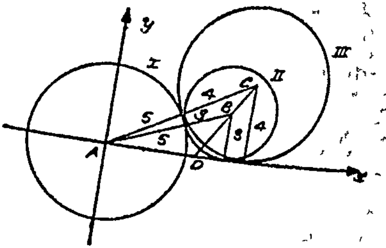


Fig. 22



$$h=\frac{\sqrt{(55)}\times 4-\sqrt{(65)}\times}{4-3}=4\sqrt{(55)}-3\sqrt{(65)} \quad ...(1)$$

Now any line through $(h, 0)$ is

$y-0=m(x-h)$ or $mx-y-mh=0$

It will touch the circle I if,

$$\frac{m\sqrt{(55)}-3-mh}{\sqrt{(1+m^2)}}=\pm 3 \text{ or } [m(\sqrt{(55)}-h)-3]^2=9(1+m^2)$$

or $m^2[\sqrt{(55)}-h]^2+9-6m[\sqrt{(55)}-h]=9+9m^2$

or $m^2[(\sqrt{(55)}-h)^2-9]-6m(\sqrt{(55)}-h)=0$

$m=0$ and on putting the value of h from (1), we get

$m[(\sqrt{(55)}-4\sqrt{(55)}+3\sqrt{(65)})^2-9]=6[\sqrt{(55)}-4\sqrt{(55}+3\sqrt{(65)}]$

or $9m[\{\sqrt{(65)}-\sqrt{(55)}\}^2-1)]=18[\sqrt{(65)}-\sqrt{(55)}]$

or $m=\dfrac{2[\sqrt{(65)}-\sqrt{(55)}]}{119-10\sqrt{(143)}}$

Hence the two common tangents are

$y=0$ and $y=\dfrac{2[\sqrt{(65)}-\sqrt{(55)}]}{119-10\sqrt{143}}[(x-4\sqrt{55}+3\sqrt{(65)}]$

18 (a) You may proceed as in Q 17 and alternative method is given below

Any tangent to the first circle $x^2+y^2=25$ is

$y=mx\pm a\sqrt{(1+m^2)}$ or $mx-y+5\sqrt{(1+m^2)}=0 \quad ...(1)$

If it is a tangent to 2nd circle $(x-12)^2+y^2=9$, then perpendicular from centre $(12,0)$ should be equal to radius 3,

$$\frac{12m+5\sqrt{(1+m^2)}}{\pm\sqrt{(m^2+1)}}=3$$

(f) Chord with a given middle point Let L be the middle point of the chord AB and C be the centre of the circle $x^2+y^2=a^2$, then CL is perpendicular to AB

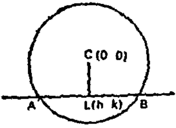


Fig 24

Slope of $CL=\frac{k-0}{h-0}=\frac{k}{h}$

Slope of $AB=-h/k$ $[m_1m_2=-1]$

Equation to AB which passes through L (h, k) and whose slope is $-h/k$

$$y-k=\frac{-h}{k}(x-h)$$

or $$hx+ky=h^2+k^2 \tag{6}$$

It is of the form $T=S_1$

where $T=hx+ky-a^2$, $S_1=h^2+k^2-a^2$,

(g) Length of tangent from a given point Let the circle be

$x^2+y^2+2gx+2fy+c=0$

and the given point P i e (x_1, y_1) and PT be the tangent drawn from P

Now if C $(-g, -f)$ be the centre then CT is perpendicular to PT

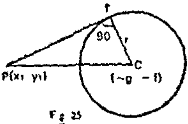


Fig 25

$$CP^2=PT^2+CT^2$$

or $$(x_1+g)^2+(y_1+f)^2=PT^2+[\sqrt{(g^2+f^2-c)}]^2$$

$$x_1^2+y_1^2+2gx_1+2fy_1+c=PT^2$$

Rule Put down the coordinates of the point P (x_1, y_1) in the equation of the circle and you get the square of the length of tangent drawn from P

Note The equation of the circle should be in standard form i e coefficient of x^2 and y^2 should be each unity If they be not unity then first make them unity and then apply the above rule

(h) Orthogonal intersection of two circles Two circles are said to intersect orthogonally if the angle between the tangents to them at their common point of intersection is a right angle Clearly from the figure the tangent to one will pass through

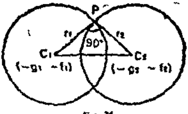


Fig 26

19 Equation of the chord AB of the circle

$$x^2+y^2=a^2$$

whose mid point is (h, k) is

$$hx+ky=h^2+k^2$$ Rule (f) P 359

(i) It subtends a right angle at the centre C $(0, 0)$ of the circle The lines AC and BC are at right angles But these are the lines joining origin to the points of intersection of chord with the circle and their equation is obtained by making the equation of the circle homogeneous by the help of the equation of the chord

$$\frac{hx+ky}{h^2+k^2}=1$$

Hence $$x^2+y^2=a^2\left(\frac{hx+ky}{h^2+k^2}\right)^2$$

or $(h^2+k^2)^2(x^2+y^2)=a^2(h^2x^2+k^2y^2+2hkxy)$

or $x^2[(h^2+k^2)^2-a^2h^2]-2a^2hkxy+y^2[(h^2+k^2)^2-a^2k^2]=0$

Since the lines given by above are at right angles therefore sum of the coefficients of x^2 and y^2 is zero

$$(h^2+k^2)^2-a^2h^2+(h^2+k^2)^2-a^2k^2=0$$

$$2(h^2+k^2)^2=a^2(h^2+k^2) \text{ or } h^2+k^2=a^2/2$$

Hence the locus of the mid point is $x^2+y^2=a^2/2$

(ii) If the chord passes through a fixed point (x_1, y_1), then

$$hx_1+ky_1=h^2+k^2$$

Hence the locus of the mid point in this case is

$$xx_1+yy_1=x^2+y^2 \text{ or } x^2+y^2-xx_1-yy_1=0$$

Above respresents a circle whose centre is $(x_1/2, y_1/2)$

20 Refer fig 24 Page 359

Centre of the circle $x^2+y^2-2x-6y+10=0$ is $C(1, 3)$ and if the mid point of the chord be $L(h, k)$ then slope of

$$CL=\frac{k-3}{h-1}$$

Also chord AB whose mid point is (h, k) passes through $(0, 0)$ as given so that its slope is $\frac{k-0}{h-0}=\frac{k}{h}$

Again CL and AB are perpendicular to each other so that product of their slopes is -1

$$\frac{k-3}{h-1}\cdot\frac{k}{h}=-1 \text{ or } h(h-1)+k(k-3)=0$$

Hence we obtain the following rule to obtain the equations of common tangents to two circles

Let A and B be the centres and a, b the radii of two given circles Find the point C and D say, which respectively divide AB externally and internally in the ratio a b Then *for the direct common tangents find the straight lines through C which are at a distance a from A and for the transverse common tangents find the straight lines through D at a distance a from A*

Note The points C and D which divide the line of centres AB internally and externally in the ratio of the radii are called the centres of similitude of the two circles

Problem Set (B)

1 Prove that the line $y=mx+c$ will be a tangent to the circle

$$(x-a)^2+(y-b)^2=r^2 \quad \text{if}$$

$$m^2(a^2-r^2)+2ma(c-b)+(c-b)^2=r^2$$

2 Show that straight line $7y-x=5$ touches the circle

$$x^2+y^2-5x+5y=0$$

and find the coordinates of the point of contact Show that the other parallel tangent is $7y-x=-45$

3 Find the equation of those tangents to the circles

$$x^2+y^2-6x+4y-12=0$$

which are parallel to the line

$$4x+3y+5=0$$

4 (*a*) Find the equation of the tangents to the circle

$$x^2+y^2-2x-4y-20=0$$

which pass through the point (8, 1)

(*b*) Find the equation of the two tangents from the point (0, 1) to the circle $x^2+y^2-2x+4y=0$ **(Roorkee 79)**

5 Find the equations of tangents to the circle

$$x^2+y^2+20(x+y)+20=0$$

which pass through the origin

6 Find the equation of the tangent line to the circle

$$(x+2)^2+(y-3)^2=25$$

at the point $A(-5, 7)$

7 Show that the equation to the tangent at A where $\theta=\alpha$ to the circle

$(x-a)^2+(y-b)^2=r^2$ is $(x-a)\cos\alpha+(y-b)\sin\alpha=r$

Since it pass through $(0, a)$ and $(0, -a)$

$$a^2+2fa+c=0,\ a^2-2fa+c=0$$

Subtracting we get $4fa=0$, $\quad f=0 \quad c=-a^2$

Hence the circle becomes $x^2+y^2+2gx-a^2=0$

Centre is $(-g, 0)$ and radius $\sqrt{(g^2-c)}=\sqrt{(g^2+a^2)}$

It is given that the circle touches the line

$$y=mx+c \quad \text{or} \quad mx-y+c=0,$$

Hence perpendicular from centre should be equal to radius

$$\frac{-mg+c}{\sqrt{(m^2+1)}}=\sqrt{(g^2+a^2)}$$

Square $\quad (c-mg)^2=(g^2+a^2)(m^2+1)$

or $\quad g^2+2gmc+\{a^2(1+m^2)-c^2\}=0 \qquad (1)$

$$g_1g_2=a^2(1+m^2)-c^2$$

Above is a quadratic in g and gives us two values of g showing that there will be two circles which satisfy the given conditions If the two values of g be g_1 and g_2 then the two circles are

$$x^2+y^2+2g_1x-a^2=0 \quad \text{and} \quad x^2+y^2+2g_2x-a^2=0$$

These two will intersect orthogonally if

$$2g_1g_2+2f_1f_2=c_1+c_2$$

$$2\{a^2(1+m^2)-c^2\}+0=-a^2-a^2=-2a^2$$

or $\quad a^2(1+m^2)-c^2=-a^2 \quad$ or $\quad a^2(2+m^2)=c^2$

is the required condition

25 (a) Equation of the chord of contact AB of tangents drawn from the point $P(h, k)$ to the given circle is

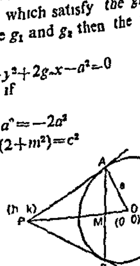


Fig 33

$$hx+ky-a^2=0$$

$OM=\perp$ distance of $(0\ 0)$ from AB

$$OM=\frac{-a^2}{\sqrt{(h^2+k^2)}}$$

$$AM^2=OA^2-OM^2=a^2-\frac{a^4}{h^2+k^2}=\frac{a^2(h^2+k^2-a^2)}{(h^2+k^2)}$$

$$AB=2AM=2\sqrt{\left(\frac{h^2+k^2-a^2}{h^2+k^2}\right)}$$

$PM=\perp$ distance of $(h\ k)$ from AB

$$PM=\frac{h^2+k^2-a^2}{\sqrt{(h^2+k^2)}}$$

right of A Find the equation of any two common tangents to the circle II and III (Roorkee 83)

18 (a) Find the equation to the four common tangents to the circles $x^2+y^2=25$
and $(x-12)^2+y^2=9$

(b) A straight line AB is divided at C so that $AC=3CB$ Circles are described on AC and CB as diameters and a common tangent meets AB produced at D, Show that BD is equal to the radius of the smaller circle (I I T 72)

19 Find the locus of the middle points of the chords of the circle $x^2+y^2=a$ (i) which subtend a right angle at the centre (ii) which pass through a given point (x_1, y_1) (I I T 1983)

20 Find the locus of the middle points of the chords of the circle

$$x^2+y^2-2x-6y-10=0$$

which pass through the origin

21 If tangents be drawn to the circle $x^2+y^2=12$ at its points of intersection with the circle

$$x^2+y^2-5x+3y-2=0$$

find the coordinates of their point of intersection

22 Find the equation of the circle which cuts the circles

$$x^2+y^2-9x+14=0$$

and

$$x^2+y^2+15x+14=0$$

orthogonally and passes through the point (2, 5)

23 (a) Find the equation of the circle which passes through origin and cuts orthogonally each of the circles

$$x^2+y^2-8y+12=0$$

and

$$x^2+y^2-4x-6y-3=0$$

(b) Show that the locus of the centres of the circles which cut the circles

$$x^2+y^2+4x-6y+9=0$$

and

$$x^2+y^2-4x+6y+4=0$$

orthogonally is $8x-12y+5=0$

24 Prove that the two circles which pass through the points $(0, a)$ and $(0, -a)$ and touch the line $y=mx+c$ will cut orthogonally if $c^2=a^2(2+m^2)$

25 (a) Find the area of the triangle formed by the tangents from the points (h, k) to the circle $x^2+y^2=a^2$ and their chord of contact [M N R 1980]

(b) The area of the triangle formed by tangents from the point

Compare with $4l^2-5m^2+6l+1=0$

$a^2-r^2=4$, $b^2-r^2=-5$, $ab=0$, $2a=6$, $2b=0$

$a=3$, $b=0$ $\quad r^2=5$

Hence the circle is

$$(x-3)^2+y^2=5 \quad \text{or} \quad x^2+y^2-6x+4=0$$

29 (a) The given circle has its centre (a, a) and radius a so that this circle touches both the axes

Let the third side be

$$\frac{x}{p}+\frac{y}{q}=1$$

so that P is $(p, 0)$ and Q is $(0, q)$ The line PQ touches the given circle at the point R

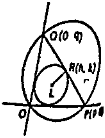


Fig 31

Since $\angle POQ$ is a right angle therefore PQ is diameter of the circle passing through the points O, P and Q and its centre is mid point R of PQ

R is $(p/2 \;\; q/2)=(h, k)$ say

$p=2h$, $q=2k$

Now the line PQ, $x/p+y/q=1$

or $\quad qx+py-pq=0$

touches the given circle

$$\frac{qa+pa-pq}{\sqrt{(p^2+q^2)}}=a \quad \text{Square}$$

or $\quad a^2(p+q)^2+p^2q^2-2pqa(p+q)=a^2(p^2+q^2)$

or $\quad a^2\, 2pq+p^2q^2-2pqa(p+q)=0 \quad$ cancel pq

or $\quad 2a^2-2a(p+q)+pq=0$

$2a^2-2a(2h+2k)+2h\, 2k=0$

Locus of centre (h, k) is

$$a^2-2a(x+y)+2xy=0$$

(b) It is same question as part (a) if we choose $a=2$ The condition of tangency gives

$$\frac{qa+pa-pq}{\pm\sqrt{(p^2+q^2)}}=a \quad \text{put } a=2,\ p=2h \text{ and } q=2k$$

$(2k)\,2+(2h)2-4hk=\pm 2\sqrt{(4h^2+4k^2)}$

or $\quad h+k-hk\pm\sqrt{(h^2+k^2)}=0$

Locus is $x+y-xy\pm\sqrt{(x^2+y^2)}=0$

Let the circle be $\qquad k=\pm 1$

$$x^2+y^2+2gx+2fy+c=0$$

34 The chord of contact of tangents drawn form any point on the circle $x^2+y^2=a^2$ to the circle $x^2+y^2=b^2$ touches the circle $x^2+y^2=c^2$. Show that a, b, c are in G P

35 Find the equations of the circles passing through $(-4, 3)$ and touching the lines $x+y=2$ and $x-y=2$ *(I I T 82)*

36 AB is a diameter of a circle CD is a chord parallel to AB and $2CD=AB$ The tangent at B meets the line AC produced at E Prove that $AE=2AB$ *(I I T 80)*

37 Find the equation of a circle whose centre is the point (3, 1) and which cuts off a chord of length 6 on the line

$$2x-5y+18=0$$ *(Roorkee 77)*

38 Prove that the radical axis of three circles taken in pairs are concurrent What is the point of concurrency called

39 Find the coodinates of the point from which the length of tangents to the following three circles be equal

$$3x^2+3y^2+4x-6y-1=0,\ 2x^2+2y^2-3x-2y-4=0,$$
$$2x^2+2y^2-x+y-1=0$$

40 Find the general equation of all circles, so that the radical axis of any two of them is the same as those of the circles

$$x^2+y^2=4 \text{ and } x^2+y^2+2x+4y-6=0$$

41 Find the equation of the circle which passes through the point $(2a\ 0)$ and whose radical axis w r t the circle $x^2+y^2=a^2$ is the line $x=a/2$

42 Prove that the locus of a point which moves such that the difference of the squares of the lengths of tangents drawn from it to two given circles is constant is a line parallel to the radical axis of the given circles

43 If two circles cut a third circle orthogonally then prove that their radical axis or their common chord will pass through the centre of the third circle

44 Define coaxial circles and deduce their equation in simplest form

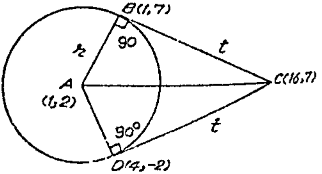
45 If $S_1=0$, $S_2=0$ be the equations of two circles and $P=0$ be the equation of a line then interpret the equations

$$S_1+\lambda S_2=0 \quad \text{and} \quad S_1+\lambda P=0$$

46 Find the equation of a circle which is coaxial with the circles

$$2x^2+2y^2-2x+6y-3=0 \text{ and } x^2+y^2+4x+2y+1=0$$

It is given that the centre of the circle to be determined lies on the radical axis of these circles (Roorkee 84)



From Ist and 3rd, we get $k = -5/2$

Hence the required point is $(-2, -5/2)$ and $t^2 = 37/4$

34 $(a\cos\theta, a\sin\theta)$ is on $x^2 + y^2 = a^2$

C C is $ax\cos\theta + ay\sin\theta = b$

It touches $x^2 + y^2 = c$ $\quad \dfrac{b^2}{a\sqrt{(\cos^2\theta + \sin^2\theta)}} = c$ or $b^2 = ac$

a, b, c are in G P

35 Let the circle be $x^2 + y^2 + 2gx + 2fy + c = 0$ It passes through (1)

$(-4, 3)$ $\quad 25 - 8g + 6f + c = 0$

It touches the given lines and hence the condition of perpendicularity gives

$$\frac{-g-f-2}{\sqrt{2}} = \sqrt{(g^2+f^2-c)} = \frac{-g+f-2}{\sqrt{2}}$$

$f = 0$ from Ist and last (2)

$(g+2)^2 = 2(g^2 - c)$ or $g^2 - 4g - 4 - 2c = 0$ (3)

Also form (1) on putting $f = 0$ we get $c = 8g - 25$

Putting in (2) we get $\quad g^2 - 2g + 46 = 0$

$g = 10 \pm 3\sqrt{6}$ and hence $c = 55 \pm 24\sqrt{6}$ and $f = 0$

The required circles are

$x^2 + y^2 + 2(10 \pm 3\sqrt{6})x + (55 \pm 24\sqrt{6}) = 0$

36 Let the circle be $x^2 + y^2 = a^2$ and the diameter AB be chosen along x axis so that A is $(a, 0)$ and B is $(-a, 0)$ Since CD is parallel to AB and half its length therefore x coordinate of C will be $a/2$ and that of D will be $-a/2$

$$C \text{ is } \left(\frac{a}{2}, \; \frac{\sqrt{3}}{2}a\right) \quad (1)$$

Equation to AC is $y - 0 = \sqrt{3}(x - a)$

Tangent at B is $\quad x = -a$

(1) and (2) meet at E whose coordinates are $(-a, -2\sqrt{3}a)$

(*b*) If a circle passes through the point (a, b) and cuts the circle $x^2+y^2=k^2$ orthogonally, then the locus of its centre is $2ax+2by-(a^2+b^2+k^2)=0$ (I I T 88)

55 The lengths of tangents from a fixed point to three circles of a co axial system are t_1 t_2, t_3 respectively If P, Q, R be the centres of the three circles, show that

$$QRt_1^2+RPt_2^2+PQt_3^2=0$$

(*b*) P Q R are the centres and r_1, r_2, r_3 are the radii respectively of three coaxial circles show that

$$r_1^2\, QR+r_2^2\, RP+r_3^2\, PQ=-PQ\, QR\, RP$$

56 A straight rod AB of fixed length c moves so that its extremities A, B lie on two fixed mutually perpendicular lines Prove that the locus of the circumcircle of triangle OAB is a circle, where O is the point of intersection of mutually perpendicular straight lines

57 Show that the locus of the point the tangents from which to the circle $x^2+y^2=a^2$ include a constant angle α is

$$(x^2+y^2-2a^2)^2 \tan^2\alpha=4a^2\,(x^2+y^2-a^2)$$

58 How are the following points $(0, 1)$, $(3\ 1)$ and $(1, 3)$ situated w r t the circle

$$x^2+y^2-2x-4y+3=0$$ (M N R 80)

59 Show that the general equation of a circle which passes through (x_1, y_1) and (x_2, y_2) may be written as

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+\lambda\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix}=0$$

and hence deduce the diameter form of the equation of a circle

60 Lines $5x+12y-10=0$ and $5x-12y-40=0$ touch circle c_1 of diameter 6 If the centre of c_1 lies in the first quadrant, find the equation of the circle c_2 which is concentric with c_1 and cuts intercepts of length 8 on these lines (I I T 86)

61 Let $S=x^2+y^2+2gx+2fy+c=0$ be a given circle Find the locus of the foot of the perpendicular drawn from the origin upon any chord which subtends a right angle at the origin ((I I T

$\frac{17}{6}x-y+\frac{5}{3}=0$, $x-\frac{3}{2}y-\frac{3}{2}=0$ and $-\frac{11}{6}x+\frac{5}{2}y-\frac{1}{6}=0$

Solving any two we get the point $\left(\frac{-16}{21},\frac{-31}{63}\right)$ which satisfies the third also. This point is called the radical centre and by definition the length of the tangents from it to the three circles are equal.

40 If the two circles interesect at P and Q then PQ will be their radical axis. Now of all those circles which pass through the points P and Q, any pair have its radical axis the line PQ. Hence the general equation of the circles are the circles which pass through the intersection of the given circles and their equation will be $S_1+\lambda S_2=0$

or $(x^2+y^2-4)+\lambda\,(x^2+y^2+2x+4y-6)=0$

such a system is called coaxial system of circles as their common chord or radical axis is the line PQ

2nd Form

The common chord PQ is given by

$S_1-S=0$ or $P=0$ $\quad S_1+\lambda P=0$

also represents a family of circles each of which passes through the point of intersection of

$S_1=0$ and $S_2=0$ or $S_1=0$ and $P=0$

$P=0$ is $S_1-S_2=0$ or $-2x-4y+2=0$ or $x+2y-1=0$

Requird family is given by $S_1+\lambda P=0$

$(x^2+y^2-4)+\lambda\,(x+2y-1)=0$

41 Radical axis PQ is $x-a/2=0$ which cuts the circle $x^2+y^2-a^2$ in P and Q. Now any circle which passes through the points P and Q will have radical axis the line PQ with respect to $x^2+y^2-a^2=0$. Hence its equation is the equation of the circle through the points of intersection of the circle

$x^2+y^2-a^2=0$

and the line $x=a/2=0$ and is given by $S+\lambda P=0$

or $(x^2+y^2-a^2)+\lambda\,(x-a/2)=0$

As it passes through the point $(2a, 0)$,

$(4a^2-a^2)+\lambda\,(2a-a/2)=0$ or $\lambda=-2a$

Hence the required circle is $(x^2+y^2-a^2)-2a\,(x-a/2)=0$

or $x^2+y^2-2ax=0$

42 $S_1-S_2=\lambda$. It is a straight line parallel to radical axis

$S_1-S_2=0$

$14p-35=-2q-5$ or $7p+q=15$

and $2q+5=-7q+7p$ or $7p-9q=5$

Solving the above two we get $p=2$, $q=1$

Hence the point of contact is $(2, 1)$

3 Proceed as in Q 2

Ans $4x+3y+19=0$ $4x+3y-31=0$

4 Any line through the point $(8, 1)$ is

$$y-1=m(x-8) \quad \text{or} \quad mx-y+(1-8m)=0 \qquad (1)$$

If it is a tangent then perpendicular from centre $(1, 2)$ is equal to radius $\sqrt{(1+4+20)}=5$

$$\frac{m-2+(1-8m)}{\sqrt{(m^2+1)}}=5 \quad \text{or} \quad (-7m-1)^2=25(m^2+1)$$

or $49m^2+14m+1=25m^2+25$

or $24m^2+14m-24=0$

or $12m^2+7m-12=0$

or $12m^2+16m-9m-12=0$

$(3m+4)(4m-3)=0$ $\quad m=-4/3,\ 3/4$

Putting the values of m in (1) the required tangents are

$4x+3y-35=0$ and $3x-4y-20=0$

(b) Proceed as above

$m=2\ -\frac{1}{2}$ $\quad 2x-y+1=0$ $\quad x+2y-2=0$

5 Proceed as in Q 4

$x+2y=0,\ 2x+y=0$

6 The given circle is

$$x^2+y^2+4x-6y-12=0$$

Tangent at $(-5, 7)$ is

$x(-5)+y(7)+2(x-5)-3(y+7)-12=0$

or $-3x+4y+43=0$

or $3x-4y-43=0$

7 *Shift the origin to the point (a, b) by writing $x+a$ for x and $y+b$ for y so that the equation of the circle becomes*

$$x^2+y^2=r^2$$

Tangent at $\theta=\alpha$ i e $(r\cos\alpha\ \ r\sin\alpha)$ is

$$x(r\cos\alpha)+y(r\sin\alpha)=r^2$$

or $x\cos\alpha+y\sin\alpha=r$

Shift the origin back to $(a\ \ b)$ by writing $x-a$ for x and $y-b$ for y

$(x-a)\cos\alpha+(y-b)\sin\alpha=r$ is the required tangent

Circle coaxial with the given circles is $S_1+\lambda S_2=0$

$$(x^2+y^2-x-3y-\tfrac{3}{2})+\lambda(x^2+y^2+4x+2)+1)=0,$$

or $$(x^2+y^2)(1+\lambda)-(1-4\lambda)x+(3+2\lambda)y-\tfrac{3}{2}+\lambda=0$$

or $$x^2+y^2-\frac{1-4\lambda}{1+\lambda}x-\frac{3+2\lambda}{1+\lambda}y+\frac{\lambda-\frac{3}{2}}{1+\lambda}=0 \quad (2)$$

Its centre $\left(\frac{1}{2}\frac{1-4\lambda}{1+\lambda}, -\frac{1}{2}\frac{3+2\lambda}{1+\lambda}\right)$ i e $(-g, -f)$ lies on (1)

$$\frac{5(1-4\lambda)}{1+\lambda}+\frac{3+2\lambda}{1+\lambda}+5=0 \qquad \lambda=1$$

or $5-20\lambda+3+2\lambda+5+5\lambda=0$ or $13-13\lambda=0$

Putting in (2) the required circle is

$$x^2+y^2+\tfrac{3}{2}x+\tfrac{5}{2}y-\tfrac{1}{4}=0$$

or $$4x^2+4y^2+6x+10y-1=0$$

47 The second circle can be reduced to standard form on dividing by 2 so that its equation is

$$S_2=x^2+y^2+\tfrac{3}{2}x+4y+c=0$$

The radical axis of circle is $S_1=0$ and $S_2=0$ is given by

$$S_1-S_2=0 \text{ or } 2x(g-3/4)+2y(f-2)=0$$

or $$x(g-3/4)+y(f-2)=0$$

It touches the circle $S_3=0$ whose centre is $(-1, -1)$ and radius 1. The condition of tangency i e $p=r$ gives

$$\frac{-1(g-3/4)-(f-2)1}{\sqrt{[(g-3/4)^2+(f-2)^2]}}=1$$

Squaring we get

$$(g-3/4)^2+(f-2)^2+2(g-3/4)(f-2)=(g-3/4)^2+(f-2)^2$$

$$2(g-3/4)(f-2)=0,$$

Hence either $g-3/4=0$ or $f-2=0$

Either $g=3/4$ or $f=2$

48 Let the given circles be denoted by $S_1=0$ and $S_2=0$ and the points $PQRS$ lie on the circle say $S_3=0$. PQ intersects both S_1 and S_3 and RS intersects both S_2 and S_3

PQ is radical axis of S_1 and S_3 and RS is radical axis of S_2 and S_3

$Ax+By+C=0$ Radical axis of S_1 and S_3

$A'x+B'y+C'=0$ Radical axis of S_2 and S_3

Also radical axis of S_1 and S_2 is given by $S_1-S_2=0$

or $(a-a')x+(b-b')y+(c-c')=0$

or $\qquad 9(k+3)^2+(k+3)^2=r^2$

or $\qquad r^2=10\,(k+3)^2$

Hence the required equation of the circle is

$$(x-h)^2+(y-k)^2=r^2$$

or $\qquad (x+3k+8)^2+(y-k)^2=10\,(k+3)^2 \qquad (1)$

For different values of k above equation represents a family of circles

In case $r=2\sqrt{(10)}$ then $r^2=40=10\,(k+3)^2 \qquad k+3=\pm 2$

$k=-5,\,-1$ and hence $h=-3k-8=7,\,-5$

Hence the centres are $(7,\,-5)$ and $(-5,\,-1)$ and $r^2=40$

The circles are

$$(x-7)^2+(y+5)^2=40 \text{ and } (x+5)^2+(y+1)^2=40$$

12 Let the parametric angles be α and $\alpha+\pi/3$ and the equation of the circle is $\qquad x^2+y^2=a^2$

Tangent at α i e $(a\cos\alpha,\, a\sin\alpha)$ is

$$x\cos\alpha+y\sin\alpha=a \qquad (1)$$

Tangent at $\alpha+\pi/3$ is

$x\cos(\alpha+\pi/3)+y\sin(\alpha+\pi/3)=a$

or $\quad x(\cos\alpha\cos\pi/3-\sin\alpha\sin\pi/3)$

$$+y(\sin\alpha\cos\pi/3+\cos\alpha\sin\pi/3)=a$$

or $\quad \frac{1}{2}(x\cos\alpha+y\sin\alpha)-\frac{\sqrt{3}}{2}(x\sin\alpha-y\cos\alpha)=a$

or $\quad \frac{1}{2}a-\frac{\sqrt{3}}{2}(x\sin\alpha-y\cos\alpha)=a$ by (1)

or $\qquad x\sin\alpha-y\cos\alpha=\frac{-a}{\sqrt{3}} \qquad (2)$

In order to find the locus of the point of intersection we have to *eliminate the parameter α for which we square and add* (1) and (2),

$$x^2(\cos^2\alpha+\sin^2\alpha)+y^2(\sin^2\alpha+\cos^2\alpha)=a^2+\frac{a^2}{3}=\frac{4a^2}{3}$$

or $\qquad 3x^2+3y^2=4a$

When the parametric angles differ by $\pi/2$, then the locus can easily be found to be $\quad x^2+y^2=2a^2$

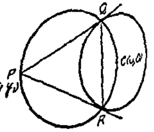
13 Any tangent to the circle $x^2+y^2=a^2$ is

$$y=mx\pm a\sqrt{(1+m^2)}$$

Its intercepts on the axes are

$$OA=h=\pm(a/m)\sqrt{(1+m^2)} \text{ and } k=\pm a\sqrt{(1+m^2)}$$

51 Let P be (x_1, y_1) so that QR is chord of contact whose equation is $xx_1+yy_1-a^2=0$. The circle PQR thus passes through the intersection of given circle and chord QR and hence by $S-\lambda P=0$ its equation is



(I)

$(x^2+y^2-a^2)-\lambda(xx_1+yy_1-a^2)=0$

As it passes through $P(x_1, y_1)$

$(x_1^2+y_1^2-a^2)-\lambda(x_1^2+y_1^2-a^2)=0$ $\quad \lambda=1$

Putting the value of λ in (1) the required circle is $x^2+y^2-xx_1-yy_1=0$ which clearly passes through the centre $(0, 0)$ of the given circle.

Another form The equation of the circle can also be written as $(x-0)(x-x_1)+(y-0)(y-y_1)=0$ i.e. it is on CP as diameter.

52 The equation of the conic through the four points of intersection of $S_1=0$ and $S_2=0$ is given by $S_1-\lambda S_2=0$

If it be a circle then

coeff of x^2 = coeff of y^2 $\quad a-\lambda a=b-\lambda b'$ $\quad$ (1)

or $\quad a-b=\lambda(a-b)$

Coeff of $xy=0$ $\quad 2h-\lambda 2h=0$ $\quad$ (2)

$h=\lambda h$

Eliminating λ we get $\frac{a-b}{h}=\frac{a-b}{h}$ $\quad$ Proved

53 Apply the condition that $S-\lambda xy=0$ represents a circle where $xy=0$ represents the equation of axes $\quad \lambda=-5$

Ans $2x^2+2y^2+7x-5y+3=0$ centre $(-7/4, 5/4)$

54 (a) For the sake of convenience let the line of centres be chosen as axis of x and distance between them be $2a$ and mid point of the centres be chosen as origin so that the centres are $(a, 0)$ radius r for S and $(-a, 0)$ radius R for S

$S=(x-a)^2+y^2=r^2$ or $x^2+y^2-2ax+a^2-r^2=0$

$S=(x+a)^2+y^2=R^2$ or $x^2+y^2+2ax+a^2-R^2=0$

The two circles are $\frac{S}{r}\pm\frac{S}{R}=0$ or $RS\pm rS=0$

Circle $RS+rS=0$ becomes

$(R+r)(x^2+y^2+a^2)-2ax(R-r)-rR(r+R)=0$

16 Condition of tangency gives $l^2+m^2=1/a^2$ and hence the point (l, m) lies on the circle $x^2+y^2=1/a^2$

17 (*a*) The centres of the circles are

C_1 (1, 3) and $r_1=\sqrt{(1+9-9)}=1$

C_2 (−3, 1) and $r_2=\sqrt{(9+1-1)}=3$

1 : 3

(1, 3)——— ———(−3, 1)

C_1 P C_2

Transverse common tangents are tangents drawn from the point P which divides C_1C_2 internally in the ratio of radii 1 : 3

Direct common tangents are tangents drawn from the point Q which divides C_1C_2 externally in the ratio 1 : 3

Coordinates of P are $\left(\frac{1\,(-3)+3\,1}{1+3},\ \frac{1\,1+3\,3}{1+3}\right)$ *i e* (0, 5/2)

Coordinates of Q are $\left(\frac{1\,(-3)-3\,(1)}{1-3},\ \frac{1\,1-3\,(3)}{1-3}\right)$ *i e* (3, 4)

Transverse tangents are tangents through the point (0, 5/2)

Any line through (0, 5/2) is

$$y-5/2=mx \qquad (1)$$

or

$$mx-y+5/2=0$$

Apply the usual condition of tangency to any of the circle

$$\frac{m\,1-3+5/2}{\sqrt{(m^2+1)}}=1 \quad \text{or} \quad (m-1/2)^2=m^2+1$$

or $-m-3/4=0$ or $0m^2-m-3/4=0$

Hence $m=-3/4$ and ∞ as coeff of m^2 is zero

Therefore from (1) $\frac{y-5/2}{x}=m=\infty$ and $-3/4$

∴ $x=0$ is a tangent and $y-5/2=-3x/4$

or $3x+4y-10=0$ is another tangent

Direct tangents are tangents drawn from the point Q (3, 4)

Now proceeding as for transverse tangents their equations are

$$y=4,\ 4x-3y=0$$

(*b*) The centre of the circle I is the origin A and Ax, Ay are the coordinate axes From the figure it is evident that the coordinates of the centres B and C of circles II and III are $(\sqrt{(55)}, 3)$ and $(\sqrt{(65)}, 4)$ and their radii are 3 and 4 respectively Clearly one of the common tangents is x-axis, *i e*, $y=0$ Produce CB to meet x axis in D Then D will divides CB externally in the ratio of the radii

Hence if $(h, 0)$ are the coordinates of D, then

$= -(g_1-g_2)(g_2-g_3)(g_3-g_1) - c\,0$

$= -PQ\,QR\,RP$

56 Let $A(a, 0)$, $B(0, b)$ be the points on the axes then $\angle AOB = \pi/2$ therefore circle OAB is on AB as diameter whose equation is $(x-a)(x-0)+(y-0)(y-b)=0$

or $x^2+y^2-ax-by=0$

If (h, k) be the centre then $h=a/2$, $k=b/2$

Since $AB=c$ $\quad a^2+b^2=c^2$ or $4h^2+4k^2=c^2$

Locus of centre is $x^2+y^2=c^2/4$

57 Equation of any tangent to the circle $x^2+y^2=a^2$ is $y=mx\pm a\sqrt{(1+m^2)}$ If it passes through the point (h, k) then $(k-mh)^2=a^2(1+m^2)$

or $m^2(h^2-a^2)-2mhk+(k^2-a^2)=0$ (1)

Above is a quadratic in m showing that there will be two tangents passing through the point (h, k) and whose slopes are given by (1)

$$m_1+m_2=\frac{2hk}{h^2-a^2},\quad m_1m_2=\frac{k^2-a^2}{h^2-a^2} \qquad (2)$$

If α be the angle between the tangents then

$$\tan\alpha=\frac{m_1-m_2}{1+m_1m_2}$$

or $(1+m_1m_2)^2\tan^2\alpha=(m_1+m_2)^2-4m_1m_2$

or $\left(1+\frac{k^2-a^2}{h^2-a^2}\right)^2\tan^2\alpha=\frac{4h^2k^2}{(h^2-a^2)^2}-4\frac{(k^2-a^2)}{(h^2-a^2)}$

or $(h^2+k^2-2a^2)^2\tan^2\alpha=4a^2(h^2+k^2-a^2)$

Hence the required locus of the point is

$(x^2+y^2-2a^2)^2\tan^2\alpha=4a^2(x^2+y^2-a^2)$

Particular case If the tangents be at right angles then

$$m_1m_2=-1 \quad\text{or}\quad \frac{k^2-a^2}{h^2-a^2}=-1$$

or $h^2+k^2=2a^2$ Required locus is $x^2+y^2=2a^2$

58 A point P is outside, on or inside a curve $S=0$ according as $S >=< 0$

$(0, 1)$ is on the circle, $(3, 1)$ is outside and $(1, 3)$ inside

59 The first part represents a circle $S=0$ on $P(x_1, y_1)$ and $Q(x_2, y_2)$ as diameter The second part i e determinant is the equation of a straight line $P=0$ which passes through the points P and Q The given equation is now of

or $\qquad 12m+5\sqrt{(1+m^2)}=\pm 3\sqrt{(1+m^2)}$

$\therefore \quad 12m=-8\sqrt{(1+m^2)}$ or $12m=-2\sqrt{(1+m^2)}$

Hence $9m^2=4+4m^2$ or $36m^2=1+m^2$

$5m^2=4$ or $35m^2=1$

$m=\pm 2/\sqrt{5}$ or $m=\pm 1/\sqrt{(35)}$

Putting in (1) the required tangents are

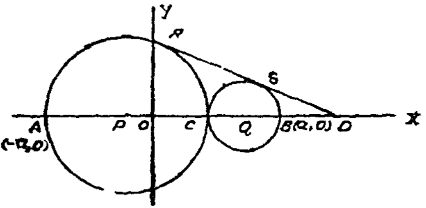
$y=\pm 2x/\sqrt{5}+5\sqrt{(1+4/5)}$ or $\sqrt{5}y=\pm 2x+15$

and $y=\pm x/\sqrt{(35)}+5\sqrt{(1+1/35)}$ or $\sqrt{(35)}\,y=\pm x+30$

(b) We take the coordinates of A and B as $(-a, 0)$ and $(a, 0)$ respectively so that the mid point of AB is the origin and AB the x axis. If $(h, 0)$ are the coordinates of C, then since $AC=3CB$, we have

$$h=\frac{3\cdot a+1\cdot(-a)}{3+1}=\tfrac{1}{2}a$$

On AC and CB as diameters we draw two circles so that they touch each other at $C\,(\tfrac{1}{2}a, 0)$. Their centres are $P\,(-\tfrac{1}{4}a, 0)$ and $Q\,(\tfrac{3}{4}a, 0)$ and radii are $\tfrac{3}{4}a$ and $\tfrac{1}{4}a$ respectively. Let a common tangent RS meet AB produced in D. Let $(\alpha, 0)$ be the coordinates of D. Then since



D divides PQ externally in the ratio of the radii $\tfrac{3}{4}a : \tfrac{1}{4}a$ i.e. $3 : 1$ (See § 2) (j), we have

$$\alpha=\frac{3\cdot(\tfrac{3}{4}a)-1\cdot(-\tfrac{1}{4}a)}{3-1}=\tfrac{5}{4}a$$

$OD=\tfrac{5}{4}a$ and $OB=a$ so that $BD=\tfrac{5}{4}a-a=\tfrac{1}{4}a$

Hence BD is equal to the radius of the smaller circle

circle are at right angles Making the equation of circle homogeneous by the help of line (I) and applying the condition $a+b=0$ of perpendicularity we get (2)

$$c\,(l^2+m^2)+2\,(gl+fm)+2=0 \quad (3)$$

Any line through origin ⊥ to (I) is $mx-ly=0$

or $\frac{x}{l}=\frac{y}{m}=\frac{x^2+y^2}{lx+my}=\frac{x^2+y^2}{1}$ by (I)

$l=\frac{x}{x^2+y^2}$, $m=\frac{y}{x^2+y^2}$ We have solved I and (3)

In order to find the locus of foot of perpendicular we have to eliminate the variables l and m Putting the values of l and m from above in (2) the required locus is

$$c\,\frac{1}{(x^2+y^2)}+2\,\frac{(gx+fy)}{x^2+y^2}+2=0$$

or $$x^2+y^2+gx+fy+\tfrac{1}{2}c=0$$

OBJECTIVE QUESTIONS

(Circle)

1 The locus of the centre of a circle of radius 2 which rolls on the outside of the circle $x^2+y^2+3x-6y-9=0$ is

(i) $x^2+y^2+3x-6y+5=0$
(ii) $x^2+y^2+3x-6y-31=0$
(iii) $x^2+y^2+3x-6y+\frac{29}{4}=0$

2 A square is inscribed in the circle $x^2+y^2-2x+4y+3=0$, Its sides are parallel to the coordinate axes Then one vertex of the square is

(a) $(1+\sqrt{2}, -2)$, (b) $(1-\sqrt{2}, -2)$,
(c) $(1, -2+\sqrt{2})$, (d) None of these

(I I T 80)

3 Two circles $x^2+y^2=6$ and $x^2+y^2-6x+8=0$ are given Then the equation of the circle through their point of intersection and the point (1, 1) is

(a) $x^2+y^2-6x+4=0$, (b) $x^2+y^2-3x+1=0$
(c) $x^2+y^2-4y+2=0$, (d) None of these

(I.I T 80)

4 (a) The equation of the circle through M (5, 4) and touching the x axis at L (2, 0) is

(b) The area of circle centred at (1, 2) and passing through (4, 6) is

Hence the locus of the mid point (h, k) is

$$x^2+y^2-x-3y=0$$

21 $S\equiv x^2+y^2-12=0$, $S'=x^2+y^2-5x+3y-2=0$

Equation of the common chord is $S-S'=0$

or $5x-3y-10=0$ or $5x-3y=10$ (1)

If the tangents to the circle $x^2+y^2=12$ at the extermities of the chord (1) intersect at the point (h, k) then this chord is the chord of contact of the point (h, k) w r t the circle $x^2+y^2=12$ whose equation is

$$hx+ky=12 \quad (2)$$

Comparing (1) and (2) we get

$$\frac{h}{5}=\frac{k}{-3}=\frac{12}{10} \qquad h=6,\ k=-\frac{18}{5}$$

Hence the required point is $(6, -18/5)$.

22 Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

Since it passes through the point $(2, 5)$

$$4g+10f+c=29 \quad (1)$$

Also it cuts the two given circles orthogonally and hence applying the condition $2g_1g_2+2f_1f_2=c_1+c_2$ we get

$$2g\left(-\frac{9}{2}\right)+2f\,(0)=c+14 \quad (2)$$

$$2g\left(\frac{15}{2}\right)+2f\,(0)=c+14 \quad (3)$$

Subtracting (2) and (3) we get $2g\,(12)=0$ $\quad g=0$

Hence from (2), $c+14=0$ or $c=-14$

Therefore from (1) on putting for g and c we get

$$10f=-15 \text{ or } f=-3/2$$

The required circle is $x^2+y^2-3y-14=0$

23 (*a*) Here $c=0$ as the circle passes through $(0, 0)$ and applying the condition of orthogonality as in Q 22 we get

$$f=-3/2,\ g=3$$

Hence the circle is $x^2+y^2+6x-3y=0$

(*b*) Apply the condition of othogonality with each of the circles and subtract Replace $-g$ by x and $-f$ by y

24 Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0$$

$$\Delta PAB = \tfrac{1}{2} PM \cdot AB = \tfrac{1}{2} \frac{h^2+k^2-a^2}{\sqrt{(h^2+k^2)}} \cdot 2a\sqrt{\left(\frac{h^2+k^2-a^2}{h^2+k^2}\right)}$$

$$= a\,\frac{(h^2+k^2-a^2)^{3/2}}{h^2+k^2} \qquad (2)$$

(*b*) Do yourself

Suppose the coordinates of the pole are (x_1, y_1) and as it lies on

$$x^2+y^2=9c^2 \qquad x_1^2+y_1^2=9c^2 \qquad (1)$$

The polar of this point w r t the circle $x^2+y^2=c^2$ is

$$xx_1+yy_1=c^2$$

If this polar touches the circle $x^2+y^2=c^2/9$ then perpendicular from centre (0, 0) should be equal to radius $c/3$

$$\frac{0+0-c^2}{\sqrt{(x_1^2+y_1^2)}}=c/3 \quad \text{or} \quad 9c^2=x_1^2+y_1^2$$

Above is true by the help of (1)

Let the equation of the circle be $x^2+y^2=a^2$ and the coordinates of any two points are $P\ (x_1, y_1)$ and $Q\ (x_2, y_2)$ The centre of the circle is (0, 0)

$$\frac{OP}{OQ}=\frac{\sqrt{(x_1^2+y_1^2)}}{\sqrt{(x_2^2+y_2^2)}} \qquad (1)$$

Polar of $P(x_1, y_1)$ is $xx_1+yy_1-a^2=0$

Its distance from $Q\ (x_2, y_2)$ is

$$\frac{x_1x_2+y_1y_2-a^2}{\sqrt{(x_1^2+y_1^2)}} \qquad (2)$$

Polar of $Q\ (x_2, y_2)$ is $xx_2+yy_2-a^2=0$

Its distance from $P\ (x_1, y_1)$ is

$$\frac{x_1x_2+y_1y_2-a^2}{\sqrt{(x_2^2+y_2^2)}} \qquad (3)$$

Ratio of these distances

$$=\frac{\sqrt{(x_2^2+y_2^2)}}{\sqrt{(x_1^2+y_1^2)}}=\frac{OQ}{OP} \quad \text{by (1)}$$

Hence proved

3 Let the circle be $(x-a)^2+(y-b)^2=r^2$

Condition of tangency gives

$$\frac{(al+bm+1)}{\sqrt{(l^2+m^2)}}=r$$

or $(al+bm)^2+1+2\,(al+bm)=r^2\,(l^2+m^2)$

or $l^2\,(a^2-r^2)+m^2\,(b^2-r^2)+2ablm+2al+2bm+1=0 \qquad (1)$

(a) the area of $\triangle ABC$ is maximum when it is isosceles
(b) the area of $\triangle ABC$ is minimum when it is isosceles
(c) the perimeter of $\triangle ABC$ is maximum when it is isosceles
(d) None of these (IIT 83)

13 A and B are points in the plane such that

$$PA/PB=k \text{ (constant)}$$

for all P on a circle, then the value of k cannot be equal to (IIT 83)

14 The centre of a circle passing through the point (0, 1) and touching the curve $y=x^2$ at (2, 4) is

(a) $\left(\frac{-16}{5}, \frac{27}{10}\right)$, (b) $\left(\frac{-16}{7}, \frac{5}{10}\right)$,

(c) $\left(\frac{-16}{5}, \frac{53}{10}\right)$, (d) None of these (IIT 83)

15 The locus of the mid points of a chord of the circle $x^2+y^2=4$ which subtends a right angle at the origin is
(a) $x+y=2$, (b) $x^2+y^2=1$,
(c) $x^2+y^2=2$, (d) $x+y=1$ (IIT 84)

16 The lines $3x-4y+4=0$ and $6x-8y-7=0$ are tangents to the same circle The radius of this circle is (IIT 84)

17 The pole of the straight line $9x+y-28=0$ with respect to the circle $2x^2+2y^2-3x+5y-7=0$ is
(a) (3, 1), (b) (1, 3) (c) (3, −1), (d) (−3, 1) (MNR 84)

18 The equation of the circle passing though (4, 5) having the centre at (2, 2) is
(a) $x^2+y^2+4x+4y-5=0$ (b) $x^2+y^2-4x-4y-5=0$
(c) $x^2+y^2-4x=13$ (d) $x^2+y^2-4x-4y+5=0$ (MNR 85)

19 The equation of tangents drawn from the origin to the circle $x^2+y^2-2rx-2hy+h^2=0$ are
(a) $x=0$ (b) $y=0$, (c) $(h^2-r^2)x-2rhy=0$
(d) $(h^2-r^2)x+2rhy=0$ (IIT 88)

Solution

1 Ans (a)
Let (h, k) be the centre of the circle which rolls on the outside of the given circle The centre of the given circle is $\left(\frac{-3}{2}, \frac{3}{2}\right)$ and its radius $=\sqrt{\left[\left(\frac{9}{4}+9+9\right)\right]}=\frac{9}{2}$ Clearly ...

Its centre $(-g, -f)$ lies on

$$2x-2y+9=0$$

$$-2g+2f+9=0 \quad (1)$$

It cuts orthogonally the circle

$$x^2+y^2-4=0$$

$$2g_1g_2+2f_1f_2=c_1+c_2$$

or $\quad +0=c-4 \qquad c=4$

and from (1) $\quad 2g=9+2f$

Putting the values of c and g we get the equation of the circle as $\quad x^2+y^2+(9+2f)x+2fy+4=0$

or $\quad (x^2+y^2+9x+4)+2f(x+y)=0$

Above is of the form $S+\lambda P=0$

Above represents a family of circles which passes through the points of intersection of $S=0$ and $P=0$

Solving $x^2+y^2+9x+4=0$ and $x+y=0$ we get the fixed points as $(-1/2, 1/2)$ and $(-4, 4)$

31 Both the circles pass through the origin and hence if they touch they must have the same tangent at the point (0, 0)

$$x\cdot 0+y\cdot 0+g(x+0)+f(y+0)=0$$

or $\quad gx+fy+0 \quad (1)$

and similarly $\quad g'x+f'y=0 \quad (2)$

Comparing (1) and (2) we get

$$g/g'=f/f' \quad \text{or} \quad f/g=f'/g' \qquad \text{(c) is correct}$$

32 $A(1, 2)$ is centre and $\sqrt{(1-4+20)}=5$ is the radius AB or AD which are perpendicular to tangents at B and D respectively. Tangent at $B(1, 7)$ is

$$x\cdot 1+y\cdot 7-1(x+1)-2(y+7)-20=0 \quad \text{or} \quad y=7$$

Tangent at $D(4, -2)$ is $\quad 3x-4y-20=0$

They meet at C which is $(16, 7)$

$t=CB=CD=$ length of tangent from C to given circle

$$=[16^2+7^2-2\cdot 16-4\cdot 7-20]^{1/2}=\sqrt{(225)}=15$$

Note $\quad CB$ or CD can also be found by distance formula

Quadrilateral $ABCD=\triangle ABC+\triangle ADC$

$\frac{1}{2}rt+\frac{1}{2}rt=rt=5\cdot 15=75$ sq unit

Both the triangles are right angled and area is

$\frac{1}{2}$ base $\times$ height $=\frac{1}{2}rt$

33 Let the point be (h, k) then by the given condition

$$t^2=h^2+k^2-1=h^2+k^2+8h+15=h^2+k^2+10k+24$$

From Ist and 2nd, we get $h=-2$

$C_1C_2 < r_1+r_2$ and $C_1C_2 > r_1-r_2$
$C_1=(0, 0)$, $C_2=(5, 0)$, $r_1=r$, $r_2=3$
$5 < r+3$ and $5 > r-3$
or $2 < r$ and $8 > r$ i e $r < 8$
Hence the required condition is $2 < r < 8$
i e (a) is correct answer

10 The line which passes through centre $(3, -1)$
(i) is correct

11 $2\sqrt{(f^2-c)}=\sqrt{(4f^2-4c)}=\sqrt{\left(4\times\frac{49}{4}-4\times 12\right)}=1$
(i) is correct

12 Ans (i)

13 $k\neq 1$

14 Ans (c)

15 Ans (c)

16 Ans 3/4

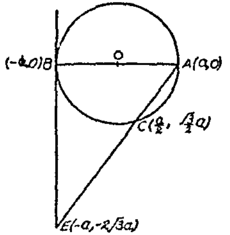
17 Ans (c)

18 Clearly (ii) or (iii) have centre (2, 2) but only (ii) is satisfied by the pt (4, 5) and hence it is correct

18 The centre is (r, h) and radius is r and hence the circle touches y axis i.e $x=0$ is a tangent through the origin Now any line through origin is $y-mx=0$ Apply the condition of tangency

$$\frac{h-mr}{\sqrt{(1+m^2)}}=r \quad \text{or} \quad h^2+m^2r^2-2mhr=r^2+m^2r^2$$

or $0m^2+2mhr+(r^2-h^2)=0$
$m=\infty$, $m=(h^2-r^2)/2hr$ Put $m=y/x$
Tangents are $x=0$ (Already found otherwise) and
$(h^2-r^2)\,x-2rhy=0$
Hence (a) and (b) are correct answers



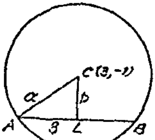
$AE^2=(a+a)^2+(2\sqrt{3}a)^2=4a^2+12a^2=16a^2$

$AB=4a=2AB$

37 C is $(3,-1)$, AB is

$2x-5y+18=0$

$$p=CL=\frac{6+5+18}{\sqrt{(29)}}=\sqrt{(29)}$$



Also $AL=\frac{1}{2}AB=3$

$a^2=AL^2+CL^2=9+29=38$

Circle is

$(x-3)^2+(y-1)^2=38$

38 Let in usual notations the three circles be $S_1=0, S_2=0, S_3=0$ Their radical axes taken in pairs are given by

$S_1-S_2=0,\ S_2-S_3=0$ and $S_3-S_1=0$

i e $\quad 2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0 \quad (1)$

$2x(g_2-g_3)+2y(f_2-f_3)+c_2-c_3=0 \quad (2)$

$2x(g_3-g_1)+2y(f_3-f_1)+c_3-c_1=0 \quad (3)$

Adding (1), (2) and (3) we get $\quad 0x+0y+0=0$

Hence the three radical axis are concurrent by Rule 21, 2nd Method Page 266

The point of concurrency is called the radical centre of the three circles and the length of the tangents drawn from this point to the three circles are equal in length

39 Here we have to find the radical centre of the three circles First reduce them to standard form in which coefficients of x^2 and y^2 be each unity Subtracting in pairs the three radical axes are

43 Let $S_1=0$ and $S_2=0$ both cut $S_3=0$ orthogonally

$$2g_1g_3+2f_1f_3=c_1+c_3$$
$$2g_2g_3+2f_2f_3=c_2+c_3$$

Subtracting we get

$$2g_3(g_1-g_2)+2f_3(f_1-f_2)=c_1-c_2 \quad (1)$$

Again the radical axis of $S_1=0$ and $S_2=0$ is given by $S_1-S_2=0$

$$2x(g_1-g_2)+2y(f_1-f_2)+c_1-c_2=0$$

It will pass through the centre $(-g_3, -f_3)$ of $S_3=0$ if

$$-2g_3(g_1-g_2)-2f_3(f_1-f_2)=-(c_1-c_2)$$

or $\quad 2g_3(g_1-g_2)+2f_3(f_1-f_2)=c_1-c_2$

Above is true by (1)

44 A system of circles is said to be co axial if every pair of the system has the same radical axis Circles passing through two fixed points form a coaxial system The equations $S+\lambda P=0$, $S_1+\lambda S_2=0$ represent a family of coaxial circles

Equation of coaxial circles in simplest form

Let the common radical axis be chosen along y axis and the line of centres which will be perpendicular to radical axis be along x axis Hence the equation of any circle will be

$$x^2+y^2+2gx+c=0 \quad (1)$$

Let any other circle of the system be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad (2)$$

The radical axis of (1) and (2) is

$$2x(g-g_1)-2f_1y+(c-c_1)=0 \quad (3)$$

But we are given that radical axis is y axis *i e* $x=0$ (4)

Comparing (3) and (4), we get

$f_1=0$ and $c-c_1=0 \quad c_1=c$

Hence any other circle of the system will have its equation of the form $\quad x^2+y^2+2g_1x+c=0$

Thus the system of equation $\quad x^2+y^2+2g_r x+c=0$

where c is constant and g_r a parameter represents a family of coaxial circles whose common radical axis is y axis *i e* $x=0$

45 Refer Q 40

46 The two given circles in standard form are

$$S_1=0 \text{ or } x^2+y^2-x+3y-\tfrac{3}{2}=0$$
$$S_2=0 \text{ or } x^2+y^2+4x+2y+1=0$$

Radical axis is given by $\quad S_1-S_2=0$

$$-5x+y-\tfrac{5}{2}=0 \text{ or } 10x-2y+5=0 \quad (1)$$

8 $\frac{dy}{dx}\frac{dx}{dy}=1$ or $\frac{dy}{dx}=\frac{1}{dx/dy}$

9 **Differentiation of one function w r t another function**

Let $y=f(x)$, $z=\phi(x)$ and we have to differentiate $f(x)$ w r t $\phi(x)$ i e y w r t z so that we have to find the value of

$$\frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=\frac{dy}{dx}\frac{dx}{dz}$$

10 **Logarithmic Differentiation**

If $y=[f_1(x)]^{f_2(x)}$ or $y=f_1(x)f_2(x)f_3(x)$

or
$$y=\frac{f_1(x)f_2(x)}{\phi_1(x)\phi_2(x)}$$

Then it will be convenient to take log of both sides before performing differentiation

81 **Parametric Equations**

If $x=f(t)$, $y=\phi(t)$, then $\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}$

12 **Implicit function** $f(x,y)=c$

Differentiate each term w r t x and note that

$$\frac{d}{dx}(\phi(y))=\frac{d}{dy}(\phi(y))\frac{dy}{dx}$$

13 **Trigonometrical transformations and expansions**

Sometimes by a trigonometrical transformation the derivatives of various functions are calculated in a much simpler way than otherwise The following trigonometrical expansions and formula be remembered!

(i) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-$

$\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+$

$\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+$

(ii) $\tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}$

$\tan^{-1}a-\tan^{-1}b=\tan^{-1}\frac{a-b}{1+ab}$

$\sin^{-1}x+\cos^{-1}x=\tan^{-1}x+\cot^{-1}x=\sec^{-1}x+\operatorname{cosec}^{-1}x=\pi/2$

$\sin^{-1}x=\operatorname{cosec}^{-1}(1/x)$ $\tan^{-1}x=\cot^{-1}(1/x)$

$\cos^{-1}x = \sec^{-1}\frac{1}{x}$)

Again from question 38 we know that the radical axis of three circles taken in pairs are concurrent Hence by rule 21 P 265, we have

$$\begin{vmatrix} a-a & b-b' & c-c \\ A & B & C \\ A' & B & C \end{vmatrix}=0$$

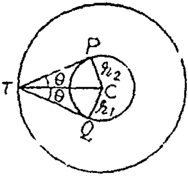
49 Centre of the first circle is $(-g, -f)$ and its radius

$CT=r_1=\sqrt{(g^2+f^2-c)}$

The centre of the second circle is also $(-g, -f)$ but its radius is

$CP=r_2=\sqrt{\{g^2+f^2-c \sin^2 \alpha}$
$-(g^2+f^2)\cos^2 \alpha\}$

or $r_2=\sqrt{(g^2+f^2-c)} \sin \alpha$
$=r_1 \sin \alpha$ (1)



Since $\sin \alpha$ is less than 1 therefore $r_2 < r_1$ and as such the second circle is inner circle concentric with outer circle Now if θ be the angle PTC, then from right angled triangle

$$\sin \theta=\frac{r_2}{r_1}=\frac{r_1 \sin \alpha}{r_1}=\sin \alpha$$

$\theta=\alpha$ $\angle PTQ=2\theta=2\alpha$ **Proved**

50 If λ_1 λ λ_3 be the three values of λ then $\lambda_2^2=\lambda_1\lambda_3$ (1)

If (h, k), be any point on the circle $x^2+y^2=c^2$ then
$h+k^2=c^2$ (2)

Now if t_1 t_2 t_3 be the lengths of tangents from (h, k) to the circle $x^2+y^2-2\lambda x-c=0$ then $t_1^2=h+k^2-2\lambda_1 h-c^2=-2\lambda_1 h$ by (2)

Similarly $t_2^2=-2\lambda_2 h$, $t_3^2=-2\lambda_3 h$ Now t_1 t_2, t_3 will be in G P if t_1^2, t_2^2 and t_3^2 are also in G P

or $(-2\lambda_2 h)^2=(-2\lambda_1 h)(-2\lambda_3 h)$

or $\lambda_2^2=\lambda_1\lambda_3$ which is true by (1)

11 $y=\sin^{-1} x$, $\frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}$

12 $y=\cos^{-1} x$, $\frac{dy}{dx}=\frac{-1}{\sqrt{(1-x^2)}}$

13 $y=\tan^{-1} x$, $\frac{dy}{dx}=\frac{1}{(1+x^2)}$

14 $y=\cot^{-1} x$, $\frac{dy}{dx}=-\frac{1}{(1+x^2)}$

Problem Set (A)

1 Differentiate from first principles from 1 to 35

1 $\sin x$, (M N R 81) 2 $\cos x$, 3 $\tan x$ (I I T 75)
4 $\cot x$, 5 $\sec x$, 6 $\operatorname{cosec} x$,
7 x^n, 8 e^x, 9 $\log x$
10 $(ax+b)^n$, 11 $\sqrt{(x)}$, 12 $\sqrt{\frac{1}{x+a}}$ (Roorkee 74)
13 $\sin^{-1} x$, 14 $\cos^{-1} x$, 15 (a) $\tan^{-1} x$, (b) $\cot^{-1} x$,
16 $\sec^{-1} x$, 17 $\operatorname{cosec}^{-1} x$, 18 $\sin^2 x$,
19 $\sin x^2$, 20 $\sqrt{(\sin x)}$, 21 $\sqrt{(\tan x)}$
(I I T 70, Roorkee 76)
22 $\frac{\sin x}{x}$, 23 $\log \cos x$, 24 $\sin \log x$,
25 $\log \sin^{-1} x$, 26 $\cos^2 (\log x)$, 27 $\tan (1/x)$,
28 $e^{\tan x}$ 29 $e^{\sin x}$, 30 $x^2 \log x$ (b) xe^x, (I I T 71)
31 $\sin \sqrt{(x)}$, 32 $\tan^2 ax$ (I I T 78)
33 $\cos (ax^2+bx+c)$ 34 $\sin (x^2+1)$ (I I T 77)
35 (a) $x^2 \cos x$,
35 (b) $\underset{h\to 0}{\text{Lt}} \frac{(a+h)^2 \sin(a+h)-a^2 \sin a}{h}=2a \sin a+a^2 \cos a$ (M N R 87)

36 $f(x)=\frac{x-1}{2x^2-7x+5}$ when $x\neq 1$ $f(x)=-\frac{1}{3}$ when $x=1$, find $f'(1)$ (I I T 79)

37 If $f(x)=x \tan^{-1} x$, find $f'(1)$

38 If $y=\frac{1}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}$, find $\frac{dy}{dx}$

39 If $y=\sec (x^o+30^o)$ find dy/dx

40 If $y=\cos (2 \sin^2 x^2)$ find dy/dx

41 If $y=\frac{5x}{^3\sqrt{(1-x)^2}}+\cos^2 (2x+1)$ find dy/dx (I I T 80)

or $x^2+y^2-2a\,\dfrac{R-r}{R+r}\,x+(a^2-rR)=0$ (1)

Replacing r by $-r$ the circle $RS-rS$ becomes

$x^2+y^2-2a\,\dfrac{R+r}{R-r}\,x+(a^2+rR)=0$ (2)

The circle (1) and (2) will cut orthogonally if

$2g_1g_2+2f_1f_2=c_1+c_2$

or $2\left\{-a\,\dfrac{R-r}{R+r}\right\}\left\{-a\,\dfrac{R+r}{R-r}\right\}+0=(a^2-rR)+(a^2+rR)$

or $2a^2=2a^2$ which is true

Hence the circles $\dfrac{S}{r}\pm\dfrac{S}{R}$ cut each other orthogonally

(*b*) Let the circle be $x^2+y^2+2gx+2fy+c=0$

Since it cuts $x^2+y^2-k^2=0$ orthogonally therefore

$2g(0)+2f(0)=c-k^2=0 \qquad c=k^2$

Again the circle C_2 passes through (a, b)

$a^2+b^2+2ga+2fb+k^2=0 \qquad c=k^2$

Locus of centre $(-g, -f)$ is $a^2+b^2+2a\,(-x)$

$+2b\,(-y)+k^2=0$

or $2ax+2by-(a^2+b^2+k^2)=0$

55 (*a*) By special choice of axes the equation of coaxial system of circles can be put in the form

$x^2+y^2+2g_r\,x+c=0$ where $r=1, 2, 3$

The co-ordinates of the centres of the three circle are

$P\,(-g_1, 0)$, $Q\,(-g_2, 0)$, $R\,(-g_3\ 0)$

$QR=(-g_3)-(-g_2)=g_2-g_3$

$RP=g_3-g_1$ and $PQ=g_1-g_2$

If the fixed point from where the tangents be drawn be (h, k) then

$t_1^2=h^2+k^2+2g_1h+c$ etc

$QRt_1^2+RPt_2^2+PQt_3^2$

$=(g_2-g_3)\,\{h^2+k^2+c+2g_1h\}+\quad+\quad=0$

or $(h^2+k^2+c)\,\Sigma\,(g_2-g_3)+2h\Sigma g_1(g_2-g_3)=0$

$\Sigma\,(g_2-g_3)=0$ and $\Sigma g_1\,(g_2-g_3)=0$

(*b*) From part (*a*) $r_1^2=g_1^2-c$ etc

$r_1^2\,QR+r_2^2\,RP+r_3^2\,PQ$

$=(g_1^2-c)\,(g_2-g_3)+(g_2^2-c)\,(g_3-g_1)+(g_3^2-c)\,(g_1-g_2)$

$=\Sigma g_1^2\,(g_2-g_3)-c\Sigma(g_2-g_3)$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sin \delta x}{\delta x}\ \frac{1}{\cos x \cos (x+\delta x)}=1\ \sec^2 x=\sec^2 x$$

4 Do yourself

5 $y=\sec x$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sec (x+\delta x)-\sec x}{\delta x}\ \text{(change to cos)}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\cos x-\cos (x+\delta x)}{\delta x \cos (x+\delta x)\cos x}$$

$$=\sin x\ \frac{1}{\cos x \cos x}=\sec x \tan x$$

The limit $\sin x$ of $\frac{\cos x-\cos (x+\delta x)}{\delta x}$ is to be calculated

6 Do yourself

7 $y=x^n$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{(x+\delta x)^n-x^n}{\delta x}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x^n\left[\left(1+\frac{\delta x}{x}\right)^n-1\right]}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x^n\left[1+n\ \frac{\delta x}{x}+\frac{n(n-1)}{1\ 2}\left(\frac{\delta x}{x}\right)^2+\ \ \right]-x^n}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\delta x}{\delta x}\left[nx^{n-1}+\frac{n(n-1)}{1\ 2}x^{n-2}\,\delta x+\text{higher powers of }\delta x\right]$$

$$=nx^{n-1}$$

8 Do yourself

9 Do yourself

10 Do yourself

11 $y=\sqrt{x}$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sqrt{(x+\delta x)}-\sqrt{x}}{\delta x}$$

Multiply above and below by $\sqrt{(x+\delta x)}+\sqrt{x}$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{(x+\delta x)-x}{\delta x\,[\sqrt{(x+\delta x)}+\sqrt{x}]}=\frac{1}{\sqrt{x}+\sqrt{x}}=\frac{1}{2\sqrt{x}}$$

12 $y=\frac{1}{\sqrt{(x+a)}}$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{1}{\delta x}\left[\frac{1}{\sqrt{(x+\delta x+a)}}-\frac{1}{\sqrt{(x+a)}}\right]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{1}{\delta x}\ \frac{\sqrt{(x+a)}-\sqrt{(x+a+\delta x)}}{\sqrt{(x+a)}\sqrt{(x+a+\delta x)}}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{1}{\delta x}\ \frac{(x+a)-(x+a+\delta x)}{\sqrt{(x+a)}\sqrt{(x+a+\delta x)}\,[\sqrt{(x+a)}+\sqrt{(x+a+\delta x)}]}$$

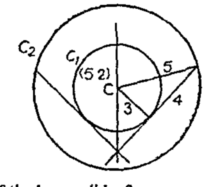
(as in Q

the form $S+P=0$ i e $S+\lambda P=0$ $(\lambda=1)$ represents a circle passing through the intersection of $S=0$ and $P=0$ i e the points P and Q. In case PQ is a diameter then the centre $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ will lie on the line PQ and hence the determinant becomes

$$\begin{vmatrix} \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} \text{ zero which is}$$

because R_2+R_3 becomes identical with R_1. The equation $S+P=0$ in this case reduces to $S=0$ which therefore represents the circle described on PQ as diameter

60 If C be the centre of the circle C_1 then it lies on the bisectors of the given lines which are



$$\frac{5x+12y-10}{13}=\pm\frac{5x-12y-40}{13}$$

These give $x=5$ and $y=-5/4$

Since centre lies in Ist quadrant therefore $y=-5/4$ is ruled out

Let the centre be $(5, k)$ then its perpendicular distance from each of the lines will be 3

$$\frac{25+12k-10}{13}=3 \text{ and } \frac{25-12k-40}{13}=3$$

$$k=2 \text{ or } k=-\frac{54}{12}=-\frac{9}{2}$$

The value $-9/2$ of k is ruled out as centre C lies in Ist quadrant C is $(5, 2)$

Now the circle C_2 cuts off intercepts 8 from these lines and if its radius be r then

$$r^2=3^2+4^2=25 \text{ or } r=5$$

Circle C_2 is $(x-5)^2+(y-2)^2=5^2$

or $x^2+y^2-10x-4y+4=0$

61 Any chord of the circle be $lx+my=1$ (1)

It subtends a right angle at the origin. In other words the lines joining origin to the points of intersection of chord and

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{2\cos(x^2+x\delta x+\frac{1}{2}\delta x^2)\sin(x\delta x+\frac{1}{2}\delta x^2)}{\delta x}$$

$$=2\cos x^2\ \underset{\delta x\to 0}{\text{Lt}}\ \frac{x\delta x+\frac{1}{2}\delta x^2}{\delta x}=2x\cos x^2$$

20 $y=\sqrt{(\sin x)}$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sqrt{\sin(x+\delta x)}-\sqrt{(\sin x)}}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sin(x+\delta x)-\sin x}{\delta x}\quad\frac{1}{\sqrt{\{\sin(x+\delta x)+\sqrt{(\sin x)}\}}}$$

$$=\cos x\ \frac{1}{2\sqrt{(\sin x)}}$$

21 Do yourself (Lt is to be calculated as in Q 1 for sin x).

22 $y=\frac{\sin x}{x}$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{1}{\delta x}\left[\frac{\sin(x+\delta x)}{x+\delta x}-\frac{\sin x}{x}\right]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x\sin(x+\delta x)-(x+\delta x)\sin x}{x(x+\delta x)\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\left[\frac{x\{\sin(x+\delta x)-\sin x\}}{x(x+\delta x)\delta x}-\frac{\delta x\sin x}{\delta x\,x(x+\delta x)}\right]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\left[\frac{1}{x+\delta x}\ \frac{\sin(x+\delta x)-\sin x}{\delta x}-\frac{\sin x}{x(x+\delta x)}\right]$$

$$=\frac{1}{x}\ \cos x-\frac{\sin x}{x^2}=\frac{x\cos x-\sin x}{x^2}$$

The limit cos x is to be calculated as in Q 1 for sin x

3. $y=\log\cos x$, $\qquad e^y=\cos x$

$y+\delta y=\log\cos(x+\delta x)\qquad e^{y+\delta y}=\cos(x+\delta x)$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\delta y}{\delta x}$$

Now $\frac{e^{y+\delta y}-e^y}{\delta x}=\frac{\cos(x+\delta x)-\cos x}{\delta x}$ by above

or $\frac{e^{y+\delta y}-e^y}{\delta y}\ \frac{\delta y}{\delta x}=\frac{\cos(x+\delta x)-\cos x}{\delta x}$

Now take limits of both the sides when $\delta x\to 0$ and consequently $\delta y\to 0$, as in Q 2 for cos x and Q 8 for e^x and $\text{Lt}\frac{\delta y}{\delta x}=\frac{dy}{dx}$

$$e^y\ \frac{dy}{dx}=-\sin x\ \text{ or }\ \frac{dy}{dx}=-\frac{\sin x}{e^y}=-\frac{\sin x}{\cos x}=-\tan x$$

(i) 5π (ii) 10π (iii) 25π

(iv) None of these (M N R 82)

5 A point P moves so that the length of the tangent from P to the circle $x^2+y^2-2x-4y+1=0$ is three times the distance of P from the point $(1, -2)$. Then the locus of P is a straight line,

(a) True (b) false

6 The equation of the circle of radius 5 in the first quadrant which touches the x axis and the line $4y=3x$ is

7 (a) Given the circles

$x^2+y^2-4x-5=0$ and $x^2+y^2+6x-2y+6=0$

Let P be a point (α, β) such that the tangents from P to both the circles are equal. Then

(i) $2\alpha+10\beta+11=0$, (ii) $2\alpha-10\beta+11=0$

(iii) $10\alpha-2\beta+11=0$, (iv) $10\alpha+2\beta+11=0$

(b) The length of tangent from $(5, 1)$ to the circle $x^2+y^2+6x-4y-3=0$ is

(i) 81 (ii) 29 (iii) 7 (iv) 21

(M N R 81)

8 If the equation $ax^2+by^2+2hxy+2gx+2fy+c=0$ represents a circle the condition will be

(a) $a=b$ and $c=0$ (b) $f=g$ and $h=0$

(c) $a=b$ and $h=0$ (d) $f=g$ and $c=0$

(M N R 79)

9 Given the equation of two circles

$x^2+y^2=r^2$ and $x^2+y^2-10x+16=0$,

the value of r such that they intersect in real and distinct points is given by

(a) $2<r<8$, (b) $r=2$ or $r=8$,

(c) $r<2$ or $r>8$, (d) None of the above

10 The equation of the circle passing through origin is $x^2+y^2-6x+2y=0$. The equation of one of its diameters is

(a) $x+3y=0$ (b) $x+y=0$

(c) $x=y$ (d) $3x+y=0$

11 The circle $x^2+y^2+4x-7y+12=0$ cuts an intercept on y axis equal to

(a) 1, (b) 3 (c) 4, (d) 7

12 AB is a diameter of a circle and C is any point on the circumference of the circle. Then

$$=\text{Lt}\,\frac{\sin\left(\frac{1}{x+\delta x}-\frac{1}{x}\right)}{\delta x\cos\frac{1}{x}\,\cos\frac{1}{x+\delta x}}$$

$$=\text{Lt}\left[\frac{\sin\frac{-\delta x}{x(x+\delta x)}}{\delta x}\right]\left[\frac{1}{\cos\frac{1}{x}\cos\frac{1}{x+\delta x}}\right]$$

Now $\text{Lt}\sin\theta=\theta$ when $\theta\to 0$ Here $\frac{\delta x}{x(x+\delta x)}\to 0$

$$\frac{dy}{dx}=\text{Lt}\,\frac{-\delta x}{\delta x.x(x+\delta x)}\;\frac{1}{\cos\frac{1}{x}\cos\frac{1}{x+\delta x}}$$

$$=-\frac{1}{x^2}\;\frac{1}{\cos^2\frac{1}{x}}=-\frac{1}{x^2}\sec^2\frac{1}{x}$$

28 $y=e^{\tan x}$ $\qquad\log y=\tan x$

$y+\delta y=e^{\tan(x+\delta x)}$ $\qquad\log(y+\delta y)=\tan(x+\delta x)$

$$\frac{\log(y+\delta y)-\log y}{\delta y}=\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

or $\frac{\log(y+\delta y)-\log y}{\delta y}\,\frac{\delta y}{\delta x}=\frac{\tan(x+\delta x)-\tan x}{\delta x}$

Both the above limits have been calculated in Q 3 and
When $\delta x\to 0$, δy also $\to 0$

$$\frac{1}{y}\,\frac{dy}{dx}=\sec^2 x \text{ or } \frac{dy}{dx}=y\sec^2 x=e^{\tan x}\sec^2 x$$

Proceed as above $\frac{dy}{dx}=e^{\sin x}\cos x$

(a) $y=x^2\log x$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\,\frac{(x+\delta x)^2\log(x+\delta x)-x^2\log x}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\,x^2\left[\frac{\log(x+\delta x)-\log x}{\delta x}\right]+\frac{2x\delta x\log(x+\delta x)}{\delta x}+\frac{\delta x^2\log(x+\delta x)}{\delta x}$$

$=x^2\,\frac{1}{x}+2x\log x+0=x+2x\log x=x(1+2\log$
e limit has been calculated as in Q 9

$y=xe^x$

$$=\underset{\delta x\to 0}{\text{Lt}}\,\frac{(x+\delta x)\,e^{x+\delta x}-xe^x}{\delta x}$$

is always at a distance equal to the sum $\left(\frac{9}{2}+2\right)=\frac{13}{2}$ of the radii of two circles

$$\left(h+\frac{3}{2}\right)^2+(k-3)^2=\left(\frac{13}{2}\right)^2$$

$$h^2+k^2+3h-6k+\frac{9}{4}+9-\frac{169}{4}=0$$

locus of (h, k) is

$$x^2+y^2+3x-6y-31=0$$

2 Ans (d)
The centre of the given circle is $(1, -2)$ and sides of the inscribed square are parallel to the coordinate axes Hence no vertex of the square can have its x coordinate equal to 1 and no vertex can have its y coordinate equal to -2
Hence none of the points given in (a), (b), (c) can be the vertex of the square

3 (a) Ans (b)
Equation of any circle through the intersection of given circles is

$$x^2-y^2-6x+8+\lambda(x^2+y-6)=0$$

It passes through (1, 1) if

$$1+1-6+8+\lambda(1+1-6)=0 \text{ or } \lambda=1$$

Hence the equation of required circle is

$$x^2+y^2-6x+8+x+y^2-6=0$$

or $2x^2+2y^2-6x+2=0$

or $x^2+y^2-3x+1=0$

(b) Proceed as in part (a), $\lambda=2$
Ans (b)

4 (a) $4x^2+4y-16x-25y+16=0$
(b) (iii)

5 Ans (b) Locus is a circle

6 Ans $x^2+y-30x-10y+225=0$

7 (a) Ans (c)
(b) Ans (c)

8 Ans (iii)

9 Ans (a)
We know that the two circles touch if

$C_1C_2=r_1+r_2$ [Externally]

$C_1C_2=r_1-r_2$ [Internally]

In case they intersect in real point then

$$=\text{Lt}\ \frac{\sin\left(\frac{1}{x+\delta x}-\frac{1}{x}\right)}{\delta x\cos\frac{1}{x}\ \cos\frac{1}{x+\delta x}}$$

$$=\text{Lt}\left[\frac{\sin\frac{-\delta x}{x(x+\delta x)}}{\delta x}\right]\left[\frac{1}{\cos\frac{1}{x}\cos\frac{1}{x+\delta x}}\right]$$

Now Lt $\sin\theta=\theta$ when $\theta\to 0$ Here $\frac{\delta x}{x(x+\delta x)}\to 0$

$$\frac{dy}{dx}=\text{Lt}\ \frac{-\delta x}{\delta x.x(x+\delta x)}\ \frac{1}{\cos\frac{1}{x}\ \cos\frac{1}{x+\delta x}}$$

$$=-\frac{1}{x^2}\ \frac{1}{\cos^2\frac{1}{x}}=-\frac{1}{x^2}\sec^2\frac{1}{x}$$

28 $y=e^{\tan x}$ $\log y=\tan x$

$y+\delta y=e^{\tan(x+\delta x)}$ $\log(y+\delta y)=\tan(x+\delta x)$

$$\therefore\ \frac{\log(y+\delta y)-\log y}{\delta y}=\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

or $\frac{\log(y+\delta y)-\log y}{\delta y}\ \frac{\delta y}{\delta x}=\frac{\tan(x+\delta x)-\tan x}{\delta x}$

Both the above limits have been calculated in Q 3 and 9

When $\delta x\to 0$, δy also $\to 0$

$$\frac{1}{y}\ \frac{dy}{dx}=\sec^2 x \text{ or } \frac{dy}{dx}=y\sec^2 x=e^{\tan x}\sec^2 x$$

29 Proceed as above $\frac{dy}{dx}=e^{\sin x}\cos x$

30 (a) $y=x^2\log x$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{(x+\delta x)^2\log(x+\delta x)-x^2\log x}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ x^2\left[\frac{\log(x+\delta x)-\log x}{\delta x}\right]+\frac{2x\delta x\log(x+\delta x)}{\delta x}+\frac{\delta x^2\log(x+\delta x)}{\delta x}$$

$$=x^2\ \frac{1}{x}+2x\log x+0=x+2x\log x=x(1+2\log x)$$

The limit has been calculated as in Q 9

(b) $y=xe^x$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{(x+\delta x)e^{x+\delta x}-xe^x}{\delta x}$$

is always at a distance equal to the sum $\left(\frac{9}{2}+2\right)=\frac{13}{2}$ of the radii of two circles

$$\left(h+\frac{3}{2}\right)^2+(k-3)^2=\left(\frac{13}{2}\right)^2$$

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Equation of any circle through the intersection of given circles is

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$$1+1-6+8+\lambda(1+1-6)=0 \text{ or } \lambda=1$$

Hence the equation of required circle is

$$x^2+y^2-6x+8+x^2+y^2-6=0$$

or $2x^2+2y^2-6x+2=0$

or $x^2+y^2-3x+1=0$

(*b*) Proceed as in part (a), $\lambda=2$

Ans (b)

4 (*a*) $4x^2+4y^2-16x-25y+16=0$

(b) (iii)

5 Ans (b) Locus is a circle

6 Ans $x^2+y^2-30x-10y+225=0$

7 (*a*) Ans (c)

(*b*) Ans (c)

8 Ans (iii)

9 Ans (a)

We know that the two circles touch if

$C_1C_2=r_1+r_2$ [Externally]

$C_1C_2=r_1-r_2$ [Internally]

In case they intersect in real point then

Also as $\delta x \to 0$, $\delta t \to 0$

$$\frac{dy}{dx} = \text{Lt}\,\frac{\delta y}{\delta x} = \text{Lt}\,\frac{\cos(t+\delta t)-\cos t}{\delta t}\cdot\frac{a\,2x\,\delta x + a\delta x^2 + b\delta x}{\delta x}$$

$$= -\sin t\,(2ax+b+0)$$

$$= -(2ax+b)\sin(ax^2+bx+c) \qquad \text{As in Q}$$

34 Proceed as above $\frac{dy}{dx} = 2x\cos(x^2+1)$

35 (a) $y = x^2 \cos x$

$$\frac{dy}{dx} = \underset{\delta x\to 0}{\text{Lt}}\ \frac{(x+\delta x)^2\cos(x+\delta x) - x^2\cos x}{\delta x}$$

$$\underset{\delta x\to 0}{\text{Lt}}\ x^2\left[\frac{\cos(x+\delta x)-\cos x}{\delta x}\right] + \frac{2x\,\delta x\cos(x+\delta x)}{\delta x} + \frac{(\delta x)^2}{\delta x}\cos(x+\delta x)$$

$$= x^2(-\sin x) + 2x\cos x - 0 = -x^2\sin x + 2x\cos x$$

(b) Above limit is differential coefficient of $a^2 \sin a$ w r t. a, and it can be shown to be $2a\sin a + a^2\cos a$ as in part (a).

36 $f(x) = \frac{x-1}{2x^2-7x+5} = \frac{x-1}{(x-1)(2x-5)} = \frac{1}{2x-5}$ as $x \neq 1$

Also $f(1) = -1/3$ (given)

$$f'(1) = \underset{h\to 0}{\text{Lt}}\ \frac{f(1+h)-f(1)}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{\frac{1}{2(1+h)-5} - \left(-\frac{1}{3}\right)}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{\frac{1}{2h-3}+\frac{1}{3}}{h} = \underset{h\to 0}{\text{Lt}}\ \frac{2h}{3h(2h-3)} = \underset{h\to 0}{\text{Lt}}\ \frac{2}{3(2h-3)}$$

$$= -\frac{2}{9}$$

37 $f(x) = x\tan^{-1}(x)$, $\quad f(1) = 1\tan^{-1} 1$

$$f'(1) = \underset{h\to 0}{\text{Lt}}\ \frac{(1+h)\tan^{-1}(1+h) - \tan^{-1} 1}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{\tan^{-1}(1+h) - \tan^{-1} 1}{h} + \frac{h\tan^{-1}(1+h)}{h}$$

$$= \underset{h\to 0}{\text{Lt}}\ \frac{\tan^{-1}\frac{(1+h)-1}{1+(1+h)1}}{h} + \tan^{-1}(1+h)$$

$$\underset{h\to 0}{\text{Lt}}\ \frac{\tan^{-1}\{h/(2+h)\}}{(2+h)\{h/(2+h)\}} + \tan^{-1}(1+h)$$

Now let $\frac{h}{2+h} = x$ and when $h \to 0$, $x \to 0$,

$$\underset{x\to 0}{\text{Lt}}\ \frac{\tan^{-1} x}{x} = \underset{\theta\to 0}{\text{Lt}}\ \frac{\theta}{\tan\theta} \text{ where } x = \tan\theta \text{ Also when } x\to 0$$

III
DIFFERENTIAL AND INTEGRAL CALCULUS

$$+\frac{a^2}{2}\left[\left(\frac{1}{x+\sqrt{(x^2+a^2)}}\right)\left(1+\frac{2x}{2\sqrt{(x^2+a^2)}}\right)\right]$$

$$=\frac{1}{2\sqrt{(a^2+x^2)}}(a^2+2x^2)+\frac{a^2}{2}\frac{1}{\sqrt{(a^2+x^2)}}$$

$$=\frac{1}{2\sqrt{(a^2+x^2)}}(a^2+2x^2+a^2)=\frac{1}{2\sqrt{(a^2+x^2)}}\,2(a^2 \ldots$$

$$=\sqrt{(a^2+x^2)}$$

43 We know that $\frac{d}{dx}\log x=\frac{1}{x}$ i e the base must be e H… the question the base is 7 which should be changed …
Also $\log_a x=\log_e x \log_a e$

$$y=\log_7(\log_7 x)=\log_e(\log_7 x)\log_7 e$$

$$y=\log_7 e\,\log_e(\log_e x\,\log_7 e)=k\log(k\log x)\text{ where }k=\log \ldots$$

$$\frac{dy}{dx}=k\frac{1}{(k\log x)}k\frac{1}{x}=\frac{\log_7 e}{x\log_e x}$$

44 $y=\log_3\left[\frac{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}\right]$

$$=\log_e\left[\frac{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}\right]\log_3 e$$

$$=\log_3 e\,[\log\{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}\}-\log\{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}\}]$$

$$\frac{dy}{dx}=\log_3 e\left[\frac{1}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}\left\{\frac{2x}{2\sqrt{(x^2+a^2)}}+\frac{2x}{2\sqrt{(x^2+b^2)}}\right\}\right]$$

$$-\log_3 e\left[\frac{1}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}\left\{\frac{2x}{2\sqrt{(x^2+a^2)}}-\frac{2x}{2\sqrt{(x^2+b^2)}}\right\}\right]$$

$$=\log_3 e\left[\frac{x}{\sqrt{(x^2+a^2)}\sqrt{(x^2+b^2)}}-\frac{-x}{\sqrt{(x^2+a^2)}\sqrt{(x^2+b^2)}}\right]$$

$$=\frac{2x}{\sqrt{(x^2+a^2)}\sqrt{(x^2+b^2)}}\log_3 e$$

$y=\log_{10}x+\log_x 10+\log_x x+\log_{10}10$
Just as in Q 43 we can change the base to e

$$y=\log_e x\,\log_{10}e+\log_e 10\,\log_x e+1+1$$

$$=\log x\,\log_{10}e+\frac{\log_e 10}{\log_e x}+2$$

$$\frac{dy}{dx}=\frac{1}{x}\log_{10}e+\log_e 10\left[-\left(\frac{1}{\log x}\right)^2\right]\frac{1}{x}+0$$

1
Differentiation

§ 1 *Basic formulae*

1 Differentiation from first Principle

If $y=f(x)$, then $\frac{dy}{dx}=\underset{\delta x\Rightarrow 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}=f'(x)$

Also $\left(\frac{dy}{dx}\right)_{x=a}=\underset{\delta x\Rightarrow 0}{\text{Lt}}\frac{f(a+\delta x)-f(a)}{\delta x}$

In evaluating the limits the following should be noted

$\underset{\theta\Rightarrow 0}{\text{Lt}}\frac{\sin\theta}{\theta}=1$ $\lim_{\theta\to 0}\cos\theta=1$, $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}=1$

Limit of sum = sum of Limits,

Limit of product = Product of Limits

$\underset{h\Rightarrow 0}{\text{Lt}}(1+h)^{1/h}=e$, $\underset{h\Rightarrow 0}{\text{Lt}}\left(1+\frac{h}{2}\right)^{1/h}=e^{1/2}$ etc

Fundamental Theorems Let u, v, w be functions of x whose derivatives exist

1 Differential coefficient of constant is zero, *i e* $\frac{d}{dx}(k)=0$

2 $\frac{d}{dx}(ku)=k\frac{du}{dx}$

3 $\frac{d}{dx}(u\pm v)=\frac{du}{dx}\pm\frac{dv}{dx}$ **Sum or difference**

4 $\frac{d}{dx}(uv)=u\frac{dv}{dx}+v\frac{du}{dx}$ ***Product***

5 $\frac{d}{dx}\left(\frac{u}{v}\right)=\frac{v\frac{du}{dx}-u\frac{dv}{dx}}{v^2}$ **Quotient**

6 If $y=f(t)$ and $t=\phi(x)$ **function of function**

then $\frac{dy}{dx}=\frac{dy}{dt}\cdot\frac{dt}{dx}$

7 If $u=f(y)$ then $\frac{du}{dx}=\frac{du}{dy}\cdot\frac{dy}{dx}=f'(y)\frac{dy}{dx}$

10 (i) $\sin^{-1}\frac{2x}{1+x^2}$, (ii) $\cos^{-1}\frac{1-x^2}{1+x^2}$, (iii) $\tan^{-1}\frac{2x}{1-x^2}$

(vi) $\cos^{-1}\frac{x^{2n}-1}{x^{2n}+1}$, (v) $\cos^{-1}\frac{(x-x^{-1})}{(x+x^{-1})}$

11 $\sin^{-1}(3x-4x^3)$, $\cos^{-1}(4x^3-3x)$ (Roorkee 63)

12 (a) $\sin^{-1}x+\sin^{-1}\sqrt{(1-x^2)}$,

(b) $\sin^{-1}\sqrt{(1-x)}+\cos^{-1}\sqrt{(x)}$

13 (a) $\sin^{-1}[x\sqrt{(1-x)}-\sqrt{(x)}\sqrt{(1-x^2)}]$ (Roorkee 81)

(b) $\sin^{-1}\frac{\sqrt{(1+x)}+\sqrt{(1-x)}}{2}$

14 Defferentiate $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ w r t $\tan^{-1}x$,

15 (a) Differentiate $\tan^{-1}\frac{2x}{1-x^2}$ w r t $\sin^{-1}\frac{2x}{1+x}$ (Roorkee 66)

(b) Differentiate $\sin^{-1}\frac{1-x}{1+x}$ with respect to $\sqrt{x}$

16 (a) Differentiate $\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$ w r t $\sec^{-1}\frac{1}{2x^2-1}$, (Roorkee 1984)

(b) Differentiate $\sin^{-1}x$ w r t $\cos^{-1}\sqrt{(1-x^2)}$

17 (a) Differentiate $\sec^{-1}\frac{1}{2x^2-1}$ w r t $\sqrt{(1-x^2)}$ (M N.R 83)

(b) Differentiate $\frac{\tan^{-1}x}{1+\tan^{-1}x}$ w r t $\tan^{-1}x$

18 If $\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y)$ show that

$$\frac{dy}{dx}=\sqrt{\left(\frac{1-y^2}{1-x^2}\right)}$$

If $\tan y=\frac{2t}{1-t^2}$ and $\sin x=\frac{2t}{1+t^2}$ prove that $\frac{dy}{dx}=1$ (M.N.R 81)

If $y=\tan^{-1}\frac{4x}{1+5x^2}+\tan^{-1}\frac{2+3x}{3-2x}$ prove that

$$\frac{dy}{dx}=\frac{5}{1+25x^2}$$

If $y=\sin^{-1}\frac{2x}{1+x^2}+\sec^{-1}\frac{1+x^2}{1-x^2}$ then $\frac{dy}{dx}=\frac{4}{1+x^2}$

If $y=\sin^{-1}2x\sqrt{(1-x^2)}+\sec^{-1}\frac{1}{\sqrt{(1-x^2)}}$, then

$$\frac{dy}{dx}=\frac{3}{\sqrt{(1-x^2)}}$$

$\sin^{-1}\cos x=\sin^{-1}\sin(\frac{1}{2}\pi-x)=\frac{1}{2}\pi-x$

$\tan^{-1}(\tan\theta)=\theta,\ \sin^{-1}(\sin\theta)=\theta,\ \cos^{-1}(\cos\theta)=\theta$

(*i i*) **Important result of trigonometry**

$$\frac{1-\cos x}{1+\cos x}=\frac{2\sin^2 x/2}{2\cos^2 x/2}=\tan^2 x/2,\quad \frac{1+\cos x}{1-\cos x}=\cot^2 x/2$$

$$\frac{1-\cos x}{\sin x}=\frac{2\sin^2 x/2}{2\sin x/2\cos x/2}=\tan x/2,\quad \frac{1+\cos x}{\sin x}=\cot x/2$$

$$\sqrt{(1\pm\sin x)}=(\cos^2 x/2+\sin^2 x/2\pm 2\sin x/2\cos x/2)^{1/2}$$
$$=[(\cos x/2\pm\sin x/2)^2]^{1/2}=\cos x/2\pm\sin x/2$$

$$\tan A\pm\tan B=\frac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B}=\frac{\sin(A\pm B)}{\cos A\cos B}$$

$$\tan(\tfrac{1}{4}\pi+\theta)=\frac{1+\tan\theta}{1-\tan\theta},\quad \tan(\tfrac{1}{4}\pi-\theta)=\frac{1-\tan\theta}{1+\tan\theta}$$

14 Standard results

1 $y=x^n,\ \frac{dy}{dx}=nx^{n-1},\ y=u^n,\ \frac{dy}{dx}=n\,u^{n-1}\frac{du}{dx}$

Particular case $y=\sqrt{x},\ \frac{dy}{dx}=\frac{1}{2\sqrt{x}}$ **(Imp.)**

$y=\frac{1}{x^n},\ \frac{dy}{dx}=-\frac{n}{x^{n+1}}$

2 $y=e^x,\ \frac{dy}{dx}=e^x,\ y=e^u,\ \frac{dy}{dx}=e^u\frac{du}{dx}$

3 $y=\log x,\ \frac{dy}{dx}=\frac{1}{x},\ y=\log u,\ \frac{dy}{dx}=\frac{1}{u}\frac{du}{dx}$

4 $y=a^x,\ \frac{dy}{dx}=a^x\log a,\ y=a^u,\ \frac{dy}{dx}=a^u\log a\frac{du}{dx}$

5 $y=\sin x,\ \frac{dy}{dx}=\cos x,\ y=\sin u,\ \frac{dy}{dx}=\cos u\frac{du}{dx}$

6 $y=\cos x,\ \frac{dy}{dx}=-\sin x,\ y=\cos u,\ \frac{dy}{dx}=-\sin u\frac{du}{dx}$

7 $y=\tan x,\ \frac{dy}{dx}=\sec^2 x,\ y=\tan u\ \frac{dy}{dx}=\sec^2 u\frac{du}{dx}$

8 $y=\cot x,\ \frac{dy}{dx}=-\text{cosec}^2 x,\ y=\cot u,\ \frac{dy}{dx}=-\text{cosec}^2 u\frac{du}{dx}$

9 $y=\sec x,\ \frac{dy}{dx}=\sec x\tan x$

10 $y=\text{cosec}\,x,\ \frac{dy}{dx}=-\text{cosec}\,x\cot x$

42 If $y=\frac{x}{2}\sqrt{(a^2+x^2)}+\frac{a^2}{2}\log(x+\sqrt{(x^2+a^2)})$, prove that

$$\frac{dy}{dx}=\sqrt{(x^2+a^2)}$$

43 If $y=\log_7(\log_7 x)$, prove that $\frac{dy}{dx}=\frac{\log_7 e}{x\log_e x}$

44 If $y=\log_3\frac{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}$, prove that

$$\frac{dy}{dx}=\frac{2x}{\log_e 3\sqrt{\{(x^2+a^2)(x^2+b^2)\}}}$$

45 If $y=\log_{10}x+\log_x 10+\log_x x+\log_{10}10$, find dy/dx

46 If $y=\log x^x$ prove $\frac{dy}{dx}=\log(ex)$ (M N R 78)

47 Find $\frac{dy}{dx}$ if $y=[x+\sqrt{\{x+\sqrt{(x)}\}}]^{1/2}$

48 Find $\frac{dy}{dx}$ if $y=\log|x|$

Solutions to Problems set (A)

1 $y=\sin x=f(x)$, $y+\delta y=\sin(x+\delta x)=f(x+\delta x)$

$$\delta y=\sin(x+\delta x)-\sin x=f(x+\delta x)-f(x)$$

$$\frac{\delta y}{\delta x}=\frac{\sin(x+\delta x)-\sin x}{\delta x}=\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin(x+\delta x)-\sin x}{\delta x}=\underset{\delta x\to 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{2\cos(x+\frac{1}{2}\delta x)\sin\frac{1}{2}\delta x}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\cos\left(x+\frac{\delta x}{2}\right)\frac{\sin(\delta x/2)}{(\delta x/2)}$$

$$=\cos x\cdot 1=\cos x\quad\left(\underset{\theta\to 0}{\text{Lt}}\frac{\sin\theta}{\theta}=1\right)$$

Note In rest of the questions we will simply write that

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{f(x+\delta x)-f(x)}{\delta x}$$

2 Do yourself

3 $y=\tan x$,

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\tan(x+\delta x)-\tan x}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin(x+\delta x-x)}{\delta x\cos x\cos(x+\delta x)}\text{ (change to sin and cos)}$$

or $y=\sin^{-1} x+\sin^{-1}\sqrt{(x)}$

$$\frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}+\frac{1}{\sqrt{(1-x)}}\cdot\frac{1}{2\sqrt{(x)}}$$

(b) Put $x=\cos\theta$, $1+\cos\theta=2\cos^2\theta/2$, $1-\cos\theta=2\sin^2\theta/2$

$$y=\sin^{-1}\frac{1}{\sqrt{(2)}}(\cos\theta/2+\sin\theta/2)$$

$$=\sin^{-1}\sin\left(\frac{\theta}{2}+\pi/4\right)=\theta/2+\pi/4$$

$y=\frac{1}{2}\cos^{-1}x+\frac{\pi}{4}$ $\qquad \frac{dy}{dx}=\frac{-1}{2\sqrt{(1-x^2)}}$

14 Let $y=\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$ and $z=\tan^{-1}x$

and we have to find $\frac{dy}{dz}$

$$\frac{dy}{dx}=\frac{1}{2}\cdot\frac{1}{1+x^2} \text{ and } \frac{dz}{dx}=\frac{1}{1+x^2}$$

$$\frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=\frac{1}{2}$$

15 (a) Ans 1 by Q 10

(b) Ans $\qquad -\frac{2}{1+x}$

16 (a) Let $y=\tan^{-1}\frac{x}{\sqrt{(1-x^2)}}$, $z=\sec^{-1}\frac{1}{2x^2-1}$

Put $x=\sin\theta$ in 1st and $x=\cos\phi$ in 2nd

$\frac{dy}{dx}=\frac{1}{\sqrt{(1-x^2)}}$ by Q 3 (c)

$$z=\sec^{-1}\frac{1}{2\cos^2\phi-1}=\sec^{-1}\sec 2\phi=2\phi=2\cos^{-1}x$$

$\frac{dz}{dx}=-\frac{2}{\sqrt{(1-x^2)}}$ $\qquad \frac{dy}{dz}=\frac{dy}{dx}\div\frac{dz}{dx}=-\frac{1}{2}$

(b) $z=\cos^{-1}\sqrt{(1-x^2)}=\sin^{-1}x$ $\qquad y=z$ and $\frac{dy}{dz}=1$

(a) $y=\sec^{-1}\frac{1}{2x^2-1}$ $\quad z=\sqrt{(1-x^2)}$

$\frac{dy}{dx}=-\frac{2}{\sqrt{(1-x^2)}}$ as in Q 16, $\frac{dz}{dx}=\frac{1}{2\sqrt{(1-x^2)}}(-2x)$

$$=-\frac{x}{\sqrt{(1-x^2)}}$$

$$=\frac{-1}{\sqrt{(x+a)}\sqrt{(x+a)}\,[2\sqrt{(x+a)}]}=\frac{-1}{2(x+a)^{3/2}}$$

13 $y=\sin^{-1} x, \qquad x=\sin y$

$y+\delta y=\sin^{-1}(x+\delta x) \qquad (x+\delta x)=\sin(y+\delta y)$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta y}{\sin(y+\delta y)-\sin y}=\frac{1}{\cos y}$$

$$=\frac{1}{\sqrt{(1-\sin^2 y)}}=\frac{1}{\sqrt{(1-x^2)}}$$

The limit is evaluated as in Q 1 for $\sin x$

Note that when $\delta x\to 0$, δy also tends to zero

14 Proceed as above $y=\cos^{-1} x$, $\dfrac{dy}{dx}=\dfrac{-1}{\sqrt{(1-x^2)}}$

15 (a) $y=\tan^{-1} x \qquad x=\tan y$

$y+\delta y=\tan^{-1}(x+\delta x) \qquad x+\delta x=\tan(y+\delta y)$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\delta y}{\delta x}=\underset{\delta y\to 0}{\text{Lt}}\frac{\delta y}{\tan(y+\delta y)-\tan y}=\frac{1}{\sec^2 y}$$

$$=\frac{1}{1+\tan^2 y}=\frac{1}{1+x^2}$$

The limit is to be calculated as in Q 3 for $\tan x$

(b) Proceed as above, $y=\cot^{-1} x$, $\dfrac{dy}{dx}=-\dfrac{1}{1+x^2}$

16 Do yourself

$$\frac{dy}{dx}=\frac{1}{\sec y\sqrt{(\sec^2 y-1)}}=\frac{1}{x\sqrt{(x^2-1)}}$$

The limit is to be calculated as in Q 5 for $\sec x$

17 Do yourself

18 $y=\sin^2 x$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin^2(x+\delta x)-\sin^2 x}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin(x+\delta x+x)\sin(x+\delta x-x)}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\sin(2x+\delta x)\frac{\sin\delta x}{\delta x}=\sin 2x$$

We have used $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$

19 $y=\sin x^2$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin(x+\delta x)^2-\sin x^2}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\frac{\sin(x^2+2x\,\delta x+\delta x^2)-\sin x^2}{\delta x}$$

23 (a) Put $x=\tan\theta$

$$y=\pi/2-\theta+\theta/2=\pi/2-\tfrac{1}{2}\tan^{-1}x$$

$$\frac{dy}{dx}=-\frac{1}{2}\,\frac{1}{1+x}\text{ by Q 4 (c)}$$

24 (a) We know $\sec^{-1}z=\cos^{-1}1/z$ and $\cos^{-1}y+\sin^{-1}y=\pi/2$

$$y=\cos^{-1}\frac{\sqrt{x-1}}{\sqrt{x+1}}+\sin^{-1}\frac{\sqrt{x-1}}{\sqrt{x+1}}=\frac{\pi}{2}\qquad\frac{dy}{dx}=0$$

(b) $\cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}$

$$y=\sin^{-1}x+\cos^{-1}x=\frac{\pi}{2}\qquad\frac{dy}{dx}=0$$

25 Put $2=r\cos\alpha$, $3=r\sin\alpha$ $\qquad r=\sqrt{(13)}$, $\tan\alpha=\frac{3}{2}$

$$y=\cos^{-1}\frac{r(\cos x\cos\alpha+\sin x\sin\alpha)}{\sqrt{(13)}}$$

$$=\cos^{-1}\cos(x-\alpha)=x-\alpha$$

or

$$y=x-\tan^{-1}\tfrac{3}{2}\qquad\frac{dy}{dx}=1$$

26 Put $x=\sin\theta$ and proceed as above

27 (a) $y=\sin(2\sin^{-1}x)$ $\qquad\sin^{-1}y=2\sin^{-1}x$

Differentiate both sides w r t x

$$\frac{1}{\sqrt{(1-y^2)}}\frac{dy}{dx}=2\,\frac{1}{\sqrt{(1-x^2)}}\qquad\frac{dy}{dx}=2\sqrt{\left(\frac{1-y^2}{1-x^2}\right)}$$

(b) $\dfrac{dy}{dx}=\dfrac{1}{\tan x/2}\sec^2\dfrac{x}{2}\,\dfrac{1}{2}-1$

[$\sin^{-1}\cos x=\sin^{-1}\sin(\pi/2-x)=\pi/2-x$]

$$=\frac{1}{2\dfrac{\sin x/2}{\cos x/2}\cos^2\dfrac{x}{2}}-1=\frac{1}{2\sin x/2\cos x/2}-1=\operatorname{cosec}x-1$$

8 Do yourself

$$\frac{dx}{dt}=a\left(-\sin t+\frac{1}{\sin t}\right)$$

$$=\frac{a(1-\sin^2 t)}{\sin t}=\frac{a\cos^2 t}{\sin t}$$

$$\frac{dy}{dt}=a\cos t\qquad\frac{dy}{dx}=a\cos t\,\frac{\sin t}{a\cos^2 t}=\tan t$$

$$\frac{dx}{dt}=2(\sin 2t-\sin t)\quad\frac{dy}{dt}=(2\cos t-\cos 2t)$$

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{2\sin\dfrac{3t}{2}\sin\dfrac{t}{2}}{2\cos\dfrac{3t}{2}\sin\dfrac{t}{2}}=\tan\frac{3t}{2}$$

Note $\log \sec x = \log (\cos x)^{-1} = -\log \cos x$

Hence $\frac{d}{dx} \log (\sec x) = \tan x$

24 $y = \sin \log x$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\sin \log (x+\delta x) - \sin \log x}{\delta x}$$

Put $\log x = z$ $\log (x+\delta x) = z + \delta z$ and as $\delta x \to 0$, δz also tends to zero

$$\frac{dy}{dx} = \underset{\delta z \to 0}{\text{Lt}} \frac{\sin (z+\delta z) - \sin z}{\delta z} \cdot \frac{\delta z}{\delta x}$$

$$= \underset{\delta z \to 0}{\text{Lt}} \frac{\sin (z+\delta z) - \sin z}{\delta z} \cdot \underset{\delta x \to 0}{\text{Lt}} \frac{\log (x+\delta x) - \log x}{\delta x}$$

Both the above limits have been calculated in Q 1 and 9

$$\frac{dy}{dx} = \cos z \cdot \frac{1}{x} = \frac{1}{x} \cos (\log x)$$

Similarly if $y = \cos \log (x)$, then $\frac{dy}{dx} = -\frac{1}{x} \sin (\log x)$

25, $y = \log \sin^{-1} x$ Put $\sin^{-1} x = t$

$y = \log t$ and $x = \sin t$

$y + \delta y = \log (t+\delta t)$ $x + \delta x = \sin (t+\delta t)$

$$\frac{\delta y}{\delta t} = \frac{\log (t+\delta t) - \log t}{\delta t}, \quad \frac{\delta x}{\delta t} = \frac{\sin (t+\delta t) - \sin t}{\delta t}$$

Now $\frac{\delta y}{\delta t} = \frac{\delta y}{\delta x} \cdot \frac{\delta x}{\delta t}$

$$\frac{\log (t+\delta t) - \log t}{\delta t} = \frac{\delta y}{\delta x} \cdot \frac{\sin (t+\delta t) - \sin t}{\delta t},$$

Both the above limits have been calculated in Q 1 and 9

Hence proceeding to limits, we get

$$\frac{1}{t} = \frac{dy}{dx} \cos t \qquad \frac{dy}{dx} = \frac{1}{t \cos t} = \frac{1}{t\sqrt{(1-\sin^2 t)}}$$

or $$\frac{dy}{dx} = \frac{1}{\sin^{-1} x \sqrt{(1-x^2)}}$$

26 Use $\cos^2 A - \cos^2 B = \sin (A+B) \sin (B-A)$

Ans $(-1/x) \sin (2 \log x)$

27 $y = \tan (1/x)$

$$\frac{dy}{dx} = \underset{\delta x \to 0}{\text{Lt}} \frac{\tan \frac{1}{x+\delta x} - \tan \frac{1}{x}}{\delta x} \text{ (change into sin and cos)}$$

3 (a) $y=(\cos x)^{\log x}+(\log x)^x$ (b) $e^{x \sin x^3}+(\tan x)^x$ (I I T 81)

4 $y=(1+1/x)^x+x^{1+1/x}$,

5 If $y^x=x^{\sin y}$, find dy/dx

6 (a) If $x^y+y^x=a^b$, find dy/dx (Roorkee 87)

(b) If $(\tan^{-1} x)^y+y^{\cot x}=1$, find dy/dx

7 (i) If $x^y=y^x$, prove that $dy/dx=\dfrac{y\,(x \log y-y)}{x\,(y \log x-x)}$

(ii) Find $\dfrac{dy}{dx}$ if (a) $y^x\, x^y=1$, (b) $x^y+y^x=1$

8 If $y=(x^x)^x$, prove that $dy/dx=x(x^x)^x\,(1+2 \log x)$

9 If $y=x^{(x^x)}$, $dy/dx=x^{x+x^x}\,[1/x+\log x\,(1+\log x)]$

10 If $y=x^{(\log x)^{\log \log x}}$, prove $dy/dx=y/x\,(\log x)^{\log \log x}\,(2 \log \log x+1)$

11 If $y=e^{x^{e^x}}+x^{e^{e^x}}+e^{x^{x^e}}$, find dy/dx

12 If $y=e^{x+e^{x+e^{x}\cdots\infty}}$, prove that $dy/dx=\dfrac{y}{1-y}$

13 (i) If $y=\sqrt{(x)}^{\sqrt{(x)}^{\sqrt{(x)}^{\sqrt{(x)}\cdots\infty}}}$, then $x\,dy/dx=\dfrac{y^2}{2-y \log x}$

(ii) If $x^{x^{x}\cdots\infty}$, prove that $\dfrac{dy}{dx}=\dfrac{y^2}{1-y \log x}$

14 If $y=(\sin x)^{(\sin x)^{(\sin x)}\cdots\infty}$, then $dy/dx=\dfrac{y^2 \cot x}{1-y \log \sin x}$

15 If $y=a^{x^{a^{x}\cdots\infty}}$, then $dy/dx=\dfrac{y^2 \log y}{x\,(1-y \log x \log y)}$

16 If $x^y=e^{x+y}$, then $dy/dx=\dfrac{\log x}{(1+\log x)^2}=\log x\,(\log ex)^{-2}$ (Roorkee 54)

17 If $x^p\, y^q=(x+y)^{p+q}$ prove that $dy/dx=y/x$

18 If $y=\dfrac{x \sin^{-1} x}{\sqrt{(1-x^2)}}+\log \sqrt{(1-x^2)}$ find dy/dx

19 If $e^y=\dfrac{e^{\sin x}(1+\cos x)^n}{(1-\cos x)^m}$ find dy/dx

20 If $y=[\log \{\log \sin x^a\}]^7$, find dy/dx

21 If $y=x^n \log x+x\,(\log x)^n$ find dy/dx

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{x\,[e^{x+\delta x}-e^{x}]}{\delta x}+\frac{\delta x\, e^{x+\delta x}}{\delta x}=xe^{x}+e^{x}$$

The limit is to be calculated as in Q 8

31 $y=\sin\sqrt{x}$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sin\sqrt{(x+\delta x)}-\sin\sqrt{x}}{\delta x}$$

$$=\frac{2\cos\frac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}\sin\frac{\sqrt{(x+\delta x)}-\sqrt{x}}{2}}{(x+\delta x)-x}$$

(Note the D^r)

$$=\text{Lt}\left[\frac{2\cos\frac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}}{\sqrt{(x+\delta x)}+\sqrt{x}}\right]\left[\frac{\sin\frac{\sqrt{(x+\delta x)}-x}{2}}{\sqrt{(x+\delta x)}-\sqrt{x}}\right]$$

(Note the D^r)

Now when $\delta x\to 0$, $\frac{\sqrt{(x+\delta x)}-\sqrt{x}}{2}\to 0$ and when $\theta\to 0$, $\sin\theta=\theta$

$$=\text{Lt}\left[\frac{2\cos\frac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}}{\sqrt{(x+\delta x)}+\sqrt{x}}\right]\left(\frac{[\sqrt{(x+\delta x)}-\sqrt{(x)}]}{[\sqrt{(x+\delta x)}-\sqrt{(x)}]}\right)\tfrac{1}{2}$$

$$=\frac{2\cos\frac{2\sqrt{x}}{2}}{2\sqrt{x}}\ 1\ \tfrac{1}{2}=\frac{1}{2\sqrt{x}}\cos\sqrt{x}$$

32 $y=\tan^2 ax$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\tan^2 a\,(x+\delta x)-\tan^2 ax}{\delta x}$$

$$=\underset{\delta x\to 0}{\text{Lt}}\left[\frac{\tan a\,(x+\delta x)-\tan ax}{\delta x}\right][\tan a\,(x+\delta x)+\tan ax]$$

$$\left[\text{Note that }\tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}\right]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ \frac{\sin(ax+a\delta x-ax)}{\delta x\cos(ax+a\delta x)\cos ax}\ [\tan a\,(x+\delta x)+\tan ax]$$

$$=\underset{\delta x\to 0}{\text{Lt}}\ a\,\frac{\sin(a\delta x)}{a\delta x}\left[\frac{\tan a\,(x+\delta x)+\tan ax}{\cos a\,(x+\delta x)\cos ax}\right]$$

$$=a\ 1\ \frac{2\tan ax}{\cos^2 ax}=2a\tan ax\sec^2 ax$$

33 $y=\cos(ax^2+bx+c)$

$y=\cos t$ where $t=ax^2+bx+c$

$y+\delta y=\cos(t+\delta t)$ $\quad t+\delta t=a\,(x+\delta x)^2+b\,(x+\delta x)+c$

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta t}\ \frac{\delta t}{\delta x}$$

or $$\frac{\delta y}{\delta x}=\frac{\cos(t+\delta t)-\cos t}{\delta t}\ \frac{a\,[(x+\delta x)^2-x^2]+b\,\delta x}{\delta x}$$

$=(\sin^{-1} x)^{x^2} \log \sin^{-1} x \; (2x)$

$\frac{dy}{dx}=$ Sum of the results (1) and (2)

$=2x\,(\sin^{-1} x)^{x^2} \log \sin^{-1} x + x^2 (\sin^{-1} x)^{x^2-1} \frac{1}{\sqrt{(1-x^2)}}$

2 (b, c, d) Try yourself

$y=(\cot x)^{\sin x}+(\tan x)^{\cos x}$

Here students should proceed as follows

Put $y=u+v$ so that $\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$ (1)

where $u=(\cot x)^{\sin x}$, $v=(\tan x)^{\cos x}$

Now $\frac{du}{dx}$ and $\frac{dv}{dx}$ may be calculated directly as

$$\frac{du}{dx}=\sin x\,(\cot x)^{\sin x-1}\,(-\operatorname{cosec}^2 x) + (\cot x)^{\sin x}\,(\log \cot x)\cos x$$

$$\frac{dv}{dx}=\cos x\,(\tan x)^{\cos x-1}\sec^2 x + (\tan x)^{\cos x}\log(\tan x)\,(-\sin x)$$

Hence from (1) we get the value of $\frac{dy}{dx}$

3 Proceed as in Q 2

$$\frac{dy}{dx}=\left\{\log x\,(\cos x)^{\log x-1}(-\sin x) + (\cos x)^{\log x}\log(\cos x)\,\frac{1}{x}\right\} + \left\{x\,(\log x)^{x-1}\frac{1}{x}+(\log x)^x\log(\log x)\right\}$$

$$=(\cos x)^{\log x}\left[\log x\,(-\tan x)+\frac{1}{x}\log\cos x\right] + (\log x)^x\left[\frac{1}{\log x}+\log(\log x)\right]$$

(b) $\frac{dy}{dx}=e^{x\sin x^3}\{1-\sin x^3+3x^3\cos x^3) + \{x\tan^{x-1} x\sec^2 x+(\tan x)^x\log\tan x\}$

$\left(1+\frac{1}{x}\right)^x\left[\log\left(1+\frac{1}{x}\right)-\frac{1}{x+1}\right]+x^{1+1/x}\left[\frac{x+1-\log x}{x^2}\right]$

$y^x=x^{\sin y}$ Take log of both sides

$x\log y=\sin y\log x$ Differentiate w r t x

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{x\,[e^{x+\delta x}-e^{x}]}{\delta x}+\frac{\delta x\ e^{x+\delta x}}{\delta x}=xe^{x}+e^{x}$$

The limit is to be calculated as in Q 8

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$$=\frac{2\cos\dfrac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}\ \sin\dfrac{\sqrt{(x+\delta x)}-\sqrt{x}}{2}}{(x+\delta x)-x}$$

(Note the D^r)

$$=\mathrm{Lt}\left[\frac{2\cos\dfrac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}}{\sqrt{(x+\delta x)}+\sqrt{x}}\right]\left[\frac{\sin\dfrac{\sqrt{(x+\delta x)}-x}{2}}{\sqrt{(x+\delta x)}-\sqrt{x}}\right]$$

(Note the D^r)

Now when $\delta x\to 0$, $\dfrac{\sqrt{(x+\delta x)}-\sqrt{x}}{2}\to 0$ and when $\theta\to 0$, $\sin\theta=\theta$

$$=\mathrm{Lt}\left[\frac{2\cos\dfrac{\sqrt{(x+\delta x)}+\sqrt{x}}{2}}{\sqrt{(x+\delta x)}+\sqrt{x}}\right]\left(\frac{[\sqrt{(x+\delta x)}-\sqrt{(x)}]}{[\sqrt{(x+\delta x)}-\sqrt{(x)}]}\right)\tfrac{1}{2}$$

$$=\frac{2\cos\dfrac{2\sqrt{x}}{2}}{2\sqrt{x}}\ 1\ \tfrac{1}{2}=\frac{1}{2\sqrt{x}}\ \cos\sqrt{x}$$

32 $y=\tan^2 ax$

$$\frac{dy}{dx}=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\tan^2 a\,(x+\delta x)-\tan^2 ax}{\delta x}$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\left[\frac{\tan a\,(x+\delta x)-\tan ax}{\delta x}\right][\tan a\,(x+\delta x)+\tan ax]$$

$$\left[\text{Note that }\tan A-\tan B=\frac{\sin(A-B)}{\cos A\cos B}\right]$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\ \frac{\sin(ax+a\delta x-ax)}{\delta x\cos(ax+a\delta x)\cos ax}\ [\tan a\,(x+\delta x)+\tan ax]$$

$$=\underset{\delta x\to 0}{\mathrm{Lt}}\ a\,\frac{\sin(a\delta x)}{a\delta x}\left[\frac{\tan a\,(x+\delta x)+\tan ax}{\cos a\,(x+\delta x)\cos ax}\right]$$

$$=a\ 1\ \frac{2\tan ax}{\cos^2 ax}=2a\tan ax\sec^2 ax$$

33 $y=\cos(ax^2+bx+c)$

$y=\cos t$ where $t=ax^2+bx+c$

$y+\delta y=\cos(t+\delta t)$ $t+\delta t=a\,(x+\delta x)^2+b\,(x+\delta x)+c$

$$\frac{\delta y}{\delta x}=\frac{\delta y}{\delta t}\ \frac{\delta t}{\delta x}$$

or $$\frac{\delta y}{\delta x}=\frac{\cos(t+\delta t)-\cos t}{\delta t}\ \frac{a\,[(x+\delta x)^2-x^2]+b\,\delta x}{\delta x}$$

$$=(\sin^{-1} x)^{x^2} \log \sin^{-1} x \ (2x) \qquad (2)$$

$\therefore \dfrac{dy}{dx}=$ Sum of the results (1) and (2)

$$=2x (\sin^{-1} x)^{x^2} \log \sin^{-1} x + x^2 (\sin^{-1} x)^{x^2-1} \frac{1}{\sqrt{(1-x^2)}}$$

(b, c, d) Try yourself

2 $y=(\cot x)^{\sin x}+(\tan x)^{\cos x}$

Here students should proceed as follows

Put $y=u+v$ so that $\dfrac{dy}{dx}=\dfrac{du}{dx}+\dfrac{dv}{dx}$ (1)

where $u=(\cot x)^{\sin x}$, $v=(\tan x)^{\cos x}$

Now $\dfrac{du}{dx}$ and $\dfrac{dv}{dx}$ may be calculated directly as

$$\frac{du}{dx}=\sin x (\cot x)^{\sin x-1} (-\operatorname{cosec}^2 x) + (\cot x)^{\sin x} (\log \cot x) \cos x$$

$$\frac{dv}{dx}=\cos x (\tan x)^{\cos x-1} \sec^2 x + (\tan x)^{\cos x} \log (\tan x) (-\sin x)$$

Hence from (1), we get the value of $\dfrac{dy}{dx}$

3 Proceed as in Q 2

$$\frac{dy}{dx}=\left\{ \log x (\cos x)^{\log x-1} (-\sin x) + (\cos x)^{\log x} \log (\cos x) \frac{1}{x} \right\} + \left\{ x (\log x)^{x-1} \frac{1}{x} + (\log x)^x \log (\log x) \right\}$$

$$=(\cos x)^{\log x} \left[\log x (-\tan x) + \frac{1}{x} \log \cos x \right] + (\log x)^x \left[\frac{1}{\log x} + \log (\log x) \right]$$

(b) $\dfrac{dy}{dx}=e^{x \sin x^3} \{1-\sin x^3+3x^3 \cos x^3\} + \{x \tan^{x-1} x \sec^2 x + (\tan x)^x \log \tan x \}$

4 $\left(1+\dfrac{1}{x}\right)^x \left[\log \left(1+\dfrac{1}{x}\right) - \dfrac{1}{x+1} \right] + x^{1+1/x} \left[\dfrac{x+1-\log x}{x^2} \right]$

5 $y^x=x^{\sin y}$ Take log of both sides

$x \log y = \sin y \log x$ Differentiate w r t x

$$= \underset{\theta\to 0}{\text{Lt}} \frac{\theta}{\sin\theta} \cos\theta = 1 \quad 1 = 1$$

$$f'(1) = \text{Lt} \frac{1}{2+h} \quad 1 + \tan^{-1}(1+h) = \tfrac{1}{2} + \tan^{-1} 1 = \tfrac{1}{2} + \pi/4$$

38 $y = \dfrac{1}{\sqrt{(x^2+a^2)}+\sqrt{(x^2+b^2)}} \times \dfrac{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}$

or $y = \dfrac{\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}}{(x^2+a^2)-(x^2+b^2)} = \dfrac{1}{(a^2-b^2)}\left[\sqrt{(x^2+a^2)}-\sqrt{(x^2+b^2)}\right]$

Now $\dfrac{d}{dx}\sqrt{u} = \dfrac{1}{2\sqrt{u}} \dfrac{du}{dx}$

$$\frac{dy}{dx} = \frac{1}{(a^2-b^2)}\left[\frac{1}{2\sqrt{(x^2+a^2)}} 2x - \frac{1}{2\sqrt{(x^2+b^2)}} 2x\right]$$

$$= \frac{x}{(a^2-b^2)}\left[\frac{1}{\sqrt{(x^2+a^2)}} - \frac{1}{\sqrt{(x^2+b^2)}}\right]$$

39 If $y = \sec x$, then $\dfrac{dy}{dx} = \sec x \tan x$ provided x is measured in radians

Now $180° = \pi$ radians $\qquad (x°+30°) = \dfrac{\pi}{180}(x+30)$ radians

$$= \left(\frac{\pi x}{180} + \pi/6\right) \text{ radians}$$

$$y = \sec\left(\frac{\pi x}{180} + \pi/6\right)$$

$$\frac{dy}{dx} = \sec\left(\frac{\pi x}{180} + \pi/6\right)\tan\left(\frac{\pi x}{180} + \pi/6\right) \quad (\pi/180)$$

$$= \frac{\pi}{180} \sec(x°+30°) \tan(x°+30°)$$

40 Do yourself

41 $y = 5x(1-x)^{-2/3} + \cos^2(2x+1)$

$$\frac{dy}{dx} = 5 \cdot 1 \, (1-x)^{-2/3} + 5x\left(-\frac{2}{3}\right)(1-x)^{-2/3-1}(-1)$$
$$+ 2\cos(2x+1)\{-\sin(2x+1)\}2$$

$$= \frac{5}{(1-x)^{2/3}}\left[1 + \frac{2x}{3(1-x)}\right] - 2\sin(4x+2)$$

[$\sin 2A = 2\sin A \cos A$]

$$= \frac{5}{3(1-x)^{5/3}}(3-x) - 2\sin(4x+2)$$

42 $y = \dfrac{x}{2}\sqrt{(a^2+x^2)} + \dfrac{a^2}{2}\log[x+\sqrt{(x^2+a^2)}]$

$$\frac{dy}{dx} = \left[\tfrac{1}{2}\sqrt{(a^2+x^2)} + \frac{x}{2}\frac{1}{2\sqrt{(a^2+x^2)}} 2x\right]$$

Substituting the values of $\frac{du}{dx}$ and $\frac{dv}{dx}$, this gives

$$x^y\left[y/x+\frac{dy}{dx}\log x\right]+y^x\left[(x/y)\frac{dy}{dx}+\log y\right]=0$$

or $\frac{dy}{dx}=-\frac{y\;x^{y-1}+y^x\log y}{x^y\log x+xy^{x-1}}$

8 $y=(x^x)^x \qquad \log y=x\log x^x=x^2\log x$

$$\frac{1}{y}\frac{dy}{dx}=2x\log x+x^2\frac{1}{x}=x(1+2\log x)$$

$$\frac{dy}{dx}=x\;(x^x)^x\;(1+2\log x)$$

(i)

9 $y=x^{(x^x)}, \qquad \log y=x^x\log x$

Take log again

$\log(\log y)=\log[x^x\log x]=\log x^x+\log(\log x)$

$=x\log x+\log(\log x)$

Differentiate both sides w r t x

$$\frac{1}{\log y}\;\frac{1}{y}\;\frac{dy}{dx}=x\frac{1}{x}+1\;\log x+\frac{1}{\log x}\;\frac{1}{x}$$

$$\frac{dy}{dx}=y\log y\left[1+\log x+\frac{1}{x\log x}\right]$$

$$=x^{(x^x)}\;x^x\log x\left[1+\log x+\frac{1}{x\log x}\right] \qquad \text{by (i)}$$

$$=x^{(x^x)+x}\left[\log x\,(1+\log x)+\frac{1}{x}\right]$$

10 $y=x^{(\log x)^{\log\log x}}$ Take log of both sides

$\log y=(\log x)^{\log\log x}\log x \qquad \frac{\log y}{\log x}=(\log x)^{\log\log x}$ (i)

Taking log again, we get

$\log(\log y)=\log(\log x)[\log\log x]+\log(\log x)$

or $\log(\log y)=\log(\log x)[\log\log x+1]$

Differentiate both sides w r t x, we get

$$\frac{1}{\log y}\;\frac{1}{y}\frac{dy}{dx}=\frac{1}{\log x}\;\frac{1}{x}[\log\log x+1]+\log(\log x)\;\frac{1}{\log x}\;\frac{1}{x}$$

$$=\frac{1}{x\log x}[2\log\log x+1]$$

$$\frac{dy}{dx}=\frac{y}{x}\;\frac{\log y}{\log x}[2\log\log x+1]$$

$$=\frac{1}{x\log_e 10}-\frac{\log_e 10}{x(\log_e x)^2}$$

46 $y=x\log x \qquad \frac{dy}{dx}=x\,\frac{1}{x}+1 \quad \log x=\log e+\log x=\log ex$

47 $\frac{dy}{dx}=\frac{1}{2\sqrt{[x+\sqrt{(x+\sqrt{x})}]}}\left[1+\frac{1}{2\sqrt{(x+\sqrt{x})}}\left(1+\frac{1}{2\sqrt{x}}\right)\right]$

48 The function $y=\log|x|$ for all real x except $x=0$ and
$\log|x|=\begin{matrix}\log x \text{ if } x>0\\ \log(-x) \text{ if } x<0\end{matrix}$

Hence $\frac{d}{dx}\log|x|=\left\{\frac{1}{x}, x>0, =\left(\frac{1}{-x}\right)(-1)=\frac{1}{x} \text{ if } x<0\right\}$

thus $\frac{d}{dx}\log|x|=\frac{1}{x}, x\neq 0$

Problem Set (B)

Trigonometric Transformations

Differentiate the following

1 $\tan^{-1}\frac{\sqrt{x}-x}{1+x^{3/2}}$

2 (a) $\tan^{-1}\sqrt{\left(\frac{1-\cos x}{1+\cos x}\right)}$ (b) $\tan^{-1}\sqrt{\left(\frac{1+\sin x}{1-\sin x}\right)}$

3 (a) $\tan^{-1}\frac{\sin x}{1+\cos x}$ (b) $\tan^{-1}\frac{\cos x}{1+\sin x}$ (M N R 83)

(c) $\tan^{-1}\frac{x}{\sqrt{(a^2-x^2)}}$

4 (a) $\tan^{-1}\frac{\cos x-\sin x}{\cos x+\sin x}$ (b) $\tan^{-1}\frac{a\cos x-b\sin x}{b\cos x+a\sin x}$

(c) $\tan^{-1}\frac{\sqrt{(1+x^2)}-1}{x}$

5 (a) $\tan^{-1}\frac{x}{1+\sqrt{(1-x^2)}}+\sin\left\{2\tan^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)}\right\}$

(b) $\tan x^n+\tan^n x-\tan^{-1}\frac{a+x^n}{1-ax^n}$

6 $\cot^{-1}\frac{\sqrt{(1+\sin x)}+\sqrt{(1-\sin x)}}{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}$

7 $\tan^{-1}\frac{5ax}{a^2-6x^2}$ 8 $\tan^{-1}\frac{3a^2x-x^3}{a(a^2-3x^2)}$

9 $\tan^{-1}\frac{\sqrt{(1+x^2)}+\sqrt{(1-x^2)}}{\sqrt{(1+x^2)}-\sqrt{(1-x^2)}}$ (Roorkee 80)

(b) $\tan^{-1}\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}$ (Roorkee 60)

$$\left(\frac{1}{y\log y}-\log x\right)\frac{dy}{dx}=y/x \qquad \frac{dy}{dx}=\frac{y^2\log y}{x(1-y\log x}$$

16 $x^y=e^{x-y}$ Taking log we get

$y\log x=(x-y)\log e=x-y$

$$y(1+\log x)=x \quad \text{or} \quad y=\frac{x}{1+\log x}$$

$$\frac{dy}{dx}=\frac{(1+\log x)\,1-x\,\frac{1}{x}}{(1+\log x)^2}=\frac{\log x}{(1+\log x)^2}$$

$$=\frac{\log x}{(\log e+\log x)^2}=\log x\,(\log ex)^{-2}$$

17 Taking log we get

$p\log x+q\log y=(p+q)\log(x+y)$

$$p\,\frac{1}{x}+q\,\frac{1}{y}\frac{dy}{dx}=(p+q)\,\frac{1}{x+y}\left(1+\frac{dy}{dx}\right)$$

$$\frac{p}{x}-\frac{p+q}{x+y}=\left(\frac{p+q}{x+y}-\frac{q}{y}\right)\frac{dy}{dx}$$

or $$\frac{py-qx}{x(x+y)}=\frac{py-qx}{y(x+y)}\frac{dy}{dx} \qquad \frac{dy}{dx}=\frac{y}{x}$$

18 $y=u+v$ $$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}$$

$u=\dfrac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$ Taking log

$\log u=\log x+\log\sin^{-1}x-\frac{1}{2}\log(1-x^2)$

$$\frac{1}{u}\frac{du}{dx}=\frac{1}{x}+\frac{1}{\sin^{-1}x}\,\frac{1}{\sqrt{(1-x^2)}}-\frac{1}{2}\,\frac{1}{1-x^2}\,(-2x)$$

$$\frac{du}{dx}=\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\left[\frac{1}{x}+\frac{1}{\sqrt{(1-x^2)}\sin^{-1}x}+\frac{x}{1-x^2}\right]$$

$v=\log\sqrt{[(1-x^2)]}=\frac{1}{2}\log(1-x^2)$ $\qquad \dfrac{dv}{dx}=-\dfrac{x}{1-x^2}$

$$\frac{dy}{dx}=\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}+\frac{x}{1-x^2}+\frac{x^2\sin^{-1}x}{(1-x^2)^{3/2}}+\frac{-x}{1-x^2}$$

$$=\frac{\sin^{-1}x}{(1-x^2)^{3/2}}[1-x^2+x^2]=\frac{\sin^{-1}x}{(1-x^2)^{3/2}}$$

19 $e^y=e^{\sin x}\dfrac{(1+\cos x)^n}{(1-\cos x)^m}$ Take log of both sides

$y\log e=\sin x\log e+n\log(1+\cos x)-m\log(1-\cos x)$

$$\frac{dy}{dx}=\cos x-n\,\frac{\sin x}{1+\cos x}+m\,\frac{\sin x}{1-\cos x}$$

23 If $y=\sin^{-1}\dfrac{1}{\sqrt{(1+x^2)}}+\tan^{-1}\dfrac{\sqrt{(1+x^2)}-1}{x}$, then

$$\frac{dy}{dx}=\frac{-1}{2\,(1+x^2)}$$

24 (*a*) If $y=\sec^{-1}\left(\dfrac{\sqrt{x}+1}{\sqrt{x}-1}\right)+\sin^{-1}\left(\dfrac{\sqrt{x}-1}{\sqrt{x}+1}\right)$, then $\dfrac{dy}{dx}=0$

(Roorkee 79, 67)

(*b*) If $y=\sin^{-1}x+\sin^{-1}\sqrt{(1-x^2)}$, then prove that $dy/dx=0$

25 If $y=\cos^{-1}\left(\dfrac{2\cos x+3\sin x}{\sqrt{(13)}}\right)$, then $\dfrac{dy}{dx}=1$

26 If $y=\sin^{-1}\left(\dfrac{5x+12\sqrt{(1-x^2)}}{13}\right)$, $\dfrac{dy}{dx}=\dfrac{1}{\sqrt{(1-x^2)}}$

27 (*a*) If $y=\sin\,(2\sin^{-1}x)$, $\dfrac{dy}{dx}=2\sqrt{\left(\dfrac{1-y^2}{1-x^2}\right)}$

(*b*) If $y=\log\tan\dfrac{x}{2}+\sin^{-1}(\cos x)$ $\dfrac{dy}{dx}=\operatorname{cosec}x-1$

28 If $x=a\,(t-\sin t)$, $y=a\,(1-\cos t)$, find dy/dx **(Roorkee 64)**

29 If $x=a\,(\cos t+\log\tan t/2)$ $y=a\sin t$, find dy/dx

30 If $x=2\cos t-\cos 2t$, $y=2\sin t-\sin 2t$, find the value of dy/dx

31 If $x=a\sin 2\theta\,(1+\cos 2\theta)$, $y=b\cos 2\theta\,(1-\cos 2\theta)$
prove that $dy/dx=(b\tan\theta)/a$

32 If $u=\sin^{-1}(x-y)$, $x=3t$, $y=4t^3$, then $\dfrac{du}{dt}=\dfrac{3}{\sqrt{(1-t^2)}}$

33 If $x=\cos^{-1}\dfrac{1}{\sqrt{(t^2+1)}}$, $y=\sin^{-1}\dfrac{t}{\sqrt{(1+t^2)}}$, show that
dy/dx is independent of t

34 (*a*) If $x=\dfrac{\sin^3 t}{\sqrt{(\cos 2t)}}$, $y=\dfrac{\cos^3 t}{\sqrt{(\cos 2t)}}$, find dy/dx at $t=\pi/6$

(*b*) If $x=\sin\theta\sqrt{(\cos 2\theta)}$, $y=\cos\theta\sqrt{(\sin 2\theta)}$, find
dy/dx at $\theta=\pi/4$

Solutions to Problem Set (B)

1 $y=\tan^{-1}\dfrac{\sqrt{x}-x}{1+\sqrt{(x)}\,x}=\tan^{-1}\sqrt{(x)}-\tan^{-1}x$

$$\frac{dy}{dx}=\frac{1}{1+x}\,\frac{1}{2\sqrt{(x)}}-\frac{1}{1+x^2}$$

2 (*a*) $y=\tan^{-1}\left(\tan\dfrac{x}{2}\right)=\dfrac{x}{2}$ $\qquad\dfrac{dy}{dx}=\frac{1}{2}$

24 $y\sqrt{(x^2+1)}=\log(\sqrt{(x^2+1)}-x)$ or $y=\frac{\log(}{}$

Differentiate by quotient formula

$$\frac{dy}{dx}=\frac{1}{(x^2+1)}\left[\sqrt{(x^2+1)}\ \frac{1}{\sqrt{(x^2+1)}-x}\left\{\frac{}{2\sqrt{(}}\right.\right.$$

$$-\frac{2x}{2\sqrt{(x^2+1)}}\log$$

$$(x^2+1)\frac{dy}{dx}=-1-\frac{x}{\sqrt{(x^2+1)}}\,y\sqrt{}$$

$$(x^2+1)\frac{dy}{dx}+xy+1=0$$

25 (a) $y=\frac{1}{2}\{\log(\sqrt{(1+x)}+\sqrt{(1-x)})-\log(\sqrt{}$

$$\frac{dy}{dx}=\frac{1}{2}\left[\frac{1}{\sqrt{(1+x)}+\sqrt{(1-x)}}\left\{\frac{1}{2\sqrt{(1+x)}}-\right.\right.$$

$$-\frac{1}{\sqrt{(1+x)}-\sqrt{(1-x)}}\left\{\frac{1}{2\sqrt{(1+x)}}+\frac{}{2}\right.$$

$$=\frac{-1}{4\sqrt{(1-x^2)}}\left[\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{\sqrt{(1+x)}+\sqrt{(1-x)}}+\frac{\sqrt{(1+x)}}{\sqrt{(1+x)}}\right.$$

$$=-\frac{1}{4\sqrt{(1-x^2)}}\left[\frac{2(1+x+1-x)}{(1+x)-(1-x)}\right]$$

$(a-b)^2+(a$

$$=-\frac{1}{4\sqrt{(1-x^2)}}\cdot\frac{4}{2x}=-\frac{1}{2x\sqrt{(1-x^2)}}$$

(b) Do yourself

(c) Put $x=\tan\theta$ in the value of y

$$y=\log\left(\frac{\sec\theta-1}{\sec\theta+1}\right)+\frac{\tan\theta}{\sec\theta}$$

$$=\log\frac{1-\cos\theta}{1+\cos\theta}+\sin\theta=\log\tan^2\frac{\theta}{2}+\sin\theta$$

or $y=2\log\tan\theta/2+\sin\theta$

$$\frac{dy}{d\theta}=2\ \frac{1}{\tan\theta/2}\sec^2(\theta/2)\ \frac{1}{2}+\cos\theta$$

or $\frac{dy}{d\theta}=2\ \frac{1}{2\sin\theta/2\cos\theta/2}+\cos\theta=\frac{2}{\sin\theta}+\cos$

$$\frac{dx}{d\theta}=2\tan\theta\sec\theta=2\frac{\sin\theta}{}$$

6. $y=\cot^{-1}\dfrac{(\cos x/2+\sin x/2)+(\cos x/2-\sin x/2)}{(\cos x/2+\sin x/2)-(\cos x/2-\sin x/2)}$

$=\cot^{-1}\dfrac{2\cos(x/2)}{2\sin(x/2)}$

or $y=\cos^{-1}\cot(x/2)=x/2$ $\qquad \dfrac{dy}{dx}=\frac{1}{2}$ $\qquad$ Ref 13 5

7 Do yourself

8 Do yourself

9 Put $x^2=\cos 2\theta$, $1+\cos 2\theta=2\cos^2\theta$, $1-\cos 2\theta=2\sin^2\theta$

$y=\tan^{-1}\dfrac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}=\tan^{-1}\tan(\pi/4+\theta)=\pi/4+\theta$

$y=\pi/4+\frac{1}{2}\cos^{-1}x^2 \qquad \dfrac{dy}{dx}=\frac{1}{2}\,\dfrac{-1}{\sqrt{(1-x^4)}}\,2x$

$=-\dfrac{x}{\sqrt{(1-x^4)}}$

10 (i), (ii) (iii) each is $2\tan^{-1}x$ if we put $x=\tan\theta$ by 23 2

$\dfrac{dy}{dx}=2\,\dfrac{1}{1+x^2}$

(iv) $y=\cos^{-1}\dfrac{x^{2n}-1}{x^{2n}+1}=\cos^{-1}\left\{-\left(\dfrac{1-x^{2n}}{1+x^{2n}}\right)\right\}=\pi-\cos^{-1}\dfrac{1-x^{2n}}{1+x^{2n}}$

Put $x^n=\tan\theta$, $\qquad y=\pi-\cos^{-1}\cos 2\theta=\pi-2\theta$

or $y=\pi-2\tan^{-1}x^n \qquad \dfrac{dy}{dx}=-\dfrac{2}{1+x^{2n}}\,nx^{n-1}$

11 Put $x=\sin\theta$ in 1st and $x=\cos\theta$ in 2nd

$y=3\theta=3\sin^{-1}x$ or $3\cos^{-1}x$

$\dfrac{dy}{dx}=\dfrac{3}{\sqrt{(1-x^2)}}$ or $-\dfrac{3}{\sqrt{(1-x^2)}}$

12 (a) Put $x=\sin\theta$

$y=\theta+\sin^{-1}\cos\theta=\theta+\sin^{-1}\sin(\pi/2-\theta)$

$=\theta+\pi/2-\theta=\pi/2$

$\dfrac{dy}{dx}=0$

(b) Put $\sqrt{x}=\cos\theta \qquad x=\cos^2\theta$

$y=\sin^{-1}\sin\theta+\theta=\theta+\theta=2\theta=2\cos^{-1}\sqrt{(x)}$

$\dfrac{dy}{dx}=-\dfrac{2}{\sqrt{(1-x)}}\;\dfrac{1}{2\sqrt{(x)}}=-\dfrac{1}{\sqrt{(x)}\sqrt{(1-x)}}$

13 (a) Put $y=\sin\theta$, and $\sqrt{(x)}=\sin\phi$

$y=\sin^{-1}[(\sin\theta\sqrt{(1-\sin^2\phi)}+\sin\phi\sqrt{(1-\sin^2\theta)}]$

or $y=\sin^{-1}(\sin\theta\cos\phi+\cos\theta\sin\phi)$

$=\sin^{-1}\sin(\theta+\phi)=\theta+\phi$

$= t\,\frac{(1-2t^2+1+t^2)}{1-2t^2} = \frac{t(2-t^2)}{1-2t^2}$

31 $\sin y = x\sin(a+y)$, $\quad x = \frac{\sin y}{\sin(a+y)}$

Differentiate w r t x

$$1 = \frac{\sin(a+y)\cos y - \sin y\cos(a+y)}{\sin^2(a+y)}\,\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin(a+y-y)} = \frac{\sin^2(a+y)}{\sin a}$$

32 Combining first two and the result with 3rd and again the result with 4th term we get

$$y = \frac{x^3}{(x-c_1)(x-c_2)(x-c_3)} = \frac{x}{x-c_1}\cdot\frac{x}{x-c_2}\cdot\frac{x}{x-c_3}$$

$$\log y = \log\frac{x}{x-c_1} + \log\frac{x}{x-c_2} + \log\frac{x}{x-c_3}$$

or $\log y = \{\log x - \log(x-c_1)\} + \{\quad\} + \{\quad\}$

$$\frac{1}{y}\frac{dy}{dx} = \left\{\frac{1}{x} - \frac{1}{x-c_1}\right\} + \{\quad\} + \{\quad\}$$

$$= -\frac{c_1}{x(x-c_1)} - \frac{c_2}{x(x-c_2)} - \frac{c_3}{x(x-c_3)}$$

$$\frac{dy}{dx} = \frac{y}{x}\left\{\frac{c_1}{c_1-x} + \frac{c_2}{c_2-x} + \frac{c_3}{c_3-x}\right\}$$

33 Taking log of both sides, we get

$$\log\cos\frac{x}{2} + \log\cos\frac{x}{2^2} + \log\cos\frac{x}{2^3} + \ldots \log\cos\frac{x}{2^n}$$

$$= \log\sin x - \log 2^n - \log\sin\frac{x}{2^n}$$

Differentiate w r t x

$$-\frac{1}{2}\tan\frac{x}{2} - \frac{1}{2^2}\tan\frac{x}{2^2} - \frac{1}{2^3}\tan\frac{x}{2^3} \ldots = \cot x - \frac{1}{2^n}\cot\frac{x}{2}$$

$$\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3} + \ldots = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x$$

Differentiate again w r t x

$$\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + \ldots = \operatorname{cosec}^2 x - \frac{1}{2^{2n}}\operatorname{cosec}^2\frac{x}{2^n}$$

When $n\to\infty$ $\quad 2^n\to\infty \quad \frac{x}{2^n}\to 0$

and $\quad \cos\frac{x}{2^n}\to 1 \quad \sin\frac{x}{2^n}\to\frac{x}{2}$

$$\frac{dy}{dz}=\frac{2}{x}$$

(b) $y=\dfrac{\tan^{-1} x}{1+\tan^{-1} x}$, $z=\tan^{-1} x$

$$y=\frac{z}{1+z}=\frac{z+1-1}{1+z}=1-\frac{1}{1+z}$$

$$\frac{dy}{dz}=\frac{1}{(1+z)^2}=\frac{1}{(1+\tan^{-1} x)^2}$$

18 Put $x=\sin\theta$ and $y=\sin\phi$ in the given equation

$$\cos\theta+\cos\phi=a(\sin\theta-\sin\phi)$$

or $$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=2a\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}$$

or $$\cos\frac{\theta+\phi}{2}\left[\cos\frac{\theta-\phi}{2}-a\sin\frac{\theta-\phi}{2}\right]=0$$

If $\cos\dfrac{\theta+\phi}{2}=0$, then $\dfrac{\theta+\phi}{2}=\dfrac{\pi}{2}$ $\quad\theta=\pi-\phi$ or $\sin\theta=\sin\phi$

or $x=y$,

but if we put $x=y$ in the given equation it is not satisfied and hence we must have

$$\cos\frac{\theta-\phi}{2}-a\sin\frac{\theta-\phi}{2}=0 \quad\text{or}\quad \cot\frac{\theta-\phi}{2}=a$$

$$\theta-\phi=2\cot^{-1}a \quad\text{or}\quad \sin^{-1}x-\sin^{-1}y=2\cot^{-1}a$$

Differentiate w r t x, $\dfrac{1}{\sqrt{(1-x^2)}}-\dfrac{1}{\sqrt{(1-y^2)}}\dfrac{dy}{dx}=0$

$$\frac{dy}{dx}=\sqrt{\left(\frac{1-y^2}{1-x^2}\right)}$$

19 $y=\tan^{-1}\dfrac{2t}{1-t^2}=2\tan^{-1}t$, $x=\sin^{-1}\dfrac{2t}{1+t^2}=2\tan^{-1}t$

Hence $y=x$ $\qquad \dfrac{dy}{dx}=1$

20 $y=\tan^{-1}\dfrac{5x-x}{1+5x\,x}+\tan^{-1}\dfrac{\frac{2}{3}+x}{1-\frac{2}{3}x}$

$y=\tan^{-1}5x-\tan^{-1}x+\tan^{-1}\frac{2}{3}+\tan^{-1}x$

or $y=\tan^{-1}5x+\tan^{-1}\frac{2}{3}$ $\qquad \dfrac{dy}{dx}=\dfrac{5}{1+25x^2}$

21 $y=2\tan^{-1}x+2\tan^{-1}x=4\tan^{-1}x$ $\qquad \dfrac{dy}{dx}=\dfrac{4}{1+x^2}$

22, Put $x=\sin\theta$, then $y=2\theta+\theta=3\theta=3\sin^{-1}x$ etc

$$\frac{dy}{dx}=\frac{2x}{2-\frac{x^2}{x^2+\frac{1}{x^2+\ldots}}}$$

(b) Proceed as in part (a)

37 (a) $\cos y=\frac{a+b\cos x}{b+a\cos x}\ (b>a)$

$\log\cos y=\log(a+b\cos x)-\log(b+a\cos x)$

$$\frac{1}{\cos y}(-\sin y)\frac{dy}{dx}=\frac{-b\sin x}{a+b\cos x}+\frac{a\sin x}{b+a\cos x}$$

$$=\frac{(a^2-b^2)\sin x}{(a+b\cos x)(b+a\cos x)}$$

or $\qquad \tan y\frac{dy}{dx}=\frac{(b^2-a^2)\sin x}{(a+b\cos x)(b+a\cos x)}$ (1)

We have to put the value of $\tan y$ in (1)

$$\tan y=\frac{\sqrt{(1-\cos^2 y)}}{\cos y}=\frac{\sqrt{\{(b+a\cos x)^2-(a+b\cos x)^2\}}}{a+b\cos x}$$

$$=\frac{\sqrt{\{(b^2-a^2)-(b^2-a^2)\cos^2 x\}}}{a+b\cos x}=\sqrt{(b^2-a^2)}\ \frac{\sin x}{a+b\cos x}$$

Hence from (1), we get

$$\sqrt{(b^2-a^2)}\frac{\sin x}{a+b\cos x}\frac{dy}{dx}=\frac{(b^2-a^2)\sin x}{(a+b\cos x)(b+a\cos x)}$$

$$\frac{dy}{dx}=\frac{\sqrt{(b^2-a^2)}}{b+a\cos x}$$

(b) $y=\frac{2}{\sqrt{(a^2-b^2)}}\tan^{-1}\left\{\sqrt{\left(\frac{a-b}{a+b}\right)}\tan\frac{x}{2}\right\}$

$$\frac{dy}{dx}=\frac{2}{\sqrt{(a^2-b^2)}}\ \frac{1}{\left(1+\frac{a-b}{a+b}\frac{\sin^2 x/2}{\cos^2 x/2}\right)}\ \sqrt{\left(\frac{a-b}{a+b}\right)}\ \text{sc}$$

$$=\left(\frac{1}{a+b}\right)\left(\frac{(a+b)\cos^2 x/2\sec^2 x/2}{a(\cos^2 x/2+\sin^2 x/2)+b(\cos^2 x/2-\sin}\right.$$

$$=\frac{1}{a+b\cos x}$$

Differentiating again, we get

$$\frac{d^2y}{dx^2}=-\frac{1}{(a+b\cos x)^2}(-b\sin x)=\frac{b\sin x}{(a+b\cos x)^2}$$

38 Proceed as above

39 $y=\log(1+x^2+x\sqrt{2})-\log(1+x^2-x\sqrt{2})+2\tan^{-1}\frac{x\sqrt{2}}{1-x^2}$

31 $x=a(\sin 2\theta+\frac{1}{2}\sin 2\theta)$, $y=b[\cos 2\theta-\frac{1}{2}(1+\cos 4\theta)]$

We have used $2\sin A\cos A=\sin 2A$, $2\cos^2 A=1+\cos 2A$

$$\frac{dx}{d\theta}=2a(\cos 2\theta+\cos 4\theta)=2a\,2\cos 3\theta\cos\theta$$

$$\frac{dy}{d\theta}=2b(\sin 4\theta-\sin\theta)=2b\,2\cos 3\theta\sin\theta$$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=\frac{b}{a}\tan\theta$$

32 $\sin u=x-y=3t-4t^3=3\sin\theta-4\sin^3\theta$

where $t=\sin\theta$, say

$$\sin u=\sin 3\theta \text{ or } u=3\theta=3\sin^{-1}t \qquad \frac{du}{dt}=\frac{3}{\sqrt{(1-t^2)}}$$

33 Put $t=\tan\theta$, then

$$x=\theta,\ y=\theta \qquad y=x \text{ or } \frac{dy}{dx}=1$$

34 From $x=\dfrac{\sin^3 t}{\sqrt{(\cos 2t)}}$, we have

$$\frac{dx}{dt}=\frac{3\sin^2 t\cos t\sqrt{(\cos 2t)}-\frac{1}{2\sqrt{(\cos 2t)}}(-2\sin 2t)\sin^3 t}{\cos 2t}$$

$$=\frac{3\sin^2 t\cos t\cos 2t+\sin 2t\sin^3 t}{(\cos 2t)^{3/2}}$$

And form $y=\dfrac{\cos^3 t}{\sqrt{(\cos 2t)}}$, we have

$$\frac{dy}{dt}=\frac{-3\cos^2 t\sin t\sqrt{(\cos 2t)}-\frac{1}{2\sqrt{(\cos 2t)}}(-2\sin 2t)\cos^3 t}{\cos 2t}$$

$$=\frac{-3\cos^2 t\sin t\cos 2t+\sin 2t\cos^3 t}{(\cos 2t)^{3/2}}.$$

$$\frac{dy}{dx}=\frac{-3\cos^2 t\sin t\cos 2t+\sin 2t\cos^3 t}{3\sin^2 t\cos t\cos 2t+\sin 2t\sin^3 t}.$$

Hence at $t=\pi/6$, we obtain

$$\frac{dy}{dx}=\frac{-3\,\frac{3}{4}\,\frac{1}{2}\,\frac{1}{2}+\frac{\sqrt{3}}{2}\,\frac{3\sqrt{3}}{8}}{3\,\frac{1}{4}\,\frac{\sqrt{3}}{2}\,\frac{1}{2}+\frac{\sqrt{3}}{2}\,\frac{1}{8}}=\frac{-\frac{9}{16}+\frac{9}{16}}{\frac{3\sqrt{3}}{16}+\frac{\sqrt{3}}{16}}=0$$

Problem Set (C)

Find dy/dx in Q 1 to 4

1 (a) $y=(\sin^{-1}x)^{x^2}$ (b) $y=x^x$ (M N R 79)

(c) $(\tan x)^{\log x}$ (d) $(x\log x)^{\log\log x}$ (Roorkee 81)

2 $y=(\cot x)^{\ln x}+(\tan x)^{c\ \ x}$

Problem Set (D)
Objective Questions

1 Let $y=\sin^{-1}\frac{2x}{1+x^2}$ where $0 < x < 1$ and $0 < y < \pi/2$, then $\frac{dy}{dx}$ is equal to (i) $\frac{2}{1+x^2}$ (ii) $\frac{2x}{1+x^2}$ (iii) $\frac{-2}{1+x^2}$, (iv) $\frac{-x}{1+x^2}$ (NCRT 78)

2 If $y=\sin^n x \cos nx$, then $\frac{dy}{dx}$

(i) $n \sin^{n-1} x \cos (n+1) x$ (ii) $n \sin^{n-1} x \sin (n+1) x$
(iii) $n \sin^{n-1} x \cos (n-1) x$, (iv) $n \sin^{n-1} x \cos nx$

3 If $x = t+1/t$, $y=t-1/t$ then $\frac{d^2y}{dx^2}$ is equal to

(i) $-4t(t^2-1)^{-2}$, (ii) $-4t^3(t^2-1)^{-3}$,
(iii) $(t^2+1)(t^2-1)^{-1}$, (iv) $-4t^2(t^2-1)^{-2}$

4 Given the parametric equations $x=f(t)$, $y=g(t)$, then $\frac{d^2y}{dx^2}$ equals

(i) $\dfrac{\frac{d^2y}{dt^2}\frac{dx}{dt}-\frac{dy}{dt}\frac{d^2x}{dt^2}}{(dx/dt)^2}$ (ii) $\dfrac{\frac{dx}{dt}\frac{d^2y}{dt^2}-\frac{d^2x}{dt^2}\frac{dy}{dt}}{(dx/dt)^3}$,

(iii) $\frac{d^2y}{dt^2}\Big/\frac{d^2x}{dt^2}$

5 If $2^x+2^y=2^{x+y}$, then $\frac{dy}{dx}$ is equal to

(i) $(2^x+2^y)/(2^x-2^y)$, (ii) $(2^x+2^y)/(1+2^{x+y})$,
(iii) $2^{x-y}\frac{2^y-1}{1-2^x}$, (iv) $(2^{x+y}-2^x)/2^y$

6 If $y=f\left(\frac{2x-1}{x^2+1}\right)$ and $f'(x)=\sin x^2$, then $\frac{dy}{dx}=$ (IIT 82)

7 The differential coefficient of log tan x is

(a) $2 \sec 2x$ (b) $2 \operatorname{cosec} 2x$
(c) $2 \sec^2 x$ (d) $2 \operatorname{cosec}^2 x$ (MNR 86)

8 The derivative of $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{(1-x^2)}$ at $x=\frac{1}{2}$ is (IIT 86)

22 (a) If $y=\dfrac{(x+1)^2\sqrt{(x-1)}}{(x+4)^2 e^x}$, find dy/dx **(Roorkee 78)**

(b) $y=\dfrac{2(x-\sin x)^{3/2}}{\sqrt{x}}$, find $\dfrac{dy}{dx}$ **(Roorkee 71)**

(c) $y=\dfrac{\cos^2 x\, 2x^2(x^3-2x+1)}{\tan x \cosh x\, e^{x-2}}$, find $\dfrac{dy}{dx}$ **(Roorkee 82)**

23 $y=x\log\left(\dfrac{x}{a+bx}\right)$, show that $x^3\dfrac{d^2y}{dx^2}=\left(x\dfrac{dy}{dx}-y\right)$

(Roorkee 76)

or $(a+bx)\,e^{y/x}=x$, then $x^3\dfrac{d^2y}{dx^2}=\left(x\dfrac{dy}{dx}-y\right)^2$ **(I I T 84)**

24 If $y\sqrt{(x^2+1)}=\log\{\sqrt{(x^2+1)}-x\}$, show that

$(x^2+1)\,dy/dx+xy+1=0$ **(Roorkee 78)**

25 (a) If $y=\log\left[\dfrac{\sqrt{(1+x)}+\sqrt{(1-x)}}{\sqrt{(1+x)}-\sqrt{(1-x)}}\right]^{1/2}$, then

$$\frac{dy}{dx}=-\frac{1}{2x\sqrt{(1-x^2)}}$$

(b) If $y=\log\dfrac{1+\sqrt{(x)}}{1-\sqrt{(x)}}$, then $dy/dx=\dfrac{1}{\sqrt{(x)}\,(1-x)}$

(c) If $y=\log\left(\dfrac{\sqrt{(x+1)}-1}{\sqrt{(x+1)}+1}\right)+\dfrac{\sqrt{(x)}}{\sqrt{(x+1)}}$, find dy/dx

(Roorkee 80)

26 If $y=\{x+\sqrt{(x^2+a^2)}\}^n$, then $dy/dx=\dfrac{ny}{\sqrt{(x^2+a^2)}}$

27 If $y=\dfrac{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}}$, $\dfrac{dy}{dx}=-\dfrac{2a^2}{x^3}\left\{1+\dfrac{a^2}{\sqrt{(a^4-x^4)}}\right\}$

28 If $x\sqrt{(1+y)}+y\sqrt{(1+x)}=0$, then $dy/dx=-\dfrac{1}{(1+x)^2}$

29 If $x^2+y^2=t-\dfrac{1}{t}$, $x^4+y^4=t^2+\dfrac{1}{t^2}$, then that $x^3y\,dy/dx=1$

30 If $x=\dfrac{3at}{1+t^3}$ $y=\dfrac{3at^2}{1+t^3}$, $dy/dx=\dfrac{t(2-t^3)}{1-2t^3}$

31 If $\sin y=x\sin(a+y)$, then $dy/dx=\dfrac{\sin^2(a+y)}{\sin a}$

32 If $y=1+\dfrac{c_1}{x-c_1}+\dfrac{c_2x}{(x-c_1)(x-c_2)}+\dfrac{c_3x^2}{(x-c_1)(x-c_2)(x-c_3)}$

show that $dy/dx=\dfrac{y}{x}\left\{\dfrac{c_1}{c_1-x}+\dfrac{c_2}{c_2-x}+\dfrac{c_3}{c_3-x}\right\}$

$$= \frac{\left(\frac{dx}{dt}\right)\left(\frac{d^2y}{dt^2}\right) - \left(\frac{d^2x}{dt^2}\right)\left(\frac{dy}{dt}\right)}{(dx/dt)^3}$$

5 Ans (iii)

6 Let $\frac{2x-1}{x^2+1} = z$ Then $y = f(z)$

$$\frac{dy}{dx} = \frac{d}{dz} f(z) \frac{dz}{dx} = f'(z) \frac{dz}{dx}$$

$$= \sin z^2 \frac{dz}{dx} \quad [\because f'(z) = \sin z^2]$$

$$= \sin \left(\frac{2x-1}{x^2+1}\right)^2 \times \frac{d}{dx}\left(\frac{2x-1}{x^2+1}\right)$$

$$= \sin \left(\frac{2x-1}{x^2+1}\right)^2 \times \frac{2(x^2+1) - 2x(2x-1)}{(x^2+1)^2}$$

$$= \sin \left(\frac{2x-1}{x^2+1}\right)^2 \times \frac{2(1+x-x^2)}{(x^2+1)^2}$$

7 Ans (b) 8 Ans 4 Put $x = \cos\theta$ etc

9 Ans 1 10 Ans (c) 11 Ans $1/e$

First note that $ln\ x = \log_e x$

$$f(x) = \log_x x \log_e x = \frac{\log_e \log_e x}{\log_e x}$$

$$\text{Hence } f'(x) = \frac{\left(\frac{1}{\log_e x} \cdot \frac{1}{x}\right) \log_e x - \frac{1}{x} \log_e \log_e x}{(\log_e x)^2}$$

$$= \frac{1}{x} \cdot \frac{(1 - \log_e \log_e x)}{(\log_e x)^2}$$

$$f'(e) = \frac{1}{e} \cdot \frac{1 - \log_e \log_e e}{(\log_e e)^2} = \frac{1}{e} \cdot \frac{1 - \log_e 1}{1}$$

$$= \frac{1}{e}(1 - 0) = 1/e \quad [\because \log_e 1 = 0]$$

12 Ans (c)

Solution, we have

$P'(x) = 2yy'$, $P''(x) = 2yy'' + 2y'^2$

and $P'''(x) = 2yy''' + 6y'y''$

Also $2\frac{d}{dx}(y^3 y'')$

$= 2[y^3 y''' + 3y^2 y' y'']$

$= y^2[2yy''' + 6y'y'']$

$= P(x) P'''(x)$

$$1\cdot\log y + x\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\sin y + (\log x)\cos y\frac{dy}{dx}$$

$$\left(\log y - \frac{\sin y}{x}\right) = \frac{dy}{dx}\left[\cos y\,\log x - \frac{x}{y}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x\log y - \sin y}{y\log x\cos y - x}\right]$$

6 (a) $x^y + y^x = a^b$ or $u + v = a^b$, $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$yx^{y-1} + x^y\log x\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x\log y = 0$$

by 2nd Method

$$\frac{dy}{dx} = -\frac{yx^{y-1} + y^x\log y}{xy^{x-1} + x^y\log x}$$

(b) Here $u + v = 1$

$$\frac{dy}{dx} = \frac{v\,\text{cosec}^2 x\log y - \frac{uy}{(1+x^2)\tan^{-1}x}}{u\log\tan^{-1}x + \frac{v\cot x}{y}}$$

7 (i) Take log and differentiate

(ii) Taking logarithm, we get

$x\log y + y\log x = \log 1 = 0$

Differentiating w r t x we get

$$x\frac{1}{y}\frac{dy}{dx} + 1\cdot\log y + y\frac{1}{x} + \frac{dy}{dx}\log x = 0$$

$$\frac{dy}{dx} = -\frac{\log y + y/x}{x/y + \log x}$$

(iii) Let $x^y = u$ and $y^x = v$ Then $u + v = 1$

Taking log, we get

$y\log x = \log u$ and $x\log y = \log v$

$$y\frac{1}{x} + \frac{dy}{dx}\log x = \frac{1}{u}\frac{du}{dx}$$

and $x\frac{1}{y}\frac{dy}{dx} + 1\cdot\log y = \frac{1}{v}\frac{dv}{dx}$

or $\frac{du}{dx} = x^y\left[\frac{y}{x} + \frac{dy}{dx}\log x\right]$

and $\frac{dv}{dx} = y^x\left[(x/y)\frac{dy}{dx} + \log y\right]$

Now $u + v = 1$ gives $\frac{du}{dx} + \frac{dv}{dx} = 0$

Two functions $f: X \to Y$ and $g: X \to Y$ are said to be equal iff $f(x)=g(x)$ for every $x \in X$ and written as $f=g$. A function $f: X \to Y$ is called a **constant function** if, for some $y_0 \in Y$, we have $f(x)=y_0$ for every $x \in X$. Let $I_x: X \to X$ be defined by $I_x(x)=x\ \forall\ x \in X$. Then I_x is called the identity function on X. If $B \subset A$, then the function

$$i: B \to A \quad i(b)=b \quad \forall\ b \in B$$

is called the **inclusion function**. If $f: X \to Y$ and $A \subset X$, then the mapping $g: A \to X$ defined by $g(x)=f(x)\ \forall\ x \in A$ is called the restriction of f to A and is denoted by f/A or f_A. It is evident that $f/A=f \cap (A \times Y)$. Also then f is called an extension of g. The mapping $f: X \to Y$ is said to be many one if two or more different elements in X have the same f-image in Y. The mapping f is said to be one one if different elements in X have different images in Y i.e. if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$ or equivalently

$$f(x_1)=f(x_2) \Rightarrow x_1=x_2$$

One-one mapping are also called **injection**.

The mapping f is said to be **into** if there is at least one element in Y which is not the f-image of any element in X. Note that in this case the range of f is a proper subset of Y, that is $f[x] \subset Y$ and $f[x] \neq Y$.

The mapping f is said to be **onto** if every element in Y is the f image of at least one element in X. In this case the range of f is equal to Y, that is $f[x]=Y$. Onto mappings are also called **surjection**. One one and onto mappings are called **bijection**.

Sequences. *If A is any set, then any function $s: N \to A$ is called a sequence in A where* N *denotes the set of all natural numbers.*

If s is a sequence in A the image $s(n)$ of any $n \in$ N is usually denoted by s_n. It is customary to denote the sequence s by the classical symbol $< s_n >_{n=1}^{\infty}$ or simply by $< s_n >$.

Sometimes we write it is

$$< s_1, s_2, \ldots, s_n, \ldots >$$

The image s_n of n is called the n^{th} term of the sequence. Note that the terms of a sequence need not be distinct and so the range set of a sequence may be finite or infinite. For example, the range set of the sequence $< (-1)^n >$ is $\{1, -1\}$ which consists of two

$$\therefore \log y + x\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}\sin y + (\log x)\cos y\,\frac{dy}{dx}$$

$$\left(\log y - \frac{\sin y}{x}\right) = \frac{dy}{dx}\left[\cos y\,\log x - \frac{x}{y}\right]$$

$$\frac{dy}{dx} = \frac{y}{x}\left[\frac{x\log y - \sin y}{y\log x\cos y - x}\right]$$

6 (a) $x^y + y^x = a^b$ or $u + v = a^b$, $\frac{du}{dx} + \frac{dv}{dx} = 0$

$$yx^{y-1} + x^y\log x\,\frac{dy}{dx} + xy^{x-1}\frac{dy}{dx} + y^x\log y = 0$$

by 2nd Method

$$\frac{dy}{dx} = -\frac{yx^{y-1} + y^x\log y}{xy^{x-1} + x^y\log x}$$

(b) Here $u + v = 1$

$$\frac{dy}{dx} = \frac{v\,\mathrm{cosec}^2 x\log y - \dfrac{uv}{(1+x^2)\tan^{-1}x}}{u\log\tan^{-1}x + \dfrac{v\cot x}{y}}$$

7 (i) Take log and differentiate

(ii) Taking logarithm, we get

$x\log y + y\log x = \log 1 = 0$

Differentiating w r t x, we get

$$x\frac{1}{y}\frac{dy}{dx} + 1\cdot\log y + y\frac{1}{x} + \frac{dy}{dx}\log x = 0$$

$$\frac{dy}{dx} = -\frac{\log y + y/x}{x/y + \log x}$$

(iii) Let $x^y = u$ and $y^x = v$ Then $u + v = 1$

Taking log, we get

$y\log x = \log u$ and $x\log y = \log v$

$$y\frac{1}{x} + \frac{dy}{dx}\log x = \frac{1}{u}\frac{du}{dx}$$

and $x\frac{1}{y}\frac{dy}{dx} + 1\cdot\log y = \frac{1}{v}\frac{dv}{dx}$

or $\frac{du}{dx} = x^y\left[\frac{y}{x} + \frac{dy}{dx}\log x\right]$

and $\frac{dv}{dx} = y^x\left[(x/y)\frac{dy}{dx} + \log y\right]$

Now $u + v = 1$ gives $\frac{du}{dx} + \frac{dv}{dx} = 0$

We shall use the letter y to denote a dependent variable

Domain of a variable If we give the independent variable x only those values which lie between $x=a$ and $x=b$, then all the numerical values taken collectively will be called the domain or interval of the variable The domain is said to be closed if a and b are included in it and is denoted by the symbol $[a,b]$ An open domain is denoted by (a, b) or $]a, b[$ Similarly the symbol $[a\ b)$ and $(a\ b]$ stand for semi open domains

Function If y depends upon x in such a manner that for every value of x in its domain of variation there corresponds a definite and unique value of y, then y is said to be a single valued function of x and is denoted by $y=f(x)$, f denoting the kind of dependence or relationship that exists between x and y

This relationship is often called functional relationship

$f(x_1)$ $f(x_2)$, , $f(x_r)$ are called functional values of $f(x)$ for $x=x_1$ x_2, x_r respectively

The essential thing about the definition of a function is that for each value of x there must correspond a definite value of $f(x)$ We must be in possession of a set of rules which determines for each value of x, in a certain interval, a definite value of the function These rules may take the shape of a single compact formula such as $f(x)=\sin x$ or a number of such formulae which apply to different parts of the domain of x For example

$$\left.\begin{aligned} f(x)&=\sin x \text{ for } 0\leqslant x\leqslant\frac{\pi}{2} \\ f(x)&=x \text{ for } \frac{\pi}{2}<x<\pi \\ f(x)&=\cos x \text{ for } x\geqslant\pi \end{aligned}\right\} \qquad (1)$$

In the first case $f(x)=\sin x$ is defined for values of x in any interval

In the second case $f(x)$ given by (1) is defined in the interval $(0,\infty)$

The above definition of a function of f brings about (1) idea of dependence of the function on x (2) idea of definiteness of the value of function for each value of x (3) idea of single valuedness of the function (4) idea of domain of variable x

We are accustomed to think that every function is capable of graphical representation Majority of functions are certainly capable of graphical representation but there are many functions which cannot be represented by a graph The function defined as follows is such a function

$$=\frac{y}{x}(\log x)^{\log\log x}[2\log\log x+1] \text{ by (i)}$$

11 $y=e^{x^{e^x}}+x^{e^{e^x}}+e^{x^{x^e}}=u+v+w$

$$\frac{dy}{dx}=\frac{du}{dx}+\frac{dv}{dx}+\frac{dw}{dx}$$

$u=e^{x^{e^x}}$ $\log u=x^{e^x}$ $\log e=x^{e^x}$

$\log(\log u)=e^x\log x$ Differentiate

$$\frac{1}{\log u}\frac{1}{u}\frac{du}{dx}=e^x\frac{1}{x}+e^x\log x$$

$$\frac{du}{dx}=u\log u\left[\frac{e^x}{x}+e^x\log x\right]$$

$$=e^{x^{e^x}}x^{e^x}\left[\frac{e^x}{x}+e^x\log x\right]$$

Similarly $\frac{dv}{dx}=x^{e^{e^x}}e^{e^x}\left[\frac{1}{x}+e^x\log x\right]$

$$\frac{dw}{dx}=e^{x^{x^e}}x^{x^e}x^{e-1}(1+e\log x)$$

Putting the values, we get the value of dy/dx

12 $y=e^{x+e^{x+e^{x+\cdots}}}=e^{x+y}$

$\log y=(x+y)\log e=x+y$ Hence $\frac{1}{y}\frac{dy}{dx}=1+\frac{dy}{dx}$

or $\frac{dy}{dx}\left(\frac{1}{y}-1\right)=1$ or $\frac{dy}{dx}=\frac{y}{1-y}$

13 (i) $y=(\sqrt{x})^y$ as above $\log y=y\log\sqrt{x}=\frac{1}{2}y\log x$

$$\frac{1}{y}\frac{dy}{dx}=\frac{1}{2}\left(\frac{dy}{dx}\log x+y\frac{1}{x}\right)$$

$\left(\frac{1}{y}-\frac{1}{2}\log x\right)\frac{dy}{dx}=\frac{1}{2}\frac{y}{x}$ $\therefore\ \frac{dy}{dx}=\frac{y^2}{2-y\log x}$

(ii) Proceed as in (i)

14 Proceed as above

15 $y=(a^x)^y=a^{x^y}$

$\log y=x^y\log a$ $\log\log y=y\log x+\log(\log a)$

$$\frac{1}{\log y}\frac{1}{y}\frac{dy}{dx}=\frac{dy}{dx}\log x+y\frac{1}{x}$$

think that they can get the value of $\frac{x^2-1}{x-1}$ by first dividing the numerator by denominator and then putting $x=1$, for we can divide by $x-1$ only when x is not equal to 1

Bounded and unbounded functions If for all values of x in a given interval, $f(x)$ is never greater than some fixed number M, the number M is said to be an upper bound for $f(x)$ in that interval whereas if $f(x)$ is never less than some number m, then m is called a lower bound

If one or both of the upper and lower bounds of a function are infinite the function is said to be unbounded otherwise it is said to be bounded

A rational integral function or a polynomial, is a function of the form

$$a_0x^n+a_1x^{n-1}+ \quad +a_{n-1}x+a_n$$

where a_0 a_1 a_n are constants and n is a positive integer

A rational function is defined as the quotient of one polynomial by another For example

$$\frac{3x+5}{2x^2+3x+6}$$

is a rational function

An algebraical function is a function which can be expressed as the root of an equation of the form

$$y^n+A_1y^{n-1}+A_2y^{n-2}+ \quad +A_{n-1}y+A_n=0$$

where A_1 A A_n are rational functions of x In particular rational function is also algebraical

A transcendental function is a function which is not algebraical Trigonometrical, exponential and logarithmic functions are examples of transcendental functions Thus

$$\sin x \quad \tan^{-1} x \quad e^{ax} \quad \log_b (ax+b)$$

are transcendental functions

Montonic functions The function $y=f(x)$ is said to be Monotonically increasing if corresponding to an increase in the value of x the value of y never decreases This means that if x_1 x_2 are any two values of x in the interval in which $f(x)$ is defined such that $x_2 > x_1$ then we have $f(x_2) \geqslant f(x_1)$ $f(x)$ is said to be strictly increasing if $x_2 > x_1 \Rightarrow f(x_2) > f(x_1)$

$=\cos x-n\tan x/2+m\cot x/2$ (13 5)

20 Change x^0 to $\frac{\pi x}{180}$ radians

$$y=\left[\log\left\{\log\sin\frac{\pi x}{180}\right\}\right]^7$$

$$\frac{dy}{dx}=7\left[\log\left\{\log\sin\frac{\pi x}{180}\right\}\right]^6\frac{1}{\log\sin\frac{\pi x}{180^\circ}}\ \frac{1}{\sin\frac{\pi x}{180^\circ}}$$

$$\times\cos\frac{\pi x}{180^\circ}\ \frac{\pi}{180^\circ}$$

$$=\frac{7\pi}{180^\circ}\left[\log\{\log\sin x^0\}\right]^6\ \frac{\cot x^0}{\log\sin x^0}$$

21 $y=x^n\log x+x(\log x)^n$

$$\frac{dy}{dx}=nx^{n-1}\log x+x^n\ \frac{1}{x}+xn(\log x)^{n-1}\ \frac{1}{x}+1(\log x)^n$$

$$=x^{n-1}(1+n\log x)+(\log x)^{n-1}[n+\log x]$$

22 (a) Take log etc

$$\frac{dy}{dx}=\frac{(x+1)^2\sqrt{(x-1)}}{(x+4)e^x}\left[\frac{5x-3}{2(x^2-1)}-\frac{x+6}{x+4}\right]$$

(b) $\frac{dy}{dx}=y\left\{\frac{3}{2}\ \frac{1-\cos x}{x-\sin x}-\frac{1}{2}x\right\}$

(c) $\frac{dy}{dx}=y\left[-2\tan x+2x\log 2+\frac{3x^2-2}{x^3-2x+1}\right.$

$$\left.-\frac{\sec^2 x}{\tan x}-\frac{\sinh x}{\cosh x}-1\right]$$

23 From the given relation

$y/x=\log x-\log(a+bx)$ Differentiate

$$\left(x\frac{dy}{dx}-y\right)\Big/x^2=\frac{1}{x}-\frac{1}{a+bx}\ b=\frac{a}{x(a+bx)}$$

$$x\frac{dy}{dx}-y=\frac{ax}{a+bx}\qquad (1)$$

Differentiate again w r t x

$$x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}=\frac{(a+bx)a-ax\,b}{(a+bx)^2}$$

or $\quad x\frac{d^2y}{dx^2}=\frac{a^2}{(a+bx)^2}$

$$x^3\frac{d^2y}{dx^2}=\frac{a^2x^2}{(a+bx)^2}=\left(x\frac{dy}{dx}-y\right)^2\ \text{by (1)}$$

Thus $f(x+a\lambda_1)=f(x)$, showing that $a\lambda_1$ is a period o
But since λ is the fundamental period of $f(x)$ we have

$$\lambda \leqslant a\lambda_1 \quad \text{or} \quad \lambda_1 \geqslant \lambda/a$$

Finally (1) and (2) show that λ/a is the fundamental peri

$$f(ax+b)$$

Remark The periodic function $f(x)=A \sin(\omega x+\rho)$, wh A, ω, ρ are constants, is called a *harmonic* with amplitude $|A$ frequency ω and initial phase ρ. Since the function $\sin x$ has period 2π, the function $A \sin(\omega x+\rho)$ has a period $2\pi/\omega$

§ 2 Limits

Consider the function $y=\dfrac{x^2-1}{x-1}$ The value of the function at $x=1$ is of the form 0/0 which is meaningless. In this case we cannot divide numerator by denominator since $x-1$ is zero

Now suppose x is not actually equal to 1 but very n equal to 1. Then $x-1$ is not equal to zero. Hence in this we can divide the numerator by denominator

$$\frac{x^2-1}{x-1}=x+1$$

Now if x is little greater than 1, then value of y will b greater than 2 and as x gets nearer to 1, y comes nearer to 2. Now the difference between y and 2 is

$$\frac{x^2-1}{x-1}-2=\frac{x^2-2x+1}{x-1}=\frac{(x-1)^2}{x-1}=x-1$$

This difference $(x-1)$ can be made as small as we ple letting x tend to 1

Thus we see that when x has fixed value 1, the value c meaningless but when x tends to 1, y tends to 2 and we say the limit of y is 2 when x tends to 1. This we write as

$$\lim_{x\to 1} \frac{x^2-1}{x-1}=2.$$

Definition of limit *The number A is said to be the limit of $f(x)$ at $x=a$ if for any arbitrarily chosen positive number ϵ however small but not zero, there exists a corresponding number δ greater than zero such that*

$$|f(x)-A| < \epsilon$$

for all values of x for which

$$=\frac{\cos^3\theta}{\sin^2\theta}+\frac{\cos^4\theta}{2\sin\theta}$$

$$=\cos\theta\cot^2\theta+\tfrac{1}{2}\cos^2\theta\cot\theta\cos\theta$$

$$=\cos\theta\,[\cot^2\theta+\tfrac{1}{2}\cot\theta\cos^2\theta]$$

$$=\frac{1}{\sqrt{(1+x)}}\left[\frac{1}{x}+\frac{1}{2}\frac{1}{\sqrt{x}}\frac{1}{1+x}\right]$$

26 Do yourself

27 $y=\dfrac{\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}-\sqrt{(a^2-x^2)}}$

Multiply above and below by $\sqrt{(a^2+x^2)}+\sqrt{(a^2-x^2)}$

$$y=\frac{(a^2+x^2)+(a^2-x^2)+2\sqrt{(a^4-x^4)}}{(a^2+x^2)-(a^2-x^2)}=\frac{a^2}{x^2}+\frac{\sqrt{(a^4-x^4)}}{x^2}$$

Now differentiate etc , 2nd term by quotient formula

28 $x\sqrt{(1+y)}+y\sqrt{(1+x)}=0 \qquad x\sqrt{(1+y)}=-y\sqrt{(1+x)}$

Square $\quad x^2(1+y)=y^2(1+x)$

or $\quad (x^2-y^2)+xy\,(x-y)=0$

or $\quad (x-y)\,(x+y+xy)=0$

Since $y=x$ does not satisfy the given equation, we have

$$x+y+xy=0, \qquad y=-\frac{x}{1+x}=-\left[1-\frac{1}{1+x}\right]$$

$$\frac{dy}{dx}=-\frac{1}{(1+x)^2}$$

29 $x^2+y^2=t-1/t,\ x^4+y^4=t^2+1/t^2$

$x^4+y^4=(t-1/t)^2+2=(x^2+y^2)^2+2, \qquad 2x^2y^2+2=0$

or $\quad x^2y^2=-1$ Differentiate

$$2xy^2+x^2\,2y\frac{dy}{dx}=0, \qquad x^2y\frac{dy}{dx}=xy^2$$

or $\quad x^3y\dfrac{dy}{dx}=x^2y^2$ or $x^3y\dfrac{dy}{dx}=-1, \qquad x^2y^2=-1$

30 $x=\dfrac{3at}{1+t^3},\ y=\dfrac{3at^2}{1+t^3}$

Clearly $y=tx$ Differentiate w r t x

$$\frac{dy}{dx}=t\cdot 1+x\,\frac{dt}{dx} \tag{1}$$

Now $\dfrac{dx}{dt}=3a\,\dfrac{1+t^3-t\,3t^2}{(1+t^3)^2}=\dfrac{3a\,(1-2t^3)}{(1+t^3)^2}$ (2)

$$\frac{dy}{dx}=t+\frac{3at}{1+t^3}\cdot\frac{(1+t^3)^2}{3a\,(1-2t^3)} \text{ by (2)}$$

But $|f(x)-A| > |A| - |f(x)|$

Then (1) gives $|A| - |f(x)| < k$ or $2k - |f(x)| < k$

or $k < |f(x)|$ as desired

§ 3 Algebra of limits

Theorem I *The limit of a sum is equal to the sum of the limits*

Proof Let $\lim_{x\to a} f(x)=A$, and $\lim_{x\to a} \phi(x)=B$

Then we have to prove that

$$\lim_{x\to a} [f(x)+\phi(x)]=A+B$$

The proof of this theorem follows directly from the definition of the limit We have simply to show that for any pre assigned positive number ϵ, a number δ can be determined such that

$$|f(x)+\phi(x)-A-B| < \epsilon$$

whenever x lies in the interval $]a-\delta, a+\delta[$ Now by hypothesis,

$\lim_{x\to a} f(x)=A$ so that

$|f(x)-A| < \epsilon/2$, whenever $0<|x-a|<\delta_1$ (1)

Similarly, $|\phi(x)-B| < \epsilon/2$, whenever $0 < |x-a| < \delta_2$ (2)

Choosing δ to be smaller of the two numbers δ_1 and δ_2, it follows from (1) and (2) that*

$$|f(x)+\phi(x)-A-B| = |f(x)-A+\phi(x)-B|$$
$$\leqslant |f(x)-A| + |\phi(x)-B| < \epsilon/2+\epsilon/2=\epsilon$$

whenever $0 < |x-a| < \delta$

Hence $\lim_{x\to a} [f(x)+\phi(x)]=A+B$

The theorem can evidently be extended to any finite number of functions In the same way, we can prove that

$$\lim_{x\to a} [f(x)-\phi(x)]=A-B$$

Theorem II *The limit of a product is equal to the product of the limits*

Proof Using the notation of theorem I we have to prove in this case that $|f(x)\phi(x)-AB| < \epsilon$

whenever $0 < |x-a| < \delta$

Now $|f(x)\phi(x)-AB|$

$$= |f(x)\phi(x)-A\phi(x)+A\phi(x)-AB|$$
$$\leqslant |f(x)\phi(x)-A\phi(x)| + |A\phi(x)-AB|$$
$$= |\phi(x)|\,|f(x)-A| + |A|\,|\phi(x)-B|$$

By hypothesis $\lim_{x\to a} \phi(x)=B$ so that ϕ is surely bounded in the neighbourhood of $x=a$ Hence $|\phi(x)| < M$ for all the values of x such that $0<|x-a|<\delta$

Then $|f(x)\phi(x)-AB| < M|f(x)-A| + |A|\,|\phi(x)-B|$

Hence corresponding results for infinity are

$$\cot x-\frac{1}{x/2^n}\ \frac{1}{2^n}=\frac{1}{x}-\cot x$$

and $$\operatorname{cosec}^2 x-\frac{1}{2^{2n}}\ \frac{2^{2n}}{x^2}=\operatorname{cosec}^2 x-\frac{1}{x^2}$$

34 $y=\sqrt{(\sin x+y)}$, $\quad y^2=\sin x+y$

or $\quad y^2-y=\sin x$, $\quad (2y-1)\dfrac{dy}{dx}=\cos x$

or $$\frac{dy}{dx}=\frac{\cos x}{2y-1}$$

35 (*a*) We know that when $x<1$, then $\underset{n\to\infty}{\text{Lt}}\ x^n=0$

Now $\quad (1-x)(1+x)=1-x^2$,

$(1-x)(1+x)(1+x^2)=1-x^4$

$(1-x)(1+x)(1+x^2)(1+x^4)=1-x^8$

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots(1+x^{2^{n-1}})=1-x^{2^n}$$

Making $n\to\infty$, we get

$$(1-x)(1+x)(1+x^2)(1+x^4)(1+x^8)\ldots=1$$

Taking log of both sides and differentiating, we get

$$-\frac{1}{1-x}+\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots=0$$

$$\frac{1}{1+x}+\frac{2x}{1+x^2}+\frac{4x^3}{1+x^4}+\frac{8x^7}{1+x^8}+\ldots=\frac{1}{1-x}$$

(*b*) $(1+x+x^2)(1-x+x^2)=(1+x^2)^2-x^2=1+x^2+x^4$,

$(1+x+x^2)(1-x+x^2)(1-x^2+x^4)=(1+x^2+x^4)(1-x^2+x^4)$

$=1+x^4+x^8$ etc as in part (a)

36 (*a*) y is given in terms of continued fraction

$$y=x^2+\cfrac{1}{x^2+\cfrac{1}{x^2+\ldots}}$$

$$y=x^2+\frac{1}{y}\quad\text{or}\quad y^2=x^2y+1$$

Differentiate

$$2y\frac{dy}{dx}=x^2\frac{dy}{dx}+2xy,\qquad \frac{dy}{dx}=\frac{2xy}{2y-x^2}=\frac{2x}{2-\dfrac{x^2}{y}}$$

§ 4 Composite Functions

Given two functions f and g such that Rang f ⊂ Dom g, we form a new function, called the composite function (g o f) of g and f defined as follows for every x in the domain of f,

$$(g \circ f)(x) = g(f(x))$$

Inverse Function *Let the functions $y=f(x)$ be defined on a set X and let $Y=f(X)$ be the range set of f i e Y is the set of values $f(x)$ for all $x \in X$ Let f be injective (i e one-one) Then we can define a function f^{-1} (called the inverse of f) on the set Y as follows $f(x)=y$ if and only if $f^{-1}(y)=x$*

It is clear from the above definition that if $x=f^{-1}(y)$ is an inverse of the function $y=f(x)$, then the function $y=f(x)$ is an inverse of the function $x=f^{-1}(y)$ Therefore, the functions $y=f(x)$ and $x=f^{-1}(y)$ are also called **mutually inverse**

For example, consider the function $f(x)=3x$ defined on the closed interval [0 1] The interval [3, 0] is the set of values of f The function $f^{-1}(y)=\frac{1}{3}y$ defined on [0, 3] is then the inverse of the given function f

As another example, on the closed interval [0, 1], consider the function f defined as follows

$$y=f(x)=\begin{cases} x \text{ if } x \text{ is rational,} \\ 1-x \text{ if } x \text{ is irrational} \end{cases}$$

The function $x=f^{-1}(y)$ defined on [0, 1] by

$$x=f^{-1}(y)=\begin{cases} y \text{ if } y \text{ is rational} \\ 1-y \text{ if } y \text{ is irrational} \end{cases}$$

is the inverse of f

Remark *A sufficient condition for the existence of an inverse function is a strict monotony of the original function If the function increases (decreases) then the inverse function also increases (decreases)*

§ 5 Continuity

The intuitive concept of continuity is derived from geometrical considerations If the graph of the function $y=f(x)$ is a continuous curve it is natural to call the function continuous This requires that there should be no sudden changes in the value of the function A small change in x should produce only a small change in y Moreover for the graph to be a continuous running

$$\frac{dy}{dx}=\frac{2x+\sqrt{2}}{1+x^2+x\sqrt{2}}-\frac{2x-\sqrt{2}}{1+x^2-x\sqrt{2}}+2\ \frac{1}{1+\frac{2x^2}{(1-x^2)^2}}\text{ d c of }\frac{x\sqrt{2}}{1-x^2}$$

$$=\frac{2x(-2\sqrt{2}x)+\sqrt{2}\,2\ (1+x^2)}{(1+x^2)^2-2x^2}+\frac{2(1+x^2)^2}{1+x^4}\ \frac{\sqrt{2}(1+x^2)}{(\quad x^2)}$$

$$=\frac{2\sqrt{2}(1-x^2)}{1+x^4}+\frac{2\sqrt{2}(1+x^2)}{1+x^4}=\frac{4\sqrt{2}}{1+x^4}$$

40 $y=\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x-1}{\sqrt{3}}$

$$\frac{dy}{dx}=\frac{1}{3}\ \frac{1}{x+1}-\frac{1}{6}\ \frac{2x-1}{x^2-x+1}+\frac{1}{\sqrt{3}}\ \frac{1}{1+\frac{(2x-1)^2}{3}}\ \frac{2}{\sqrt{3}}$$

$$=\frac{1}{6}\ \frac{2(x^2-x+1)-(x+1)(2x-1)}{x^3+1}+\frac{2}{3}\ \frac{3}{3+4x^2-4x+1}$$

$$=\frac{1}{6}\ \frac{-3x+3}{x^3+1}+\frac{1}{2(x^2-x+1)}$$

$$=\frac{1}{2}\ \frac{1-x}{x^3+1}+\frac{x+1}{2(x+1)(x^2-x+1)}$$

$$=\frac{1}{2(x^3+1)}\ [1-x+x+1]=\frac{1}{x^3+1}$$

41 Let $y=(\log_{\cos x}\sin x)(\log_{\sin x}\cos x)^{-1}+\sin^{-1}\frac{2x}{1+x^2}$

$$=(\log_{\cos x}\sin x)^2+2\tan^{-1}x\left[\quad \log_b a=\frac{1}{\log_a b}\right]$$

$$=\left(\frac{\log_e \sin x}{\log_e \cos x}\right)^2+2\tan^{-1}x$$

$$\frac{dy}{dx}=2\left(\frac{\log\sin x}{\log\cos x}\right)\ \frac{\cot x\log\cos x+\tan x\log\sin x}{(\log\cos x)^2}+\frac{2}{1+x^2}$$

Hence at $x=\frac{\pi}{4}$ we have

$$\frac{dy}{dx}=2\ \frac{\log(1/\sqrt{2})}{\log(1/\sqrt{2})}\ \frac{1\ \log(1/\sqrt{2})+1\ \log(1/\sqrt{2})}{\{\log(1/\sqrt{2})\}^2}+\frac{2}{1+(\pi^2/16)}$$

$$=\frac{-8}{\log 2}+\frac{32}{\pi^2+16}=8\left(\frac{4}{\pi^2+16}-\frac{1}{\log 2}\right)$$

42 $y=e^x(a\cos x+b\sin x)$ (1)

$y_1=e^x(-a\sin x+b\cos x)+e^x(a\cos x+b\sin x)$

$y_1=e^x(-a\sin x+b\cos x)+y$ (2)

$y_2=e^x(-a\cos x-b\sin x)+e^x(-a\sin x+b\cos x)+y_1$

or $y_2=-y+(y_1-y)+y_1$ by (1) and (2)

or $y_2-2y_1+2y=0$

43 Do yourself

Remark We here make some important remarks about continuity

(i) For a function $f(x)$ to be continuous at $x=a$, it is necessary that $\lim_{x \to a} f(x)$ must exist

(ii) The function must be defined at the point of continuity

(iii) The value of δ depends upon the values of ϵ and a

(iv) The interval I may be any one of the forms

$$]a, b[\,,\ ]-\infty, b\,[\,,\ ]\,a, \infty\,[\,,\ ]-\infty, \infty\,[=R$$

Continuity from left and continuity from right

Let f be a function defined on an open interval I and let $a \in I$ We say that f is continuous from the left at a if $\lim_{x \to a-0} f(x)$ exists and is equal to $f(a)$ Similarly f is said to be continuous from the right at a if $\lim_{x \to a+0} f(x)$ exists and is equal to $f(a)$

Continuity in an open interval A function f is said to be continuous in the open interval $]\,a, b\,[$ if it is continuous at each point of the interval.

continuity in a closed interval

Let f be a function defined on the closed interval $[a, b]$ We say that f is continuous at a if it is continuous from the right at a and also that f is continuous at b if it is continuous from the left at b Further, f is said to be continuous on the closed interval $[a, b]$, if, (i) it is continuous from the right at a (ii) continuous from the left at b and (iii) continuous on the open interval $]a, b[$

Similar definitions can be given for continuity in the intervals of the form $[a, \infty, [\,,\ ]-\infty\ \ b\,]\,,\ [a, b\,[\,,\ ]\,a, b\,]$

An alternative definition of continuity

A function f is said to be continuous at $a \in I$ iff $\lim_{x \to a} f(x)$ exists, is finite and is equal to $f(a)$ otherwise the function is discontinuous at $x=a$

This definition of continuity follows immediately from the definition of the limit and the definition of continuity Thus a function is said to be continuous at a if

$$f(a+h)=f(a-h)=f(a)$$

This is a working formula for testing the continuity of a function at a given point

Definition 2 *If a function is not continuous at a point, then it*

9 The derivative of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ is

10 The differential coefficient of $f(\log(x))$ where $f(x)=\log x$ is

(a) $x/\log x$ (b) $\log x/x$,

(c) $(x \log x)^{-1}$ (d) None of these

11 If $f(x)=\log_x(\ln x)$ Then $f'(x)$ at $x=e$ is ((I I T 85)

12 If $y^2=P(x)$, a polynomial of degree 3, then

$$2\frac{d}{dx}\left(y^3\frac{d^2y}{dx^2}\right)$$

equals

(a) $P'''(x)+P'(x)$ (b) $P''(x)\,P'''(x)$,

(c) $P(x)\,P'''(x)$ (d) a constant (I I T 88)

Hints and Answers

Problem Set (D)

1 Ans (i)

Put $x=\tan\theta$ Then $y=\sin^{-1}\frac{2\tan\theta}{1+\tan^2\theta}=\sin^{-1}\sin 2\theta$

$$y=2\theta=2\tan^{-1}x \text{ and so } \frac{dy}{dx}=\frac{2}{1+x^2}$$

2 Ans (i)

3 Ans (ii) We have

$$\frac{dx}{dt}=1-1/t^2,\ \frac{dy}{dt}=1+1/t^2$$

$$\frac{dy}{dx}=\frac{1+(1/t^2)}{1-(1/t^2)}=\frac{t^2+1}{t^2-1}$$

and $\frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{t^2+1}{t^2-1}\right)\frac{dt}{dx}$

$$=\frac{2t(t^2-1)-2t(t^2+1)}{(t^2-1)^2}\times\frac{t^2}{t^2-1}$$

$$=-4t^3/(t^2-1)^3$$

4 Ans (ii)

$$\frac{dy}{dx}=\frac{dy/dt}{dx/dt}$$

$$\therefore \frac{d^2y}{dx^2}=\frac{d}{dt}\left(\frac{dy/dt}{dx/dt}\right)\frac{dt}{dx}$$

$$=\frac{(d^2y/dt^2)\left(\frac{dx}{dt}\right)-\left(\frac{d^2x}{dt^2}\right)\frac{dy}{dt}}{(dx/dt)^2}\times\frac{1}{(dx/dt)}$$

Proof This theorem can easily be proved by applying the definition of continuity and theorems on limits proved earlier. Thus to prove (i), we proceed as follows

Since f and g are continuous at a, we have

$$\lim_{x\to a} f(x)=f(a), \lim_{x\to a} g(x)=g(a)$$

Then

$$\lim_{x\to a} [f(x)+g(x)]=f(a)+g(a)$$

or $$\lim_{x\to a} (f+g)(x)=(f+g)(a)$$

This shows that $f+g$ is continuous at a

We can prove the other parts of the theorem in the same manner

Theorem II *If f is continuous, then so is $|f|$*

Proof Let f be continuous at a point a of its domain. Then given $\epsilon>0$, there exists $\delta>0$ such that

$$|x-a|<\delta \Rightarrow |f(x)-f(a)|<\epsilon \quad (1)$$

But $||f(x)|-|f(a)|| \leqslant |f(x)-f(a)|$

that is, $||f|(x)-|f|(a)| \leqslant |f(x)-f(a)| \quad (2)$

From (1) and (2), we see that

$$|x-a|<\delta \Rightarrow ||f|(x)-|f|(a)|<\epsilon \quad (3)$$

Thus we see from (3) that $|f|$ is continuous at a

(C) Continuity of the composite and inverse functions

Continuity of the composite function *If the function $y=f(x)$ is continuous at the point $x=x_0$ and the function $v=\phi(y)$ is continuous at the point $y_0=f(x_0)$ then the composite function $v=\phi[f(x)]$ is continuous at the point $x=x_0$*

Continuity of an Inverse function *If a function $y=f(x)$ is defined continuous and strictly monotonic on the interval $X=[a,b]$ then there exists a single-valued inverse function $x=\phi(y)$ defined, continuous and strictly monotonic on the range $f(X)$ of the function $y=f(x)$*

§ 5 Differentiability

Derivative at a point **Definition** *Let I denote the open interval $]a, b[$ in R and let $x_0 \in I$ Then a function*

2

Limits, Continuity and Differentiability

§ 1 Functions on any set Ordered pair and Cartesian Product

We denote an ordered pair by the symbol (a, b) Two ordered pairs (a, b) and (c, d) are equal if and only if $a=c$ and $b=d$ Cartesian Product of two sets A and B denoted by $A \times B$ is the set of all ordered pairs (a, b) such that $a \in A$ and $b \in B$, that is,

$$A \times B = \{(a, b) \quad a \in A \text{ and } b \in B\}$$

Definition of defferent types of functions *Let X and Y be two non empty sets A subest f of X × Y is called a function from X to Y iff to* each $x \in X$, *there exists a* unique y in Y such that $(x, y) \in f$

The other terms used for functions are "mappings", transformations and "operators" We denote this mapping by

$$f \quad X \Rightarrow Y \text{ or } X \overset{f}{\Rightarrow} Y$$

We call X, the domain of f and Y the co domain of f The unique element y in Y assigned to $x \in X$ is called the image of x under f or the value of f at x and is denoted by $f(x)$ Also x is called a pre-image of y Note that there may be more than one pre images of y The graph of f is the subset of $X \times Y$ defined by $\{(x, f(x) \in X)\}$ The range of is the set of all images under f and is denoted by $f(Y)$ Thus

$$f[X] = \{y \in Y \quad y = f(x) \text{ for some } x \in X\} = f(x) \quad x \in X\}$$

If $A \subset X$, then the set $\{f(x) \quad x \in A\}$ is called the image of A under f and is denoted by $f[A]$ If $B \subset Y$, then the set $\{x \in X \quad f(x) \in B\}$ is called the inverse image of B under f and is denoted by $f^{-1}[B]$

elements only whereas the range of the sequence $\left\langle \frac{n}{n+1} \right\rangle$ i e of the sequence $< \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}$ > is the infinite set

Real valued functions *If A is any set, then a function from A into* R (*the set of reals*) *is called a real valued function on A*

We denote the collection of all real valued functions on A by R^A The collection R^A inherits an algebraic structure from R in as much as we can define addition, multiplication and scalar multiplication as follows

If $\quad a \in A, r \in R, f, g \in R^A,$

then $\quad (f+g)(a) = f(a) + g(a)$

$(fg)(a) = f(a)\, g(a)$

$(rf)(a) = r\,[f(a)]$

Characteristic function *Let A be a subset of a set X The characteristic function* x_A *of the set A is defined by*

$$x_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A, \text{ i e }, x \in X - A \end{cases}$$

A function which is 0 everywhere is the characteristic function of the empty set and the function which is identically 1 on X is the characteristic function of X

It is evident that two sets have the same characteristic function iff they are identical i e

$$x_A = x_B \Leftrightarrow A = B$$

§ 2 **Real Valued Functions of a real Variable**

Constant A symbol which retains the same value throughout a set of operations is called a constant

Variable A symbol which can take a number of values is called a variable *A continuous variable is one which can take* every numerical value or every numerical value from one given number to another

Independent variable A variable which may take up any arbitrary value assigned to it is called an independent variable We shall usually denote the independent variable by x

Dependent variable A variable which assumes its value as a result of a second variable or system of variables which take up any set of arbitrary values assigned to them is called a dependent variable

$f(x)$ is indeterminate at $x=a$. Other indeterminate forms are ∞/∞, $\infty-\infty$, $0\times\infty$, 1^∞, 0^0, ∞^0.

L Hospital's Rule If $\phi(x)$ and $\psi(x)$ are functions such that $\phi(a)=0$, and $\psi(a)=0$ Then

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)}=\lim_{x\to a}\frac{\phi'(x)}{\psi'(x)}$$

Important Note While applying L'Hospital's rule, we are not to differentiate $\frac{\phi(x)}{\psi(x)}$ by the rule for finding the differential coefficient of the quotient of two functions But we are to differentiate the numerator and denominator separately

The L-Hospital rule is also applicable to the form ∞/∞

Some important expansions

(i) $\sin x=x-\frac{x^3}{3!}+\frac{x^5}{5!}-$

(ii) $\cos x=1-\frac{x^2}{2!}+\frac{x^4}{4!}-$

(iii) $\tan x=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\ldots$

In $\sin x$, $\cos x$, $\tan x$, x is measured in radians in the above expansions

(iv) $e^x=1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+$

(v) If $|x|<1$, then $\log_e(1+x)=x-\frac{1}{2}x^2+\frac{1}{3}x^3-\frac{1}{4}x^4+$

Problem Set (A)

1 If $f(x)=ax^2+bx+c$ find $f(0)$, $f(a)$ and $f(b)$

2 If $f(x)=x^2-1/x^2$ prove that $f(x)=-f(1/x)$

3 If $f(x)=(x-1)/x$ for all real numbers except $x=0$ and $g(u)=u^2+1$ for all real numbers u find (i) $g[f(1)]$ and (ii) $f[g(-1)]$

4 If $f(x)=\frac{x^2+2}{x-1}$ when $x<3$ and $f(x)=\frac{\sin x}{x-3}$ when $x>3$, for what values of x is the function not defined

5 Let $f(x)=\begin{cases}1, & \text{if } x \text{ is a rational number}\\ 0, & \text{if } x \text{ is an irrational number}\end{cases}$

be a function from R to R where R is the set of real numbers

$$f(x)=1 \text{ when } x \text{ is rational}$$
$$f(x)=0 \text{ when } x \text{ is irrational}$$

The functions defined as above are generally known as **single valued functions** When y has more than one value for a value of x we call y a **multiple valued function of x** For example

$$y=x^2,\ y^2=x,\ (x>0),\ y=\sin^{-1}x\ (-1 \leqslant x \leqslant 1)$$

Here y given by the first equation is a single valued function of x, y given by the second equation is a double valued function of x, and y given by third equation is a many valued function of x

Explicit and Implicit functions A function is said to be explicit when expressed directly in terms of the independent variable or variables *e g*

$$y=e^{-x}\ x^n \quad y=r \sin \theta$$

But if the function cannot be expresssd directly in terms of the independent variable (or variables), the function is said to be implicit, *e g*

$$x^2y^2+4xy+3y+5x+6=0$$

Here y is implicit function of x

Odd and Even functions A function is said to be odd if it changes sign when the sign of variable is changed,

i e if $\quad f(-x)=-f(x)$

For example, $\sin^3 x \cos x$ is an odd function since

$$\sin^3(-x)\cos(-x)=-\sin^3 x \cos x$$

A function is said to be even if its sign does not change when the sign of the variable is changed,

i e $\quad f(-x)=f(x)$

For example ax^4+bx^2+c is an even function since its sign is unaltered by changing x into $-x$

Note —In the definition of a variable only real values are meant In the definition of a function y is required to have definite and finite value The values like

$$\frac{0}{0}, \frac{5}{0}, \frac{a}{0},$$

are inadmissible They have no meanings

The function $y=\dfrac{x^2-1}{x-1}$ is not defined for $x=1$ since y is of the form $0/0$ when $x=1$ which has no meaning The students must not

14 Is $\cos\sqrt{t}$ a periodic function? If yes, then find the period. If not, then give reasons to explain your answer (Roorkee 82)

15 (a) Is the function $\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}$ defined for all values of x? Indicate the values of x for which it is defined and real. Find the limit as $x \to 0$ (Roorkee 79)

(b) Evaluate $\underset{h\to 0}{\text{Lt}} \frac{\sqrt{(x+h)}-\sqrt{x}}{h}$ (Roorkee 83)

(c) Use the formula $\underset{x\to 0}{\text{Lt}} \frac{a^x-1}{x}=\log a$ to find

$$\underset{x\to 0}{\text{Lt}} \frac{2^x-1}{(1+x)^{1/2}-1}$$ (IIT 82)

16 Evaluate

$$\underset{h\to 0}{\text{Lim}} \frac{(a+h)^2 \sin(a+h)-a^2 \sin a}{h}$$ (IIT 80, MNR 85)

17 Evaluate the following limits

(a) $\underset{x\to 0}{\text{Lim}} \frac{\sin x}{x}$ (IIT 75, Roorkee 78)

(b) $\underset{x\to \pi/2}{\text{Lim}} \frac{\cos x}{x-\pi/2}$

(c) $\underset{x\to 1}{\text{Lim}} \frac{x-1}{2x^2-7x+5}$ (IIT 76)

$\underset{\theta\to \pi/2}{\text{Lim}} (\sec\theta-\tan\theta)$ (IIT 7?)

(e) $\underset{x\to \infty}{\text{Lim}} [x-\sqrt{(x^2+x)}]$ (IIT 7?)

(f) $\underset{x\to 0}{\text{Lim}} \frac{\sqrt{(1+\sin x)}-\sqrt{(1-\sin x)}}{x}$

(g) $\underset{x\to 0}{\text{Lim}} \frac{\sqrt[3]{(1+\sin x)}-\sqrt[3]{(1+\sin x)}}{x}$

18 Evaluate the following limits

(a) $\underset{x\to a}{\text{Lim}} \frac{\sqrt{(a+2x)}-\sqrt{(3x)}}{\sqrt{(3a+x)}-2\sqrt{x}}$ $(a\neq 0)$ (IIT ?)

(c) $\underset{x\to 1}{\text{Lim}} (1-x)\tan(\pi x/2)$ (IIT ?)

(c) $\underset{\alpha\to \pi/4}{\text{Lim}} \frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$ (IIT ?)

(d) $\underset{x\to 1}{\text{Lim}} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3}$ (IIT ?)

Similarly the function $f(x)$ is said to be Monotonically decreasing if $f(x_2) \leqslant f(x_1)$ whenever $x_2 > x_1$ and strictly decreasing if $x_2 > x_1 \Rightarrow f(x_2) < f(x_1)$ In this case y never increases for an increase in the value of x The graph of such functions either always rises or always falls For example the function $y = \sin x$ is monotonically increasing in the interval $0 < x \leqslant \frac{1}{2}\pi$ and monotonically decreasing in the interval

$$\tfrac{1}{2}\pi \leqslant x \leqslant \pi$$

Test for monotonocity If $f'(x) > 0$ in $[a, b]$, then $f(x)$ is increasing in $[a, b]$ if $f'(x) < 0$ in $[a, b]$, then $f(x)$ is decreasing in $[a, b]$

Periodic Functions A function $f(x)$ is said to be periodic if it is defined for all real x and if there exists a non zero real number λ such that $f(x+\lambda)=f(x)$ for every x We call then λ the period of $f(x)$ If λ is a period of $f(x)$ then any number $n\lambda$ where $n=\pm 1, \pm 2, \pm 3$ is also a period of $f(x)$ The smallest positive period of $f(x)$ (if such a period exists) is called the fundamental period or primitive period Furthermore if $f(x)$ and $g(x)$ each have period λ, then the function $af(x)+bg(x)$ also has period λ For example, the function $\sin x$ is periodic The period is found as follows Let λ be the period of $\sin x$ Then $\sin(x+\lambda)=\sin x$ for every x Putting $x=0$ we see that λ must satisfy $\sin \lambda = 0$ This gives $\lambda = n\pi$ Again

$$\sin(n\pi + x) = (-1)^n \sin x = \sin x$$

only if n is even Hence the period of $\sin x$ is $2n\pi$ where n is a non zero integer The number 2π is the primitive period or simply period of $\sin x$

Theorem *If $f(x)$ is a periodic function with fundamental period λ, then the function $f(ax+b)$ where $a > 0$ is periodic with fundamental period λ/a*

Proof Firstly $f[a(x+\lambda/a)+b] = f[(ax+b)+\lambda] = f(ax+b)$ since λ is the period of the function $f(x)$ This shows that λ/a is the period of the function $f(ax+b)$ To show that λ/a is the fundamental period of $f(ax+b)$ let λ_1 be any positive period of $f(ax+b)$ so that $f[a(x+\lambda_1)+b] = f(ax+b)$ (1)

Let us take an arbitrary point x from the domain of definition of the function $f(x)$ and put $x = (x-b)/a$

$$\text{Then } f(ax+b) = f\left(a\,\frac{x-b}{a}+b\right) = f(x) = f[a(x+\lambda_1)+b]$$

$$= f(ax+b+a\lambda_1) = f(x+a\lambda_1)$$

(b) Evaluate the limit $\lim_{x\to 0} \frac{\sin x}{x}$

A function $f(x)$ is defined as

$$f(x)=\begin{cases}1 \text{ when } x\neq 0\\ 2 \text{ when } x=0\end{cases}$$

Does the limit of $f(x)$ as $x\to 0$ exist? Explain your answer (Roorkee

26 (a) A function f is defined as

$$f(x)=\frac{x^2-4x+3}{x^2-1} \text{ for } x\neq 1,$$
$$=2 \text{ for } x=1$$

Test the continuity of the function at $x=1$ (IIT

(b) Determine the values of a, b, c for which the function

$$f(x)=\frac{\sin(a+1)x+\sin x}{x} \text{ for } x<0$$
$$= c \quad \text{for } x=0$$
$$=\frac{(x+bx^2)^{1/2}-x^{1/2}}{bx^{3/2}} \text{ for } x>0$$

is continuous at $x=0$ (IIT 83)

27 Test the following functions for continuity

(a) $f(x)=x\sin 1/x$, $x\neq 0$, $f(0)=0$ at $x=0$

(b) $f(x)=2^{1/x}$ when $x\neq 0$ $f(0)=3$ at $x=0$

(c) $f(x)=1/(1-e^{-1/x})$, $x\neq 0$, $f(0)=0$ at $x=0$

28 (i) Show that the function ϕ defined as

$$\phi(x)=\begin{cases}0 & \text{for } x=0\\ \frac{1}{2}-x & \text{for } 0<x<\frac{1}{2}\\ \frac{1}{2} & \text{for } x=\frac{1}{2}\\ \frac{3}{2}-x & \text{for } \frac{1}{2}<x<1\\ 1 & \text{for } x=1\end{cases}$$

has three points of discontinuity which you are required to find Also draw the graph of the function

(ii) Contruct the graph of the function given below

$$f(x)=\begin{cases}x-1, & x<0\\ \frac{1}{4} & x=0\\ x^2, & x>0\end{cases}$$

Find $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$ Discuss the continuity of $f(x)$ at $x=0$, (Roorkee 83)

29 (a) Consider the function f defined by

$$f(x)=x-[x]$$

$$0 < |x-a| < \delta$$

where $|x-a|$ means the absolute value of $x-a$ without any regard to sign

Meaning of $|x-a| < \delta$

Since $|x-a|$ means the absolute value of $x-a$ without regard to sign the inequality $|x-a| < \delta$ means that the difference between x and a taken positively is less than δ Thus

(*i*) If $x > a$, then $x-a < \delta$, (*ii*) if $x < a$, then $a-x < \delta$

In other words if $x > a$ then $x < a+\delta$ and if $x < a$, then $x > a-\delta$ *Hence $|x-a| < \delta$ means that x can be assigned any value between $a-\delta$ and $a+\delta$*

Right hand and left hand limits

If x approaches a from the right, that is, from larger values of x than a, the limit of f as defined before is called the right hand limit of $f(x)$ and is written as

$$\lim_{x\to a+0} f(x) \quad \text{or} \quad f(a+0)$$

The working rule for finding the right hand limit is

"Put $a+h$ for x in $f(x)$ and make h approach zero"

In short, we have $\quad f(a+0)=\lim_{h\to 0} f(a+h)$

Similarly if x approaches a from the left that is, from smaller values of x than a the limit of f is called the left hand limit and is written as $\quad \lim_{x\to a-0} f(x) \quad$ or $\quad f(a-0)$

In this case, we have $f(a-0)=\lim_{h\to 0} f(a-h)$,

If both right hand and left hand limits of f, as $x\to a$, exist and are equal in value, their common value, evidently will be the limit of f as $x\to a$, If, however either or both of these limits do not exist, the limit of f as $x\to a$ does not exist Even if both these limits exist but are not equal in value then also the limit of f as $x\to a$ does not exist

Theorem I *If a function f possesses a finite limit at a point a, then a neighbourhood of a exists on which f is bounded*

Proof It is an immediate consequence of the definition of limit of a function at a point

Theorem II *Let a function f have a limit A at a point a and let $A\neq 0$ Then there exist numbers $k > 0$ and $\delta > 0$ such that*

$$|f(x)| > k \text{ for all } x \in\ ]a-\delta, a+\delta[$$

Proof Let $k=\frac{1}{2}|A|$ Then a number $\delta > 0$ can be found such that for all x in $]a-\delta, a+\delta[$, we have

$$|f(x)-A| < k \qquad (1)$$

(v) Let $f(x)=1+x$ for $0 \leq x \leq 2$
$=3-x$ for $2 < x \leq 3$
Determine the form of $g(x)=f(f(x))$ and hence find the points of discontinuity of g, if any (I.I.T. 83)

(vi) Let $f(x)=\frac{x^2}{2}$, $0 \leq x \leq 1$
$=2x^2-3x+3/2$, $1 \leq x \leq 2$
Discuss the continuity of f, f' and f'' on $[0, 2]$ (I.I.T. 83)

(vii) Given the function $f(x)=1/(1-x)$. Find the points of discontinuity of the function $f(x)$ and the composite functions $f\{f(x)\}$ and $f[f\{f(x)\}]$

34 Let a function $f: R \to R$ satisfy the equation $f(x+y)=f(x)+f(y)$ for all $x, y, \in R$, show that
(i) if f is continuous at the point $x=a$, then it is continuous for all $x \in R$
(ii) if f is continuous at $x=0$, then it is continuous for all $x \in R$ (I.I.T. 81)
(iii) if f is continuous, then $f(x)=x f(1)$ for all $x \in R$

35 Suppose the function f satisfies the conditions
(a) $f(x+y)=f(x) f(y)$ for all x and y
(b) $f(x)=1+x\, g(x)$ where $\lim_{x\to 0} g(x)=1$
Show that the derivative $f'(x)$ exists and $f'(x)=f(x)$ for all x

36 Show that the function f defined on R by $f(x)=1$ when x is rational and $f(x)=-1$ when x is irrational is discontinuous for all $x \in R$

37 If $f(x)=(1/x) \sin x^2$, $x\neq 0$, $f(0)=0$, discuss the continuity and differentiability of $f(x)$ at $x=0$

38 (a) If f is twice differentiable function such that
$f''(x)=-f(x)$, and $f'(x)=g(x)$,
$h(x)=[f(x)]^2+[g(x)]^2$, find $h(10)$ if $h(5)=11$ (I.I.T. 82)
(b) Let R be the real numbers and $f: R\to R$ such that for all x and y in R, $|f(x)-f(y)| \leq |x-y|^3$
Prove that $f(x)$ is a constant (I.I.T. 8?)

39 Let $f(x+y)=f(x) \cdot f(y)$ for all x and y. Suppose $f(5)=2$ and $f'(0)=3$. Find $f'(5)$ (I.I.T. 8?)

40 (a) Find the derivative of

Since $\lim_{x\to a} f(x)=A$ and $\lim_{x\to a} \phi(x)=B$,
corresponding to any $\epsilon>0$, we can find a positive number $\delta<\delta$ such that

$$|f(x)-A|<\epsilon/2M \text{ and } |\phi(x)-B|<\epsilon/2(|A|+1)$$

whenever $0<|x-a|<\delta$. Thus

$$|f(x)\phi(x)-AB|<M(\epsilon/2M)+|A|\,\epsilon/2(|A|+1)<\epsilon,$$

whenever $0<|x-a|<\delta$

Hence $\lim_{x\to a} f(x)\phi(x)=AB$

The above result can immediately be generalized to any finite number of functions

Theorem III *The limit of a quotient is equal to the quotient of the limits provided the limit of the denominator is not zero*

Proof Let $\lim_{x\to a} f(x)=A$ and $\lim_{x\to a} \phi(x)=B\neq 0$

$$\text{Now } \left|\frac{f(x)}{\phi(x)}-\frac{A}{B}\right|=\left|\left\{\frac{f(x)}{\phi(x)}-\frac{f(x)}{B}\right\}+\left\{\frac{f(x)}{B}-\frac{A}{B}\right\}\right|$$

$$=\left|\frac{f(x)}{B\phi(x)}\{B-\phi(x)\}+\frac{1}{B}\{f(x)-A\}\right|$$

$$\leqslant^{*}\frac{|f(x)|}{|B|\,|\phi(x)|}\,|B-\phi(x)|+\frac{1}{|B|}\,|f(x)-A| \qquad (1)$$

By hypothesis $\lim_{x\to a} f(x)=A$, and $\lim_{x\to a} \phi(x)=B$

The functions f and ϕ are surely bounded in the neighbourhood of the point $x=a$. Let M be the upper bound of f and m the lower bound of ϕ so that $|f(x)|\leqslant M$ and $|\phi(x)|>m$

We may then write (1) as

$$\left|\frac{f(x)}{\phi(x)}-\frac{A}{B}\right|\leqslant\frac{M}{m|B|}\,|B-\phi(x)|+\frac{1}{|B|}\,|f(x)-A| \qquad (2)$$

Since $\lim_{x\to a} f(x)=A$ and $\lim_{x\to a} \phi(x)=B$
corresponding to any $\epsilon>0$, we can find numbers δ_1 and δ_2 such that

$$|f(x)-A|<|B|(\epsilon/2) \text{ whenever } 0<|x-a|<\delta_1$$

$$\text{and } |\phi(x)-B|<\frac{m|B|}{M}\,\epsilon/2 \text{ whenever } 0<|x-a|<\delta_2$$

Choosing δ to be smaller than δ_1 and δ_2 we see from (2) that

$$\left|\frac{f(x)}{\phi(x)}-\frac{A}{B}\right|<\epsilon/2+\epsilon/2=\epsilon \text{ whenever } 0<|x-a|<\delta$$

Hence $\lim_{x\to a}\dfrac{f(x)}{\phi(x)}=\dfrac{A}{B}$, provided $B\neq 0$

*Since the sum of the absolute values is not less than the absolute value of the sum

47 Examine the continuity and differentiability in $-\infty < x < \infty$, of the following function

$f(x) = 1$ in $-\infty < x < 0$

$f(x) = 1 + \sin x$ in $0 \leqslant x < \pi/2$

$f(x) = 2 + (x - \pi/2)^2$ in $\pi/2 \leqslant x < \infty$

48 A function f is defined as follows

$f(x) = x$ for $0 \leqslant x \leqslant 1$ and $f(x) = 2 - x$ for $x \geqslant 1$

Test the character of the function at $x = 1$ as regards its continuity and differentiability. Also draw the graph.

49 Show that the function f defined by

$f(x) = x\{1 + \frac{1}{3}\sin(\log x^2)\}$ $x \neq 0$, and $f(0) = 0$,

is everywhere continuous but has no differential coefficient at the origin.

50 A function f is defined as follows

$f(x) = x^p \cos(1/x)$, $x \neq 0$, $f(0) = 0$

What conditions should be imposed on p so that

(i) f may be continuous at $x = 0$

(ii) f may have a differential coefficient at $x = 0$?

51 Let $f(x) = \sqrt{(x)}\{1 + x\sin(1/x)\}$ for $x > 0$

$f(x) = -\sqrt{(-x)}\{1 + x\sin(1/x)\}$ for $x < 0$,

$f(0) = 0$,

Show that $f'(x)$ exists everywhere and is finite except at $x = 0$ where its value is $+\infty$.

52 Draw the graph of the function

$$y = |x - 1| + |x - 2|$$

in the interval [0, 3] and discuss the continuity and differentiability of the function in this interval.

53 Separate the intervals in which the polynomial

$$2x^3 - 15x^2 - 36x + 1$$

is increasing or decreasing.

54 Show that $x^3 - 2x^2 + 2x + 12$

is monotonically increasing in every interval.

55 (a) Show that

$$x/(1+x) < \log(1+x) < x \text{ for } x > 0$$

(b) Show that $1 + x\log\{x + \sqrt{(x^2+1)}\} \geqslant \sqrt{(1+x^2)}$ for all $x \geqslant 0$ (IIT 83)

56 Show that the function $f: R \to R$ defined by

$$f(x) = x[1 + \tfrac{1}{3}\sin\log x^2]$$

curve, it should possess a definite direction at each point But the continuity as defined in pure analysis is quite distinct from the intuitive or the geometrical concept of the term Sometimes drawing a graph is difficult We now give the arithmetical definition of continuity given by Cauchy

(A) **Cauchy's definition of continuity**

Definition I *A real valued function f defined on an open interval I is said to be continuous at $a \in I$ iff for any arbitrarily chosen positive number ϵ, however small, we can find a corresponding number δ such* $|f(x)-f(a)|<\epsilon$ (1)

for all values of x for which $|x-a|<\delta$

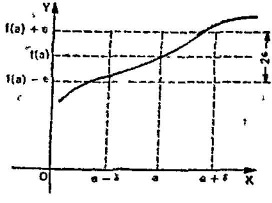
Also we say that f is a continuous function if it is continuous at every $x \in I$ In other words f is continuous at a if given $\epsilon>0$, we can find a $\delta>0$ such that

$$|x-a|<\delta \Rightarrow |f(x) \quad f(a)|<\epsilon$$

This means that the function f will be continuous at $x=a$ if the difference between $f(a)$ and the value of $f(x)$ at any point in the interval $]a-\delta, a+\delta[$ can be made less than a pre assigned positive quantity ϵ Notice that we choose δ after we have chosen ϵ

A geometrical interpretation of the above definition is immediate Correspoding to any pre assigned positive number ϵ, we can determine an interval of width 2δ about the point $x=a$ (see the figure) such that for any point x lying in the interval $]a-\delta \ a+\delta[$, $f(x)$ is confined to lie between $f(a)-\epsilon$ and $f(a)+\epsilon$

The inequality (1) may be written in the form of an equality



as $f(x)=f(a)+\eta$, where $|\eta|<\epsilon$

(h) $f(x)=\begin{cases} x^4, & x^2 < 1 \\ x, & x^2 \geqslant 1 \end{cases}$

Discuss the existence of limit at $x=1$ and $x=-1$ (Roorkee 87)

Solutions

$f(0)=a\cdot 0+b\cdot 0+c=c,$

$f(a)=a\cdot a^2+b\cdot a+c=a^3+ab+c$

$f(b)=a\cdot b^2+b\cdot b+c=ab^2+b^2+c$

2 $f(1/x)=(1/x)^2-1/(1/x)^2=1/x^2-x^2=-f(x)$

3 (i) $g[f(1)]=g\left[\frac{1-1}{1}\right]=g(0)=0+1=1,$

(ii) $f[g(-1)]=f[(-1)^2+1]=f(2)=\frac{2-1}{2}=\frac{1}{2}$

4 Ans The function is not defined at $x=1$ and $x=3$

5 (i) Since $\frac{1}{3}$ is a rational number, by definition of f, we have $f(\frac{1}{3})=1$

(ii) Since $\sqrt{7}$ is an irrational number, we have by definition of f $f(\sqrt{7})=0$

(iii) $(f\,o\,f)(1.4327)=f(f(1.4327))$ [∵ 1.4327 is rational]

$=f(1)$ [∵ 1 is rational

$=1$ since 1 is rational

(iv) $(f\,o\,f)(\sqrt{3})=f(f(\sqrt{3}))=f(0)$ [∵ $\sqrt{3}$ is irrational]

$=1$ [∵ 0 is rational]

6 We have, $f(x)=(a^x+a^{-x})/2$ $f(y)=(a^y+a^{-y})/2$

$f(x+y)=\{a^{x+y}+a^{-(x+y)}\}/2, f(x-y)=\{a^{x-y}+a^{-(x-y)}\}/2.$

$f(x+y)+f(x-y)=\frac{1}{2}[a^{x+y}+a^{-x-y}+a^{x-y}+a^{-x+y}]$

$=\frac{1}{2}[a^x(a^y+a^{-y})+a^{-x}(a^y+a^{-y})]$

$=\frac{1}{2}(a^x+a^{-x})(a^y+a^{-y})=2f(x)f(y)$

7 $f(-1)=3^{-(-1)}-1=3-1=2.$

$f(\pi/6)=\tan(\pi/12)=\tan 15°=2-\sqrt{3}$ since $0<\pi/6<\pi$,

$f(2\pi/3)=\tan(\pi/3)=\sqrt{3}$, since $0<2\pi/3<\pi$,

$f(4)=\frac{4}{4^2-2}=\frac{2}{7}$ and $f(6)=\frac{6}{6^2-2}=\frac{3}{17}$,

since both the points $x=4$, $x=6$ belong to the closed interv[al] $[\pi, 6]$

8 $\varphi(\psi(x))=\varphi(3^x)=(3^x)^2+1=3^{2x}+1$

and $\psi(\varphi(x))=\psi(x^2+1)=3^{x^2+1}$

9 (a) (i) The function is defined for all positive values of x except unity This means that the domain of definition of the function consists of the interval $]0, 1[$ and $]1, \infty[$

is said to be **discontinuous** *at the point and the point is called a point of discontinuity of this function*

Kinds of discontinuities

The point a is called a *discontinuity of the first kind* of the function $f(x)$ if both $f(a-0)$ and $f(a+0)$ exist but are not equal. More specifically, in this case the continuity is said to be *non removable of first kind*.

If both $f(a-0)$ and $f(a+0)$ exist such that

$$f(a-0)=f(a+0)\neq f(a),$$

then a is called a point of *removable discontinuity* of $f(x)$.

The point a is called a *discontinuity of second kind* of the function $f(x)$ if at least one of the limits $f(a-0)$ and $f(a+0)$ does not exist. Recall that the limits $f(a-0)$ and $f(a+0)$ are said to exist if they are finite and definite. Thus, if one or both of the limits $f(a-0)$ and $f(a+0)$ is infinite, then a is a point of discontinuity of second kind. In this case we sometimes say that $f(x)$ has an *infinite discontinuity* at $x=a$.

Note. Continuity of a function f at a point a is called a *local property* of f since it depends on the behaviour of f only in the immediate *nhd* of a. A property of f which concerns the whole domain of f is called a *global property*. Thus continuity of f on its domain is a global property.

Jump of a function at a point. If $f(a+0)$ and $f(a-0)$ both exist, then their non negative difference $f(a+0)\sim f(a-0)$ is called the jump in the function at a. A function having a finite number of jumps in a given interval is called **piecewise continuous or sectionally continuous**.

(B) *Heine's definition of continuity.* Let a function f be defined on some neighbourhood of a point a. Then f is said to be continuous at a iff for every sequence $a_1, a_2, \ldots, a_n, \ldots$ of real numbers for which

$$\lim_{n\to\infty} a_n=a, \text{ we have } \lim_{n\to\infty} f(a_n)=f(a)$$

Theorem I. *If the real valued functions f and g defined on I are continuous at $a\in I$, then so are*

(i) $f+g$, (ii) $f-g$, (iii) $cf\ (c\in\mathbb{R})$, (iv) fg, (v) $\frac{f}{g}$

provided $g(a)\neq 0$

(v) No $f(x)$ is defined only for $x > 3$ and $p(x)$ is defined for $x > 3$ and for $x < 2$

10 (i) Since $\sin x$ has period 2π the function $\sin(x/3)$ has the period 6π,

(ii) Since $\tan x$ has period π, the function $\tan 4x$ has period $\pi/4$

(iii) Ans Period is 1

(iv) $|\cos x| = \sqrt{(\cos^2 x)} = \sqrt{\left(\frac{1+\cos 2x}{2}\right)}$

Now since the function $\cos 2x$ has period π, the given function has also the period π

(v) $\sin^4 x + \cos^4 x = (\sin^2 x + \cos^2 x)^2 + 2\sin^2 x \cos^2 x$
$= 1 - \frac{1}{2}\sin^2 2x$

$= 1 - \frac{1}{4}(1 - \cos 4x) = \frac{3}{4} + \frac{1}{4}\cos 4x$

Since the function $\cos 4x$ has period $2\pi/4 = \pi/2$, the given function also has period $\pi/2$

(vi) Ans Period $= 6\pi$ (Hint Use Theorem of § 1)

11 Yes, for consider the function λ defined by

$$\lambda(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

For any rational number r, we have

$$\lambda(r+x) = \lambda(x) = \begin{cases} 1 & \text{for rational } x \\ 0 & \text{for irrational } x \end{cases}$$

But there is no least number in the set of positive rational numbers

As another example any constant function $f(x) = c$ (c a constant) is such a function since any real number α is a period but there is no least number in the set of positive real numbers

12 (i) Suppose, if possible, $\cos x^2$ has a period λ Then we must have $\cos(\lambda + x)^2 = \cos x^2$ for all real x

This gives $(\lambda + x)^2 = 2n\pi \pm x^2$ (1)

or $x^2 + 2\lambda x + \lambda^2 \mp x^2 = 2n\pi$

But the relation (1) is impossible, since its right hand side is an integral multiple of 2π, whereas the left hand member contains a linear of quadratic function of a continuous variable x and so (1) cannot hold for real values of x

For example, for $x = 0$, and $x = \lambda$, (1) reduces to

$\lambda^2 = 2n\pi$ and $4\lambda^2 \mp \lambda^2 = 2n\pi$

whence dividing, get $4 \mp 1 = 1$ which is false

$$f : I \to \mathbf{R}$$

is said to be differentiable (*or* derivable) *at* x_0 *iff*

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

or equivalently $\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$

exists and is denoted by $f'(x_0)$ *or by* $Df(x_0)$.

Progressive and regressive derivatives

Definition The progressive derivative of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \; h > 0$$

and is denoted by $Rf'(x_0)$ or by $f'(x_0 + 0)$.

The regressive derivative of f at $x = x_0$ is given by

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{-h}, \; h > 0$$

and is denoted by $Lf'(x_0)$ or by $f'(x_0 - 0)$.

Progressive and regressive derivatives are also called right hand and left hand differential coefficients of f at $x = x_0$.

It is easy to see that $f'(x_0)$ *exists iff* $Rf'(x_0)$ *and* $Lf'(x_0)$ *exist and are equal.*

Differentiability in [a, b] *A function* $f : [a, b] \to \mathbf{R}$ *is said to be differentiable at a iff* $Rf'(a)$ *exists, differentiable at b iff* $Lf'(b)$ *exists, f is said to be differentiable in* $[a, b]$ *iff it is differentiable at every point of* $[a, b]$.

Meaning of the sign of derivative

Let $f'(c) > 0$ where c is an interior point of the domain of the function f, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = f'(c) > 0$$

If $\epsilon > 0$ be any number $< f'(c)$ there exists $\delta > 0$ such that

$$|x - c| \leqslant \delta \Rightarrow \left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| < \epsilon$$

that is, $\quad x \in [c - \delta, c + \delta]$,

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \in \,]f'(c) - \epsilon, f'(c) + \epsilon[$$

Since ϵ was chosen smaller than $f'(c)$, we conclude from (1) that

$$\frac{f(x) - f(c)}{x - c} > 0 \text{ when } x \in [c - \delta, c + \delta] \; x \neq c$$

$$\lim_{x\to 0}\frac{\sqrt{(1+x)}-\sqrt{(1-x)}}{x}=\lim_{x\to 0}\frac{(1+x)-(1-x)}{x[\sqrt{(1+x)}+\sqrt{(1-x)}]}$$

$$=\lim_{x\to 0}\frac{2}{\sqrt{(1+x)}+\sqrt{(1-x)}}=\frac{2}{1+1}=1$$

(*b*) Hint, Rationalize the N^r Ans $\dfrac{1}{2\sqrt{x}}$

(*c*) $\lim_{x\to 0}\dfrac{2^x-1}{(1+x)^{1/2}-1}=\lim_{x\to 0}\dfrac{(2^x-1)\{(1+x)^{1/2}+1\}}{\{(1+x)^{1/2}-1\}\{(1+x)^{1/2}+1\}}$

$$=\lim_{x\to 0}\frac{2^x-1}{1+x-1}\{(1+x)^{1/2}+1\}$$

$$=\lim_{x\to 0}\frac{2^x-1}{x}\lim_{x\to 0}\{(1+x)^{1/2}+1\}$$

$$=(\log 2)\ 2$$

$$=2\log 2$$

16 $\lim_{h\to 0}\dfrac{(a+h)^2\sin(a+h)-a^2\sin a}{h}$

$$=\lim_{h\to 0}\frac{(a+h)^2[\sin(a+h)-\sin a]+\sin a\,[(a+h)^2-a^2]}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}(a+h)^2\,2\cos(a+h/2)\sin(h/2)$$

$$+\lim_{h\to 0}\frac{1}{h}(\sin a)\,[a^2+2ah+h^2-a^2]$$

$$\lim_{h\to 0}(a+h)^2\cos(a+h/2)\frac{\sin(h/2)}{(h/2)}+\lim_{h\to 0}(\sin a)(2a+h)$$

$$=a^2\cos a\cdot 1+(\sin a)(2a)$$

$$=a(a\cos a+2\sin a)$$

17 (*a*) Let the circular measure of each of the angles DOA and DOB be x where $0<x<\frac{1}{2}\pi$. Let the tangents at A and B meet OD produced in E and let the chord AB meet OD in C, O being the centre of the circle BDA. We shall assume now as an axiom that the chord

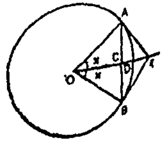


Fig.

$ACB <$ the arc $ADB < AE+BE$

Dividing by OA we have

$$\frac{\text{chord } ACB}{OA}<\frac{\text{arc } ADB}{OA}<\frac{AE+BE}{OA}$$

or $\dfrac{2AC}{OA}<\dfrac{2\text{ arc }AD}{OA}<\dfrac{2AE}{OA}$ [$AE=BE$]

As $a \to 0$ the point Q moving along the curve approaches the point P, the chord PQ approaches the tangent line TP as its limiting position and the angle α approaches the angle ψ. Hence taking limits as $h \to 0$, the equation (1) gives

$$\tan \psi = f'(a)$$

Thus $f'(a)$ is the tangent of the angle which the tangent line to the curve $y=f(x)$ at the point $P\,[a, f(a)]$ makes with x-axis.

A necessary condition for the existence of a finite derivative

Theorem. *Continuity is a necessary but not a sufficient condition for the existence of a finite derivative.*

Proof. Let f have a finite derivative $f'(x_0)$ at $x=x_0$, then

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)$$

$$\frac{f(x_0+h)-f(x_0)}{h} = f'(x_0)+\eta \qquad (1)$$

where $\qquad \lim_{h\to 0} \eta = 0$

Rewriting (1), we get $f(x_0+h)-f(x_0)=hf'(x_0)+h\eta$

Hence $|f(x_0+h)-f(x_0)| = |hf'(x_0)+h\eta| \leqslant |hf'(x_0)| + |h\eta|$

Now if we choose any $\epsilon>0$, then $|hf'(x_0)| + |h\eta|$ can be made less than ϵ by taking h sufficiently small. Then for any value of x in the interval $]x_0-h, x_0+h[$

$$|f(x)-f(x_0)| < \epsilon \qquad (2)$$

The condition (2) shows that $f(x)$ is necessarily continuous at $x=x_0$.

The converse is not however, true as can be seen by considering the follwing example. Let

$$f(x)=x \sin(1/x),\ x \neq 0, f(0)=0$$

We have already seen that this function is continuous at $x=0$. Let us see whether this function is differentiable at $x=0$ or not.

Now $Rf'(x) = \lim_{h\to 0} \frac{(0+h)\sin\{1/(0+h)\}-0}{h} = \lim_{h\to 0} \sin 1/h$, which does not exist. Similarly $Lf'(0)$ does not exist. Hence f is not differentiable at $x=0$.

§ 6. Indeterminate forms and L-Hospital's Rule

If a function $f(x)$ takes the form $\frac{0}{0}$ at $x=a$, then we say that (*f*) Ans 1

Hint Rationalize and use the limit of part (a),

(g) Ans $\frac{2}{3}$ Hint Rationalize

18 (*a*) $\lim_{x\to a} \frac{\sqrt{(a+2x)}-\sqrt{(3x)}}{\sqrt{(3a+x)}-2\sqrt{(x)}}$

$= \lim_{x\to a} \frac{(a+2x)-3x}{(3a+x)-4x} \times \frac{\sqrt{(3a+x)}+2\sqrt{(x)}}{\sqrt{(a+2x)}+\sqrt{(3x)}}$

$= \lim_{x\to a} \frac{a-x}{3(a-x)} \frac{\sqrt{(3a+x)}+2\sqrt{(x)}}{\sqrt{(a+2x)}+\sqrt{(3x)}}$

$= \lim_{x\to a} \frac{1}{3} \frac{\sqrt{(3a+x)}+2\sqrt{(x)}}{\sqrt{(a+2x)}+\sqrt{(3x)}}$

$= \frac{1}{3} \frac{\sqrt{(4a)}+2\sqrt{(a)}}{\sqrt{(3a)}+\sqrt{(3a)}} = \frac{1}{3} \frac{4\sqrt{(a)}}{2\sqrt{(3)}\sqrt{(a)}} = \frac{2}{3\sqrt{(3)}}$

(*b*) $\lim_{x\to 1} (1-x)\tan(\pi x/2)$

$= \lim_{h\to 0} [1-(1-h)] \tan\{(\pi/2)(1-h)\}$ putting $x=1-h$

$\lim_{h\to 0} h \cot(\pi/2)h = \lim_{h\to 0} \frac{-h/2}{\sin(-h/2)} \times \cos(-h/2) \frac{2}{\pi}$

$= 1 \times 1 \times \frac{2}{\pi} = \frac{2}{\pi}$

(*c*) $\lim_{\alpha\to\pi/4} \frac{\sin\alpha-\cos\alpha}{\alpha-\pi/4}$

$= \lim_{\alpha\to\pi/4} \sqrt{(2)} \frac{(1/\sqrt{2})\sin\alpha-(1/\sqrt{2})\cos\alpha}{\alpha-\pi/4}$

$= \lim_{\alpha\to\pi/4} \sqrt{(2)} \frac{\cos(\pi/4)\sin\alpha-\sin(\pi/4)\cos\alpha}{\alpha-\pi/4}$

$= \lim_{\alpha\to\pi/4} \sqrt{(2)} \frac{\sin(\alpha-\pi/4)}{(\alpha-\pi/4)}$

$= \lim_{h\to 0} \sqrt{2} \frac{\sin h}{h}$ [putting $\alpha=\pi/4+h$]

$= \sqrt{2} \times 1 = \sqrt{(2)}$

(*d*) $\lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{2x^2+x-3} = \lim_{x\to 1} \frac{(2x-3)(\sqrt{x}-1)}{(2x+3)(x-1)}$

$= \lim_{x\to 1} \frac{2x-3}{(2x+3)(\sqrt{x}+1)}$

$= \frac{2-3}{(2+3)(1+1)} = -\frac{1}{10}$

Find (i) $f(\frac{1}{2})$ (ii) $f(\sqrt{7})$, (iii) $(f \circ f)(1.4327)$ (iv) $(f \circ f)(\sqrt{3})$

6 Given the function $f(x)=(a^x+a^{-x})/2 \quad (a>0)$ Show that

$$f(x+y)+f(x-y)=2f(x)f(y)$$

7 Given the function

$$f(x)=\begin{cases} 3^{-x}-1 & \text{for } -1 \leqslant x<0, \\ \tan(x/2) & \text{for } 0 \leqslant x<\pi \\ x/(x^2+1) & \text{for } \pi \leqslant x \leqslant 6 \end{cases}$$

Find $f(-1)$, $f(\pi/6)$, $f(2\pi/3)$, $f(4)$, $f(6)$

8 Find $\varphi[\psi(x)]$ and $\psi[\phi(x)]$ if $\phi(x)=x^2+1$ and $\psi(x)=3^x$

9 (a) Find the domain of definition of the following functions

(i) $f(x)=\log_x e$, (ii) $f(x)=\sqrt{\left\{\log\left(\frac{5x-x^2}{6}\right)\right\}}$

(iii) $f(x)=\dfrac{1}{|x|-x}$ (iv) $f(x)=\log|4-x^2|$,

(v) $f(x)=\sqrt{\{\cos(\sin x)\}}+\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$

(b) Find the range of the functions

(i) $y=\dfrac{x}{1+x^2}$ (ii) $y=\dfrac{1}{2-\sin 3x}$

(c) Are the following functions identical?

(i) $f(x)=\dfrac{x}{x}$ and $\varphi(x)=1$

(ii) $f(x)=\log x^2$ and $\varphi(x)=2\log x$

(iii) $f(x)=1$ and $\varphi(x)=\sin^2 x+\cos^2 x$

(iv) $f(x)=x$ and $\varphi(x)=(\sqrt{x})^2$

(v) $f(x)=\log(x-2)+\log(x-3)$ and $\varphi(x)=\log(x-2)(x-3)$

10 Find the fundamental period, if any, of the following functions

(i) $\sin(x/3)$, (ii) $\tan 4x$

(iii) $\cos 2\pi x$ (iv) $|\cos x|$,

(v) $\sin^4 x+\cos^4 x$, (vi) $2\cos\frac{1}{3}(x-\pi)$

11 Does there exist a function which is periodic but has no fundamental period?

12 Prove that the following functions are non periodic

(i) $\cos x^2$ (ii) $x+\sin x$

13 If the function $f(x)=\sin x+\cos ax$ is periodic, then prove that a is rational number

$$\lim_{x\to 0} f(x)=\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}\left[\text{Form }\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{2\cos x-2\cos 2x}{3x^2}\left[\text{Form }\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{-2\sin x+4\sin 2x}{6x}\left[\text{Form }\frac{0}{0}\right]$$

$$=\lim_{x\to 0}\frac{-2\cos x+8\cos 2x}{6}=\frac{-2+8}{6}=1$$

21 Here we use the expansions

$\sin x=x-x^3/3!+x^5/5!\ldots$ and $\cos x=1-x^2/2!+x^4/4!\ldots$

Then we have

$$=\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}$$

$$=\lim_{x\to 0}\frac{x+ax(1-x^2/2!+x^4/4!\ldots)}{x^3}-\frac{b(x-x^3/3!+x^5/5!\ldots)}{x^3}$$

$$=\lim_{x\to 0}\frac{(1+a-b)x+(b/6-a/2)x^3+(a/24-b/120)x^5+\ldots}{x^3}$$

$$=\lim_{x\to 0}\frac{(1+a-b)+(b/6-a/2)x^2+(a/24-b/120)x^4+\ldots}{x^2}$$

Since the limit is given as 1, a finite quantity, we must have (1)

$$1+a-b=0 \qquad (2)$$

and

$$b/6-a/2=1$$

Solving (1) and (2), we have $a=-5/2$, $b=-3/2$

22 we have

$$f(x)=\frac{x^2-3x+2}{x^2+x-6}=\frac{(x-1)(x-2)}{(x+3)(x-2)}$$

The function f is not defined at $x=2$ and $x=-3$

Hence domain of $f=\{x : x\in R,\ x\neq -3,\ x\neq 2\}$

$$\lim_{x\to 2}\frac{x^2-3x+2}{x^2+x-6}=\lim_{x\to 2}\frac{(x-2)(x-1)}{(x-2)(x+3)}$$

$$=\lim_{x\to 2}\frac{x-1}{x+3}=\frac{2-1}{2+3}=\frac{1}{5}$$

To find the range of f we first observe that it cannot take the value $\frac{1}{5}$ since it is not defined at $x=2$. Also for $x\neq 2$, we have

$$y=f(x)=\frac{x-1}{x+3}$$

19 Evaluate the following limits

(a) $\lim_{x \to 0} \dfrac{\tan x - \sin x}{x^3}$ (I I T 74)

(b) $\lim_{x \to \pi/2} \dfrac{2x - \pi}{\cos x}$ (I I T 73)

(c) $\lim_{x \to 0} \dfrac{2 \sin^2 3x}{x^2}$ (Roorkee 82)

(d) $\lim_{x \to 0} \dfrac{\tan 2x - x}{3x - \sin x}$ (I I T 71)

20 $f(x)$ is the integral of $\dfrac{2 \sin x - \sin 2x}{x^3}$, $x \neq 0$, find

$$\lim_{x \to 0} f(x)$$ (I I T 79)

21 Find the values of a and b such that
$\lim_{x \to 0} \dfrac{x(1 + a \cos x) - b \sin x}{x^3} = 1$ (Roorkee 76)

22 What is the domain and range [Range $= \{f(x) \quad x \in$ domain of $f\}$] of the function

$$f(x) = \frac{x^2 - 3x + 2}{x^2 + x - 6}$$

Find the limit of $f(x)$ as x approaches 2 (I I T 70)

23 Evaluate the following limits

(a) $\lim_{x \to 0} \dfrac{\sin x - x + \frac{1}{6}x^3}{x^5}$ (b) $\lim_{x \to 0} \left[\dfrac{1}{x} - \dfrac{1}{x^2} \log (1 + x)\right]$

(c) $\lim_{x \to 0} x \log \sin x$

(d) $\lim_{x \to \pi/2} [x \tan x - (\pi/2) \sec x]$

(e) $\lim_{x \to 0} \left(\dfrac{\tan x}{x}\right)^{1/x^2}$

24 Do the following limits exist ? If they exist find their values

(a) $\lim_{x \to 0} e^{1/x}$ (b) $\lim_{x \to 0} \sin (1/x)$

25 (a) Draw the graph of function $f(x) = |x| / x$ Is $f(x)$ defined at $x = 0$? Does the limit of $f(x)$ exists when $x \to 0$? Explain, (Roorkee 82)

$$=\lim_{x\to\pi/2}\frac{[2\sin x+2x\cos x]}{-2\sin x}=\frac{2\cdot 1+2\cdot\pi/2\cdot 0}{-2\cdot 1}=-1$$

(v) Let $A=\lim_{x\to 0}(\tan x/x)^{1/x^2}$ $\qquad \log_e A=\lim_{x\to 0}\frac{1}{x}\log\frac{\tan x}{x}$

$$=\lim_{x\to 0}(1/x^2)\log_e[(1/x)\{x+(1/3)x^3+(2/15)x^5+\ldots\}]$$

[using the expansion of $\tan x$

$$=\lim_{x\to 0}(1/x^2)\log_e\{1+(1/3)x^2+(2/15)x^4\ldots\}$$

$$\lim_{x\to 0}(1/x^2)\left[(1/3)x^2+(2/15)x^4+\ldots\right]$$

$$=\lim_{x\to 0}\left[\frac{1}{3}+\frac{2}{15}x\ldots\right]=\tfrac{1}{3}$$

$A=e^{1/3}$

24 (i) Let $f(x)=e^{1/x}$. Then

$$f(0+0)=\lim_{h\to 0}e^{1/(0+h)}=\lim_{h\to 0}e^{1/h}=e^{\infty}=\infty$$

and $$f(0-0)=\lim_{h\to 0}e^{1/(0-h)}=\lim_{h\to 0}e^{-(1/h)}=e^{-\infty}=0$$

Since the right hand and left hand limits are unequal, the limit $\lim_{x\to 0}e^{1/x}$ does not exist

(ii) As $x\to 0$, the value of $\sin(1/x)$ oscillates between $+1$ and -1, passing through zero and intermediate values, an infinite number of times. Evidently there is no definite number A to which $\sin(1/x)$ tends as x tends to zero from the right. So the limit on the right does not exist. Similarly the limit on the left does not exist. Hence the ordinary limit of $\sin(1/x)$ as $x\to 0$ does not exist. This is also evident from the graph of the function. Considering only positive values of x we find that

$$f(x)=0,\ x=\frac{1}{\pi},\frac{1}{2\pi},\frac{1}{3\pi},\ldots\frac{1}{n\pi}\ldots$$

and between consecutive zeros $f(x)$ is alternately negative and positive. Thus there are an infinite number of oscillations which get closer and closer as we approach the origin. Each oscillation extends upto the line $y=+1$ and $y=-1$

For $x>(1/\pi)$, $f(x)$ positive. After reaching the maximum value $+1$ at $x=2/\pi$ the curve continually approaches the axis of x as x tends to infinity where x is a positive variable, and $[x]$ denotes the integral part of x and show that it is discontinuous for integral values of x, and continuous for all others Is the function periodic If periodic, what is its period ? Draw its graph

(*b*) Draw the graph of the function $F(x)$ defined as follows

$$F(x)=\begin{cases} x-[x] & \text{when } 2n \leqslant x<2n+1 \\ \frac{1}{2} & \text{when } 2n+1 \leqslant x <2n+2 \end{cases}$$

where n is an integer and $[x]$ denotes the largest integer not exceeding x Is the function F periodic ? If periodic what is its period ? What are the points of discontinuity of F ?

(*c*) Show that the function $f(x)=[x]+[-x]$ has removable discontinuity for integral values of x

30 Give an example of each of the following types of functions

(*a*) The function which possesses a limit at $x=1$ but is not defined at $x=1$

(*b*) The function which is neither defined at $x=1$ nor has a limit at $x=1$

(*c*) The function which is defined at two points but is nevertheless discontinuous at both the points

31 Let $y=E(x)$ where $E(x)$ denotes the integral part of x Prove that the function is discontinuous where x has an integral value Also draw the graph

32 Let (x) denote the positive or negative excess of x over the nearest integer and when x exceeds an integer by $\frac{1}{2}$ let $(x)=0$ What do you say about the continuity of (x) ? Draw the graph

33 Test the following functions for continuity and discontinuity

(*i*) $f(x)=\dfrac{xe^{1/x}}{1+e^{1/x}}+\sin\dfrac{1}{x}$, $x\neq 0$ $f(x)=0$ at $x=0$

(*ii*) $\phi(x)=1+x$ if $x \leqslant 2$ and
$\phi(x)=5-x$ if $x>2$ at $x=2$

(*iii*) $f(x)=\dfrac{3x+4\tan x}{x}$, $x\neq 0$ $f(x)=7$ at $x=0$

(*iv*) $f(x)=\dfrac{1}{2^n}$ for $\dfrac{1}{2^{n+1}}<x\leqslant \dfrac{1}{2^n}$

$n=0, 1, 2$ and $f(0)=0$

and left hand limits at $x=0$ Thus

$$f(0+0)=\frac{|0+h|}{0+h}=\lim_{h\to 0}\frac{h}{h}=1$$

$$f(0-0)=\lim_{h\to 0}\frac{|0-h|}{0-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

Since $f(0+0)\neq f(0-0)$, the limit of $f(x)$ as $x\to 0$ does not exist

(*b*) For the first part, see problem 9 (i) For the second part we see that

$$f(0+0)=\lim_{h\to 0}[1]=1 \quad \text{and } f(0-0)=\lim_{h\to 0}[1]=1$$

Hence the limit of $f(x)$ as $x\to 0$ exists and its values is 1

26 (*a*) We have, $f(1)=2$

$$f(1+0)=\lim_{h\to 0}\frac{(1+h)^2-4(1+h)+3}{(1+h)^2-1}$$

$$=\lim_{h\to 0}\frac{-2h+h^2}{2h+h^2}$$

$$=\lim_{h\to 0}\frac{-2+h}{2+h}=\frac{-2}{2}=-1$$

Since $f(1)\neq f(1+0)$ the function is discontinuous at $x=1$

(*b*) $$f(0-0)=\lim_{h\to 0}\frac{\sin(a+1)(0-h)+\sin(0-h)}{-h}$$

$$=\lim_{h\to 0}\frac{\sin(a+1)h+\sin h}{h}$$

$$=\lim_{h\to 0}\frac{2\sin(a/2+1)h\cos(ah/2)}{h}$$

$$=\lim_{h\to 0}\frac{\sin\{(a+2)/2\}h}{\{(a+2)/2\}h}(a+2)\cos\frac{ah}{2}$$

$$=1\cdot(a+2)\cdot 1=a+2$$

and $$f(0+0)=\lim_{h\to 0}\frac{(h+bh^2)^{1/2}-h^{1/2}}{bh^{3/2}}$$

$$=\lim_{h\to 0}\frac{(1+bh)^{1/2}-1}{bh}$$

$$=\lim_{h\to 0}\frac{1+\frac{1}{2}bh-1}{bh}=\frac{1}{2}, \text{ which is in-}$$

dependent of b and so b may have any real value Again by continuity at $x=0$ we have $a+2=1=c$

$a=-1$ and $c=1$

27 (i) Here $f(0+0)=\lim_{h\to 0}(0+h)\sin\frac{1}{0+h}=\lim_{h\to 0}h\sin\frac{1}{h}$

$$f(x)=\begin{cases}\dfrac{x-1}{2x^2-7x+5} & \text{when } x\neq 1\\ -\dfrac{1}{3} & \text{when } x=1\end{cases}$$

at $x=1$ (I I T 79)

(*b*) Investigate the following function from the point of view of its differentiability Does the differential coefficient of the function exist at $x=0$ and $x=1$?

$$f(x)=\begin{cases}-x & \text{if } x<0\\ x^2 & \text{if } 0\leqslant x\leqslant 1\\ x^3-x+1 & \text{if } x>1\end{cases}$$ (Roorkee 1982)

41 If $f(x)=\dfrac{x}{1+e^{1/x}}$ $x\neq 0, f(0)=0$

show that f is continuous at $x=0$ but $f'(0)$ does not exist (I I T 83)

42 If $f(x)=x^2 \sin\dfrac{1}{x}$ for $x\neq 0$, $f(0)=0$, show that f is continuous and differentiable everywhere and that $f'(0)=0$ Show further that the function $f'(x)$ is discontinuous at the origin

43 (*a*) Prove that the function $f(x)=|x|$ is continuous at $x=0$, but not differentiable at $x=0$ where $|x|$ means the numerical value of x

(*b*) Sketch the function $y=|x-2|$ in the interval $(-1, 3)$ Is this function (i) continuous (ii) differentiable at $x=2$? (Roorkee 84)

44 A function ϕ is defined as follows

$\phi(x)=-x$ for $x\leqslant 0$, $\phi(x)=x$ for $x\geqslant 0$

Test the character of function at $x=0$ as regards continuity and differentiability

45 Examine the following curve for continuity and differentiability

$y=x^2$ for $x\leqslant 0$, $y=1$ for $0<x\leqslant 1$

and $y=1/x$ for $x>1$

Also draw the graph of the function

46 Discuss the continuity and differentiability of the following function

$f(x)=x^2$ for $x<-2$, $f(x)=4$ for $-2\leqslant x\leqslant 2$

and $f(x)=x^2$ for $x>2$

Also draw the graph

and $f(0-0)=\lim\limits_{h\to 0}\dfrac{1}{1-e^{-1/(0-h)}}=\lim\limits_{h\to 0}\dfrac{1}{1-e^{1/h}}$

$=\lim\limits_{h\to 0}\dfrac{1}{1-e^{\infty}}=\dfrac{1}{1-\infty}=0$

Also $f(0)=0$

Hence f is discontinuous at $x=0$

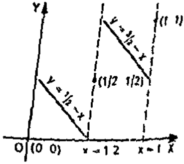
The function has a jump of one unit at 0 since

$$f(0+0)-f(0-0)=1$$

28 (i) We test the function for continuity at $x=0$, $\frac{1}{2}$ and 1

For $x=0$, we have

$\phi(0)=0$



$\phi(0+0)=\lim\limits_{h\to 0}[\frac{1}{2}-(0+h)]=\frac{1}{2}$

Since $\phi(0)\neq\phi(0+0)$, the function ϕ is discontinuous at $x=0$

For $x=\frac{1}{2}$, we have

$\phi(\frac{1}{2})=\frac{1}{2}$,

$\phi(\frac{1}{2}-0)=\lim\limits_{h\to 0}[\frac{1}{2}-(\frac{1}{2}-h)]=0$

$\phi(\frac{1}{2}+0)=\lim\limits_{h\to 0}[\frac{3}{2}-(\frac{1}{2}+h)]=1$

Since $\phi(\frac{1}{2}-0)\neq\phi(\frac{1}{2})\neq\phi(\frac{1}{2}+0)$, the function is discontinuous at $x=\frac{1}{2}$

Finally, we consider $x=1$

$\phi(1)=1$, $\phi(1-0)=\lim\limits_{h\to 0}[\frac{3}{2}-(1-h)]=\frac{1}{2}$,

Since $\phi(1-0)\neq\phi(1)$, the function is discontinuous at $x=1$

Hence the function ϕ is discontinuous at $x=0$, $\frac{1}{2}$ and 1

The graph of the function consists of the point $(0, 0)$, the segment of the line

$$y=\tfrac{1}{2}-x,\ 0<x<\tfrac{1}{2},$$

the point $(\frac{1}{2}, \frac{1}{2})$, the segment of the line $y=\frac{3}{2}-x$, $\frac{1}{2}<x<1$ and the point $(1, 1)$

Thus the graph is as shown above

From the graph it is clear that the function is discontinuo at $x=0$, $\frac{1}{2}$ and 1

(ii) The graph consists of the straight line $y=x-1$ for $x<0$ the point $(0, \frac{1}{2})$ and the parabola $y=x^2$ for $x>0$

when $x \neq 0$ and $f(0) = 0$ is everywhere continuous and monotonic but has no differential coefficient at the origin.

57 Use the function $f(x) = x^{1/x}$, $x > 0$
to determine the bigger of the two numbers e^{π} and π^{e}
(IIT 81)

58 (*a*) Find the intervals of monotonocity of the function
$y = 2x^2 - \log_e |x| \quad (x \neq 0)$ **(IIT 1983)**

(*b*) Prove that
$x - x^3/6 < \sin x < x$ for $0 < x \leqslant \pi/2$

59 Let $f(x) = x^3 - x^2 + x + 1$
and $g(x) = \max \{f(t) \; 0 \leqslant t \leqslant x\}, 0 \leqslant x \leqslant 1$
$= 3 - x, \quad 1 < x \leqslant 2$
Discuss the continuity and differentiability of the function $g(x)$ in the interval $(0, 2)$ **(IIT 85)**

60 Let $f(x)$ be defined in the interval $[-2, 2]$ such that
$f(x) = -1, \quad -2 \leqslant x \leqslant 0$
$= x - 1, \quad 0 < x \leqslant 2$
and $g(x) = f(|x|) + |f(x)|$
Test the differentiability of $g(x)$ in $(-2, 2)$ **(IIT 86)**

61 (*a*) Determine the set of all points where the function
$f(x) = \dfrac{x}{1 + |x|}$ is differentiable **(IIT 87)**

(*b*) Let f and g be increasing and decreasing functions respectively from $[0, \infty)$ to $[0, \infty)$ and let $h(x) = f[g(x)]$
If $h(0) = 0$ then $h(x) - h(1)$ is
(i) always zero, (ii) always negative
(iii) always positive, (iv) Strictly Increasing,
(v) None of these **(IIT 87)**

(*c*) The set of all x for which $\log_e (1 + x) \leqslant x$ is equal to $[0, \infty)$ **(IIT 87)**

(*d*) Let $f(x)$ be a continuous function and $g(x)$ be discontinuous function. Prove that $f(x) + g(x)$ is a discontinuous function **(IIT 87)**

(*e*) Let $f(x)$ be function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists find its value **(IIT 87)**

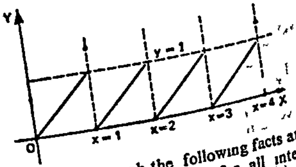
(*f*) $\underset{x \to \infty}{\text{Lt}} \dfrac{x^4 \sin (1/x) + x^2}{1 + |x|^3} = -1$

(*g*) If $y = x^2 (x - 2)$ find for what values of x, y increases
(Roorkee 87)

we see that the graph consists of the following

$y=x$ when $0<x<1$
$y=0$ when $x=1$
$y=x-1$ when $1<x<2$
$y=0$ when $x=2$
$y=x-2$ when $2<x<3$
$y=0$ when $x=3$
$y=x-3$ when $3<x<4$
$y=0$ when $x=4$

and so on. The graph is shown by thick lines from $x=0$ to $x=4$



Remark From the graph the following facts are evident —

(i) The function is discontinuous for all integral values of x and continuous for all others

(ii) In every range which includes an integer, it is bounded between 0 and 1

(iii) The lower bound zero is attained but the upper bound 1 is not attained since $f(x)\neq 1$ for any value of x whatsoever

(b) Draw the graph yourself. The function is discontinuous for all integral values of x. It is a periodic function with period 2

(c) It is easy to see that $f(x)=0$ when x is an integer and $f(x)=-1$ when x is not an integer. Hence if n is any integer, then $f(n-0)=f(n+0)=-1$ and $f(n)=0$. Hence $f(x)$ has a removable discontinuity at $x=n$, where n is an integer

30 (a) Consider the function f defined as follows $f(x)=x^2$ for $x>1$, $f(x)=x^3$ for $x<1$. function is not defined at $x=1$

(*ii*) The function is defined for those values of x for which $\log \frac{5x-x^2}{6} \geqslant 0$, which will be satisfied if

$$\frac{5x-x^2}{6} \geqslant 1 \text{ i e } x^2 \quad 5x+6 \leqslant 0$$

or $$(x-2)(x-3) \leqslant 0 \qquad (1)$$

Now the inequality (1) will hold if $2 \leqslant x \leqslant 3$

Hence the domain of definition of the function is the closed interval *[2, 3]*

(*iii*) The function is defined for all values of x for which $|x| - x > 0$, whence $|x| > x$ This inequality will hold at $x < 0$ Hence the domain of the function is $]-\infty, 0[$

(*iv*) Here $f(x)$ is defined for all real values of x except $x = \pm 2$ (why ?)

(*v*) The following inequalities must be satisfied simultaneously $\cos(\sin x) \geqslant 0, \left|\frac{1+x^2}{2x}\right| \leqslant 1$

The first inequality is satisfied for all values of x and the second for $|x| = 1$ *i e* for $x = \pm 1$ Since, $|1+x^2| \leqslant 2|x|$

$\Rightarrow 1+x^2 \leqslant 2|x| \Rightarrow |x|^2 - 2|x| + 1 \leqslant 0$

$\Rightarrow (|x|-1)^2 \leqslant 0 \Rightarrow |x| = 1 \Rightarrow x = \pm 1$

Hence the domain of definition of $f(x)$ consists only of the two points $x = -1$ and $x = 1$

(*b*) (*i*) We have $y = \frac{x}{1+x^2}$ or $yx^2 - x + y = 0$

$$x = \{1 \pm \sqrt{(1-4y^2)}\}/2y$$

Since x is real the range of the function y is determined from the relation $1-4y^2 \geqslant 0$, whence $-\frac{1}{2} \leqslant y \leqslant \frac{1}{2}$

(*ii*) We have, $2y - y \sin 3x = 1$ or $\sin 3x = \frac{2y-1}{y}$

since $-1 \leqslant \sin 3x \leqslant 1$ we have $-1 \leqslant \frac{2y-1}{y} \leqslant 1$ (1)

Since $y > 0$ (why ?) multiplying the inequality (1) by y, we obtain $-y \leqslant 2y-1 < y$ whence $\frac{1}{3} \leqslant y \leqslant 1$

(*c*) (*i*) No $\varphi(0) = 1$ and $f(0)$ is not defined

(*ii*) No $f(x)$ is defined for all $x \neq 0$ ane $\varphi(x)$ is defined only for $x > 0$

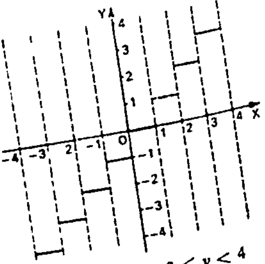
(*iii*) Yes

(*iv*) No $f(x)$ is defined for all x, and $\phi(x)$ only $x \geqslant 0$

$y=0$ when $0 \leqslant x < 1$

$y=1$ when $1 \leqslant x < 2$

$y=2$ when $2 \leqslant x < 3$



$y=3$ when $3 \leqslant y < 4$

$y=4$ when $4 \leqslant x < 5$

and so on

The graph is shown by thick lines in the figure above

32 From the definition of (x), we have

$(x)=x-n$ when $n-\frac{1}{2} < x < n+\frac{1}{2}$

$(x)=0$ when $x=n+\frac{1}{2}$

$(x)=x-(n+1)$ when $n+\frac{1}{2} < x < n+\frac{1}{2}$

where n is an integer Then $(n+\frac{1}{2})=0$,

We consider $x=n+\frac{1}{2}$

$$(n+\tfrac{1}{2}-0)=\lim_{x\to n+\frac{1}{2}-0}[x-n]$$

$$=\lim_{h\to 0}[n+\tfrac{1}{2}-h-n]=\tfrac{1}{2},$$

$$(n+\tfrac{1}{2}+0)=\lim_{x\to n+\frac{1}{2}+0}[x-(n+1)]$$

$$=\lim_{h\to 0}[n+\tfrac{1}{2}+h-(n+1)]=-\tfrac{1}{2}$$

Since $(n+\frac{1}{2}+0)\neq(n+\frac{1}{2}-0)\neq(n+\frac{1}{2})$,

the function (x) is discontinuous when $x=n+\frac{1}{2}$ where n is an integer

To draw the graph, we put

$n=\quad, -3, -2, -1, 0, 1, 2, 3,$

(ii) Suppose, if possible $x+\sin x$ has a period λ. Then

$$\lambda+x+\sin(\lambda+x)=x+\sin x \text{ for all } x$$

This gives $\sin(\lambda+x)-\sin x=-\lambda$

or $2\cos(x+\lambda/2)\sin\lambda/2=-\lambda$

or $\cos(x+\lambda/2)=-\lambda/\{2\sin(\lambda/2)\}$,

which is impossible for any constant λ, since the left side is not constant.

13. Let λ be the period of $\sin x+\cos ax$. Then

$$\sin(\lambda+x)+\cos a(\lambda+x)=\sin x+\cos ax \text{ for all } x.$$

In this identity, putting $x=0$ and $x=-\lambda$ we get,

$$\sin\lambda+\cos a\lambda=1, \quad \text{and } 1=-\sin\lambda+\cos a\lambda$$

Solving these equations we get $\sin\lambda=0$ and $\cos a\lambda=1$

Hence $\lambda=n\pi$ and $a\lambda=2m\pi$ where m, n are non zero integers

Hence $\dfrac{a\lambda}{\lambda}=\dfrac{2m\pi}{n\pi}$ or $a=\dfrac{2m}{n}$ ($\because \lambda\neq 0$),

which is a rational number.

14. Suppose $\cos\sqrt{t}$ is periodic with period λ.

Then $\cos\sqrt{(\lambda+t)}=\cos\sqrt{t}$, whence $\sqrt{(\lambda+t)}=2n\pi\pm\sqrt{t}$

Thus either $\sqrt{(\lambda+t)}+\sqrt{t}=2n\pi$ (1)

or $\sqrt{(\lambda+t)}-\sqrt{t}=2n\pi$ that is $\dfrac{\lambda}{\sqrt{(\lambda+t)}+\sqrt{t}}=2n\pi$ (2)

But the equalities (1) and (2) connot hold for all real t since their left members are functions of a continuous variable t whereas the right members are only integral multiples of 2π. For example if we put $t=0$ and $t=\lambda$ then (1) gives

$$\sqrt{\lambda}=2n\pi \text{ and } \sqrt{(2\lambda)}+\sqrt{\lambda}=2n\pi,$$

whence $\sqrt{\lambda}(\sqrt{2}+1)=\sqrt{\lambda}$

which is impossible since neither $\lambda=0$ nor $\sqrt{2}+1=1$.

Similarly (2) gives $\dfrac{\lambda}{\sqrt{\lambda}}=2n\pi$ and $\dfrac{\lambda}{\sqrt{(2\lambda)}+\sqrt{\lambda}}=2n\pi$

whence $\dfrac{\lambda}{\sqrt{\lambda}}=\dfrac{\lambda}{\sqrt{(2\lambda)}+\sqrt{\lambda}}$ which is again impossible since neither $\lambda=0$ nor $1=\dfrac{1}{\sqrt{2}+1}$.

Hence $\cos\sqrt{t}$ is not periodic.

15. (a) The given function is not defined at $x=0$ since it takes the indeterminate form 0/0 at $x=0$. The value of the function is imaginary for all values of $x>1$ and $x<-1$ and so it is also not defined for these values of x. It follows that the function is defined and real for all values of x in the interval $-1\leqslant x\leqslant 1$ except $x=0$. Now

$$g(x)=f(f(x))=1+f(x) \text{ for } 0 \leqslant f(x) \leqslant 2$$
$$=3-f(x) \text{ for } 2 < f(x) \leqslant 3$$

Now we find the form of $g(x)$ as a function of x. We consider two sub intervals I_1 $(0 \leqslant x \leqslant 2)$ and I_2 $(2 < x \leqslant 3)$ of the interval I $(0 \leqslant x \leqslant 3)$

(i) On I_1, we have $f(x)=1+x$

Now $0 \leqslant f(x) \leqslant 2 \Rightarrow 0 \leqslant 1+x \leqslant 2$

$\Rightarrow -1 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant x \leqslant 1$ on I_1

$g(x)=1+f(x)=1+(1+x)=2+x$ for $0 \leqslant x \leqslant 1$

$2 < f(x) \leqslant 3$

$\Rightarrow 2 < 1+x \leqslant 3$

$\Rightarrow 1 < x \leqslant 2$ which belong to I_1

$g(x)=3-(1+x)=2-x$, $1 < x \leqslant 2$

(ii) On I_2 we have $f(x)=3-x$

Now $0 \leqslant f(x) \leqslant 2$

$\Rightarrow 0 \leqslant 3-x \leqslant 2$

$\Rightarrow -3 \leqslant -x \leqslant -1$

$\Rightarrow 3 \geqslant x \geqslant 1 \Rightarrow 1 \leqslant x \leqslant 3$

$\Rightarrow 2 \leqslant x \leqslant 3$ on I_2

$g(x)=1+(3-x)=4-x$ for $2 < x \leqslant 3$

and $2 < f(x) \leqslant 3 \Rightarrow 2 < 3-x \leqslant 3$

$\Rightarrow -1 < -x \leqslant 0 \Rightarrow 0 \leqslant x < 1 \notin I$

$g(x)$ is not defined in this case

Thus g is defined as follows

$g(x)=2+x$ for $0 \leqslant x \leqslant 1$

$=2-x$ for $1 < x \leqslant 2$

$=4-x$ for $2 < x \leqslant 3$

We test the function g for continuity at $x=1$ and $x=2$ only

$g(1)=3$, $g(1-0)=\lim_{h \to 0} [2+(1-h)]=3$

and $g(1+0)=\lim_{h \to 0} [2-(1+h)]=1$

Hence g is discontinuous at $x=1$

$g(2)=2-2=0$, $g(2-0)=\lim_{h \to 0} [2-(2-h)]=0$

and $g(2+0)=\lim_{h \to 0} [4-(2+h)]=2$

Hence g is also discontinuous at $x=2$

33 Parts (i) to (vi) are easy and left as an exercise

(vii) The point $x=1$ is clearly a point of discontinuity of the

$$\sin x < x < \tan x$$

or $\quad 1 < x/\sin x < 1/\cos x$

Now let $x \to 0$. Then $1/\cos x \to 1$. Hence $x/\sin x$ must also $\to 1$.

Therefore $\lim_{x\to 0} \dfrac{\sin x}{x} = 1$

(*b*) Let $x = \pi/2 - h$ so that $h \to 0$ as $x \to \dfrac{\pi}{2}$

Hence $\lim_{x\to\pi/2} \dfrac{\cos x}{x-\pi/2} = \lim_{h\to 0} \dfrac{\cos(\pi/2-h)}{\pi/2-h-\pi/2}$

$= \lim_{h\to 0} -\dfrac{\sin h}{h} = -1$ by part (*i*)

Note.—We can put $x = \pi/2 + h$ instead of $x = \pi/2 - h$. The students can verify that the limit is -1 in this case also.

(c) $\lim_{x\to 1} \dfrac{x-1}{2x^2-7x+5} = \lim_{x\to 1} \dfrac{x-1}{(x-1)(2x-5)}$

$= \lim_{x\to 1} \dfrac{1}{2x-5}$ [$\because x-1 \neq 0$ when $x \to 1$]

$= \dfrac{1}{2-5} = \dfrac{1}{-3}$

(*d*) $\lim_{\theta\to\pi/2} (\sec\theta - \tan\theta) = \lim_{\theta\to\pi/2} \dfrac{1-\sin\theta}{\cos\theta}$

$= \lim_{h\to 0} \dfrac{1-\sin(\pi/2-h)}{\cos(\pi/2-h)}$ [Putting $\theta = \pi/2 - h$]

$= \lim_{h\to 0} \dfrac{1-\cos h}{\sin h} = \lim_{h\to 0} \dfrac{2\sin^2(h/2)}{2\sin(h/2)\cos(h/2)}$

$= \lim_{h\to 0} \dfrac{\sin(h/2)}{\cos(h/2)} = \lim_{h\to 0} \tan(h/2) = 0$

(*e*) Put $x = 1/h$. The $h \to 0$ as $x \to \infty$

Then $\lim_{x\to\infty} [x - \sqrt{(x^2+x)}]$

$= \lim_{h\to 0} \left\{\dfrac{1}{h} - \sqrt{\left(\dfrac{1}{h^2}+\dfrac{1}{h}\right)}\right\}$

$= \lim_{h\to 0} \dfrac{1-\sqrt{(1+h)}}{h}$

$= \lim_{h\to 0} \dfrac{1-(1+h)}{[1+\sqrt{(1+h)}]} = \lim_{h\to 0} \dfrac{-h}{h\,[1+\sqrt{(1+h)}]}$

$= \lim_{h\to 0} -\dfrac{1}{1+\sqrt{(1+h)}} = -\dfrac{1}{1+1} = -\tfrac{1}{2}$

It follows that f is continuous at α. Since α is arbitrary, the function f is continuous for all $x \in \mathbb{R}$.

(ii) It is a particular case of (i) with $a=0$.

(iii) Case I First let $x=0$.

Since $f(x+x)=f(x)+f(x)$, we have $f(0)=f(0)+f(0)$

and so $f(0)=0$

Thus $f(x)=kx$ is satisfied for all constants k in this case. In particular

$$f(x)=xf(1)$$

Case II Now let x be any positive integer. Then

$$f(x)=f(1+1+\ldots x \text{ times})$$
$$=f(1)+f(1)+\ldots x \text{ times}$$
$$=xf(1)$$

Case III Next let x be any negative integer. Put $x=-y$ so that y is a positive integer.

Now $f(0)=f(y-y)=f(y)+f(-y)$

or $f(y)+f(-y)=0$ by case I

Hence $f(x)=f(-y)=-f(y)$

$$=-yf(1) \text{ by case II}=xf(1)$$

Case IV Let x be any rational number. Put $x=\frac{p}{q}$ where q is a positive integer and p is any integer, positive, negative or zero. Then

$$f\{q(p/q)\}=f(p/q+p/q+\ldots q \text{ times})$$
$$=f(p/q)+f(p/q)+\ldots q \text{ times}$$
$$f(p)=qf(p/q)$$

But $f(p)=pf(1)$ by previous cases

Hence $qf(p/q)=pf(1)$ or $f(p/q)=(p/q)f(1)$

or $f(x)=xf(1)$ in this case also.

Case V Finally let x be any real number, and let $\{x_n\}$ be a sequence of rational numbers which represents x. Since f is continuous at x, the sequence $\{f(x_n)\}$ converges to $f(x)$.

Thus $\lim_{x\to\infty} x_n=x$ and $\lim_{x\to\infty} f(x_n)=f(x)$

Since x_n is a rational number, we have by case IV

$$f(x_n)=x_n f(1)$$

$$\lim_{x\to\infty} f(x_n)=f(1)\lim_{x\to\infty} x_n$$

or

$$f(x)=xf(1)$$

19 (a) $\lim_{x\to 0}\frac{\tan x-\sin x}{x^3}=\lim_{x\to 0}\frac{\sin x\,(1-\cos x)}{x^3\cos x}$

$=\lim_{x\to 0}\frac{2\sin(x/2)\cos(x/2)\,2\sin^2(x/2)}{x^3\cos x}$

$=\lim_{x\to 0}\frac{4\sin^3(x/2)}{x^3}\cdot\frac{\cos(x/2)}{\cos x}$

$=\lim_{x\to 0}\frac{1}{2}\left(\frac{\sin(x/2)}{x/2}\right)^3\frac{\cos(x/2)}{\cos x}=\frac{1}{2}\cdot 1\cdot 1=\frac{1}{2}$

(b) Do yourself. Ans -2

(c) $\lim_{x\to 0}\frac{2\sin^2 3x}{x^2}$

$=\lim_{x\to 0}18\frac{\sin^2 3x}{9x^2}=\lim_{x\to 0}18\left(\frac{\sin 3x}{3x}\right)^2$

$=18\qquad\left[\lim_{x\to 0}\frac{\sin 3x}{3x}=1\right]$

(d) $=\lim_{x\to 0}\frac{\tan 2x-x}{3x-\sin x}\qquad\left[\text{Form }\frac{0}{0}\right]$

$\lim_{x\to 0}\frac{\frac{d}{dx}(\tan 2x-x)}{\frac{d}{dx}(3x-\sin x)}$ By L Hospital Rule

$=\lim_{x\to 0}\frac{2\sec^2 2x-1}{3-\cos x}=\frac{2-1}{3-1}=\frac{1}{2}$

20 We are given $f(x)=\int\frac{2\sin x-\sin 2x}{x^3}\,dx,\ x\neq 0$

Then by definition, $f'(x)=\frac{2\sin x-\sin 2x}{x^3}$

$\lim_{x\to 0}f'(x)=\lim_{x\to 0}\frac{2\sin x-\sin 2x}{x^3}$

$=\lim_{x\to 0}\frac{2\sin x-2\sin x\cos x}{x^3}$

$=\lim_{x\to 0}\frac{2\sin x\,(1-\cos x)}{x^3}$

$=\lim_{x\to 0}\frac{4\sin(x/2)\cos(x/2)\,2\sin^2(x/2)}{x^3}$

$=\lim_{x\to 0}\frac{8\sin^3(x/2)\cos(x/2)}{x^3}$

$=\lim_{x\to 0}\left(\frac{\sin(x/2)}{x/2}\right)^3\cos(x/2)=1$

Alternative Using L Hospital Rule, we have and $f(0-0)=\lim_{h\to 0}\frac{\sin(0-h)^2}{0-h}=\lim_{h\to 0}\frac{\sin h^2}{h^2}(-h)$

$=1\times 0=0$

Hence $f(x)$ is continuous at $x=0$

Now $Rf'(0)=\lim_{h\to 0}\left(\frac{\sin(0+h)^2}{(0+h)}-0\right)\Big/h$

$=\lim_{h\to 0}\frac{\sin h^2}{h^2}=1$

and $Lf'(0)=\lim_{h\to 0}\left(\frac{\sin(0-h)^2-0}{0-h}\right)\Big/(-h),$

$=\lim_{h\to 0}\frac{\sin h^2}{h^2}=1$

Hence $f(x)$ is differentiable at

$x=0$ and $f'(0)=1$

38 (a) Differentiating the given relation

$h(x)=[f(x)]^2+[g(x)]^2,$

with respect to x, we get (1)

$h'(x)=2f(x)f'(x)+2g(x)g'(x)$

But we are given $f''(x)=-f(x)$ and $f'(x)=g(x)$ so that $f''(x)=g'(x)$ Then (1) may be re written as

$h'(x)=-2f''(x)f(x)+2f'(x)f''(x)=0$

Thus $h'(x)=0,$

whence by integrating, we get

$h(x)=\text{constant}=c$, say

Hence $h(x)=c$, for all x

In particular, $h(5)=c$ But we are given $h(5)=11$

It follows that $c=11$ and we have $h(x)=11$ for all x

Therefore $h(10)=11$

(b) We are given that f is a function from R to R such that

$|f(x)-f(y)|\leqslant|x-y|^2$ (1)

where (1) holds for all real numbers x and y Let x be any real number and let y be chosen arbitrarily close to x but not equal x Then writing (1) as

$\left|\frac{f(x)-f(y)}{x-y}\right|\leqslant|x-y|^2,$

and letting $y\to x$, we get

$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right|\leqslant\lim_{y\to x}|x-y|^2$ (2)

Since $\lim_{y\to x}\frac{f(x)-f(y)}{x-y}=f'(x)$ we or $\qquad xy+3y=x-1 \quad \text{or} \quad x=-\dfrac{3y+1}{y-1}$

Hence $y\neq 1$ in the domain of x. Also at $x=-3$, $y=\infty$

Thus $f(x)$ takes all real values in the domain of x except $y=\frac{1}{5}$ and $y=1$

Hence range of $f=\{y : y\in R, y\neq \frac{1}{5}, y\neq 1\}$

23 (i) $\displaystyle \lim_{x\to 0} \frac{\sin x - x + (1/6)\,x^3}{x^5}$

$\displaystyle = \lim_{x\to 0} \frac{(x-(1/3!)\,x^3+(1/5!)\,x^5-\ldots)-x+(1/6)\,x^3}{x^5}$

$\displaystyle \lim_{x\to 0} \frac{1}{x^5}\left[\frac{1}{5!}\,x^5+\text{higher powers of } x\right]$

$\displaystyle = \lim_{x\to 0} \left[\frac{1}{120}+\text{powers of } x\right]=\frac{1}{120}$

(ii) $\displaystyle \lim_{x\to 0}\left[\frac{1}{x}-\frac{1}{x^2}\log(1+x)\right]=\lim_{x\to 0}\left[\frac{x-\log(1+x)}{x^2}\right]$

$\displaystyle = \lim_{x\to 0}\frac{x-\{x-(1/2)x^2+(1/3)\,x^3 \ldots\}}{x^2}$

$\displaystyle = \lim_{x\to 0}\left(\frac{1}{2}-\frac{1}{3}x+\ldots\right)=\frac{1}{2}$

(iii) $\displaystyle \lim_{x\to 0} x\log\sin x$ $\qquad$ [Form $0\times\infty$]

$\displaystyle = \lim_{x\to 0}\frac{\log\sin x}{1/x}$ $\qquad$ $\left[\text{Form } \frac{\infty}{\infty}\right]$

$\displaystyle = \lim_{x\to 0}\frac{\frac{d}{dx}\log\sin x}{\frac{d}{dx}(1/x)}$, By L-Hospital Rule

$\displaystyle = \lim_{x\to 0}\frac{\cot x}{-(1/x^2)}=\lim_{x\to 0}-\frac{x}{\tan x}$ $\qquad$ $\left[\text{Form } \frac{0}{0}\right]$

$\displaystyle = \lim_{x\to 0}-\frac{2x}{\sec^2 x}$, by L Hospital Rule

$=0$

(iv) $\displaystyle \lim_{x\to\pi/2}\,[x\tan x-(\pi/2)\sec x]$

$\displaystyle = \lim_{x\to\pi/2}\frac{2x\sin x-\pi}{2\cos x}$ $\qquad$ $\left[\text{Form } \frac{0}{0}\right]$

$\displaystyle = \lim_{x\to\pi/2}\frac{d/dx\,[2x\sin x-\pi]}{d/dx\,[2\cos x]}$ by L-Hospital Rule

Since $Lf'(0) \neq Rf'(0)$, the function is not differentiable at $x=0$

Again $Lf'(1) = \lim_{h \to 0} \frac{f(1-h)-f(1)}{h} = \lim_{h \to 0} \frac{(1-h)^2-1}{-h}$

$= \lim_{h \to 0} \frac{-2h+h^2}{-h}$

$= \lim_{h \to 0} (2-h) = 2$

and $Rf'(1) = \lim_{h \to 0} \frac{f(1+h)-f(1)}{h}$

$= \lim_{h \to 0} \frac{(1+h)^3-(1+h)+1-1}{h}$

$= \lim_{h \to 0} \frac{2h+3h^2+h^3}{h} = \lim_{h \to 0} (2+3h+h^2) = 2$

Hence $f'(1)$ exists i e, function is differentiable at $x=1$

41 We have

$f(0+0) = \lim_{h \to 0} \frac{0+h}{1+e^{1/(0+h)}} = \lim_{h \to 0} \frac{h}{1+e^{1/h}} = 0$

Similarly $f(0-0) = \lim_{h \to 0} \frac{-h}{1+e^{-1/h}} = 0$

Since $f(0+0) = f(0) = f(0-0)$, the function is continuous at $x=0$

We now proceed to find the derivative of $f(x)$ at $x=0$ We have

$$Rf'(0) = \lim_{h \to 0} \frac{\frac{h}{1+e^{1/h}} - 0}{h} = \lim_{h \to 0} \frac{1}{1+e^{1/h}} = \frac{1}{1+e^{\infty}} = 0$$

And $$Lf'(0) = \lim_{h \to 0} \frac{\frac{-h}{1+e^{-1/h}}}{-h} = \lim_{h \to 0} \frac{1}{1+e^{-1/h}} = \frac{1}{1+e^{-\infty}}$$

$$= \frac{1}{1+0} = 1$$

Since $Rf'(0) \neq Lf'(0)$

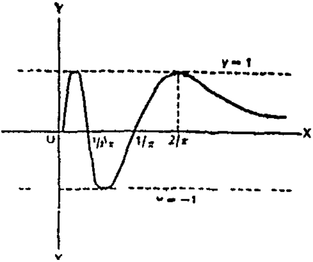
the derivative of f at $x=0$ does not exist

42 We have $f(0+0) = \lim_{h \to 0} (0+h)^2 \sin \frac{1}{0+h} = \lim_{h \to 0} h^2 \sin \frac{1}{h} = 0$

Similarly $f(0-0) = 0$

Since $f(0+0) = f(0-0) = f(0)$,

the function is continuous at $x=0$



The graph has been drawn on the right of the origin The students can verify that the curve on the left is exactly of the same type

From the graph it is clear that the function makes an infinite number of oscillations between −1 and 1 in the neighbourhood of 0 on the right Similarly it makes an infinite number of oscillations between − 1 and 1 on the left of 0 Hence the limit on the right as well as the limit on left does not exist at the origin

25 (a) To draw the graph we consider two cases (i) $x>0$ (ii) $x<0$

In case (i), we have $f(x)=(x/x)=1$

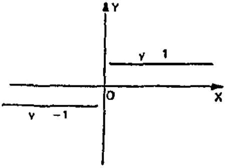
Thus in this case, graph is the ray $y=1$ parallel to x axis extending from the neighbourhood of 0 to ∞,

In case (ii) we have

$$f(x)=\frac{-x}{x}=-1$$

Hence in this case, the graph is the ray $y=-1$ extending from the neighbourhood of 0 to $-\infty$

Thus the graph is as shown in the adjoining figure



The function $f(x)$ is not defined at $x=0$ since it takes the indeterminate form 0/0 at $x=0$

To see whether $\lim_{x\to 0} f(x)$ exist or not, we find the right hand

It is therefore non differentiable also

At $x=1$, we have, $f(1)=1$

$$f(1-0)=1 \text{ and } f(1+0)=\lim_{h\to 0}\frac{1}{1+h}=1$$

Hence $f(1+0)=f(1-0)=f(1)=1$

It follows that f is continuous at $x=1$

Now $$Lf'(1)=\lim_{h\to 0}\frac{1-1}{-h}=0,$$

$$Rf'(1)=\lim_{h\to 0}\frac{1/(1+h)-1}{h}=\lim_{h\to 0}\frac{1-1-h}{h(1+h)}$$

$$=\lim_{h\to 0}\frac{-1}{1+h}=-1$$

Since $Lf'(1)\neq Rf'(1)$, the function is non differentiable at $x=1$

To draw the graph, we proceed as follows

(i) For $x \leqslant 0$, we have the parabola $y=x^2$

We construct the following table for this portion of the graph

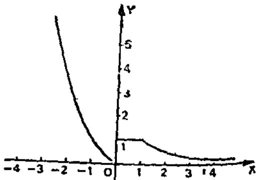
x	0	-1	-2	-3	-4
y	0	1	4	9	16

(ii) For $0 < x < 1$, we have $y=1$ which is a straight line parallel to x-axis

(iii) For $x > 1$, we have $y=1/x$ and table for this is as follows,

x	$1+h$	2	4	8	$\to\infty$
y	$\to 1$ as $h\to 0$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\to 0$

Hence the graph is as shown below



46 We consider $x=-2, 2$

At $x=-2$ we have $f(-2)=4$

$= 0 \times a$ finite quantity $= 0$

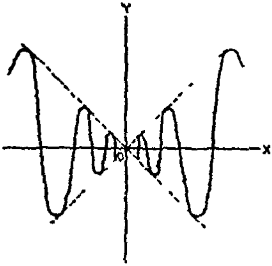
[$\sin 1/h$ is bounded lying between -1 and 1]

Similarly $f(0-0) \lim_{h \to 0} (0-h) \sin \frac{1}{0-h} = \lim_{h \to 0} h \sin 1/h$

$= 0$ as before

Also $f(0) = 0$

Since $f(0+0) = f(0-0) = f(0)$, the function $x \sin 1/x$ is continuous at $x = 0$. Although the function is continuous at the origin, yet the graph of the function in the vicinity of the origin cannot be drawn, since the function oscillatates infinitely often in any interval containing the origin (see the figure)



From the graph it is clear that the function makes an infinite number of oscillations in the neighbourhood of $x = 0$. The *oscillations*, however, go on diminishing in length as x approaches zero

(ii) Here $f(x) = 2^{1/x}$

$$f(0+0) = \lim_{h \to 0} 2^{1/(0+h)} = 2^{\infty} = \infty$$

and $f(0-0) = \lim_{h \to 0} 2^{1/(0-h)} = 2^{-\infty} = 0$

and $f(0) = 0$,

Since $f(0+0) \neq f(0-0)$

the function is dis-continuous at the origin

(iii) Here $f(x) = \frac{1}{1 - e^{-1/x}}$

$$f(0+0) = \lim_{h \to 0} \frac{1}{1 - e^{-1/(0+h)}} = \lim_{h \to 0} \frac{1}{1 - e^{-1/h}}$$

$$= \frac{1}{1 - e^{-\infty}} = \frac{1}{1-0} = 1$$

(ii) The straight line $y=4$ for $-2 \le x \le 2$
and (iii) The parabola $y=x^2$ for $x > 2$
Hence the graph is as drawn below

From the graph it is evident that the function is continuous throughout and non-differentiable at $x=-2$, since the slopes on the two sides of these points are different

47 We consider $x=0$, $\pi/2$

At $x=0$ we have $f(0)=1+\sin 0=1$

$$f(0-0)=\lim_{h\to 0} 1=1 \text{ and } f(0+0)=\lim_{h\to 0}\{1+\sin(0+h)\}=1$$

Since $f(0+0)=f(0-0)=f(0)$, the function is continuous at $x=0$

$$\text{Now}\quad Lf'(0)=\lim_{h\to 0}\frac{1-1}{-h}=0$$

$$\text{And}\quad Rf'(0)=\lim_{h\to 0}\frac{1+\sin(0+h)-1}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

Since $\quad Lf'(0)\ne Rf'(0)$,
the function is non differentiable at $x=0$
At $x=\pi/2$ we have $f(\pi/2)=2+(\pi/2-\pi/2)^2=2$

$$f(\pi/2-0)=\lim_{h\to 0}\{1+\sin(\pi/2-h)\}=\lim_{h\to 0}(1+\cos h)=2$$

$$f(\pi/2+0)=\lim_{h\to 0}\{2+(\pi/2+h-\pi/2)^2\}=2$$

Since $\quad f(\pi/2+0)=f(\pi/2-0)=f(\pi/2)$,
the function is continuous at $x=\pi/2$

$$\text{Now}\quad Lf'(\pi/2)=\lim_{h\to 0}\frac{1+\sin(\pi/2-h)-2}{-h}$$

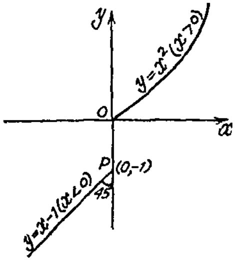
$$=\lim_{h\to 0}\frac{1-\cos h}{h}=\lim_{h\to 0}\frac{2\sin^2(h/2)}{h}$$

$$=\lim_{h\to 0}\frac{\sin(h/2)}{(h/2)}\sin(h/2)=1\times 0=0$$

$$\text{And}\quad Rf'(\pi/2)=\lim_{h\to 0}\frac{2+(\pi/2+h-\pi/2)^2-2}{h}=\lim_{h\to 0} h=0$$

Since $\quad Rf'(\pi/2)=Lf'(\pi/2)$,
the function is differentiable at $x=\pi/2$

48 We have $\quad f(1)=1$



Now $\lim\limits_{x\to 0+} f(x)=\lim\limits_{h\to 0}(0+h)^2=0$

and $\lim\limits_{x\to 0-} f(x)=\lim\limits_{h\to 0}(0-h-1)=-1$

Since $\lim\limits_{x\to 0+} f(x)\neq \lim\limits_{x\to 0-} f(x)$

the function $f(x)$ is discontinuous at $x=0$

29 (a) From the definition of the function it follows that

$f(x)=x-(n-1)$ for $n-1<x<n$,

$f(x)=0$ for $x=n$,

$f(x)=x-n$ for $n<x<n+1$,

where n is an integer

We test the function for continuity at $x=n$ We have

$f(n)=0$.

$f(n-0)=\lim\limits_{x\to n-0}[x-(n-1)]$

$=\lim\limits_{h\to 0}[(n-h)-(n-1)]=1$

and $f(n+0)=\lim\limits_{x\to n+0}(x-n)=\lim\limits_{h\to 0}(n+h-n)=0$

Hence f is discontinuous at $x=n$ i.e. for all integral values of x It is obviously continuous for all other values

Since x is a positive variable, putting

$n=1, 2, 3, 4, 5 ..$

$$=\lim_{h\to 0}\,[1+\tfrac{1}{3}\sin\log h]$$

Now $\sin\log h^2$ oscillates between -1 and 1 as $h\to 0$

so that $\lim_{h\to 0}\sin(\log h^2)$ does not exist

Hence $Lf'(0)$ does not exist

Similarly $Rf'(0)=\lim_{h\to 0}\dfrac{(0+h)\{1+\frac{1}{3}\sin\log(0+h)\}-0}{h}$

$$=\lim_{h\to 0}\{1+\tfrac{1}{3}\sin\log h\}$$

which does not exist as before

Hence f has no differential coefficient at $x=0$

50 $f(0+0)=\lim_{h\to 0}(0+h)^p\cos\dfrac{1}{0+h}=\lim_{h\to 0}h^p\cos\dfrac{1}{h}$ (1)

And $f(0-0)=\lim_{h\to 0}(-h)^p\cos\left(\dfrac{1}{-h}\right)=\lim_{h\to 0}(-h)^p\cos\dfrac{1}{h}$ (2)

Now in order that the function may be continuous at $x=0$, the limits given in (1) and (2) must both tend to zero This will be the case if $p>0$ which is the required condition

Again $Rf'(0)=\lim_{h\to 0}\dfrac{h^p\cos(1/h)-0}{h}=\lim_{h\to 0}h^{p-1}\cos\dfrac{1}{h}$ (3)

And $Lf'(0)=\lim_{h\to 0}\dfrac{(-h)^p\cos(-1/h)-0}{-h}$

$$=\lim_{h\to 0}-(-1)^p h^{p-1}\cos(1/h) \qquad (4)$$

Now in order that $f'(0)$ may exist, it is necessary that both the limits in (3) and (4) must tend to the same quantity This will be the case when $p>1$ for in that case both $Rf'(0)$ and $Lf'(0)$ will be zero Hence in order that f may have a differential coefficient at $x=0$, p should be greater than 1

51 $Rf'(0)=\lim_{h\to 0}\dfrac{\sqrt{h}\{1+h\sin(1/h)\}-0}{h}$

$$=\lim_{h\to 0}\left[\frac{1}{\sqrt{h}}+\sqrt{h}\sin(1/h)\right]=\infty+0=+\infty$$

And $Lf'(0)=\lim_{h\to 0}\dfrac{-\sqrt{[-(0-h)]}\left[1+(0-h)\sin\dfrac{1}{0-h}\right]-0}{-h}$

$$=\lim_{h\to 0}\left[\frac{1}{\sqrt{h}}-\sqrt{h}\sin\frac{1}{h}\right]=\infty-0=+\infty$$

Now $\quad f(1+0) = \lim\limits_{h\to 0} (1+h) = 1,$

and $\quad f(1-0) = \lim\limits_{h\to 0} (1-h)^2 = 1$

Hence $\quad f(1+0) = f(1-0) = 1$

This shows that the function has the limit 1 at $x=1$

(*b*) Consider the function ϕ defined as follows

$\phi(x) = -x^2$ for $x < 1$, $\phi(x) = x^2$ for $x > 1$

This function is not defined at $x=1$. Moreover

$$\phi(1-0) = \lim_{h\to 0} -(1-h)^2 = -1$$

and $\phi(1+0) = \lim\limits_{h\to 0} (1+h)^2 = 1$

Since

$$\phi(1-0) \neq \phi(1+0),$$

the limit does not exist at $x=1$

(c) Consider the function ψ defined as follows

$\psi(x) = 0 \qquad$ for $x \leq 0$,

$\psi(x) = \frac{1}{2} - x$ for $0 < x \leqslant \frac{1}{2}$,

$\psi(x) = \frac{3}{2} + x$ for $x > \frac{1}{2}$

The function is defined for $x=0$ and $x=\frac{1}{2}$ but is discontinuous at both these points as can be easily seen

31 From the definition of $E(x)$ we have

$E(x) = n-1 \qquad$ for $n-1 \leqslant x < n$

$E(x) = n \qquad$ for $\quad n \leqslant x < n+1$

$E(x) = n+1 \qquad$ for $n+1 \leqslant x < n+2$

and so on where n is an integer

We consider $x=n$. We have $E(n) = n$

$E(n-0) = n-1$ and $E(n+0) = n+1$

Since $\qquad E(n+0) \neq E(n-0) \neq E(n)$,

The function $E(x)$ is discontinuous when $x=n$ i.e. when x has an integral value

To draw the graph, we put

$n = \ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4,$

$y = -5$ when $-5 \leqslant x < -4$

$y = -4$ when $-4 \leqslant x < -3$

$y = -3$ when $-3 \leqslant x < -2$

$y = -2$ when $-2 \leqslant x < -1$

$y = -1$ when $-1 \leqslant x < 0$

At $x=2$, we have $f(2)=1$, $f(2-0)=\lim_{h\to 0} 1=1$

And $f(2+0)=\lim_{h\to 0}[2(2+h)-3]=1$

Since $f(2+0)=f(2-0)=f(2)$, the function is continuous at $x=2$

Now $Lf'(2)=\lim_{h\to 0}\frac{1-1}{-h}=0$

And $Rf(2)=\lim_{h\to 0}\frac{2(2+h)-3-1}{h}=\lim_{h\to 0}\frac{2h}{h}=2$

Since $Lf'(2)\neq Rf'(2)$, the function is not differentiable at $x=2$

53 Let $f(x)=2x^3-15x^2+36x+1$,

Then $f'(x)=6x^2-30x+36=6(x-2)(x-3)$

Now $f'(x)>0$ for $x<2$,

$f'(x)<0$ for $2<x<3$

$f'(x)>0$ for $x>3$

$f'(x)=0$ for $x=2$ and $x=3$

Hence $f'(x)$ is positive in the interval $]-\infty, 2[$ and $]3, \infty[$ and negative in the interval $]2, 3[$

Hence f is monotonically increasing in the intervals $]-\infty, 2[$ $]3, \infty[$ and monotonically decreasing in the interval $[2, 3]$

54 Do yourself

55 (a) Let $f(x)=\log(1+x)-\frac{x}{1+x}$

$f'(x)=\frac{1}{1+x}-\frac{1}{(1+x)}+\frac{x}{(1+x)^2}=\frac{x}{(1+x)^2}$

Then $f'(x)>0$ when $x>0$

and $=0$ when $x=0$

Thus $f(x)$ is monotonically increasing in the interval $[0, \infty[$

Again $f(0)=0$

$f(x)>f(0)=0$ when $x>0$

Hence $f(x)$ is positive for every positive value of x so that

$\log(1+x)>\frac{x}{1+x}$ when $x>0$ (i)

Now let $\phi(x)=x-\log(1+x)$

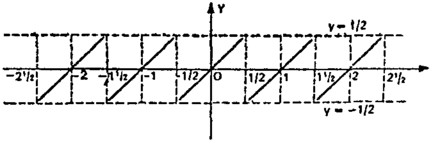
so that $\phi'(x)=1-\frac{1}{1+x}=\frac{x}{1+x}$

The putting $(x)=1$, we have

$y=x+3$	when	$-3\frac{1}{2}<x<-2\frac{1}{2}$
$y=0$	when	$x=-2\frac{1}{2}$
$y=x+2$	when	$-2\frac{1}{2}<x<-1\frac{1}{2}$
$y=0$	when	$x=-1\frac{1}{2}$
$y=x+1$	when	$-1\frac{1}{2}<x<-\frac{1}{2}$
$y=0$	when	$x=-\frac{1}{2}$
$y=x$	when	$-\frac{1}{2}<x<\frac{1}{2}$
$y=0$	when	$x=\frac{1}{2}$
$y=x-1$	when	$\frac{1}{2}<x<1\frac{1}{2}$
$y=0$	when	$x=1\frac{1}{2}$
$y=x-2$	when	$1\frac{1}{2}<x<2\frac{1}{2}$
$y=0$	when	$x=2\frac{1}{2}$
$y=x-3$	when	$2\frac{1}{2}<x<3\frac{1}{2}$
$y=0$	when	$x=3\frac{1}{2}$

and so on The following graph consists of segments of parallel straight lines

$x-y=-2,\ x-y=-1,\ x-y=0$
$x-y=1,\ x-y=2,$



and so on, between open intervals of

$]-2\frac{1}{2},-1\frac{1}{2}[,\quad]-1\frac{1}{2},-\frac{1}{2}[,\quad]-\frac{1}{2},\frac{1}{2}[,$
$]\frac{1}{2},1\frac{1}{2}[,\quad]1\frac{1}{2},2\frac{1}{2}[$

and so on and the points

$x=\pm\frac{1}{2},\ \pm1\frac{1}{2},\ \pm2\frac{1}{2}$ on x axis

The graph is shown by thick lines and thick dots or, x axis From the graph, it is evident that the function (x) is discontinuous at

$$x=\pm\tfrac{1}{2},\ \pm\tfrac{3}{2},\ \pm\tfrac{5}{2},$$

and so on

(v) By definitions of the functions f and g, we have

58 (*a*) We are given

$$y = 2x^2 - \log |x| \qquad (x \neq 0)$$

Then by def of $|x|$, we have

$$y = 2x^2 - \log(-x) \quad \text{for } x < 0$$

and $\quad y = 2x^2 - \log x \quad \text{for } x > 0$

Hence $\dfrac{dy}{dx} = 4x - \dfrac{1}{(-x)}(-1)$ for $x < 0$

and $\dfrac{dy}{dx} = 4x - \dfrac{1}{x}$ for $x > 0$

Hence $\dfrac{dy}{dx} = 4x - \dfrac{1}{x} = \dfrac{(2x-1)(2x+1)}{x}$ for $x \neq 0$

Now $\dfrac{dy}{dx} > 0$ for $-\frac{1}{2} < x < 0$ or $x > \frac{1}{2}$

and $\dfrac{dy}{dx} < 0$ for $x < -\frac{1}{2}$ or $0 < x < \frac{1}{2}$

Hence y increases in the intervals $(-\frac{1}{2}, 0)$ and $(\frac{1}{2}, \infty)$ and decreases in the intervals $(-\infty, -\frac{1}{2})$ and $(0, \frac{1}{2})$

(*b*) First let $f(x) = x - x^3/6 - \sin x$

Then $f'(x) = 1 - \frac{1}{6}\, 3x^2 - \cos x = 1 - \frac{1}{2}x^2 - \cos x$

We now find the sign of $f'(x)$ in the interval $0 < x \leqslant \pi/2$ We write $g(x) = 1 - \frac{1}{2}x^2 - \cos x$

Then $g'(x) = -x + \sin x < 0$ for $0 < x \leqslant \pi/2$

[See inequality (1) of Q 17 (*i*)] Hence $g(x)$ is decreasing for $0 < x < \pi/2$ But

$g(0) = 1 - 0 - 1 = 0$ It follows that

$g(x) < 0$ for $0 < x \leqslant \pi/2$, that is,

$f'(x) = 1 - \frac{1}{2}x^2 - \cos x < 0$ for $0 < x \leqslant \pi/2$

$f(x)$ is decreasing for $0 < x \leqslant \pi/2$

Hence $f(x) < f(0)$ for $0 < x \leqslant \pi/2$

or $x - \frac{1}{6}x^3 - \sin x < 0$ for $0 < x \leqslant \pi/2$

[Note that $f(0) = 0$] (1)

Thus $x - x^3/6 < \sin x$ for $0 < x \leqslant \pi/2$

Again let $g(x) = \sin x - x$ Then

$g'(x) = \cos x - 1 < 0$ for $0 < x \leqslant \pi/2$

Hence $g(x)$ is decreasing for $0 < x \leqslant \pi/2$

$g(x) < g(0)$ for $0 < x \leqslant \pi/2$

or $g(x) < 0$ for $0 < x \leqslant \pi/2$, $\quad g(0) = 0$ (2)

or $\sin x - x < 0 \Rightarrow \sin x < x$

function $y=f(x)=\dfrac{1}{1-x}$

If $x\neq 1$, then

$$v(x)=f\{f(x)\}=f\left(\frac{1}{1-x}\right)=\frac{1}{1-\{1/(1-x)\}}=\frac{x-1}{x}$$

Hence, the point $x=0$ is a discontinuity of the function v

If $x\neq 0$, $x\neq 1$, then

$$u(x)=f[f\{f(x)\}]=f\left[f\left(\frac{1}{1-x}\right)\right]=f\left(\frac{x-1}{x}\right)$$

$$=\frac{1}{1-(x-1)/x}=x$$

Hence u is clearly continuous everywhere. Thus, the points of discontinuity of the composite function $f[f\{f(x)\}]$ are $x=0$, $x=1$ and the composite function $f\{f(x)\}$ has a discontinuity at $x=1$ only.

34 (*i*) Since f is continuous at a, we have

$$f(a)=f(a+0)=\lim_{h\to 0} f(a+h)=\lim_{h\to 0}[f(a)+f(h)]$$

[See definition of f]

$$=\lim_{h\to 0} f(a)+\lim_{h\to 0} f(h)=f(a)+\lim_{h\to 0} f(h)$$

Hence $\displaystyle\lim_{h\to 0} f(h)=0$ (1)

Similarly $\displaystyle f(a)=f(a-0)=\lim_{h\to 0} f(a-h)=f(a)+\lim_{h\to 0} f(-h)$

$$\lim_{h\to 0} f(-h)=0 \quad (2)$$

Now let α be any member of R. To show that f is continuous at α. We have

$$f(\alpha+0)=f(\alpha+h)=\lim_{h\to 0}[f(\alpha)+f(h)]$$

$$=\lim_{h\to 0} f(\alpha)+\lim_{h\to 0} f(h)=f(\alpha)+\lim_{h\to 0} f(h)=f(\alpha)$$

by (1)

Similarly by using (2), we shall have

$$f(\alpha-0)=f(\alpha)$$

Hence $f(\alpha+0)=f(\alpha)=f(\alpha-0)$

60 First let $-2 \leqslant x \leqslant 0$ Then

$0 \leqslant |x| \leqslant 2$

$f(|x|)=|x|-1$ by def of f

and $|f(x)|=|-1|=1$, by def of f

Thus when $-2 \leqslant x \leqslant 0$, $g(x)=|x|-1+1=|x|$

Now let $0 < x \leqslant 2$ Then $|x|=x$

$f(|x|)=f(x)=x-1$ and $|f(x)|=|x-1|$

When $0 < x \leqslant 2$ $g(x)=(x-1)+|x-1|$

Thus we have

$g(x)=|x|$ when $-2 \leqslant x \leqslant 0$

and $g(x)=(x-1)+|x-1|$ when $0 < x \leqslant 2$

We need only test for differentiability at $x=0$ [$g(x)$ is clearly continuous at $x=0$] we have

$g(0)=|0|=0$

$$Lg'(0)=\lim_{h\to 0}\frac{|0-h|-0}{-h}=\lim_{h\to 0}\frac{h}{-h}=-1$$

and
$$Rg'(0)=\lim_{h\to 0}\frac{(0+h-1)+|0+h-1|-0}{h}$$

$$=\lim_{h\to 0}\frac{h-1+1-h}{h} \quad [\text{For small } h,\ 1-h>0]$$

$=0$

Since $Lg'(0) \neq Rg'(0)$ the function $g(x)$ is not differentiable at $x=0$

(a) $|x|=x,\ x>0$ $|x|=-x,\ x<0,$ $|x|=0,\ x=0$

$$f(x)=\frac{x}{1-x}\ x<0\quad f(x)=0,\ x=0,\ f(x)=\frac{x}{1+x}\ x>0$$

Consider $x=0$

For continuity $Lt\, f(0-h)=Lt\, f(0+h)=f(0)$

$$Lt\ \frac{-h}{1+h}=0 \quad Lt\ \frac{h}{1+h}=0 \quad f(0)=0$$

Since limit=value function is continuous

For Differentiability

$$Lt\ \frac{f(0-h)-f(0)}{-h}=Lt\ \frac{f(0+h)-f(0)}{h}$$

$$Lt\ \frac{1}{1+h}=1 \qquad Lt\ \frac{1}{1+h}=1$$

Hence both left hand and right hand derivatives exist so the function is differentiable at $x=0$ and hence for all other real values of x it is differentiable Therefore it is differentiable in the interval $(-\infty, \infty)$

Thus we have shown that

$$f(x)=xf(1) \ \forall \ x \in \mathbf{R}$$

35 Writing δx for y in the given condition (*a*), we have

$$f(x+\delta x)=f(x)\,f(\delta x)$$

Then $\quad f(x+\delta x)-f(x)=f(x)f(\delta x)-f(x)$

or $\quad \dfrac{f(x+\delta x)-f(x)}{\delta x}=\dfrac{f(x)\,[f(\delta x)-1]}{\delta x}$

$=\dfrac{f(x)\,\delta x\,g(\delta x)}{\delta x}$ by condition (*b*)

$=f(x)\,g(\delta x)$

Hence $\displaystyle\lim_{\delta x\to 0}\frac{f(x+\delta x)-f(x)}{\delta x}=\lim_{\delta x\to 0}f(x)\,g(\delta x)$

$=f(x)\cdot 1$, since by hypothesis $\displaystyle\lim_{\delta x\to 0} g(\delta x)=1$

It follows that $\quad f'(x)=f(x)$

Since $f(x)$ exists $f'(x)$ also exists and $f'(x)=f(x)$

36 Let α be any arbitrary real number We shall show that $f(x)$ is discontinuous at α Let $\{x_n\}$ be any sequence of rational numbers such that $x_n\to\alpha$ as $n\to\infty$ Then by definition of f, we have $f(x_n)=1$ for all positive integers n

Hence $f(x_n)\to 1$ as $n\to\infty$

Again let $\{y_n\}$ be a sequence of irrational numbers such that

$$y_n\to\alpha \quad \text{as} \quad n\to\infty$$

Then by definition of f, we have $f(y_n)=-1$ for all positive integers n It follows that $f(y_n)\to -1$ as $n\to\infty$

Now either (i) $f(\alpha)=1$ when α is rational

or $\quad$ (ii) $f(\alpha)=-1$ when α is irrational

In case (i) $\displaystyle\lim_{n\to\infty} f(y_n)=-1\neq f(\alpha)$

and in case (ii), $\displaystyle\lim_{n\to\infty} f(y_n)=1\neq f(\alpha)$

Hence by Heine s definition of continuity, $f(x)$ is discontinuous at $x=\alpha$ Since α is an arbitrary real number, it follows that $f(x)$ is discontinuous for all real numbers x

37 $\quad f(0)=0$

$$f(0+0)=\lim_{h\to 0}\frac{\sin(0+h)^2}{(0+h)}=\lim_{h\to 0}\frac{\sin h^2}{h^2}\,h=1\times 0=0$$

$$=2x(x-2)(x-2+x)=4x(x-1)(x-2)$$

Clearly $\frac{dy}{dx}=-\text{ive} \quad x<0$

$=+\text{ive} \quad 0<x<1$

$=-\text{ive} \quad 1<x<2$

$=+\text{ive} \quad x>2$

Hence y is an increasing function when $0<x<1$ or $x>2$

(h) $x^2<1 \Rightarrow x^2-1=-\text{ive}\ (x-(-1))(x-1)=-\text{ive}$

x lies between -1 and 1 i e $-1<x<1$ (1)

$x^2 \geqslant 1 \Rightarrow (x-(-1))(x-1) \geqslant 0$ i e $+\text{ive}$

x does not lie between -1 and 1

$x \leqslant -1$ or $x \geqslant 1$ (2)

$f(x)=x^4 \quad x^2<1$
$\quad =x \quad x^2 \geqslant 1$ may be defined as

$f(x)=x^4 \quad -1<x<1$ by 1 A
$\quad =x \quad x \leqslant -1 \quad x \geqslant 1$ by 2 B

Lt at $x=1$

$\text{Lt} f(1-h)=\text{Lt}(1-h)^4=1$ by (A)

$\text{Lt} f(1+h)=\text{Lt}(1+h)=1$

Since right hand and left hand limits are equal therefore limit exists and is equal to 1

Lt at $x=-1$

$\text{Lt} f(-1-h) \quad \text{Lt}(-1-h)=-1$ from B

$\text{Lt}(-1+h) \quad (-1+h)^4=1$ from A

Since these limits are not equal therefore Limit does exist at $x=-1$

Problem Set (B)

(Objective Questions)

1 $\lim_{x\to 0} \frac{1-\cos 2x}{x}$ is (M N R 83)

(a) 0 (b) 1 (c) 2, (d) 4

2 $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x$ equals

(a) e, (b) ∞, (c) e^2, (d) $1/e$

3 $\lim_{x\to 0}\frac{\log\cos x}{x}$ is equal to

(a) 0 (b) ∞, (c) 1 (d) None of these see from (2) That

$$|f'(x)|=0 \Rightarrow f'(x)=0$$

Hence $f(x)$ is constant

39 We are given

$$f(x+y)=f(x)\,f(y) \qquad (1)$$

for all x and y

In the identity (1), we put $x=5$, $y=0$,

Then $f(5)=f(5)\,f(0)$

This gives $f(0)=1$ $\quad [\because f(5)=2\neq 0]$

Now $f'(5)=\lim_{h\to 0}\dfrac{f(5+h)-f(5)}{h}$

$=\lim_{h\to 0}\dfrac{f(5)\,f(h)-f(5)}{h}$ by the given identity

$=f(5)\lim_{h\to 0}\dfrac{f(h)-1}{h}$

$=f(5)\lim_{h\to 0}\dfrac{f(h)-f(0)}{h}$ $\quad [\because f(0)=1]$

$=f(5)\,f'(0)$, by definition of differential coefficient

$=2\times 3=6$

40 (*a*) By definition of a derivative at a point we have

$$f'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\left[\frac{1+h-1}{2(1+h)^2-7(1+h)+5}-\left(-\frac{1}{3}\right)\right]\Big/h$$

$$=\lim_{h\to 0}\frac{3h+2(1+h)^2-7(1+h)+5}{3h\,[2(1+h)^2-7(1+h)+5]}$$

$$=\lim_{h\to 0}\frac{2h^2}{3h\,[-3h+2h^2]}=\lim_{h\to 0}\frac{2}{-9+6h}=-\frac{2}{9}$$

Hence differential coefficient of $f(x)$ at $x=1$ is is $-2/9$

(*b*) We test the function $f(x)$ for differentiability at $x=0$ and $x=1$ only For other values of x, $f(x)$ can be easily seen to be differentiable Students can check that $f(x)$ is continuous at $x=0$ and $x=1$

We now first test the differentiability at $x=0$

$$Lf'(0)=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\frac{-(0-h)-0}{-h}=-1$$

and $$Rf'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{(0+h)^2-0}{h}=\lim_{h\to 0}h=0$$

$$f(x)=\frac{\tan(\pi[x-\pi])}{1+[x]^2} \text{ is}$$

(a) discontinuous at some x,
(b) continuous at all x, but the derivative $f'(x)$ does not exist for some x
(c) $f'(x)$ exists for all x but second derivative $f''(x)$ does not exist (I I T 81)
(d) $f''(x)$ exists for all x

13 The function $f(x)=-\frac{1}{x}$ on its domain is
(a) increasing (b) decreasing,
(c) constant (d) information insufficient

14 There exists a function $f(x)$ satisfying
$f(0)=1$, $f'(0)=-1$, $f(x)>0$ for all x and $f''(x)<0$ for all x,
(i) $f''(x)<0$ for all x (ii) $-1<f'(x)<0$ for all x,
(iii) $-2 \leqslant f'(x) \leqslant -1$ for all x (I I T 82)
(iv) $f'(x) < -2$ for all x

15 If $f(x)=\cos(\log x)$, then
$f(x)f(y)-\frac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$ has the value
(A) -1 (B) $\frac{1}{2}$ (C) -2 (D) None of these (I I T 83)

16 If $f(x)=\sqrt{\left(\frac{x-\sin x}{x+\cos^2 x}\right)}$ then $\underset{x\to\infty}{\text{Lt}} f(x)$ is
(A) 0 (B) ∞ (C) 1 (D) None of these (I I T 79)

17 If $f(a)=2, f'(a)=1, g(a)=-1, g'(a)=2$ then the value of
$\underset{x\to a}{\text{Lt}} \frac{g(x)f(a)-g(a)f(x)}{x-a}$ is
(A) -5 (B) $\frac{1}{5}$ (C) 5 (D) None of these (I I T 83)

18 If $G(x)=-\sqrt{(25-x^2)}$ then $\underset{x\to 1}{\text{Lt}} \frac{G(x)-G(1)}{x-1}$ has the value
(a) $\frac{1}{24}$ (b) $\frac{1}{5}$ (c) $-\sqrt{(24)}$ (d) None of these (I I T 83)

19 Which of the following functions is periodic
(A) $f(x)=x-[x]$ where $[x]$ denotes the largest integer less than or equal to the real number x

Now $Rf'(0)=\lim_{h\to 0}\dfrac{(0+h)^2 \sin\left(\dfrac{1}{0+h}\right)-0}{h}$

$=\lim_{h\to 0} h \sin\dfrac{1}{h}=0$

Similarly $Lf'(0)=0$

Hence f is differentiable at $x=0$ The derivative of f at $x=0$ has the value zero At all other points f is easily seen to be continuous and differentiable

Now $f'(x)=2x\sin\dfrac{1}{x}-\cos\dfrac{1}{x}$ at $x\neq 0$ and $f'(0)=0$

$f'(0+0)=\lim_{h\to 0}\left(2h\sin\dfrac{1}{h}-\cos\dfrac{1}{h}\right)$

which does not exist

Similarly $f'(0-0)$ does not exist Hence f' is discontinuous at the origin

43 (*a*) We have $f(0)=|0|=0$

$f(0+0)=\lim_{h\to 0}|0+h|=0$ and $f(0-0)=\lim_{h\to 0}|0-h|=0$

Hence $f(x)$ is continuous at $x=0$

As regards differentiability, we have

$Rf'(0)=\lim_{h\to 0}\dfrac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\dfrac{|h|-0}{h}$

$=\lim_{h\to 0}\dfrac{h}{h}$ (h being positive)$=1$

And $Lf'(0)=\lim_{h\to 0}\dfrac{f(0-h)-f(0)}{-h}=\lim_{h\to 0}\dfrac{|-h|-0}{-h}$

$=\lim_{h\to 0}\dfrac{h}{-h}$ (h being positive)$=-1$

Since $Rf'(0)\neq Lf'(0)$, the function f is not differentiable at $x=0$

(*b*) Sketch the graph yourself Continuous but not differentiable at $x=2$

44 Do yourself

45 Let $y=f(x)$ We consider $x=0, 1$

At $x=0$ we have $f(0)=0$

$f(0-0)=\lim_{h\to 0}(0-h)=0$, $f(0+0)=\lim_{h\to 0}1=1$

Since $f(0-0)\neq f(0+0)$ the function is discontinuous at $x=0$

27 Let [x] denote the greatest integer less than or equal to x. If $f(x) = [x \sin \pi x]$, then $f[x]$ is
(A) continuous at $x=0$, (B) continuous in $(-1, 0)$
(C) differentiable at $x=1$ (D) differentiable in $(-1, 1)$,
(E) None of these (IIT 86)

28 If $f(x) = \sin x,\ x \neq n\pi,\ n = 0, \pm 1, \pm 2, \pm 3,$
$= 2$, otherwise
and $g(x) = x^2 + 1,\ x \neq 0, 2$
$= 4,\ x = 0$
$= 5,\ x = 2$
then $\lim_{x \to 0} g[f(x)]$ is (IIT 1986)

29 If $f_1(x)$ and $f_2(x)$ are defined on domains D_1 and D_2 respectively then $f_1(x) + f_2(x)$ is defined on $D_1 \cup D_2$
(a) True, (b) False (IIT 88)

30 The function

$$f(x) = \begin{cases} |x-3|, & x \geq 1 \\ x^2/4 - 3x/2 + 13/4, & x < 1 \end{cases}$$

is
(A) continuous at $x=1$, (B) continuous at $x=3$
(C) differentiable at $x=1$ (D) differentiable at $x=3$
(IIT 88)

31 If $f(9) = 9,\ f'(9) = 4$, then
$\lim_{x \to 9} \frac{\sqrt{(f(x)} - 3}{\sqrt{x} - 3}$
equals (IIT 88)

32 If $f(x) = x \sin(1/x),\ x \neq 0$
$= 0,\ x = 0$
Then $\lim_{x \to 0} f(x)$ equals
(a) 1 (b) 0
(c) -1, (d) none of these [MNR 88]

33 If $f(x) = 1$ for $x < 0$
$= 1 + \sin x$ for $0 \leq x < \pi/2$,
then at $x = 0$, the derivative $f(x)$
(a) is 1 (b) is 0,
(c) is infinite, (d) does not exist [MNR 88]

$f(-2-0)=\lim_{h\to 0}[-2-h]^2=4$ and $f(-2+0)=\lim_{h\to 0}4=4$

Since $f(-2+0)=f(-2-0)=f(-2)$, the function is continuous at $x=-2$

Now $Lf'(-2)=\lim_{h\to 0}\frac{(-2-h)^2-4}{-h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{-h}$

$=\lim_{h\to 0}[-4-h]=-4$

And $Rf'(-2)=\lim_{h\to 0}\frac{4-4}{h}=0$

Since $Lf'(-2)\neq Rf'(-2)$, the function is non differentiable at $x=-2$

At $x=2$, we have $f(2)=4$

$f(2-0)=\lim_{h\to 0}4=4$ and $f(2+0)=\lim_{h\to 0}[2+h]^2=4$

Hence f is continuous at $x=2$

$Lf'(2)=\lim_{h\to 0}\frac{4-4}{-h}=0$

$Rf'(2)=\lim_{h\to 0}\frac{(2+h)^2-4}{h}=\lim_{h\to 0}\frac{4+4h+h^2-4}{h}$

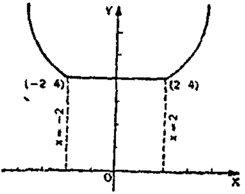
$=\lim_{h\to 0}[4+h]=4$

Since $Lf'(2)\neq Rf'(2)$ the function is non differentiable at $x=2$

Hence the function is continuous throughout and non differentiable at $x=-2, 2$

The graph consists of

(i) *The parabola $y=x^2$ for $x<-2$*



Ans (c)

$$\lim_{x\to\infty}\sqrt{\left(\frac{x-\sin x}{x+\cos^2 x}\right)}=\lim_{x\to\infty}\sqrt{\left(\frac{1-\sin x/x}{1+\cos^2 x/x}\right)}=\sqrt{\left(\frac{1-0}{1+0}\right)}=1$$

$\left[\frac{\sin x}{x}\to 0 \text{ and } \frac{\cos^2 x}{x}\to 0 \text{ as } x\to\infty\right]$

8 (a) A discontinuous function satisfying the given condition is defined by $f(x)=\sqrt{(4-x^2)}$ for $x \geqslant 0$ and $f(x)=-\sqrt{(4-x^2)}$ for $x<0$. This function is discontinuous at $x=0$ since $f(0)=2$, $f(0+0)=\lim_{h\to 0}\sqrt{\{4-(0+h)^2\}}$ $=2$ and $f(0-0)=\lim_{h\to 0}-\sqrt{\{4-(0-h)^2\}}=-2$

(b) Put $x=1+h$. Then $h\to 0$ as $x\to 1$

Hence

$$\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}=\lim_{h\to 0}\{1-(1+h)\}\tan\left\{\frac{\pi}{2}(1+h)\right\}$$

$$=\lim_{h\to 0}(-h)\left\{-\cot\left(\frac{\pi}{2}h\right)\right\}=\lim_{h\to 0}\frac{2}{\pi}\cdot\frac{\frac{\pi}{2}h\times\cos\left(\frac{\pi}{2}h\right)}{\sin\left(\frac{\pi}{2}h\right)}$$

$$=\frac{2}{\pi}\cdot 1\cdot 1=\frac{2}{\pi}$$

9 Ans $k=7$

We have

$$\lim_{x\to 2}f(x)=\lim_{x\to 2}\frac{x^3+x^2-16x+20}{(x-2)^2}$$

$$=\lim_{x\to 2}\frac{(x-2)^2(x+5)}{(x-2)^2}\quad\text{[Factorising the Denominator]}$$

$$=\lim_{x\to 2}(x+5)=7$$

Also $f(2)=k$

Since the function $f(x)$ is continuous at $x=2$, we have

$\lim_{x\to 2}f(x)=f(2)$ or $7=k$. Hence $k=7$

10 Ans (a)

We have, $f(0)=0$

$$f(0+0)=\lim_{h\to 0}\frac{\sin(0+h)^2}{0+h}=\lim_{h\to 0}\frac{\sin h}{h}$$

$f(1-0)=\lim_{h\to 0}[1-h]=1$ and $f(1+0)=\lim_{h\to 0}[2-(1+h)]=1$

Since $f(1+0)=f(1-0)=f(1)$,

the function is continuous at $x=0$

Now $Lf(1)=\lim_{h\to 0}\dfrac{(1-h)-1}{-h}=\lim_{h\to 0}\dfrac{-h}{-h}=1$

And $Rf(1)=\lim_{h\to 0}\dfrac{2-(1+h)-1}{h}=\lim_{h\to 0}\dfrac{-h}{h}=-1$

Since $Rf(1)\neq Lf(1)$,

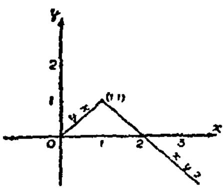
the function is non differentiable at $x=1$

The graph consists of the segments of the two straight lines

(i) $y=x$ for $0\leqslant x\leqslant 1$

(ii) $y=2-x$ for $x\geqslant 1$

Hence the graph is as drawn



From the graph, it is evident that the function is continuous at $x=1$ but non differentiable since the slopes on the two sides are different

49 We consider $x=0$ We have $f(0)=0$

$f(0+0)=\lim_{h\to 0}\{(0+h)\,(1+\frac{1}{3}\sin\log(0+h)^2\}$

$=\lim_{h\to 0}[h+(h/3)\sin\log h^2]$

$=0+0\times$a finite quantity

$=0$ [$\because$ $\sin\log h^2$ oscillates between -1 and 1 as $h\to 0$ and hence finite]

and $f(0-0)=\lim_{h\to 0}(0-h)\{1+\frac{1}{3}\sin\log(0-h)^2\}$

$=\lim_{h\to 0}[-h-(h/3)\sin\log h^2]=0$ as before

Hence f is continuous at $x=0$

Now $Lf(0)=\lim_{h\to 0}\dfrac{(0-h)\{1+\frac{1}{3}\sin(0-h)\}-0}{-h}$

$$= \lim_{h\to 0} -\frac{1}{h}\left[-(h^2+2h+1)\sin\frac{1}{(h+1)} \quad h+\sin 1\right]$$

$$= \lim_{h\to 0}\left[h\sin\frac{1}{h+1}+2\sin\frac{1}{h+1}+1+\frac{\sin\{1/(h+1)\}-\sin 1}{h}\right]$$

$$=0+2\sin 1+1+\lim_{h\to 0}\frac{2}{h}\cos\frac{1}{2}\left(\frac{1}{h+1}+1\right)\sin\frac{1}{2}\left(\frac{1}{h+1}-1\right)$$

$$=2\sin 1+1+2\cos 1\ \lim_{h\to 0}\frac{1}{h}\sin\left(\frac{-h}{2(h+1)}\right)$$

$$=2\sin 1+1-2\cos 1\ \lim_{h\to 0}\ \frac{[\sin(h/2(h+1))]}{[h/2(h+1)]}\ \frac{1}{2(h+1)}$$

$$=2\sin 1+1-2\cos 1\times 1\times\tfrac{1}{2}$$

$$=2\sin 1+1-\cos 1$$

Thus $\qquad Rf'(0)=2\sin 1-1-\cos 1$

and $\qquad Lf(0)=2\sin 1+1-\cos 1$

Since $Rf(0)\neq Lf(0)$, the function $f(x)$ is not differentiable at $x=0$

We now test at $x=1$

We have $\quad f(1)=-1$

$$Rf'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[(1+h-1)^2\sin\frac{1}{(1+h-1)}-|1+h|\quad(-1)\right]$$

$$=\lim_{h\to 0}\frac{1}{h}\left[h^2\sin\frac{1}{h}-1-h+1\right]$$

$$=\lim_{h\to 0}\left(h\sin\frac{1}{h}-1\right)=0-1=-1$$

and $Lf'(1)=\lim_{h\to 0}-\frac{1}{h}\left[(1-h-1)^2\sin\frac{1}{(1-h-1)}-|1-h|\quad(1)\right]$

$$\lim_{h\to 0}-\frac{1}{h}\left[h^2\sin\left(-\frac{1}{h}\right)-1+h+1\right]$$

$$\lim_{h\to 0}\left(h\sin\frac{1}{h}-1\right)=0-1=-1$$

Since $Rf(1)=Lf(1)$, the function is differentiable at $x=1$
Hence the only point where the function is not differentiable is at $x=0$, that is, set of points where the function is not differentiable is $\{0\}$

Since $Rf'(0)=Lf'(0)=+\infty$, the derivative of f at $x=0$ is $+\infty$

For values of x, other than zero, we have

$$f'(x)=\frac{1}{2\sqrt{x}}+\frac{3}{2}\sqrt{x}\sin\frac{1}{x}-\frac{1}{\sqrt{x}}\cos\frac{1}{x} \text{ when } x>0$$

and $$f'(x)=\frac{1}{2\sqrt{(-x)}}+\frac{3}{2}\sqrt{(-x)}\sin\frac{1}{x}-\frac{1}{\sqrt{(-x)}}\cos\frac{1}{x} \text{ when } x<0$$

Hence $f'(a)$ is finite for $a\neq 0$

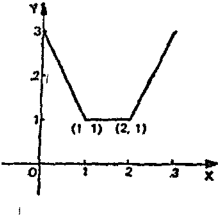
52 From the definition of the function we have

$y=1-x+2-x=3-2x$ when $x\leqslant 1$

$y=x-1+2-x=1$ when $1\leqslant x\leqslant 2$

$y=x-1+x-2=2x-3$ when $x>2$

The graph thus consists of the segments of the three straight lines $y=3-2x$, $y=1$, $y=2x-3$ corresponding to the intervals [0, 1], [1, 2], [2 3] respectively Hence the graph drawn for the interval [0, 3] is as shown



From the graph it is clear that the function is continuous throughout the interval but is not differentiable at $x=1$, 2 since the slopes at these points are different on the left hand and right hand sides

To test it analytically, we write $y=f(x)$. Then

$f(x)=3-2x$ when $x\leqslant 1$

$=1$ when $1\leqslant x\leqslant 2$

$=2x-3$ when $x>2$

At $x=1$, we have $f(1)=3-2=1$

$$f(1-0)=\lim_{h\to 0}[3-2(1-h)]=1 \text{ and } f(1+0)=\lim_{h\to 0}[1]=1$$

Hence f is continuous at $x=0$

Now $$Rf'(1)=\lim_{h\to 0}\frac{1-1}{h}=0$$

And $$Lf'(1)=\lim_{h\to 0}\frac{3-2(1-h)-1}{-h}=\lim_{h\to 0}\frac{2h}{-h}=-2$$

Since $Rf'(1)\neq Lf'(1)$ the function is not differentiable at $x=1$

First note that by def of $[x]$, we have

$[x]=-1$ when $-1\leqslant x<0$

and $[x]=0$ when $0\leqslant x<1$

Hence by def of f,

$$f(x)=\frac{\sin(-1)}{-1}=\sin 1 \text{ when } -1\leqslant x<0$$

and $f(x)=0$ when $0\leqslant x<1$

$$\therefore f(0-0)=\lim_{h\to 0}\sin 1=\sin 1$$

and $f(0+0)=\lim_{h\to 0} 0=0$

Since $f(0-0)\neq f(0+0)$, the limit of $f(x)$ at $x=0$ does not exist

23 Ans, (*B*)

It is easy to see that $f(0+0)=f(0-0)=f(0)=0$ (This is left for the students) Hence $f(x)$ is continuous at $x=0$

Now $Lf'(0)=\lim_{h\to 0}\dfrac{(0-h)[\sqrt{(0-h)}-\sqrt{(0-h+1)}]-0}{-h}$

$=\lim_{h\to 0}[\sqrt{(-h)}-\sqrt{(-h+1)}]$

$=0-\sqrt{1}=-1$

and $Rf'(0)=\lim_{h\to 0}\dfrac{(0+h)[\sqrt{(0+h)}-\sqrt{(0+h+1)}]-0}{h}$

$=\lim_{h\to 0}[\sqrt{h}-\sqrt{(h+1)}]$

$=0-\sqrt{1}=-1$

Since $Lf'(0)=Rf'(0)=-1$, the function $f(x)$ is differentiable at $x=0$

24 Ans Total no of functions $=n^n$ and no of onto functions $=n!$

Let A consist of n distinct elements $a_1, a_2, \ldots, a_n$. To count the number of all possible distinct functions $f: A\to A$, we observe that the first element a_1 has just n possible images $a_1, a_2, \ldots, a_n$ and corresponding to each of these images, there are again n choices for the image of the second element a_2 and so on Hence the total number of ways of choosing all n images is n multiplied by itself n times, that is n^n

Since A consists of finite number of elements n, all onto functions will be one-one as well Hence the no of all onto functions (i e

Then $\phi(x) > 0$, when $x > 0$
and $= 0$, when $x = 0$
Therefore ϕ is monotonically increasing in the intervals $[0, \infty[$
Also $\phi(0) = 0$ $\quad \phi(x) > \phi(0) = 0$ when $x > 0$
Hence $\phi(x)$ is positive for positive values of x
so that $\quad x > \log(1+x)$ (2)
From (1) and (2) we have

$$\frac{x}{1+x} < \log(1+x) < x \text{ when } x > 0$$

(*b*) Let $f(x) = 1 - \sqrt{(1+x^2)} + x \log\{x + \sqrt{(1+x^2)}\}$

Then $f'(x) = -\dfrac{2x}{2\sqrt{(1+x^2)}} + x \dfrac{1}{x+\sqrt{(1+x^2)}}\left\{1 + \dfrac{2x}{2\sqrt{(1+x^2)}}\right\} + \log\{x + \sqrt{(1+x^2)}\}$

$= -\dfrac{x}{\sqrt{(1+x^2)}} + \dfrac{x}{\sqrt{(1+x^2)}} + \log\{x + \sqrt{(1+x^2)}\}$

$= \log\{x + \sqrt{(1+x^2)}\}$

Hence $f'(x) \geqslant 0 \ \forall \ x \geqslant 0$

$f(x)$ is an increasing function of x for all $x \geqslant 0$. It follows that

$f(x) \geqslant f(0) = 0$ *i.e.* $1 + x \log\{x + \sqrt{(1+x^2)}\} \geqslant \sqrt{(1+x^2)}$
for all $x \geqslant 0$

56 For continuity and differentiability of f see problem 49.
We now show that f is monotonic. We have

$$f'(x) = 1 + \tfrac{1}{3} \sin \log x^2 + \tfrac{2}{3} \cos \log x^2$$
$$\geqslant 0 \text{ for all values of } x \text{ other than zero}$$

[Note that $f'(x)$ can be zero at the most]
It follows that f is monotonically increasing

57 We have, $f(x) = x^{1/x}$ $\qquad \log f(x) = \dfrac{1}{x} \log_e x$

Differentiating with respect to x, we get

$$\frac{1}{f(x)} f'(x) = \frac{1}{x} \cdot \frac{1}{x} - \frac{1}{x^2} \log_e x \text{ or } f'(x) = \frac{x^{1/x}}{x^2}[1 - \log_e x] \quad (1)$$

Since $e > 1$, for $x > e$, we have $\log_e x > 1$. Then it follows from (1) that $\quad f'(x) < 0 \quad$ for $x > e$
Hence $f(x)$ is a decreasing function of x for $x > e$
Since $\quad \pi > e$, we conclude

$f(\pi) < f(e) \Rightarrow \pi^{1/\pi} < e^{1/e} \Rightarrow (\pi^{1/\pi})^{e\pi} < (e^{1/e})^{e\pi} \Rightarrow \pi^e < e^\pi$

Thus e^π is bigger than π^e

Now $Rf'(n\pi)=\lim_{h\to 0}\frac{1+|\sin(n\pi+h)|-1}{h}(h>0)$

$=\lim_{h\to 0}\frac{|\pm\sin h|}{h}$

$=\lim_{h\to 0}\frac{\sin h}{h}$

[h is small and >0 and as $\sin h>0$]

$=1$

and $Lf'(n\pi)=\lim_{h\to 0}\frac{1+|\sin(n\pi-h)|-1}{-h}$

$=\lim_{h\to 0}\frac{|\pm\sin h|}{-h}$

$=\lim_{h\to 0}\frac{\sin h}{-h}$

$=-1$

Since $Rf'(n\pi)\neq Lf'(n\pi)$, $f(x)$ is not differentiable at $x=n\pi$, $n=0,\pm1,\pm2,\pm3,$

At all other points $f(x)$ is clearly differentiable For if a is any real number such that $n\pi<a<(n+1)\pi$ Now

Then we can write $a=n\pi+\alpha$ where $0<\alpha<\pi$

$Rf'(a)=\lim_{h\to 0}\frac{\{1+|\sin(a+h)|\}-\{1+|\sin a|\}}{h}$

$=\lim_{h\to 0}\frac{|\sin(a+h)|-|\sin a|}{h}$

$=\lim_{h\to 0}\frac{|\sin(n\pi+\alpha+h)|-|\sin(n\pi+\alpha)|}{h}$

$=\lim_{h\to 0}\frac{|\pm\sin(\alpha+h)|-|\pm\sin\alpha|}{h}$

$=\lim_{h\to 0}\frac{\sin(\alpha+h)-\sin\alpha}{h}$

Note that since h is small and $0<\alpha<\pi$, we must also have $0<\alpha+h<\pi$ so that $\sin(\alpha+h)>0$

$=\lim_{h\to 0}\frac{2\cos(\alpha+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$

$=\lim_{h\to 0}\cos(\alpha+\frac{1}{2}h)\lim_{h\to 0}\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}=\cos\alpha\cdot 1=\cos\alpha$

$=\cos(a-n\pi)$

Similarly $Lf'(a)=\cos(a-n\pi)$

Finally from (1) and (2), we get

$$x - x^3/6 < \sin x < x \text{ for } 0 < x \leqslant \pi/2$$

59 We are given $f(x) = x^3 - x^2 + x + 1$

$$f'(x) = 3x^2 - 2x + 1 = 3\left(x^2 - \tfrac{2}{3}x + \tfrac{1}{3}\right)$$

$$= 3\left\{\left(x - \tfrac{1}{3}\right)^2 - \tfrac{1}{9} + \tfrac{1}{3}\right\}$$

$$= 3\left\{\left(x - \tfrac{1}{3}\right)^2 + \tfrac{2}{9}\right\} > 0 \text{ for all real } x$$

Hence $f(x)$ is an increasing function of x for all real values of x

$$\max\{f(t) : 0 \leqslant t \leqslant x\} = x^3 - x^2 + x + 1$$

[Note that here max $f(t)$ means the greatest value of $f(t)$ in the interval $0 \leqslant t \leqslant x$ which is obtained at $t = x$ since the function $f(t)$ is increasing for all t]

Hence the function g is defined as follows

$$g(x) = x^3 - x^2 + x + 1 \quad \text{when} \quad 0 \leqslant x \leqslant 1$$

and $$g(x) = 3 - x \quad \text{when} \quad 1 < x \leqslant 2$$

It is sufficient to discuss the continuity and differentiability of $g(x)$ at $x = 1$ since for all other values of x, $g(x)$ is clearly continuous and differentiable, being a polynomial function of x. We have

$$g(1) = 2,$$

$$g(1-0) = \lim_{h \to 0} \{(1-h)^3 - (1-h)^2 + (1-h) + 1\} = 2$$

and $$g(1+0) = \lim_{h \to 0} [3 - (1+h)] = 2$$

Hence $g(x)$ is continuous at $x = 1$

Now $$Lg'(1) = \lim_{h \to 0} \frac{\{(1-h)^3 - (1-h)^2 + (1-h) + 1\} - 2}{-h}$$

$$= \lim_{h \to 0} \frac{1 - 3h + 3h^2 - h^3 - 1 + 2h - h^2 + 1 - h + 1 - 2}{-h}$$

$$= \lim_{h \to 0} \frac{-2h + 2h^2 - h^3}{-h} = \lim_{h \to 0} \{2 - 2h + h^2\} = 2$$

and $$Rg'(1) = \lim_{h \to 0} \frac{[3 - (1+h)] - 2}{h} = \lim_{h \to 0} \frac{-h}{h} = -1$$

Since $Lg'(1) \neq Rg'(1)$, the function $g(x)$ is not differentiable at $x = 1$

Hence $g(x)$ is continuous throughout the interval $(0, 2)$. It is also differentiable throughout the interval $(0, 2)$ except at $x = 1$ where it is non differentiable

$$=\lim_{x\to 9}\frac{\sqrt{x}\,f'(x)}{\sqrt{f(x)}}=\frac{3\times 4}{3}=4$$

32 Ans (b)

33 Ans (d)

We have $f(0)=1+\sin 0=1$. Then

$$Rf'(0)=\lim_{h\to 0}\frac{1+\sin(0+h)-1}{h}=\lim_{h\to 0}\frac{\sin h}{h}=1$$

and $Lf'(0)=\lim_{h\to 0}\frac{1-1}{-h}=0$

Hence $f'(0)$ does not exist

34 Ans (b)

For continuity actual value must be equal to the limiting value. Let

$$A=\lim_{x\to 0}(x+1)^{\cot x}$$

$$\log A=\lim_{x\to 0}\cot x\log(1+x)=\lim_{x\to 0}\frac{\log_e(1+x)}{\tan x}$$

$$=\lim_{x\to 0}\frac{1/(1+x)}{\sec^2 x}\quad\text{by L' Hospital Rule}$$

$$=1$$

Hence $A=e^1=e$

$f(0)$ must be defined as $f(0)=e$

(b) Let $F(x)=h(x)-h(1)=f(g(x))-h(1)$

$F'(x)=f'(g(x))\, g'(x)=(+)(-)=-$ ive

As f is increasing function $f'(g(x))$ is + ive and as g is decreasing function $g'(x)$ is $-$ ive

Since $F'(x)$ is $-$ ive therefore $F(x)$ i.e. $h(x)-h(1)$ is decreasing function

(c) Let $f(x)=\log(1+x)-x$

$$f(0)=0 \quad \text{I}$$

$$f'(x)=\frac{1}{1+x}-1=\frac{-x}{1+x}=-\text{ive for } x \geqslant 0$$

$f(x)$ is a decreasing function in the interval $[0, \infty)$

Since $f(0)=0 \qquad f(x) \leqslant 0 \quad \forall\ x \in [0, \infty)$

or $\log(1+x)-x \leqslant 0$ or $\log(1+x) \leqslant x$

(d) Let $F(x)=f(x)+g(x)$ where $f(x)$ is continuous and $g(x)$ is discontinuous. Let us suppose that $F(x)$ be continuous so that $F(x)-f(x)$ is also continuous i.e. $g(x)$ is also continuous

But it is a contradiction as $g(x)$ is given to be discontinuous

$f(x)$ must be discontinuous

(e) $f(-x)=f(x)$ given. Also Lt $f'(0)$ exists

$$\text{Lt}\ \frac{f(0-h)-f(0)}{-h}=\text{Lt}\ \frac{f(0+h)-f(0)}{h} \text{ by definition}$$

$$\text{or}\quad \text{Lt}\ \frac{f(h)-f(0)}{-h}=\text{Lt}\ \frac{f(h)-f(0)}{h} \text{ by given condition}$$

$$2\ \text{Lt}\ \frac{f(h)-f(0)}{h}=0 \text{ or } 2f'(0)=0 \qquad f'(0)=0$$

(f) $\underset{x\to 0}{\text{Lt}}\ \dfrac{x^4 \sin 1/x+x^2}{1+|x|^3}$. Put $x=-\dfrac{1}{y}$ $\quad y\to 0$

and y is + ive when $x\to-\infty$

$$\underset{y\to 0}{\text{Lt}}\ \frac{\frac{1}{y^4}\sin\left(-\frac{1}{y}\right)+\frac{1}{y^2}}{1+\left|-\frac{1}{y}\right|^3}=\underset{y\to 0}{\text{Lt}}\ \frac{\frac{y^2-\sin y}{y^4}}{\left(1+\frac{1}{y^3}\right)}$$

$|x|=-x$ when x is $-$ ive

$$\underset{y\to 0}{\text{Lt}}\ \frac{y^2-\sin y}{y^4+y}=\left(\frac{0}{0}\right)=\underset{y\to 0}{\text{Lt}}\ \frac{2y-\cos y}{4y^3+1}=\frac{0-1}{0+1}=-1$$

(g) $y=x^2(x-2)$

$$\frac{dy}{dx}=2x(x-2)^2+x^2\ 2(x-2)$$

point (x_1, y_1) then $\left(\frac{dy}{dx}\right)_{(x_1, y_1)}$ i.e. the value of dy/dx at (x_1, y_1) will represent the slope of the tangent and hence its equation in this case will be

$$y-y_1=\left(\frac{dy}{dx}\right)_{(x_1, y_1)}(x-x_1)$$

Normal $y-y_1=\frac{1}{(dy/dx)_{(x_1, y_1)}}(x-x_1)$ (4)

Slope of tangent $=m_1=dy/dx$ for its value at a given point

Slope of Normal $=-\frac{1}{m_1}=-\frac{1}{dy/dx}$ for its value at a given point

Condition for tangent to be parallel or $\perp$ to x axis

If the tangent is parallel to x axis or normal is $\perp$ to x axis then $m=0$ so that $dy/dx=0$

If the tangent is perpendicular to x axis or normal is parallel to x axis then $m=\infty$, $dy/dx=\infty$ or its reciprocal $dx/dy=0$ (M N R 79, 78)

Angle of intersection of the two curves

By angle of intersection of two curves we mean the angle between the tangents to the two curves at their common point of intersection. Hence if θ be the angle between the tangents then

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$$

where m_1=value of $(dy/dx)_I$ at the common point for Ist curve

m_2=value of $(dy/dx)_{II}$ at the common point for 2nd curve

Condition for orthogonal Intersection

Two curves are said to cut orthogonally if the angle between them is a right angle i.e. $\theta=90°$ $\tan 90°=\infty$

or $1+m_1m_2=0$ or $m_1m_2=-1$

or $(dy/dx)_I\,(dy/dx)_{II}=-1$

Condition for the two curves to touch

If the two curves touch then $\theta=0$ $\tan\theta=0$

$m_1-m_2=0$ or $m_1=m_2$

or $(dy/dx)_I=(dy/dx)_{II}$

Intercepts of tangent on the axes

Find the equation of the tangent. Put $y=0$ and find the value of x which will be intercept on axis of x. Then put $x=0$ and find the value of y which will be intercept on y axis.

4 $\lim_{x\to 0}\dfrac{x\cos x-\log(1+x)}{x^2}=$

(*a*) 1/2, (*b*) 0, (*c*) 1, (*d*) None of these

5 If $\lim_{x\to a}\dfrac{a^x-x^a}{x^x-a^a}=-1$ then

(*i*) $a=1$ (*ii*) $a=0$, (*iii*) $a=e$, (*iv*) None of these

6 If $\lim_{x\to 0}[f(x)\,g(x)]$ exists, then both $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist

(*a*) True (*b*) False (I I T 81)

7 If $f(x)=\sqrt{\left(\dfrac{x-\sin x}{x+\cos^2 x}\right)}$ then $\lim_{x\to\infty} f(x)$ is

(*a*) 0 (*b*) ∞, (*c*) 1, (*d*) None of these (I I T 79)

8 (*a*) A discontinuous function $y=f(x)$ satisfying $x^2+y^2=4$ is given by $f(x)=$ (IIT 82)

(*b*) $\lim_{x\to 1}(1-x)\tan\dfrac{\pi x}{2}=$ (I I T 84)

9 Let $f(x)=\begin{cases}\dfrac{x^3+x^2-16x+20}{(x-2)^2} & \text{if } x\neq 2\\ k & \text{if } x=2\end{cases}$

If $f(x)$ is continuous for all x, then $k=$ (I I T 81)

10 The function f defined as

$f(x)=\sin x^2/x$ for $x\neq 0$ and $f(0)=0$ is

(a) Continuous and derivable at $x=0$

(b) Neither continuous nor derivable at $x=0$,

(c) Continuous but not derivable at $x=0$

(d) None of these

11 Let $f(x)=\begin{cases}(x-1)^2\sin\dfrac{1}{x-1}-|x|, & \text{if } x\neq 1\\ -1 & \text{if } x=1,\end{cases}$

be a real valued function Then the set of points where $f(x)$ is not differentiable is (I I T 81)

12 For a real number y, let $[y]$ denote the greatest integer less than or equal to y Then

5 Tangent to the parabola $y^2=4ax$ in the form $y=mx+a/m$ where m is the slope of the tangent

6 Tangent and normal to (i) $y^2(a+x)=x^2(3a-x)$ (ii) $y^2(2a-x)=x^3$ at the points where $x=a$

7 Normal to $y=x^3-3x$ which is parallel to $2x+18y=9$

8 Tangent to $3x^2+y^2+x+2y=0$ which are perpendicular to the line $4x-2y-1$

(b) Tangent to parabola $y^2=4x+5$ which is parallel to $y=2x+7$ (M N R 79)

9 Tangent and normal to the curve

$$x=\frac{2at^2}{1+t^2},\ y=\frac{2at^3}{1+t^2}$$ at the point for which $t=1/2$

10 Tangents to $y=(x^2-1)(x-2)$ at the points where the curve cuts the x axis (Roorkee 80)

11 Normal at any point θ to the curve $x=a\cos\theta+a\theta\sin\theta$, $y=a\sin\theta-a\theta\cos\theta$. Also show that it is at a constant distance from the origin (I I T 83, Roorkee 76)

12 Tangent and normal to the curve $y(x-2)(x-3)-x+7=0$ at the point where it cuts the axis of x

13 (a) Find the equation to the tangent to $x^3=ay^2$ at $(4am^2, 8am^3)$ and also the point in which the tangent cuts the curve again.

(b) Show that the normal at the point $(3t, 4/t)$ of the curve $xy=12$ cuts the curve again at the point whose parameter t_1 is given by $t_1=-\frac{16}{9t^3}$

14 Show that the tangent to the curve $3xy^2-2x^2y=1$ at $(1, 1)$ meets the curve again at the point $(-16/5, -1/20)$

15 Find the point on the curve

$$y=x^4-6x^3+13x^2-10x+5$$

where the tangent is parallel to the line $y=2x$. Show that two of these points have the same tangent

16 Prove that all points on the curve $y^2=4a(x+a\sin x/a)$ at which the tangent is parallel to the x axis lie on the parabola $y^2=4ax$

17 Show that tangents to the folium of descartes $x^3+y^3=3axy$ at the point where it meets the parabola $y^2=ax$ are parallel to the axis of y. Also determine the point on the curve at which the tangent is parallel to x axis

(B) $f(x)=\sin\frac{1}{x}$ for $x\neq 0, f(0)=0$

(C) $f(x)=x\cos x$,

(D) None of these (I I T 83)

20 The function $\frac{\log(1+ax)-\log(1-bx)}{x}$ *is not defined at* $x=0$

The value which should be assigned to f at $x=0$ so that it is continuous at $x=0$ is

(A) $a-b$, (B) $1+b$ (C) $\log a+\log b$,

(D) None of these (I I T 83)

21 If $y=f(x)=\frac{x+2}{x-1}$, then

(A) $x=f(y)$, (B) $f(1)=3$

(C) y increases with x for $x<1$,

(D) f is a rational function of x (I I T 84)

22 If $f(x)=\frac{\sin[x]}{[x]}$, $[x]\neq 0$,

$=0$, $[x]=0$

where $[x]$ denotes the greatest integer less than or equal to x, then $\lim_{x\to 0} f(x)$ equal

(A) ' (B) 0 (C) -1 (D) None of these ((I I T 85)

23 If $f(x)=x[\sqrt{x}-\sqrt{(x+1)}]$ then—

(A) $f(x)$ is continuous but not differentiable at $x=0$

(B) $f(x)$ is differentiable at $x=0$

(C) $f(x)$ is not differentiable at $x=0$

(D) None of these (I I T 85)

24 Let A be set of n distinct elements, then the total number of distinct functions from A to A is and out of these are onto functions

25 If $f(x)=\sin\ ln\left(\frac{\sqrt{(4-x^2)}}{1-x}\right)$ then the domain of $f(x)$ is and its range is (I I T 85)

26 The function $f(x)=1+|\sin x|$ is

(A) continuous nowhere (B) continuous everywhere

(C) differentiable nowhere (D) not differentiable at $x=0$

(E) not differentiable at an infinite number of points

(I I T 86)

Deductions Tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Putting $m=2$, $\frac{X}{a}\left(\frac{x}{a}\right)^{2-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{2-1}=1$

or
$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1 \qquad (A)$$

Tangent to $\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1$

Putting $m=\frac{2}{3}$, $\frac{X}{a}\left(\frac{x}{a}\right)^{(2/3)-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{(2/3)-1}=1$

or
$$\frac{X}{a}\left(\frac{a}{x}\right)^{1/3}+\frac{Y}{b}\left(\frac{b}{y}\right)^{1/3}=1$$

or
$$\frac{X}{x^{1/3}a^{2/3}}+\frac{Y}{y^{1/3}b^{2/3}}=1 \qquad (B)$$

Note You should prove it independently as in the main question

Tangent to $x^{2/3}+y^{2/3}=a^{2/3}$

Putting $b=a$ in (B) above the equation of tangent becomes
$$\frac{X}{x^{1/3}}+\frac{Y}{y^{1/3}}=a^{2/3} \qquad (C)$$

Tangent to $\left(\frac{x}{a}\right)^{n/(n-1)}+\left(\frac{y}{b}\right)^{n/(n-1)}=1$

Putting $m=\frac{n}{n-1}$ or $m-1=\frac{n}{n-1}-1=\frac{1}{n-1}$, we get
$$\frac{X}{a}\left(\frac{x}{a}\right)^{1/(n-1)}+\frac{Y}{b}\left(\frac{y}{b}\right)^{1/(n-1)}=1 \qquad (D)$$

2 Eliminating θ we get
$$\left(\frac{x}{a}\right)^{1/3}=\sin\theta,\ \left(\frac{y}{a}\right)^{1/3}=\cos\theta$$
$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{a}\right)^{2/3}=1 \quad \text{or} \quad x^{2/3}+y^{2/3}=a^{2/3}$$

The tangent to above we have found in (C) above

Parametric form
$$x=a\sin^3\theta, \qquad y=a\cos^3\theta$$
$$\frac{dx}{d\theta}=3a\sin^2\theta\cos\theta,\quad \frac{dy}{d\theta}=-3a\cos^2\theta\sin\theta$$
$$\frac{dy}{dx}=\frac{dy}{d\theta}\div\frac{dx}{d\theta}=-\frac{\cos\theta}{\sin\theta}$$

34 In order that the function $f(x)=(x+1)^{\cot x}$ is continuous at $x=0$, $f(0)$ must be defined as

(a) $f(0)=0$ (b) $f(0)=e$,

(c) $f(0)=1/e$ (d) none of these [M N R 88]

Solutions to Problem Set (B)

1 Ans (a)

2 Ans (c)

Let $x=1/y$ so that $y\to 0$ as $x\to\infty$

Now $\lim_{x\to\infty}\left(1+\frac{2}{x}\right)^x=\lim_{y\to 0}(1+2y)^{1/y}$

$$=\lim_{y\to 0}\left[1+\frac{1}{y}\,2y+\frac{(1/y)(1/y-1)}{2!}\,2^2y^2+\frac{(1/y)(1/y-1)(1/y-2)}{3!}\,2^3y^3+\ldots\right]$$

by the binomial theorem

$$=\lim_{y\to 0}\left[1+2+\frac{(1-y)}{2!}\,2^2+\frac{(1-y)(1-2y)}{3!}\,2^3+\ldots\right]$$

$$=1+2+\frac{2^2}{2!}+\frac{2^3}{3!}+\ldots=e^2$$

3 Ans (a) [Hint Use L-Hospital Rule]

4 Ans (a)

[Hint Use the expansion of $\cos x$ and $\log(1+x)$]

5 Ans (a)

Using L-Hospital's rule, we get

$$-1=\lim_{x\to a}\frac{a^x-x^a}{x^x-a^a}=\lim_{x\to a}\frac{a^x\log_e a-ax^{a-1}}{x^x+x^x\log_e x}$$

or $$-1=\frac{a^a\log_e a-a\,a^{a-1}}{a^a+a^a\log_e a}=\frac{\log_e a-1}{\log_e a+1} \quad (1)$$

Now (1) is satisfied only when $a=1$

6 Ans, (b)

Take $f(x)=x$, $g(x)=\sin\frac{1}{x}$

Then $\lim_{x\to 0}f(x)\,g(x)=\lim_{x\to 0}x\sin\frac{1}{x}=0$

and so $\lim_{x\to 0}f(x)\,g(x)$ exists

But $\lim_{x\to 0}g(x)=\lim_{x\to 0}\sin\frac{1}{x}$ does not exist

5 Slope of the tangent to $y^2=4ax$ is

$$\frac{dy}{dx}=\frac{2a}{y}=m \text{ given}$$

$$y=\frac{2a}{m} \qquad x=\frac{y^2}{4a}=\frac{1}{4a}\cdot\frac{4a^2}{m^2}=\frac{a}{m^2}$$

Hence the tangent whose slope is m will be at the point $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ and hence its equation is

$$y-\frac{2a}{m}=m\left(x-\frac{a}{m^2}\right)$$

or
$$y=mx+\frac{a}{m}$$

6 $y^2(a+x)=x^2(3a-x)$

The point whose $x=a$, then $y^2(2a)=a^2\cdot 2a$

$$y^2=a^2 \quad \text{or} \quad y=a, -a$$

Hence the points are $P(a, a)$, $Q(a, -a)$

$$y^2=\frac{x^2(3a-x)}{a+x}$$

Take log $2\log y=2\log x+\log(3a-x)-\log(a+x)$

$$2\frac{1}{y}\frac{dy}{dx}=\frac{2}{x}-\frac{1}{3a-x}-\frac{1}{a+x}$$

We have to find the value of $\frac{dy}{dx}$ at $P(a, a)$

$$\frac{2}{a}\left(\frac{dy}{dx}\right)_{(a,\,a)}=\frac{2}{a}-\frac{1}{2a}-\frac{1}{2a}=\frac{1}{a}$$

$$\left(\frac{dy}{dx}\right)_{(a,\,a)}=\tfrac{1}{2}=\text{slope of tangent}$$

Tangent is $y-a=\frac{1}{2}(x-a)$ or $2y-x=a$

Since slope of tangent is $\frac{1}{2}$, therefore slope of normal will be -2

Normal is

$$y-a=-2(x-a) \quad \text{or} \quad y+2x=3a$$

Similarly we can write tangent and normal at $Q(a, -a)$ as

$$x+2y+a=0 \quad \text{and} \quad 2x-y-3a=0 \text{ respectively}$$

7 $y=x^3-3x \qquad \frac{dy}{dx}=3x^2-3$

Slope of normal is $-\dfrac{1}{\frac{dy}{dx}}=-\dfrac{1}{3x^2-3}$

$= \lim_{h\to 0} \frac{\sin h^2}{h^2} h = 1\times 0 = 0$

Similarly $f(0-0)=0$

Hence $f(x)$ is continuous at $x=0$

Now $Rf(0) = \lim_{h\to 0} \frac{\sin(0+h)^2/(0+h) - 0}{h} = \lim_{h\to 0} \frac{\sin h}{h} = 1$

and $Lf(0) = \lim_{h\to 0} \frac{\sin(0-h)^2/(0-h)-0}{-h} = \lim_{h\to 0} \frac{\sin h^2}{h^2} = 1$

Since $Rf(0) = Lf(0)$ the function $f(x)$ is derivable at $x=0$

11 Ans {0}

We need only test the function for differentiability at $x=0$ and $x=1$]

[We test at $x=0$ because of $|x|$ and $x=1$ because of its definition at $x=1$]

At $x=0$, we have $f(0)=(0-1)^2 \sin\frac{1}{(-1)} - |0| = -\sin 1$,

$Rf(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{1}{h}\left[(0+h-1)^2 \sin\frac{1}{(0+h-1)} - |0+h| - (-\sin 1)\right]$

$= \lim_{h\to 0} \frac{1}{h}\left[(h^2-2h+1)\sin\frac{1}{h-1} - h + \sin 1\right]$

$= \lim_{h\to 0}\left[h\sin\frac{1}{h-1} - 2\sin\frac{1}{h-1} - 1 + \frac{\sin\{1/(h-1)\}+\sin 1}{h}\right]$

$= \lim_{h\to 0} h\sin\frac{1}{h-1} - 2\lim_{h\to 0}\sin\frac{1}{h-1} - 1 + \lim_{h\to 0}\frac{\sin\{1/(h-1)\} + \sin 1\}}{h}$

$= 0 - 2\sin(-1) - 1 + \lim_{h\to 0} \frac{2\sin\frac{1}{2}\left(\frac{1}{h-1}+1\right)\cos\frac{1}{2}\left(\frac{1}{h-1}-1\right)}{h}$

$= 2\sin 1 - 1 + \lim_{h\to 0}\frac{2}{h}\sin\frac{h}{2(h-1)}\cos\frac{2-h}{2(h-1)}$

$= 2\sin 1 - 1 + \lim_{h\to 0}\frac{\sin[h/2(h-1)]}{[h/2(h-1)]}\cdot\frac{1}{(h-1)}\cos\frac{2-h}{2(h-1)}$

$= 2\sin 1 - 1 + 1\times(-1)\cos(-1) = 2\sin 1 - 1 - \cos 1$

And $Lf(0) = \lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$

$= \lim_{h\to 0} -\frac{1}{h}\left[(0-h-1)^2\sin\frac{1}{(0-h-1)} - |0-h| - (-\sin 1)\right]$

16 $y^2=4a\{x+a\sin x/a\}$ (1)

$\therefore\ 2y\ dy/dx=4a(1+\cos x/a)$ (2)

If the tangent is to be parallel to x axis then $dy/dx=0$

Hence from (2) we get $\cos x/a=-1$ $\sin x/a=0$

Putting in (1) we get $y^2=4a\ (x+0)=0$ or $y^2=4ax$

Hence all such points tangent at which are parallel to x axis will lie on the parabola $y^2=4ax$

17 $x^3+y^3-3axy=0$

$3x^2+3y^2\ dy/dx-3a\ (y+x\ dy/dx)=0$

or $dy/dx=-\dfrac{x^2-ay}{y^2-ax}$

At the points where the curve meets $y^2=ax$ the value of $\dfrac{dy}{dx}=-\dfrac{x^2-ay}{0}=\infty$ and hence tangent is perpendicular to x axis or parallel to y axis.

If the tangent is to be parallel to x axis then $dy/dx=0$,

$x^2=ay$

Solving with the given curve we get

$x^3+\dfrac{x^6}{a^3}-3x\ x^2=0$ or $x^3(x^3-2a^3)=0$

$x=0,\ 2^{1/3}a$ $y=x^2/a$ $y=0,\ 2^{2/3}a$

Hence the required points are $(0, 0)$, $(2^{1/3}a,\ 2^{2/3}a)$

18 $y=b\,e^{-x/a}$ It cuts y axis at $(0, b)$

$\dfrac{dy}{dx}=-\dfrac{b}{a}e^{-x/a}$ $\left(\dfrac{dy}{dx}\right)_{(0,\ b)}=-b/a\ 1=-b/a$

Tangent at $(0, b)$ is $y-b=-b/a\ (x-0)$

or $bx+ay=ab$ or $x/a+y/b=1$,

19 Differentiating $\left(\dfrac{x}{a}\right)^n+\left(\dfrac{y}{b}\right)^n=2$, we get

$$n\left(\frac{x}{a}\right)^{n-1}\frac{1}{a}+n\left(\frac{y}{b}\right)^{n-1}\frac{1}{b}\frac{dy}{dx}=0 \quad (1)$$

We have to find the value of dy/dx at (a, b) Hence from (1)

$$n\ \frac{1}{a}+n\ \frac{1}{b}\left(\frac{dy}{dx}\right)_{(a,\ b)}=0 \quad\text{or}\quad \left(\frac{dy}{dx}\right)_{(a,\ b)}=-\frac{b}{a}$$

Tangent at (a, b) is

$$y-b=\left(\frac{dy}{dx}\right)_{(a,\ b)}(x-a)$$

12 And (*d*)

By definition $[\pi - x]$ is an integer whatever x may be and so $\pi[x-\pi]$ is an integral multiple of π

Consequently $\tan (\pi[x-\pi])=0$ for all x

And since $1+[x]^2 \neq 0$ for any x, we conclude that $f(x)=0$

Thus $f(x)$ is constant function and so it is continuous and differentiable any number of times, that is $f'(x)$ $f''(x)$, $f'''(x)$, $f^{(n)}(x)$, , all exists for every x, their value being 0 at every point x Hence of all the given alternatives only (*d*) is correct

13 Ans (*b*),

The domain of the function $f(x)=1/x$ consists of all real numbers except 0 Now $f'(x)=-1/x^2<0$ for all $x \neq 0$ Hence $f(x)$ is a decreasing function on its domain

14 Ans (i)

$f(x)=e^{-x}$ is one such function It can be shown that there exists no function satisfying the conditions (ii), or (iii) or (iv) and the given conditions $f(0)=1$, $f'(0)=-1$, $f(x)>0$ for all x

15 Ans (*d*) Since $f(x)=\cos \log x$, we have

$$f(x)f(y)-\tfrac{1}{2}\left[f\left(\frac{x}{y}\right)+f(xy)\right]$$

$$=\cos \log x \cos \log y - \tfrac{1}{2}\left[\cos \log\left(\frac{x}{y}\right)+\cos \log xy\right]$$

$$=\cos \log x \cos \log y$$

$$-\tfrac{1}{2}\left[2 \cos \tfrac{1}{2}\left\{\log \left(\frac{x}{y}\right)+\log xy\right\}\times\cos \tfrac{1}{2}\left\{\log \left(\frac{x}{y}\right)-\log xy\right\}\right]$$

$$=\cos \log x \cos \log y - \cos \tfrac{1}{2} \log\left(\frac{x}{y}\, xy\right) \cos \tfrac{1}{2} \log \left(\frac{x}{y}\,\frac{1}{xy}\right)$$

$$=\cos \log x \cos \log y-\cos \log x \cos (-\log y)$$

$$=\cos \log x \cos \log y - \cos \log x \cos \log y=0$$

16 Ans (*A*) 17 Ans (*C*)

18 Ans (*D*). Actual value is $\dfrac{1}{\sqrt{(24)}}$

19 Ans (*A*)

20 Ans (*D*)

21 Ans (*A*)

22 Ans (*D*)

or $\frac{dy}{dx}[1+\sin(x+y)]=-\sin(x+y)$

or $\frac{dy}{dx}=-\frac{\sin(x+y)}{1+\sin(x+y)}$

From given condition,

$$-\frac{1}{2}=-\frac{\sin(x+y)}{1+\sin(x+y)}$$

or $1+\sin(x+y)=2\sin(x+y)$

or $\sin(x+y)=1=\sin\pi/2$

$$x+y=n\pi+(-1)^n\frac{\pi}{2},\ n=0,\ \pm1,\ \pm2,\ \pm3,$$

$$x+y=-\frac{7\pi}{2},\ -\frac{3\pi}{2},\ \frac{\pi}{2},\ \frac{5\pi}{2},\ \frac{9}{2}\pi,\ \frac{13}{2}\pi,\ \frac{17}{2}\pi,$$

Then $y=\cos(x+y)=0$

Since $-2\pi\leqslant x\leqslant 2\pi$, the values of x are $-\frac{3\pi}{2}$ and $\frac{\pi}{2}$ only

Hence the points are $\left(-\frac{3\pi}{2},0\right)$ and $\left(\frac{\pi}{2},0\right)$ where the tangents are parallel to the line $x+2y=0$

The equations of tangents are

$$y-0=-(\tfrac{1}{2})(x+3\pi/2) \quad\text{and}\quad y-0=-(\tfrac{1}{2})(x-\pi/2)$$

or $2x-4y+3\pi=0$ and $2x+4y-\pi=0$

Problem Set (B)

1 Find the angle of intersection of the following curves

(a) $x^2+y^2=a^2\sqrt{2}$ $x^2-y^2=a^2$

(b) $x^2y=1$ and $y=x^4$

(c) $x^2=4ay,\ 2y^2=ax$, (c₁) $y^2=2x,\ x^2=16y$ common Tangent (M N R 83)

(d) $y^2=4ax,\ x^2=4by$

(e) $y^2=8x,\ x^2=4y-12$

(f) $xy=a^2,\ x^2+y^2=2a^2$

(g) $y^2=16x,\ 2x^2+y^2=4$

(h) $y=x^2$ and $6y=7-x^2$

(i) $y=4x^2,\ y=x^2$ (Roorkee 73)

2 If the curves $\frac{x^2}{a^2}+\frac{y^2}{4}=1$ and $y^2=16x$ intersect at right angles show that $a^2=4/3$

bijective functions in this case) is the number of arrangements of n different things taken all at a time, that is, $n!$

25 Ans Dom f is given by $-2<x<1$ and Ran f is given by $-1\leqslant y\leqslant 1$

First note that $\sin x$ is defined for all real values of x and $\ln x$ is defined for $x>0$

Now $\sqrt{(4-x^2)}=\sqrt{[(2-x)(2+x)]}$ is real and +ive if $-2<x<2$ and $1-x>0$ if $x<1$ Hence $\frac{\sqrt{(4-x^2)}}{1-x}>0$ if $-2<x<1$ Hence the domain of $f(x)$ is the set of all real values x satisfying the double inequality

$$-2<x<1$$

Now we find the range of $f(x)$ We put $u=\sqrt{(4-x^2)}/(1-x)$ so that $y=f(x)=\sin \ln u$ u is clearly a continuous function in $-2<x<1$ Further we easily see that

$$\frac{du}{dx}=(4-x)/(1-x)^2\sqrt{(4-x^2)}$$

$$>0 \text{ when } -2<x<1$$

Hence u is a continuously increasing function of x in

$$-2<x<1$$

Also $u=0$ at $x=-2$ and $u\to\infty$ as $x\to 1-0$

Hence $0<u<\infty$ when $-2<x<1$

It follows that $-\infty<\ln u<\infty$

i e $\ln u$ takes all real values

Hence $-1\leqslant \sin \ln u\leqslant 1$

Thus the range of $f(x)$ is given by

$$-1\leqslant y\leqslant 1$$

26 Ans (B), (D), (E)

The function is clearly continuous everywhere For if a is any real number, then

$$f(a)=1+|\sin a|$$

$$f(a+0)=\lim_{h\to 0}[1+|\sin(a+h)|]=1+|\sin a|$$

and $$f(a-0)=\lim_{h\to 0}[1+|\sin(a-h)|]=1+|\sin a|$$

The function is not differentiable at $x=n\pi$, $n=0, \pm 1, \pm 2,$ as shown below

$$f(n\pi)=1+|\sin n\pi|=1+|0|=1$$

ratio (all the constants being positive)

[Roorkee 88]

If x_1, y_1 be the parts of the axes intercepted by the tangent at any point (x, y) on the curve

$$\left(\frac{x}{a}\right)^{2/3}+\left(\frac{y}{b}\right)^{2/3}=1, \text{ show that } \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=1$$

Prove that the portion of the tangent to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ which is intercepted between the axes is of constant length (Roorkee 75)

If x_1 and y_1 be the intercepts on the axes of x and y cut off by the tangent to the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=1$ then show that

$$\left(\frac{a}{x_1}\right)^{n/(n-1)}+\left(\frac{b}{y_1}\right)^{n/(n-1)}=1$$

Prove that the sum of intercepts of the tangent to the curve $\sqrt{(x)}+\sqrt{(y)}=\sqrt{(a)}$ upon the co ordinate axes is of constant length

Show that θ is the angle which the perpendicular from the origin on the tangent makes with the x axis for the curve whose parametric equations are $x=a \sin^3 \theta$, $y=a \cos^3 \theta$

If p_1 and p_2 be the lengths of perpendiculars from the origin on the tangent and normal to the curve $x^{2/3}+y^{2/3}=a^{2/3}$ respectively, prove that $4p_1^2+p_2^2=a^2$

(a) Find the abscissa of the point on the curve $ay=x^2$, the normal at which cuts off equal intercepts from the axes

(b) The point on the curve $y^2=x$ the tangent at which makes an angle 45° with x axis is $(\frac{1}{4}, \frac{1}{2})$ (M N R 79)

Find the abscissa of the point on the curve $xy=(c+x)^2$ the normal at which cuts off numerically equal intercepts from the axes of coordinates

In the curve $x=a$ (cos t+log tan $t/2$), $y=a$ sin t show that the portion of the tangent between the point of contact and the x axis is of constant length

Show that the portion of the tangent for the curve

$$x=\sqrt{(a^2-y^2)}+\frac{a}{2}\log\frac{a-\sqrt{(a^2-y^2)}}{a+\sqrt{(a^2-y^2)}}$$

between the point of contact and the x axis is a

Hence $f(x)$ is continuous everywhere. Also $f(x)$ is differentiable everywhere except at $x=n\pi$, $n=0, \pm 1, \pm 2, \pm 3,$

27 Ans (*A*), (*B*) and (*D*)

By def of $[x]$, we easily see that

$$f(x)=[x \sin \pi x]=0 \text{ when } -1 \leqslant x \leqslant 1$$

[Note that $-1 \leqslant x \leqslant 1 \Rightarrow 0 \leqslant x \sin \pi x \leqslant \frac{1}{2}$]

and $f(x)=[x \sin \pi x]=-1$ when $1 < x < 1+h$, (h small)

[Note $x \sin \pi x$ becomes negative and numerically less than 1 when x is slightly greater than 1 and so by def, of $[x]$, $[x \sin \pi x]=-1$ when $1 < x < [1+h]$]

Thus $f(x)$ is constant and equal to 0 in the closed interval $[-1, 1]$ and so $f(x)$ is continuous and differentiable in the open interval $(-1, 1)$. At $x=1$, $f(x)$ is clearly discontinuous since $f(1-0)=0$ and $f(1+0)=-1$ and $f(x)$ is also non-differentiable at $x=1$. Hence (*A*), (*B*), and (*D*) are correct answers

28 Ans 5

Hint $\lim_{x\to 0} g[f(x)] = \lim_{x\to 0} g(2)=5$

[Note that $x\to 0$, implies that x is never zero so that by def of $f(x)$, $f(x)=2$ when $x \neq 0$ and then by def of g, we have $g(2)=5$]

29 Ans (b)

[If $x \in D_1 \cup D_2$ is such that $x \in D_1$ but $x \notin D_2$, then $f_1(x)$ is defined but $f_2(x)$ is not defined so that $f_1(x)+f_2(x)$ is not defined at $x \in D_1 \cup D_2$. Of course $f_1(x)+f_2(x)$ is defined at $x \in D_1 \cap D_2$

30 Ans (*A*) (*B*) and (*C*)

We write the given function as

$$f(x)=\begin{cases} x-3, & x \geqslant 3 \\ 3-x & 1 \leqslant x \leqslant 3 \\ x^2/4 -3x/2+13/4, & x < 1 \end{cases}$$

Now it can be easily seen that $f(x)$ is continuous at $x=1$ and $x=3$, differentiable at $x=1$ but non differentiable at $x=3$

31 Ans 4

$$\lim_{x\to 9} \frac{\sqrt{f(x)}-3}{\sqrt{x}-3} = \lim_{x\to 9} \frac{\frac{f'(x)}{2\sqrt{f(x)}}}{\frac{1}{2\sqrt{x}}}, \text{ by L' Hospital Rule}$$

(f) Touch

(g) $y^2=16x,\quad 2x^2+y^2=4$

$$2y\frac{dy}{dx}=16 \qquad \left(\frac{dy}{dx}\right)_I=\frac{8}{y}=m_1$$

$$4x+2y\frac{dy}{dx}=0 \qquad \left(\frac{dy}{dx}\right)_{II}=\frac{-2x}{y}=m_2$$

$$m_1m_2=-\frac{16x}{y^2}=-1 \qquad y^2=16x$$

Hence the curves cut orthogonally Here we have not found the point of intersection which will lie on both the curves and hence we have used $y^2=16x$

(h) Orthogonal i e they cut at right angles

(i) The curves touch at the origin

2 $\frac{x^2}{a^2}+\frac{y^2}{4}=1,\ y^2=16x$

$$\frac{2x}{a^2}+\frac{2y}{4}\frac{dy}{dx}=0 \qquad \left(\frac{dy}{dx}\right)_I=\frac{-4x}{a\,y}=m_1$$

$$3y\cdot\frac{dy}{dx}=16 \qquad \left(\frac{dy}{dx}\right)_{II}=\frac{16}{3y^2}=m_2$$

If the curves cut at right angles then $m_1m_2=-1$

$$\left(\frac{-4x}{a\,y}\right)\frac{16}{3y}=-1 \quad \text{or} \quad 64x=3a^2y^2=3a\quad 16x$$

$$a^2=4/3$$

3 $x^3-3xy^2=-2 \qquad 3x^2y-y^3=2$

$$3x^2-3y^2-6xy\frac{dy}{dx}=0 \qquad \left(\frac{dy}{dx}\right)_I=\frac{(x^2-y^2)}{2xy}=m_1$$

$$6xy+(3x^2-3y^2)\frac{dy}{dx}=0 \qquad \left(\frac{dy}{dx}\right)_{II}=-\frac{2xy}{x^2-y^2}=m_2$$

Clearly $m_1m_2=-1$ Hence the curves cut orthogonally

4 $ax^2+by^2=1, \qquad a'x^2+b'y^2=1$

$$\left(\frac{dy}{dx}\right)_I=-\frac{ax}{by}=m_1\quad \left(\frac{dy}{dx}\right)_{II}=-\frac{a'x}{b'y}=m_2$$

If the curves cut orthogonally, then $m_1m_2=-1$

or $\left(-\frac{ax}{by}\right)\left(-\frac{a'x}{b'y}\right)=-1$ or $aa'x^2+bb'y^2=0$ (1)

We have now to put the values of x^2 and y^2 corresponding to the points of intersection of

$$ax^2+by^2-1=0$$

$$a'x^2+b'y^2-1=0$$

3

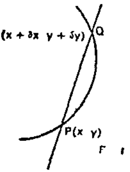
Tangents Normals and Simple Applications of the Derivative

§ 1 Tangent at (x, y) to y=f (x)

Let $y=f(x)$ be a given curve and $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ be two neighbouring points on it. Equation of the line PQ is

$$Y-y=\frac{y+\delta y-y}{x+\delta x-x}(X-x)$$

or $$Y-y=\frac{\delta y}{\delta x}(X-x) \qquad (1)$$



The line (1) will be a tangent to the given curve at P if $Q \to P$ which in turn means that $\delta x \to 0$ and we know that

$$\underset{\delta x \to 0}{\text{Lt}} \frac{\delta y}{\delta x}=dy/dx$$

Therefore the equation of the tangent is

$$Y-y=(dy/dx)(X-x) \qquad (2)$$

Normal at (x, y)

The normal at (x, y) being perpendicular to tangent (2) will have its slope as $-1\Big/\frac{dy}{dx}$ and hence its equation is

$$Y-y=-\frac{1}{dy/dx}(X-x) \qquad (3)$$

Geometrical meaning of dy/dx

From equation (1) we observe that dy/dx represents the slope of the tangent to the given curve $y=f(x)$ at any point (x, y)

$$\frac{dy}{dx}=\tan\psi$$

where ψ is the angle which the tangent to the curve makes with +ive direction of x axis. In case we are to find the tangent at any hence $bx = 2n\pi + \frac{\pi}{2}$ (1)

Now $\left(\frac{dy}{dx}\right)_I = -ae^{-ax}\sin bx + be^{-ax}\cos bx$

and $\left(\frac{dy}{dx}\right)_{II} = -ae^{-ax} = -ae^{-ax}$

Now condition for the two curves to touch is

$$\left(\frac{dy}{dx}\right)_I = \left(\frac{dy}{dx}\right)_{II} \text{ i.e. } a\sin bx - b\cos bx = a$$

or $\frac{a}{\sqrt{(a^2+b^2)}}\sin bx - \frac{b}{\sqrt{(a^2+b^2)}}\cos bx = \frac{a}{\sqrt{(a^2+b^2)}}$

or $\sin\alpha\sin bx - \cos\alpha\cos bx = \sin\alpha,$

where $\sin\alpha = \frac{a}{\sqrt{(a^2+b^2)}}$

or $-\cos(bx+\alpha) = \sin\alpha$

or $\cos(bx+\alpha) = \cos\left(\frac{\pi}{2}+\alpha\right)$

Hence $bx+\alpha = 2n\pi \pm \left(\frac{\pi}{2}+\alpha\right)$

$$bx = 2n\pi + \frac{\pi}{2} \quad \text{or} \quad bx = 2n\pi - \frac{\pi}{2} - 2\alpha \qquad (2)$$

Values common to the sets of values given by (1) and (2) are

$$bx = 2n\pi + \frac{\pi}{2}$$

which is satisfied for $bx = 2n\pi + \pi/2$

Therefore the two curves touch at the point for which

$$bx = 2n\pi + \pi/2$$

7 $\log(x^2+y^2) + k\tan^{-1} y/x$

$$\frac{1}{(x^2+y^2)}\left(2x+2y\frac{dy}{dx}\right) = k\,\frac{1}{1+y^2/x^2}\,\frac{x\,dy/dx - y}{x}\frac{1}{x}$$

or $2\left(x+y\frac{dy}{dx}\right) = k\left(x\frac{dy}{dx} - y\right)$ (1)

The tangent makes an angle ψ with x axis

$$\tan\psi = \frac{dy}{dx} \qquad (2)$$

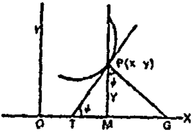
Let OP where P is (x, y) make an angle θ with x axis, then

$$\tan\theta = y/x \qquad (3)$$

Length of tangent and normal

Length of tangent $=PT$, where P is the point of contact and T is the point where tangent meets the axis of x

Length of normal $=PG$ where P is the point of contact and G is the point where normal meets the axis of x



From the figure $\frac{y}{PT}=\sin\psi$

$PT=y\ \operatorname{cosec}\psi \qquad (1)$

$\frac{y}{PG}=\cos\psi \qquad (2)$

$PG=y\sec\psi$

Now $\tan\psi=dy/dx=y'$ say

$\sec\psi=\sqrt{(1+\tan^2\psi)}=\sqrt{(1+y'^2)} \qquad (3)$

$$\operatorname{cosec}\psi=\sqrt{(1+\cot^2\psi)}=\sqrt{\left(1+\frac{1}{y'^2}\right)}=\frac{\sqrt{(1+y'^2)}}{y'} \qquad (4)$$

$PT=\frac{y}{y'}\sqrt{(1+y'^2)}$ by (1) and (4)

$PG=y\sqrt{(1+y'^2)}$ by (2) and (3)

Condition for a given line to touch a given curve

Let the line be a tangent to the given curve at (x, y) then write the equation of the tangent as

$Y-y=(dy/dx)\ (X-x)$

Compare this with the given line $aX+bY+c=0$ and then eliminate x and y

Problem Set (A)

Find the equation of the tangent and normal to the curve at any point (x, y)

1 $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ Only tangent

2 $x=a\sin^3\theta,\ y=a\cos^3\theta$, Tangent and Normal

or $x^{2/3}+y^{2/3}=a^{2/3}$ (Roorkee 75)

3 Normal to $x^{2/3}+y^{2/3}=a^{2/3}$ in the form

$y\cos\theta-x\sin\theta=a\cos 2\theta$ (Roorkee 66)

where θ is the angle which the normal makes with the axis of x

4 Normal to the parabola $y^2=4ax$ in the form

$y=mx-2am-am^3$ where m is the slope of the normal

$$\frac{\left(\frac{x}{a}\right)^{m-1}}{a\cos\alpha}=\frac{\left(\frac{y}{b}\right)^{m-1}}{b\sin\alpha}=\frac{1}{p}$$

$$\frac{x}{a}=\left(\frac{a\cos\alpha}{p}\right)^{1/(m-1)},\quad \frac{y}{b}=\left(\frac{b\sin\alpha}{p}\right)^{1/(m-1)} \tag{3}$$

But the point (x, y) lies on $\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m=1$ (4)

Putting for $\frac{x}{a}$ and $\frac{y}{b}$ from (3) in (4), we get

$$\left(\frac{a\cos\alpha}{p}\right)^{m/(m-1)}+\left(\frac{b\sin\alpha}{p}\right)^{m/(m-1)}=1$$

or $\quad (a\cos\alpha)^{m/(m-1)}+(b\sin\alpha)^{m/(m-1)}=p^{m/(m-1)}$

Note —You have to deduce the equation of the tangent as in Q 1 page 529

(b) Choosing $m=2$, the required condition is

$$a^2\cos^2\alpha+b^2\sin^2\alpha=p^2$$

(c) Choose

$$m=\frac{n}{n-1},\qquad m-1=\frac{n}{n-1}-1=\frac{1}{n-1} \text{ and } \frac{m}{m-1}=n$$

Hence the required condition is

$$(a\cos\alpha)^n+(b\sin\alpha)^n=p^n$$

Note in the examination do the question with power m as in part (a) and then in the result put $m=\frac{n}{n-1}$ as above

(d) $x^m y^n=a^{m+n}$ Take log

$m\log x+n\log y=(m+n)\log a$

$$m\frac{1}{x}+n\frac{1}{y}\frac{dy}{dx}=0 \qquad \frac{dy}{dx}=-\frac{m}{n}\frac{y}{x}$$

Hence the equation of the tangent is

$$Y-y=-\frac{m}{n}\frac{y}{x}(X-x)$$

$$myX+nxY=(m+n)xy \tag{1}$$

Compare with $X\cos\alpha+Y\sin\alpha=p$

$$\frac{my}{\cos\alpha}=\frac{nx}{\sin\alpha}=\frac{(m+n)xy}{p}$$

$$(m+n)x=\frac{pm}{\cos\alpha} \text{ and } (m+n)y=\frac{pn}{\sin\alpha}$$

18 Show that the line $\frac{x}{a}+\frac{y}{b}=1$ touches the curve $y=b\,e^{-x/a}$ at the point where the curve crosses the y-axis (M N R 81)

19 Show that the curve $\left(\frac{x}{a}\right)^n+\left(\frac{y}{b}\right)^n=2$ touches the straight line $\frac{x}{a}+\frac{y}{b}=2$ at the point (a, b) whatever the value of n may be

20 The rectangular coordinates of a point on the curve are given by $x=3\cos\theta-\cos^3\theta$, $y=3\sin\theta-\sin^3\theta$

Find the equation of the normal at any point on the curve and show that at the point P where $\theta=\pi/4$, the normal passes through the origin

21 Tangents are drawn from origin to the curve $y=\sin x$ Prove that their points of contact lie on $x^2y^2=x^2-y^2$

(M N R 79 Roorkee 86)

22 Find all the tangents to the curve

$$y=\cos(x+y),\ -2\pi\leqslant x\leqslant 2\pi$$

that are parallel to the line $x+2y=0$ (I I T 85)

Solutions to Problem Set (A)

1 $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$ (1)

Differentiating w r t x, we get

$$m\frac{x^{m-1}}{a^m}+m\frac{y^{m-1}}{b^m}\frac{dy}{dx}=0 \qquad \frac{dy}{dx}=-\frac{1}{a}\left(\frac{x}{a}\right)^{m-1} b\left(\frac{b}{y}\right)^{m-1}$$

Equation of the tangent is

$$Y-y=\frac{dy}{dx}(X-x)$$

or $$(Y-y)=-\frac{1}{a}\left(\frac{x}{a}\right)^{m-1} b\left(\frac{b}{y}\right)^{m-1}(X-x)$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{m-1}-\left(\frac{x}{a}\right)^m=-\frac{Y}{b}\left(\frac{y}{b}\right)^{m-1}+\left(\frac{y}{b}\right)^m$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{m-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{m-1}=\left(\frac{x}{a}\right)^m+\left(\frac{y}{b}\right)^m$$

or $$\frac{X}{a}\left(\frac{x}{a}\right)^{m-1}+\frac{Y}{b}\left(\frac{y}{b}\right)^{m-1}=1 \text{ by (1) (V Imp)} \qquad (2)$$

Tangent is

or $$y-a\cos^3\theta=-\frac{\cos\theta}{\sin\theta}(x-a\sin^3\theta)$$

or $$x\cos\theta+y\sin\theta=a\sin\theta\cos\theta\,(\cos^2\theta+\sin^2\theta)$$

or $$x\cos\theta+y\sin\theta=\frac{a}{2}\sin 2\theta$$

Normal Slope of the tangent is $-\cot\theta$ and hence slope of the normal will be

$$+\tan\theta=\frac{\sin\theta}{\cos\theta} \qquad m_1m_2=-1$$

Normal is $$y-a\cos^3\theta=\frac{\sin\theta}{\cos\theta}(x-a\sin^3\theta)$$

or $y\cos\theta-x\sin\theta=a(\cos^4\theta-\sin^4\theta)$

or $y\cos\theta-x\sin\theta=a\cos 2\theta$

$\cos^4\theta-\sin^4\theta=(\cos^2\theta-\sin^2\theta)(\cos^2\theta+\sin^2\theta)=\cos 2\theta$

3 $x^{2/3}+y^{2/3}=a^{2/3}$

$$\frac{2}{3}x^{-1/3}+\frac{2}{3}y^{-1/3}\frac{dy}{dx}=0 \qquad \frac{dy}{dx}=-\frac{x^{-1/3}}{y^{-1/3}}=-\frac{y^{1/3}}{x^{1/3}}$$

Slope of the normal is $\frac{x^{1/3}}{y^{1/3}}=\tan\theta$ (given)

(It is given that the normal makes an angle θ with x axis.)

$$\frac{x^{1/3}}{\sin\theta}=\frac{y^{1/3}}{\cos\theta}=\left[\frac{x^{2/3}+y^{2/3}}{\sin^2\theta+\cos^2\theta}\right]^{1/2}=a^{1/3} \text{ by (1)}$$

$$x=a\sin^3\theta,\quad y=a\cos^3\theta$$

Hence the normal whose slope is $\tan\theta$ is given by

$$y-a\cos^3\theta=\frac{\sin\theta}{\cos\theta}(x-a\sin^3\theta)$$

or $y\cos\theta-x\sin\theta=a\cos 2\theta$ as in Q 2

4 $y^2=4ax$

$$2y\frac{dy}{dx}=4a \quad \text{or} \quad \frac{dy}{dx}=\frac{2a}{y}$$

Slope of the normal $=-\dfrac{1}{dy/dx}=-\dfrac{y}{2a}=m$ given

$$y=-2am$$

Therefore $x=\dfrac{y^2}{4a}=\dfrac{4a^2m^2}{4a}=am^2$

Hence the normal whose slope is m will be at the point $(am^2, -2am)$ so that its equation is

$$y+2am=m(x-am^2)$$

or $$y=mx-2am-am^3$$

$$a \sin t = PT \sin t \qquad PT = a = \text{constant}$$

) $x = \sqrt{(a^2-y^2)} + \frac{a}{2} \log \frac{a-\sqrt{(a^2-y^2)}}{a+\sqrt{(a^2-y^2)}}$

or $x = \sqrt{(a^2-y^2)} + \frac{a}{2}[\log\{a-\sqrt{(a^2-y^2)}\} - \log\{a+\sqrt{(a^2-y^2)}\}]$

$$\frac{dx}{dy} = \frac{-2y}{2\sqrt{(a^2-y^2)}} + \frac{a}{2}\left[\frac{1}{a-\sqrt{(a^2-y^2)}} \cdot \frac{2y}{2\sqrt{(a^2-y^2)}} - \frac{1}{a+\sqrt{(a^2-y^2)}} \cdot \frac{-2y}{2\sqrt{(a^2-y^2)}}\right]$$

$$= \frac{y}{\sqrt{(a^2-y^2)}}\left[-1 + \frac{a}{2}\left\{\frac{2a}{a^2-(a^2-y^2)}\right\}\right]$$

$$= \frac{y}{\sqrt{(a^2-y^2)}}\left[-1 + \frac{a^2}{y^2}\right] = \frac{y}{\sqrt{(a^2-y^2)}} \cdot \frac{a^2-y^2}{y^2} = \frac{\sqrt{(a^2-y^2)}}{y}$$

$$\frac{dy}{dx} = \frac{y}{\sqrt{(a^2-y^2)}} = \tan\psi \quad \text{or} \quad \cot\psi = \frac{\sqrt{(a^2-y^2)}}{y}$$

From fig. of 19 $y = PT \sin\psi$

$$PT^2 = y^2 \operatorname{cosec}^2\psi = y^2(1+\cot^2\psi)$$

or $$PT^2 = y^2\left(1 + \frac{a^2-y^2}{y^2}\right) = y^2 \cdot \frac{a^2}{y^2} = a^2$$

$PT = a$ i.e. constant

11. $xy = c^2$ or $y = \frac{c^2}{x}$

$$\frac{dy}{dx} = -\frac{c^2}{x^2} = -\frac{xy}{x^2} = -\frac{y}{x}$$

Tangent is $Y - y = -\frac{y}{x}(X - x)$

or $Yy + Yx = 2xy$

or $\frac{X}{2x} + \frac{Y}{2y} = 1$

It meets the axes at $A\ (2x, 0)$ and $B\ (0, 2y)$

Mid point of AB is (x, y) i.e. the point of contact

$\therefore \triangle\ OAB$ is a right angled triangle whose two sides are $2x$ and $2y$. Hence its area $= \frac{1}{2}\ 2x \cdot 2y$

$= 2xy = 2c^2$ i.e. constant

§2 Simple Applications

(A) Derivative as a rate measure. The derivative $\frac{dy}{dx}$ represents the *rate of* change of y with respect to x. In particular, if s

Since it is parallel to $2x+18y=9$ whose slope is $-1/9$

$$\frac{-1}{3x^2-3}=-\frac{1}{9} \quad \text{or} \quad x^2-1=3 \text{ or } x^2=4 \qquad x=2,-2$$

Hence two such points are (2, 2) and (−2, −2) slope of the normals at which is $-1/9$. Hence their equations are

$$y-2=-\tfrac{1}{9}(x-2) \quad \text{and} \quad y+2=-\tfrac{1}{9}(x+2)$$

$$x+9y=20 \quad \text{and} \quad x+9y=-20$$

8 Ans $2y+x=0$ at (0,0) and $2y+x+13/3=0$ at $(-\frac{1}{3},-2)$

(b) $2y\dfrac{dy}{dx}=4 \qquad \dfrac{dy}{dx}=\dfrac{2}{y}=2$ given $y=1$ and $x=-1$

Tangent is $y-1=2(x+1)$ or $y=2x+3$

9 $x=\dfrac{2at^2}{1+t^2},\ y=\dfrac{2at^3}{1+t^2}$ at $t=\frac{1}{2}$

$x=2a/5 \quad y=a/5$

Again $y=tx$

$$\frac{dy}{dx}=t+x\frac{dt}{dx} \tag{1}$$

Now $\dfrac{dx}{dt}=2a\dfrac{(1+t^2)2t-t^2\,2t}{(1+t^2)^2}=\dfrac{4at}{(1+t^2)^2}$

$\dfrac{dy}{dx}=t+x\dfrac{(1+t^2)^2}{4at} \qquad \dfrac{dx}{dt}\dfrac{dt}{dx}=1$

$\dfrac{dy}{dx}$ at $t=\dfrac{1}{2}$ is $\dfrac{1}{2}+\dfrac{2}{5}a\dfrac{25}{16\cdot 2a}=\dfrac{1}{2}+\dfrac{5}{16}=\dfrac{13}{16}$

Slope of tangent is 13/16 and that of normal is −16/13

Tangent at $t=1/2$ is $\left(y-\frac{1}{5}a\right)=\frac{13}{16}\left(x-\frac{2}{5}a\right)$

or $\qquad 13x-16y=2a$

Normal at $t=1/2$ is $\left(y-\frac{1}{5}a\right)=-\frac{16}{13}\left(x-\frac{2}{5}a\right)$

or $\qquad 16x+13y=9a$

10 Points are (1, 0) and (2, 0)

Tangents are $y+3x=3$, $y-7x+14=0$

11 $dx/d\theta=-a\sin\theta+a\theta\cos\theta+a\sin\theta=a\theta\cos\theta$

$dy/d\theta=a\cos\theta-a\cos\theta+a\theta\sin\theta=a\theta\sin\theta$

$$\frac{dy}{dx}=\frac{dy}{d\theta}\Big/\frac{dx}{d\theta}=\frac{\sin\theta}{\cos\theta}=\text{slope of tangent}$$

or $\qquad y-b=-(b/a)\,(x-a)$

or $\qquad bx+ay=2ab \quad \text{or} \quad x/a+y/b=2$

Since the value of dy/dx at (a, b) is constant i e of n and as such the line $x/a+y/b=2$ is always ta given curve for all values of n

20 $x=3\cos\theta-\cos^3\theta,\ y=3\sin\theta-\sin^3\theta$

$dx/d\theta=-3\sin\theta+3\cos^2\theta\sin\theta=-3\sin\theta\,(1-\cos$

$dy/d\theta=3\cos\theta-3\sin^2\theta\cos\theta=3\cos\theta\,(1-\sin^2\theta$

$$\therefore \frac{dy}{dx}=-\frac{\cos^3\theta}{\sin^3\theta}=\text{slope of tangent}$$

slope of normal is

$$=\frac{-1}{dy/dx}=\frac{\sin^3\theta}{\cos^3\theta}$$

Hence the equation of the normal is

$$y-(3\sin\theta-\sin^3\theta)=\frac{\sin^3\theta}{\cos^3\theta}(x-(3\cos\theta-\cos$$

or $\quad y\cos^3\theta-x\sin^3\theta=3\sin\theta\cos^3\theta-3\cos\theta\sin^3\theta$

Dividing throughout by $\sin^3\theta\cos^3\theta$, we get

$$y\operatorname{cosec}^3\theta-x\sec^3\theta=3\,(\operatorname{cosec}^2\theta-\sec^2\theta)$$

If $\theta=\pi/4$ then $\sec\pi/4=\operatorname{cosec}\pi/4=\sqrt{2}$

Normal is $y\ 2\sqrt{2}-x\ 2\sqrt{2}=0 \quad \text{or} \quad y-x=0$

It clearly passes through the origin

21 Let the tangent be drawn at the point (x, y)

Its equation is

$$Y-y=\frac{dy}{dx}\,(X-x)$$

But $\quad y=\sin x \qquad\qquad \dfrac{dy}{dx}=\cos x$

$$Y-y=\cos x\,(X-x)$$

Since it passes through (0, 0) $\qquad -y=-x\cos$

or $$\frac{y}{x}=\cos x \text{ and } y=\sin x$$

$$\frac{y^2}{x^2}+y^2=\cos^2 x+\sin^2 x=1$$

or $\quad y^2+x^2y^2=x^2 \quad \ldots =x^2y^2$

$$\frac{dV}{dt}=4\pi r^2\frac{dr}{dt} \quad\text{and}\quad \frac{dS}{dt}=8\pi r\,\frac{dr}{dt}$$

Given $r=8$ and $\frac{dV}{dt}=40$

$$40=4\pi\,64\frac{dr}{dt} \quad\text{and}\quad \frac{dS}{dt}=8\pi\times 8\frac{dr}{dt}$$

These give $\frac{dr}{dt}=\frac{5}{32\pi}$ and $\frac{dS}{dt}=64\pi\times\frac{5}{32\pi}=10$

Thus the surface area is increasing at the rate of 10 cm /min

Again to find approximately the increase in the radius during the next ½ minute we use formula

$$\delta r=\frac{dr}{dt}\,\delta t$$

Here $\frac{dr}{dt}=\frac{5}{32\pi}$ and $\delta t=\frac{1}{2}$

Hence $\delta t=\frac{5}{32\pi}\times\frac{1}{2}=\frac{5\times 7}{64\times 22}=\frac{35}{1408}$

$$=025 \text{ cm approx}$$

8 Do yourself

9 If s denotes the length of the rope between the drum and the raft and x the distance from the raft to the bank then by hypothesis, $s=x^2+4^2$

$$2s\,\frac{ds}{dt}=2x\frac{dx}{dt} \quad\text{or}\quad \frac{dx}{dt}=\frac{s}{x}\,\frac{ds}{dt} \qquad (1)$$

We are given $\frac{ds}{dt}=3$, $x=25$,

and $s=\sqrt{(25^2+4^2)}=25\ 3$ approx

Substituting in (1), we get

$$\frac{dx}{dt}=\frac{25\ 3}{25}\times 3=3\ 03 \text{ m/min approx}$$

10 We have $12\frac{dy}{dt}=3x^2\,\frac{dx}{dt}$ or $\frac{dy}{dt}\Big/\frac{dx}{dt}=\frac{x^2}{4}$ Hence

(i) in $-2<x<2$, the ratio $\frac{dy}{dt}\Big/\frac{dx}{dt}$ is less than 1 so that the rate of ordinate is less than the rate of abscissa

(ii) at $x=\pm 2$, this ratio is 1, that is, at these points the rates of change of coordinates are equal

(iii) at $x<-2$ or $x>2$, the rate of change of ordinate exceeds that of the abscissa as can be easily shown

3 Show that the curves $x^3-3xy^2=-2$ and $3x^2y-y^3=2$ cut orthogonally

4 Find the condition that the following conics may cut orthogonally

(a) $ax^2+by^2=1$, $a'x^2+b'y^2=1$,

(b) $\frac{x^2}{a}+\frac{y^2}{b}=1$ $\frac{x^2}{a'}+\frac{y^2}{b'}=1$

(c) Show that

$$\frac{x^2}{a^2+k_1}+\frac{y^2}{b^2+k_1}=1 \text{ and } \frac{x^2}{a^2+k_2}+\frac{y^2}{b^2+k_2}=1$$

intersect orthogonally

5 Show that the curves $y^2=4ax$ and $ay^2=4x^3$ intersect each other at an angle of $\tan^{-1}\frac{1}{2}$ and also if PG_1 and PG_2 be the normals to the two curves at common point of intersection (which is not origin) meeting the axis of x in G_1 and G_2, then $G_1G_2=4a$

6 Prove that the curves $y=e^{-ax}\sin bx$ and $y=e^{-ax}$ touch at the points for which $bx=2n\pi+\pi/2$

7 Show that the angle between the tangent at any point P and the line joining P to the origin O is the same at all points of the curve $\log(x^2+y^2)=k\tan^{-1}y/x$

8 Find the condition that the line $Ax+By=1$ may be a normal to the curve $a^{n-1}y=x^n$

9 If $ax+by=1$ is a normal to the parabola $y^2=4px$ then $pa^3+2pab^2=b^2$

10 (i) Find the condition that the line $x\cos\alpha+y\sin\alpha=p$ may touch the curve

(a) $\frac{x^m}{a^m}+\frac{y^m}{b^m}=1$, (b) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$,

(c) $\left(\frac{x}{a}\right)^{n/(n-1)}+\left(\frac{y}{b}\right)^{n/(n-1)}=1$,

(d) $x^my^n=a^{m+n}$ Prove also that the portion of the tangent intercepted between the axes is divided at its point of contact in a constant ratio

(e) In the curve $x^a y^b=k^{a+b}$, prove that the portion of the tangent intercepted between the coordinate axes is divided at its point of contact into segments which are in constant

21 For the curve $xy=c^2$, prove the following

(a) The intercept between the axes on the tangent at any point is bisected at the point of contact

(b) The tangent at any point makes with coordinate axes a triangle of constant area (Roorkee 67)

Solutions to Problem Set (B)

1 (a) $x^2+y^2=a^2\sqrt{2}$, $\quad x^2-y^2=a^2$

For points of intersection $\left.\begin{array}{l}2x^2=a^2(\sqrt{2}+1)\\ 2y^2=a^2(\sqrt{2}-1)\end{array}\right\}$ (A)

$$2x+2y\left(\frac{dy}{dx}\right)_I=0, \qquad \left(\frac{dy}{dx}\right)_I=-\frac{x}{y}=m_1$$

$$2x-2y\left(\frac{dy}{dx}\right)_{II}=0, \qquad \left(\frac{dy}{dx}\right)_{II}=\frac{x}{y}=m_2$$

If θ be the angle between the curves, then

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2}=\frac{-2\dfrac{x}{y}}{1-x^2/y^2}=\frac{-2xy}{y^2-x^2}$$

Now from (A), $4x^2y^2=a^4$ ∴ $2xy=a^2$ Also $y^2-x^2=-a^2$

Hence $\tan\theta=\dfrac{-a^2}{-a^2}=1$ $\quad\theta=\pi/4$

(b) $x^2y=1$, $y=x^2$

Points of intersection are (1, 1) and (−1, −1) and

$\theta=\tan^{-1}4/3$ at both the points

(c) $x^2=4ay$, $2y^2=ax$ [Ans $\pi/2$ and $\tan^{-1}\frac{3}{5}$]

(d) $y^2=ax$, $x^2=by$

Ans $\pi/2$ at (0, 0) and $\tan^{-1}\dfrac{3\,a^{1/3}\,b^{1/3}}{2(a^{2/3}+b^{2/3})}$ at $(a^{1/3}b^{2/3}, a^{2/3}b^{1/3})$

(e) $y^2=8x$ $\quad x^2=4y-12$ Eliminate x

Clearly $y=4$ satisfies it and putting in $y^2=8x$, we get $x=2$

Point is (2, 4)

$$2y\frac{dy}{dx}=8 \qquad \left(\frac{dy}{dx}\right)_I=\frac{8}{2y}=1 \text{ at } (2, 4)$$

$$2x=4\frac{dy}{dx} \qquad \left(\frac{dy}{dx}\right)_{II}=\frac{x}{2}=1 \text{ at } (2, 4)$$

Thus their slopes are same and hence the two curves touch at the point (2, 4)

7 Ans $(\frac{1}{2}at^2, 8at^3)$

8 Ans (d) $t_2 = -t_1 - \frac{1}{t_1}$ is the correct answer

9 Ans (B) and (C)

Any point on the curve $xy=1$, may be taken as $\left(t, \frac{1}{t}\right), t \neq 0$

Now differentiating $xy=1$, we get

$x\frac{dy}{dx}+1\cdot y=0$ or $\frac{dy}{dx}=-\frac{y}{x}=-\frac{1}{t^2}$ at the point $\left(t, \frac{1}{t}\right)$

Hence the gradient of normal $=t^2$

The line $ax+by+c=0$ will be normal to $xy=1$ if

$t^2=-\frac{b}{a}$

Since $t^2>0$, we must have $a>0$ $b<0$ or $a<0, b>0$

Hence (B) and (C) are correct answers

10 Ans (iii) 11 Ans (iii)

12 Ans (iv) 13 Ans (C)

14 Ans (d) 15 Ans (d)

$$\frac{x^2}{b-b}=\frac{y^2}{a-a'}=\frac{1}{ab-a'b}$$

Putting the values of x^2 and y^2 in (1)

$$aa'\ \frac{(b-b)}{ab'-a\,b}+bb\quad\frac{(a-a')}{ab'-a\,b}=0$$

or $$\frac{b-b}{bb'}+\frac{a-a'}{aa}=0$$

or $$\left(\frac{1}{b}-\frac{1}{b}\right)+\left(\frac{1}{a'}-\frac{1}{a}\right)=0$$

or $$\frac{1}{a}-\frac{1}{b}=\frac{1}{a'}-\frac{1}{b'}$$ is the requir

(b) Proceeding as above the required condi

$$a-b=a'-b$$

(c) Put $\frac{1}{a^2+k_1}=A,\qquad \frac{1}{a+k_2}=A'$ etc

$$\frac{1}{b^2+k_1}=B\qquad \frac{1}{b^2+k_2}=B$$

Hence as in part (a) question becomes

$$Ax^2+By^2=1\qquad A'x^2+B'y^2=1$$

The condition for their orthogonal intersecti

$$\frac{1}{A}-\frac{1}{B}=\frac{1}{A}-\frac{1}{B}$$

or $(a^2+k_1)-(b^2+k_1)=(a^2+k_2)-(b^2+k_2)$

or $a^2-b^2=a^2-b^2$ which is true whatever k_1

5 The two curves meet at $(a, 2a)$ and $(0\ 0)$,

$$m_1=\left(\frac{dy}{dx}\right)_I=\frac{2a}{y}=1 \text{ at } (a, 2a)$$

$$m_2=\left(\frac{dy}{dx}\right)_{II}=\frac{6x^2}{ay}=3 \text{ at } (a, 2a)$$

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{1-3}{1+3}\right|=\tfrac{1}{2}\qquad \theta=$$

Normal to 1st $\quad y-2a=-1\,(x-a)\quad$ or y

Normal to 2nd $\quad y-2a=-\frac{1}{3}\,(x-a)\quad$ or 3.

They meet the axis $y=0$ at G_1 $(3a, 0)$ and G_2

7 Ans $(\frac{1}{2}at^2, 8at^3)$

8 Ans (*d*) $t_2 = -t_1 - \frac{1}{t_1}$ is the correct answer

9 Ans (*B*) and (*C*)

Any point on the curve $xy=1$, may be taken as $\left(t, \frac{1}{t}\right)$, $t \neq 0$

Now differentiating $xy=1$, we get

$x\frac{dy}{dx}+1 \cdot y=0$ or $\frac{dy}{dx}=-\frac{y}{x}=-\frac{1}{t^2}$ at the point $\left(t, \frac{1}{t}\right)$

Hence the gradient of normal $=t^2$

The line $ax+by+c=0$ will be normal to $xy=1$ if

$t^2=-\frac{b}{a}$

Since $t^2>0$, we must have $a>0, b<0$ or $a<0, b>0$

Hence (*B*) and (*C*) are correct answers

10 Ans (iii) 11 Ans (iii)

12 Ans (iv) 13 Ans (*C*)

14 Ans (*d*) 15 Ans (*d*)

$$\frac{x^2}{b'-b}=\frac{y^2}{a-a'}=\frac{1}{ab'-a'b}$$

Putting the values of x^2 and y^2 in (1)

$$aa'\ \frac{(b'-b)}{ab'-a'b}+bb'\ \frac{(a-a')}{ab'-a'b}=0$$

or $$\frac{b'-b}{bb'}+\frac{a-a'}{aa'}=0$$

or $$\left(\frac{1}{b}-\frac{1}{b'}\right)+\left(\frac{1}{a'}-\frac{1}{a}\right)=0$$

or $$\frac{1}{a}-\frac{1}{b}=\frac{1}{a'}-\frac{1}{b'}$$ is the required condition

(b) Proceeding as above the required condition will be

$$a-b=a'-b'$$

(c) Put $\frac{1}{a^2+k_1}=A$, $\quad \frac{1}{a^2+k_2}=A'$ etc

$$\frac{1}{b^2+k_1}=B \qquad \frac{1}{b^2+k_2}=B'$$

Hence as in part (a) question becomes

$$Ax^2+By^2=1 \qquad A'x^2+B'y^2=1$$

The condition for their orthogonal intersection is

$$\frac{1}{A}-\frac{1}{B}=\frac{1}{A'}-\frac{1}{B'}$$

or $(a^2+k_1)-(b^2+k_1)=(a^2+k_2)-(b^2+k_2)$

or $a^2-b^2=a^2-b^2$ which is true whatever k_1 and k_2 may be.

5 The two curves meet at $(a, 2a)$ and $(0, 0)$.

$$m_1=\left(\frac{dy}{dx}\right)_I=\frac{2a}{y}=1 \text{ at } (a, 2a)$$

$$m_2=\left(\frac{dy}{dx}\right)_{II}=\frac{6x^2}{ay}=3 \text{ at } (a, 2a)$$

$$\tan\theta=\left|\frac{m_1-m_2}{1+m_1m_2}\right|=\left|\frac{1-3}{1+3}\right|=\frac{1}{2} \qquad \theta=\tan^{-1}\frac{1}{2}$$

Normal to 1st $\quad y-2a=-1\ (x-a) \quad$ or $\quad y+x=3a$

Normal to 2nd $\quad y-2a=-\frac{1}{3}\ (x-a) \quad$ or $\quad 3y+x=7a$

They meet the axis $y=0$ at $G_1\ (3a,\ 0)$ and $G_2\ (7a, 0)$

$$G_1G_2=|7a-3a|=4a$$

6 $\quad y=e^{-ax}\sin bx, \qquad y=e^{-ax}$

The two curves meet at the points where $\sin bx=1$

whereas its curved surface is πrl only

Volume of a cylinder $=\pi r^2 h$

Total Surface $=2\pi r\ (r+h)$ Its curved surface is $2\pi rh$

Volume of a box $=xyz$ and surface $=2\ (xy+yz+zx)$

Greatest and least values of a function The reader will do well to bear in mind that a maximum value of $f(x)$ at $x=x_0$ in an interval $[a, b]$ does not mean that it is greatest value of $f(x)$ in that interval There may be a value of the function greater than a maximum value As a matter of fact there may exist a minimum value of the function which is greater than or equal to maximum value of the function in $[a\ b]$

However, if a function $f(x)$ is continuous in a closed interval $[a\ b]$ then greatest (least) value of $f(x)$ is attained either at critical points, or at the end points of tha interval Thus to find the greatest (least) value of the function, we have to compute its values at all the critical points on the interval $[a, b]$ and also the values $f(a)$, $f(b)$, and choose the greatest (least) one out of the numbers thus obtained If the interval is not closed it may have neither the greatest nor the least value

Sign of f(x) for small values of x The sign of $f(x)$ is governed by the lowest degree terms in $f(x)$ For example, for small x the sign of $-3+4x+7x^2$ will be governed by the constant term and hence $-$ive Similarly the sign of $x^3-7x^{11}+20x^{15}$ will be governed by x^3 and so its sign will be $+$ive or $-$ive according as x is $+$ive or $-$ive

Problem Set (A)

1 Find the max and min values of

(*a*) $x^5-5x^4+5x^3-10$

(*b*) $x^3-9x^2+15x-1$

(*c*) $2x^3-3x^2-12x+12$

(*d*) $\dfrac{x^2+x+1}{x^2-x+1}$

(*e*) $(x-2)^6\ (x-3)$

(*f*) $(x-8)^4\ (x-9)$

(*g*) $\sin x\ (1+\cos x)$ is max at $x=\pi/3$

(*h*) $\sqrt{3}\sin x+3\cos x$ is max at $x=\pi/6$

(*h*$_1$) $\sin x+\cos x$ is max at $x=\dfrac{\pi}{4}$ (M N R 83)

Normal is $y\cos\theta - x\sin\theta = a\cos 2\theta$

$$\rho_1 = \frac{\frac{a}{2}\sin 2\theta}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = \frac{a}{2}\sin 2\theta$$

$$p_2 = \frac{a\cos 2\theta}{\sqrt{(\cos^2\theta + \sin^2\theta)}} = a\cos 2\theta$$

$$4p_1^2 + p_2^2 = a^2(\sin^2 2\theta + \cos^2 2\theta) = a^2$$

17 (a) $ay^2 = x^3$ (1)

$$2ay\frac{dy}{dx} = 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2}{2ay}$$

Hence slope of the normal is

$$-\frac{1}{dy/dx} = \frac{-2ay}{3x^2}$$

Since the normal makes equal intercepts on the axes its inclination to axis of x is either 45° or 135°. In other words its slope is $\tan 45°$ or $\tan 135°$ or $1, -1$

$$-\frac{2ay}{3x^2} = \pm 1 \text{ or } 4a^2y^2 = 9x^4$$

or $\quad 4a\; x^3 = 9x^4 \quad \therefore \quad x = \frac{4a}{9}$

is the abscissa of such a point

(b) $\frac{dy}{dx} = \frac{1}{2y} = 1 \qquad y = \frac{1}{2}$ and hence $x = \frac{1}{4}$

18 Here $xy = (c+x)^2$

Slope of the normal is

$$-\frac{x^2}{x^2 - c^2} = \pm 1$$

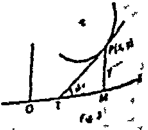
$$-x^2 = x^2 - c^2 \quad \text{or} \quad -x^2 = -(x^2 - c^2)$$

or $\quad 2x^2 = c^2 \qquad c^2 = 0$ not possible

$$x = \pm\frac{c}{\sqrt{2}}$$

19 We easily find

$\frac{dy}{dx} = \tan t = \tan\psi \qquad \psi = t$



If PT be the tangent intercepted between the curve and the axis of x then from the figure

$y = PT\sin\psi = PT\sin t$

But $\qquad y = a\sin t$

side is 2″ What must be the dimensions of the page in order that the area of the printed matter may be maximum

9 (*a*) Two towns are to get their water supply from a river Both towns are on the same side of the river at distance of 6 km and 18 km from the river bank If the distance between the points on the river bank nearest to the two towns be 10 km, find (a) *where a single pumping station* may be located requiring the least amount of pipe (b) How much pipe be needed

(*b*) A firm has a branch store in each of the three cities *A*, *B* and *C* *A* and *B* are 320 km apart and *C* is 200 km. from each of them A godown is to be built equisdistant from *A* and *B* In order to minimize the time of transportation it should be located so that sum of the distances from the godown to each of the cities is a minimum Where should the godown be built ?

(**Roorkee 87**)

10 (*a*) A factory *D* is to be connected by a road with a straight railway line on which a town *A* is situated The distance *DB* of the factory to the railway line is $5\sqrt{3}$ km Length *AB* of the railway line is 20 km Freight charges on the road are twice the charges on the railway At what point P ($AP < AB$) on the railway line should the road *DP* be connected so as to ensure minimum freight charges from the factory to the town (**Roorkee 77**)

(*b*) Two roads *OA* and *OB* intersect at an angle of 60° *A* car driver approaches O from *A* where $AO = 800$ meters, at a uniform speed of 20 meters per second Simultaneously a *runner starts running from O* towards *B* at a uniform speed of 5 meters per second Find the time when the car and the runner are closest

11 In a sub marine telegraph cable the speed of signalling varies as $x^2 \log 1/x$ where x is the ratio of the radius of the case to that of covering Show that the greatest speed is attained when this ratio is $1 \sqrt{e}$

12 The fuel charges for running a train are proportional to the square of the speed generated in miles per hour and costs *Rs* 48 *per hour at* 16 *miles per hour* What is the mos

19 An open tank is to be constructed with a square base and vertical sides so as to contain a given quantity of water Show that the expenses of lining with lead will be least if the depth is made half of the width

20 A closed rectangular box with a square base is to be made so as to contain 1000 cubic feet The cost of the material per sq foot for the bottom is 15 paisa, for the top 25 paisa and for the sides 20 paisa The labour charges for making the box are Rs 3 Find the dimensions of the box when the cost is minimum

21 A square tank of capacity 250 cubic m has to be dug out The cost of land is Rs 50 per sq m The cost of digging increases with the depth and for the whole tank is 4 0× (depth)^2 rupees *Find the dimensions of the tank for the least* total cost (Roorkee 74)

22 A box of constant volume c is to be twice as long as it is wide The material on the top and four sides cost three times as much per square meter as that in the bottom What are the most economical dimensions?

23 If 40 square feet of sheet metal are to be used in the construction of an open tank with square base, find the dimensions so that the capacity of the tank is max

24 A box is constructed from a rectangular metal sheet of 21 cm by 16 cm by cutting equal squares of sides x from the corners of the sheet and then turning up the projected portions For what value of x the volume of the box will be maximum

25 The three sides of a trapezium are equal, each being 6 long Find the area of the trapezium when it is maximum

26 LL' is the latus rectum of the parabola $y^2=4ax$ and PP' is a double ordinate drawn between the vertex and the latus rectm Show that the area of the trapezium $PP'L'L$ is maximum when the distance of PP' from vertex it $a/9$

27 (*a*) Prove that the maximum rectangle inscribed in a circle is a square

(*b*) Find the dimension of the rectangle of greatest area that can be inscribed in a semi circle of radius r

(M N R 81)

increase of 0 7 percent in the pressure prove that there is a decrease of 0 5 percent in the volume

12 The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T=2\pi\sqrt{\left(\frac{l}{g}\right)}$, where g is a constant Find the approximate error in the calculated value of T corresponding to an error of 2 percent in the value of l

Hints and Solutions to Problem Set (C)

1 (a) Since $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

i e volume increases $4\pi r^2$ as fast as the radius

(b) $V=\frac{4}{3}\pi r^3$, $\frac{dV}{dt}=4\pi r^2\frac{dr}{dt}$

We are given, $\frac{dr}{dt}=\frac{1}{7}$ and $r=\frac{7}{11}$ Hence

$$\frac{dV}{dt}=4\times\frac{22}{7}\times\frac{7\times 7}{11\times 11}\times\frac{1}{7}=\frac{8}{11}$$

Hence the volume is increasing at the rate of $\frac{8}{11}$ cm³/sec

2 If α is the semi vertical angle of the cone, then $\tan\alpha=\frac{1}{2}$. Let after time t minutes, h be the depth of water The radius of the surface of water $=h\tan\alpha=\frac{1}{2}h$ If V is the volume of water after t minutes, then

$$V=\frac{1}{3}\pi r^2 h=\frac{1}{3}\pi\,\frac{1}{4}h^2\,h=\frac{\pi}{12}h^3$$

$$\frac{dV}{dt}=\frac{\pi}{12}3h^2\frac{dh}{dt} \qquad (1)$$

We are given, $h=4$ cm and $\frac{dV}{dt}=\frac{3}{2}$ c c

Substituting in (1), we get

$$\frac{3}{2}=\frac{\pi}{4}16\frac{dh}{dt} \quad \text{or} \quad \frac{dh}{dt}=\frac{3}{8\pi}\text{ cm/min.}$$

3 Do yourself Ans (a) $\frac{bc}{a-b}$ meter per minute

(b) 2 376 km /h = 66 cm/sec

4 Do yourself Ans (b) 150 km./h.

5 11 6 (a) 26,450 (jules) (b) $\frac{68}{5}$ (erg)

7. $V=\frac{4}{3}\pi r^3$ and $S=4\pi r^2$

areas the sum of which is maximum (Roorkee 86)

39 Investigate for maxima and minima the functions

$$f(x)=\int_1^x [2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]\,dt$$ (I.I T 88)

Solutions to Problem Set (A)

1 (*a*) $y=x^5-x^4+5x^3-10$

$$\frac{dy}{dx}=5x^2(x^2-4x+3)=0 \text{ or } 5x^2(x-3)(x-1)=0$$

$x=0, 3, 1$

and at these points we shall consider max and min

Consider the point x=1

If x is slightly less than 1, $\frac{dy}{dx}=+--=+$ive

If x is slightly greater than 1, $\frac{dy}{dx}=+-+=-$ive

Above shows that $\frac{dy}{dx}$ changes sign from + ive to −ive *i e* tangent first makes acute angle and then obtuse angle Hence y is max at $x=1$

Consider the point x=3

If x is slightly less than 3 then $\frac{dy}{dx}=+-+=-$ive,

If x is slightly greater than 3 then $\frac{dy}{dx}=+++=+$ive

Above shows that $\frac{dy}{dx}$ changes sign from −ive to +ive *i e* tangent first makes an obtuse angle and then an acute angle Hence *y is* min at $x=3$

Consider the point x=0

If x is slightly less than zero, then $\frac{dy}{dx}=+--=+$ive

If x is slightly greater than zero then $\frac{dy}{dx}=+--=+$ive

Above shows that $\frac{dy}{dx}$ does not change sign and as such at $x=0$ there is neither max nor min value of y

Alternative Method

$$\frac{dy}{dx}=5x^2(x-3)(x-1)=5x^4-20x^3+15x^2=0$$

At $x=\frac{28}{11}$, $\frac{dy}{dx}$ changes from $-$ive to $+$ive Min at $x=\frac{28}{11}$

$$\text{Min } y=\left(\frac{28}{11}-2\right)^6\left(\frac{28}{11}-3\right)^5=\left(\frac{6}{11}\right)^6\left(\frac{-5}{11}\right)^5=-\frac{5^5 6^6}{11^{11}}$$

(*f*) Proceed as above

Max at $x=8$, value 3, Min at $x=\frac{76}{9}$, value $-\left(\frac{4}{9}\right)^4\left(\frac{5}{9}\right)^5$

(*g*) $y=\sin x\,(1+\cos x)=\sin x+\frac{1}{2}\sin 2x$

$$\frac{dy}{dx}=\cos x+\cos 2x=0 \qquad \cos 2x=-\cos x=\cos(\pi-x)$$

$$2x=\pi-x \quad \text{or} \quad x=\pi/3$$

$$\frac{d^2y}{dx^2}=-\sin x-2\sin 2x \text{ which is clearly } -\text{ive at } x=\pi/3$$

and hence y is Max at $x=\pi/3$ and its value is $\frac{3\sqrt{3}}{4}$

Note We are only concerned with the sign of $\frac{d^2y}{dx^2}$ and not its value

(*h*) $\sqrt{3}\,(\sin x+\sqrt{3}\cos x)$ Divide and multiply by $\sqrt{(1+3)}=2$

$$=2\sqrt{3}\left(\frac{1}{2}\sin x+\frac{\sqrt{3}}{2}\cos x\right)=2\sqrt{3}\sin\left(x+\frac{\pi}{3}\right)$$

Clearly it is Max when $x+\pi/3=\pi/2$ i e $x=\pi/6$ and Max value is $2\sqrt{3}$

(h_1) Similarly Max when $x=\pi/4$ and Max value is $\sqrt{2}$

(*i*) $y=\frac{x}{1+x\tan x}$

y will be maximum when its reciprocal

$$=\frac{1}{y}=\frac{1+x\tan x}{x} \text{ is min}$$

Let $z=\frac{1}{x}+\tan x$ $\qquad \frac{dz}{dx}=-\frac{1}{x^2}+\sec^2 x=0$

$$\frac{1}{x^2}=\sec^2 x \qquad x=\cos x$$

$$\frac{d^2z}{dx^2}=+\frac{2}{x^3}+2\sec x\sec x\tan x=+\text{ive}$$

and hence z is min when $x=\cos x$, that is, y is max at $x=\cos x$

(*j*) $y=\sin^p\theta\cos^q\theta$

Now y will be max or min according as

$$z=\log y=p\log\sin\theta+q\log\cos\theta$$

$$=(a^2+b^2+2ab)+(a^2\tan^2 x+b^2\cot^2 x-2ab)$$
$$=(a+b)+(a\tan x-b\cot x)^2$$

Above relation shows that the value of y is always greater than or equal to $(a+b)$ as $(a\tan x-b\cot x)^2$ is always non negative

Hence the value of y will be minimum when

$$(a\tan x-b\cot x)^2=0 \quad \text{or} \quad \tan^2 x=b/a$$

and in that case the minimum value will be $(a+b)^2$

(l) $y=(x-1)^2 e^x$

$$\frac{dy}{dx}=(x-1)^2 e^x+2(x-1)e^x$$
$$=e^x(x^2-2x+1+2x-2)=e^x(x^2-1)=0$$
$$x=1,\ -1$$
$$\frac{d^2y}{dx^2}=e^x(x^2-1+2x)=+\text{ive for } x=1$$
$$=-\text{ive for } x=-1$$

e and e^{-1} are both +ive

Hence min at $x=1$ and max at $x=-1$

Min value$=0$, max value is $4e^{-1}=4/e$

(m) $y=\dfrac{\log x}{x}$

$$\frac{dy}{dx}=-\frac{1}{x^2}\log x+\frac{1}{x}\cdot\frac{1}{x}=\frac{1}{x^2}(1-\log x)=0$$
$$\log x=1 \quad \text{or} \quad x=e$$
$$\frac{d^2y}{dx^2}=-\frac{2}{x^3}(1-\log x)+\frac{1}{x^2}\left(-\frac{1}{x}\right)$$
$$=0-\frac{1}{e^3}=-\text{ive for } x=e$$

Hence y is max when $x=e$ and its value is

$$\frac{\log e}{e}=\frac{1}{e}=e^{-1}$$

(n) $y=x^{1/x}$

Now will be max or min according as

$$z=\log y=\frac{1}{x}\log x \text{ is max or min}$$
$$z=\frac{1}{x}\log x \text{ and it is max when } x=e \text{ by part } (m)$$

Hence y is max when $x=e$ and its max value is $e^{1/e}$

(o) $y=x^x$

max for $\log x=-1$ or $x=e^{-1}=1/e$

15 The normal at the point (1, 1) on the curve $2y=3-x^2$, is
(a) $x+y=0$, (b) $x+y+1=0$,
(c) $x-y+1=0$, (d) $x-y=0$

(M N R 85)

Solutions

1 Ans (a)
2 (iii)
3 (iv)

We have $\frac{dy}{dx}=4x-1$ Hence the tangent is parallel to $y=3x+9$ if $4x-1=3$ or $x=1$
Putting $x=1$ in the equation of the curve, we get
$$y=2-1+1=2$$
Hence the required point is (1, 2)

4 Ans (ii)
5 Ans (iii)
6 Ans (C)

Equation of tangent and normal at P $(at^2, 2at)$ to the parabola can be easily found to be

$$ty=x+at^2 \quad (1)$$
and
$$y+tx=2at+at^3 \quad (2)$$
Putting $y=0$ in (1), we get $x=-at^2$ and putting $y=0$ in (2), we have $x=2a+at^2$
Hence coordinates of T are $(-at^2, 0)$ and coordinates of G are $(2a+at^2, 0)$
Since PT is perpendicular to PG, TG is the diameter of the circle through P T, G Hence the equation of the circle is
$$(x+at^2)(x-2a-at^2)+(y-0)(y-0)=0$$
or
$$x^2+y^2-2ax-at^2(2a+at^2)=0$$
$$2x+2y\frac{dy}{dx}-2a=0$$
or
$$\frac{dy}{dx}=\frac{a-x}{y}=\frac{a-at^2}{2at}=\frac{1-t^2}{2t} \text{ at } P$$
and from $y^2=4ax$, we get
$$\frac{dy}{dx}=\frac{2a}{y}=\frac{2a}{2at}=\frac{1}{t} \text{ at } P$$
Hence if θ is the angle between the tangents at P to the parabola and circle then
$$\tan\theta=\frac{(1/t)-(1-t^2)/2t}{1+(1/t)(1-t^2)/2t}=\frac{1+t^2}{2t}\times\frac{2t^2}{1+t^2}=t$$
or
$$\theta=\tan^{-1} t$$

We have to prove that y is max at P i e at $x=2$

Consider $x=2$

We shall not find d^2y/dx^2 but follow the other method

If x is slightly less than 2 then

$$\frac{dy}{dx}=++=+\text{ive}$$

If x is slightly greater than 2 then

$$\frac{dy}{dx}=-+=-\text{ive}$$

Since dy/dx changes sign from $+$ive to $-$ive therefore at $x=2$ the function is maximum

5 $s=\frac{1}{4}t^4-2t^3+4t^2-7$

$$v=\frac{ds}{dt}=t^3-6t^2+8t, \quad a=\frac{dv}{dt}=3t^2-12t+8$$

v is maximum at

$t=2-\frac{2}{\sqrt{3}}$, a is min when $t=2$

6 (a) Slope$=S=\frac{dv}{dx}=-3x^2+6x+2$ etc Ans 5 at (1, $-$23)

(b) Here slope $S=\frac{dy}{dx}=\{1 \ (1+x^2)-2x \ x\}/(1+x^2)^2$

$$=(1-x^2)/(1+x^2)^2$$

$$\frac{dS}{dx}=\{-2x(1+x^2)^2-2\,(1+x^2)\ 2x\,(1-x^2)\}/(1+x^2)^4$$

$$=\frac{-2x\,(1+x^2)\,(3-x^2)}{(1+x^2)^4}$$

For max or min of S $\frac{dS}{dx}=0$

This gives $x=0, \ \pm\sqrt{3}$

It can be checked that for $x=0$ $\frac{dS}{dx}$ changes from $+$ive to $-$ive

at $x=\pm\sqrt{3}$ it change from $-$ive to $+$ive

Hence slope S is maximum when $x=0$ and min when $x=\pm\sqrt{3}$ Thus for greatest slope, we have $x=0$ and $y=0$

Hence the required point is (0, 0), that is, the origin

7 Let (x, y) be on the parabola

$$y=x^2+7x+2$$

Its distance from the line

$$y=3x-3 \text{ or } 3x-y=3 \text{ is}$$

The value of x cannot be $-$ive and hence we shall consider $x=5/2$

It is easy to observe that

$$\frac{d^2p}{dx^2}=\frac{36}{(36+x^2)^{3/2}}+\frac{18}{\{18^2+(10-x)^2\}^{3/2}}$$

Above is $+$ive for $x=5/2$ and hence p is minimum

Putting $x=5/2$ in (1) the minimum length of pipe is

$$p=\sqrt{\left(36+\frac{25}{4}\right)}+\sqrt{\left(324+\frac{225}{4}\right)}$$

$$=\frac{13}{2}+\frac{39}{2}=\frac{52}{2}=26 \text{ km}$$

(b) Let G be the position of godown at a distance x each from A and B Also

$CD=\sqrt{\{200^2-160^2\}}=120$

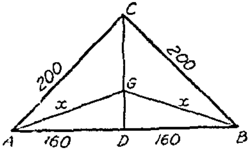
$GD=\sqrt{(x^2-162)}$

$GC=DC-DG$

$=120-\sqrt{(x^2-160^2)}$

If $y=GA+GB+GC$ then

$y=2x+120-\sqrt{(x^2-160^2)}$



$$\frac{dy}{dx}=2-\frac{1}{2\sqrt{(x^2-160^2)}}\ 2x=0 \text{ for max or min}$$

$$\frac{2}{x}=\frac{1}{\sqrt{(x^2-160^2)}}$$

or $4(x^2-160^2)=x^2 \qquad x=\frac{320}{\sqrt{3}}$ (1)

$$\frac{d^2y}{dx^2}=-\left[1\ \frac{1}{\sqrt{(x^2-160^2)}}+x\ -\frac{1}{2(x^2-160^2)^{3/2}}\ 2x\right]$$

$$=-\left[\frac{2}{x}-\frac{x^2\,8}{x^3}\right] \text{ by } (1)=\frac{6}{x}=+\text{ive}$$

y is minimum when $x=\frac{320}{\sqrt{3}}$ and G is on the perpendicular bisector of AB

10 (a) If R rupees per km be the freight for rail then that by road will be $2R$ per km If C be the total cost then

$C=R[2\sqrt{(x^2+75)}+(20-x)]$ etc

$x=5$ for C to be minimum

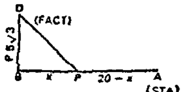


Fig 4

where $\varphi(t)$ and $\psi(t)$ have derivatives both of first and second orders within a certain interval of t, and $\varphi'(t)=0$ Let at $t=t_0$ $\psi'(t)=0$ Then

(a) if $\psi''(t_0) < 0$, $f(x)$ has a maximum at $x=x_0=\varphi(t_0)$

(b) if $\psi''(t_0) > 0$, the function $f(x)$ has a minimum at $x=x_0=\varphi(t_0)$

(c) if $\psi''(t_0)=0$ the question of the existence of an extreme value remains open

Note Sometime we also use the term *extreme values* for aximum or minimum values The points at which $f'(x)$ oes not exist are called critical points If $f'(x)=0$, we say that $=f(x)$ is stationary at $x=a$

Important Instructions

1 According to the given condition of the problem dete ne the function whose max, and min values are to be found

2 It may happen that the above function is not of single iable but contains more than one variable In such cases there l be given certain other relation between these variables with help of which by elimination we shall be able to reduce th ve function to be of single variable

A function of the form

$$k f(x) \quad k + f(x) \quad [f(x)]^k \quad \text{or} \quad [f(x)]^{1/k},$$

re k is a +ive constant will be max or min according as $f(x)$ aximum or minimum provided $f(x) > 0$

Also if y is max and min, then $\log y=z$ is also max r mum provided $y > 0$ (Note)

Also $y=f(x)$ is max or min according as $z=1/f(x)$ is m x

Commit to Memory

n usual notations

of a square $=x^2$ its perimeter $=4x$

of a rectangle $=xy$ perimeter $=2(x+y)$

of a trapezium $=\frac{1}{2}$ (sum of parallel sides) $\times$ distance betw th

ea of a circle $=\pi r^2$ Perimeter $=2\pi r$

lume of right cone $=\frac{1}{3}\pi r^2 h$, Total Surface $=\pi r(r+l)$

or $$E=S\left(\frac{3}{16}v+\frac{300}{v}\right)$$

$$\frac{dE}{dv}=S\left(\frac{3}{16}-\frac{300}{v^2}\right)=0$$

$$v^2=1600 \quad \text{or} \quad v=40$$

$$\frac{d^2E}{dv^2}=S\left(\frac{600}{v^3}\right)=+\text{ive for } v=40$$

d hence E is minimum

Hence the most economical speed is 40 m p h

Total hours is $500/x$

Diesel cost in rupees

$$=\frac{1}{300}\left(\frac{900}{x}+x\right)\times 500\times\frac{40}{100}=\frac{2}{3}\left(\frac{900}{x}+x\right)$$

Payment to driver

$$\frac{500}{x}\cdot\frac{3}{2}=\frac{750}{x}$$

If E be the expenses then

$$E=\left(\frac{600}{x}+\tfrac{2}{3}x\right)+\frac{750}{x}=\frac{1350}{x}+\frac{2}{3}x \text{ etc}$$

$x=45$ km p h is the most economical speed

(a) If the increase be Rs x per subscriber then the rate will be $300+x$ and subscribers left will be $500-x$ according to the given condition If I be the new income then

$$I=(300+x)(500-x)=150000+200x-x^2$$

$$\frac{dI}{dx}=200-2x=0, \qquad x=100$$

$$\frac{d^2I}{dx^2}=-2=-\text{ive}$$

and hence I is max for $x=100$

(b) If daily output is x sets and P the total profit then

$$P=x(50-\tfrac{1}{2}x)-(\tfrac{1}{4}x^2+35x-25)=-\tfrac{3}{4}x^2+15x-25$$

Then $\frac{dP}{dx}=-\frac{3}{2}x+15$ or $x=10$ for max or min

Also $\frac{d^2P}{dx^2}=-\frac{3}{2}<0$ max Hence for max profit, the daily output must be 10 radio sets

(i) $\frac{x}{1+x\tan x}$ is max when $x=\cos x$

(j) $\sin^p \theta \cos^q \theta$ is max when $\theta=\tan^{-1}\sqrt{(p/q)}$

(k) $\frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}$

(l) $(x-1)^2 e^x$

(m) $\frac{\log x}{x}$ (n) $x^{1/x}$ (o) x^x

(p) $f(x)=1+2\sin x+2\cos^2 x,\ 0 \leqslant x \leqslant \pi/2$ (M.N.R. 88)

2 (a) Divide 20 into two parts such that the product of one part and the cube of the other is a maximum

(b) Divide 10 into two parts such that the sum of twice of one part and square of the other is a minimum

(c) Let x and y be two variables such that $x>0$ and $xy=1$ Find the minimum value of $x+y$ (I.I.T. 81)

3 The perimeter of a rectangle is 100 meters Find the length of its sides when the area is maximum

4 The function $y=\frac{ax+b}{(x-1)(x-4)}$ has turning point at $P(2,-1)$. Find the values of a and b and show that y is maximum at P

5 A particle is moving in a straight line such that distance at any time t is given by

$$s=\tfrac{1}{4}t^4-2t^3+4t^2-7$$

Find when its velocity is maximum and acceleration minimum.

6 (a) What is the maximum slope of the curve

$$y=-x^3+3x^2+2x-27$$

(b) Find the coordinates of the point on the curve $y=\frac{x}{1+x^2}$ where the tangent to the curve has greatest slope (I.I.T. 84)

7 Find the coordinates of a point on the parabola

$$y=x^2+7x+2$$

which is closest to the straight line $y=3x-3$

8 The total area of a page is 150 square inches The combined width of the margin at the top and the bottom is 3″ and the

$$=\frac{\sqrt{3}}{4}x^2+x\left(\frac{16-3x}{2}\right)$$

$$A=8x+\left(\frac{\sqrt{3}}{4}-\frac{3}{2}\right)x^2$$

$$\frac{dA}{dx}=8+\left(\frac{\sqrt{3}}{4}-\frac{3}{2}\right)2x=0$$

or $4-\frac{(6-\sqrt{3})}{4}x=0$

$$x=\frac{16}{6-\sqrt{3}}=\frac{16(6+\sqrt{3})}{36-3}$$

$$=\frac{16(6+1\ 73)}{33}$$

$$=\frac{16(7\ 73)}{33}=\frac{123\ 68}{33}=3\ 75 \text{ nearly}$$

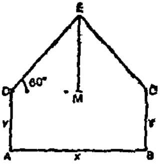


Fig 9

Clearly $\frac{d^2A}{dx^2}=-\text{ive}$ and hence A is maximum

18 $2\pi r+4x=k$, $A=\pi r^2+x^2$ etc

19 Let x be the depth and y the width of the square base so that

$V=y^2x$ (given)

$S=y^2+4\,(xy)$

base sides

$=y^2+4y\ \frac{V}{y^2}=y^2+\frac{4V}{y}$

$\frac{dS}{dy}=2y-\frac{4V}{y^2}$ $\quad y^3=2V$

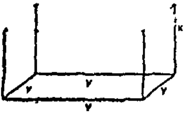


Fig. 10

$\frac{d^2S}{dy^2}=2+\frac{8V}{y^3}=+\text{ive}$ $\quad S$ is minimum when $y^3=2V$

or $y^3=2y^2x$ $\quad y=2x$ or $x=y/2$

i e depth is made half of the width

20 $x^2h=1000$ Top$=x^2$ Base$=x^2$, Sides$=4xh$

$E=15x^2\ +\ 25x^2\ +\ 20\ (4xh)\ +\ 300$

bottom top sides labour

or $E=40x^2+80x\left(\frac{1000}{x^2}\right)+300=40x^2+\frac{80\times1000}{x}+300$

$\frac{dE}{dx}=0$ $\quad 80x-\frac{80\times1000}{x^2}=0$ $\quad x^3=1000$ or $x=10$

$\frac{d^2E}{dx^2}=80+\frac{2\ \ 80\times1000}{x^3}=+\text{ive}$ $\quad$ min when $x=10$

economical speed if the fixed charges, *i e*, salaries etc amount to Rs 300 per hour

13 When travelling x km /hour, a truck burns diesel at the rate of $\frac{1}{300}\left(\frac{900}{x}+x\right)$ litres per km If the diesel oil costs 40 p per litre and driver is paid Rs 1 50 per hour, find the steady speed that will minimise the total cost of the trip of 500 km

14 (*a*) A telephone company in a town has 500 subscribers on its list and collects fixed charges of Rs 300 per subscriber per year The company proposes to increase the annual subscription and it is believed that for every increase of Rs 1/-, one subscriber will discontinue the service Find what increase will bring maximum income to the company

(*b*) The total cost of producing x pocket radio sets per day is Rs $(\frac{1}{4}x^2+35x+25)$ and the price per set at which they may be sold is Rs $(50-\frac{1}{2}x)$ What should be the daily output to obtain a maximum total profit (M N R 83)

15 Assuming the petrol burnt in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3c}{2}$ m p h

16 (*a*) A figure consists of a semi circle with a rectangle on its diameter Given that the perimeter of the figure is 20 feet, find its dimensions in order that its area may be maximum

(*b*) A window in the form of a rectangle is surmounted by a semi circular opening The total perimeter of the window is 10 cm find the dimensions of the rectangular part of the window to admit maximum light through the whole opening (Roorkee 83)

17 The section of a window consists of a rectangle surmounted by an equilateral triangle If the perimeter be given as 16 cm find the dimensions of the window in order that the maximum amount of light may be admitted

18 Given the sum of the perimeter of a square and a circle, show that sum of their areas is least when one side of the square is equal to diameter of the circle

25 $AP = BQ = 6 \sin\alpha$

$DP = QC = 6 \cos\alpha$

A = area of the trapezium

$= \frac{1}{2}[AB + DC] BQ$

$= \frac{1}{2}[6 + 6 + 12\cos\alpha)\, 6 \sin\alpha$

$= 36 (\sin\alpha + \frac{1}{2}\sin 2\alpha)$

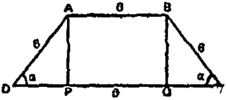


Fig. 11

$\frac{dA}{d\alpha} = 36 (\cos\alpha + \cos 2\alpha) = 0$

$\cos 2\alpha = -\cos\alpha = \cos(\pi - \alpha)$

$3\alpha = \pi$ or $\alpha = \pi/3$

$\frac{d^2A}{d\alpha^2} = 36(-\sin\alpha - 2\sin 2\alpha) = -$ive for $\alpha = \pi/3$

and hence A is max

$A = 6 (\sin 60° + \frac{1}{2}\sin 120°)$

$= 36\left[\frac{\sqrt{3}}{2} + \frac{1}{2}\frac{\sqrt{3}}{2}\right] = 27\sqrt{3}$ sq inch

26 P $(at^2, 2at)$, $L(a, 2a)$

$LM = AM - AL = a - at^2$

A = Area of trapezium $PP\,L\,L$

$= \frac{1}{2}(PP' + LL')\, LM$

$= \frac{1}{2}(4at + 4a)\, a\, (1 - t^2)$

$A = 2a^2 (1 + t)(1 - t^2)$

$= 2a^2 (1 + t - t^2 - t^3)$

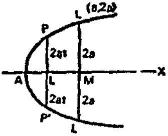


Fig. 12

$\frac{dA}{dt} = 2a^2 (1 - 2t - 3t^2) = 0,$

$2a^2 (1 - 3t)(1 + t) = 0$

$t = \frac{1}{3}, -1$

$\frac{d^2A}{dt^2} = 2a^2(-2 - 6t) = -$ive for $t = \frac{1}{3}$

Hence area is maximum when $t = \frac{1}{3}$

$LA = at^2 = a\,\frac{1}{9} = \frac{a}{9}$ Proved

27 (a) $x^2 + y^2 = a^2$

Let $A(x, y)$ when $x = a\cos\alpha$, $y = a\sin\alpha$

Area $= 2x\, 2y = 4xy = 4a^2 \sin\alpha\cos\alpha$

$A = 2a^2 \sin 2\alpha$

A is clearly maximum when $\sin 2\alpha = 1$

i e $2\alpha = \pi/2$ or $\alpha = \pi/4$

(c) Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse having its vertex coincident with one extremity of major axis

(d) Find the point on the curve $4x^2+a^2y^2=4a^2$, $4<a^2<8$ that is farthest from the point $(0, -2)$ (I I T 87)

28 If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is max when the angle between these is $\pi/3$

29 Find the volume of the greatest right angled triangle of hypotenuse 1 foot about a side

30 The strength of a beam varies as the product of its breadth and square of its depth Find the dimensions of the strongest beam which can be cut from a circular log of radius a

31 Assuming that the stiffness of a beam of rectangular cross section varies as the breadth and as the cube of the depth Prove that for the stiffest beam breadth must be equal to half the diameter of the log

32 Show that a triangle of max area that can be inscribed in a circle of radius r is an equilateral triangle

33 A running track of 440 ft is to be laid out enclosing a football field, the shape of which is a rectangle with a semi-circle at each end If the area of the rectangular portion is to be maximum find the lengths of its sides

34 The perimeter of a rectangle is 100 meters Find the length of its sides when the area is maximum

35 Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible when revolved about one of its sides

36 Find the greatest and least values of the following functions on the indicated intervals ,

(a) $f(x)=2x^3-3x^2-12x+1$ on $[-2, 5/2]$,

(b) $f(x)=x^2 \log x$ on $[1, e]$,

(c) $f(x)=x\,e^{-x}$ on $[0\ \infty]$,

37 A function $y=f(x)$ is represented parametrically as follows

$x=\varphi(t)=t^5-5t^3-20t+7$, $y=\psi(t)=4t^3-3t^2-18t+3$ $(-2<t<2)$

Find the extrema of this function

38 How should a wire 20 cms long be divided into two parts, if one part is to be bent into a circle, the other part is to be bent into a square and the two plane figures are to have

$= (4-a^2)(-2)-8=2(a^2-8)=-\text{ive}$

Hence $z=d^2$ is maximum

28 $AB+AC=\text{constant}=k$

If $AB=x$ then $AC=k-x$

$BC^2=(k-x)^2-x^2=k^2-2kx$

$\Delta=\frac{1}{2}BC\,AB=\frac{1}{2}x\sqrt{(k^2-2kx)}$

Let $Z=\Delta^2=\frac{1}{4}x^2(k^2-2kx)$

$=\frac{1}{4}(k^2x^2-2kx^3)$

Z will be max when $x=k/3$

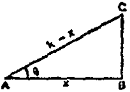


Fig 15

$$\cos\theta=\frac{x}{k-x}=\frac{k/3}{k-k/3}=\frac{1}{2}\qquad \theta=60°$$

29 Refer fig Q 28 let $AC=1$ and if its height be $AB=x$ then radius$=BC=\sqrt{(1-x^2)}=r$

$V=\frac{1}{3}\pi r^2h=\frac{1}{3}\pi(1-x^2)x$ etc.

For maximum volume

$$x=\frac{1}{\sqrt{3}}\text{ and } V=\frac{2\pi}{9\sqrt{3}}\text{ cu ft}$$

30 Let the breadth of the beam be x and depth be y where $x^2+y^2=4a^2$

Strength$=S=kxy^2$ given

or $\quad S=kx(4a^2-x^2)=k(4a^2x-x^3)$

S is max when $x=\dfrac{2a}{\sqrt{3}}$

and therefore $y=2a\sqrt{(\frac{2}{3})}$

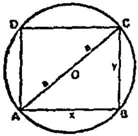


Fig 16

31 Here $S=kxy^3$ where $x^2+y^2=d^2$

$S=kx(d^2-x^2)^{3/2}$

$$\frac{dS}{dx}=k\left[(d^2-x^2)^{3/2}\,1+x\,\tfrac{3}{2}(d^2-x^2)^{1/2}(-2x)\right]$$

$$=k(d^2-x^2)^{1/2}[d^2-x^2-3x^2]=0 \qquad (1)$$

$4x^2=d^2$ or $x=d/2$ as x cannot be $-$ive and $x\neq d$

$$\frac{dS}{dx}=k\sqrt{(d^2-x^2)}(d-2x)(d+2x)\text{ by (1)}$$

If $x<\dfrac{d}{2}$, then $\dfrac{dS}{dx}=+++=+$ive

If $x>\dfrac{d}{2}$, then $\dfrac{dS}{dx}=+-+=-$ive

$\dfrac{dS}{dx}$ changes sign from $+$ive to $-$ive and hence S is maximum

$$x=0,\ 3,\ 1$$

$$\frac{d^2y}{dx^2}=20x^3-60x^2+30x=10x\,(2x^2-6x+3)$$

and $\dfrac{d^3y}{dx^3}=10\,(6x^2-12x+3)$

For $x=0$, $\dfrac{d^2y}{dx^2}=0$ and $\dfrac{d^3y}{dx^3}\neq 0$ neither max nor min.

For $x=3$, $\dfrac{d^2y}{dx^2}=30\,(18-18+3)=90=+$ive min

For $x=1$ $\dfrac{d^2y}{dx^2}=10\,(2-6+3)=-10=-$ive max

(b) Proceed as above $x=1, 5$ At $x=1$ max, $x=5$ min.

(c) At $x=-1$, max value 19 At $x=2$ min value -8

(d) $y=\dfrac{x^2+x+1}{x^2-x+1}=\dfrac{x^2-x+1+2x}{x^2-x+1}=1+2\,\dfrac{x}{x^2-x+1}$

$$\frac{dy}{dx}=2\left[\frac{(x^2-x+1)\,1-x\,(2x-1)}{(x^2-x+1)^2}\right]=-2\,\frac{(x-1)\,(x+1)}{(x^2-x+1)^2}$$

$\dfrac{dy}{dx}=0$ gives $x=1$ and -1

$$\frac{d^2y}{dx^2}=-4x\times\frac{1}{(x^2-x+1)^2}-2\,(x-1)\,(x+1)\frac{d}{dx}\frac{1}{(x^2-x+1)}$$

At $x=1$, $\dfrac{d^2y}{dx^2}<0$ and at $x=-1$, $\dfrac{d^2y}{dx^2}>0$

Hence y is max at $x=1$ and its max value is 3 and y is min at $x=-1$ and its min value is $1/3$

(e) $y=(x-2)^6\,(x-3)^5$ Taking log, we get

$\log y=6\log(x-2)+5\log(x-3)$ Differentiate

$$\frac{1}{y}\frac{dy}{dx}=\frac{6}{x-2}+\frac{5}{x-3}=\frac{11x-28}{(x-2)\,(x-3)}$$

$$\frac{dy}{dx}=\frac{y\,(11x-28)}{(x-2)\,(x-3)}=(x-2)^5\,(x-3)^4\,(11x-28)$$

Now $\dfrac{dy}{dx}=0$ gives $x=2,\ 3,\ \dfrac{28}{11}$

At $x=2$, $\dfrac{dy}{dx}$ changes from $+$ive to $-$ive Max at $x=2$

and max $y=0$

At $x=3$ $\dfrac{dy}{dx}$ changes from $+$ive to $+$ive

Neither min. nor max at $x=3$

34 $2(x+y)=110,\ A=xy=x(50-x)$

A will be maximum when $x=25$, $y=25$ *i e* it is a square

35 $2(x+y)=36$ $x+y=18$

When revolved about side y, it will generate a cylinder of height y and radius x so that $V=\pi r^2h=\pi x^2y$

$V=\pi x^2(18-x)=\pi(18x^2-x^3)$

$\frac{dV}{dx}=\pi(36x-3x^2)$ $x=12$

$\frac{d^2V}{dx^2}=\pi(36-6x)=-$ive when $x=12$ $y=6$

36 (*a*) Here $f'(x)=0$ gives $6x^2-6x-12=0$, whence $x=-1$ or 2 Both these values be inside the interval $[-2, 5/2]$ To find the greatest and least values of $f(x)$ it is necessary to compute its values at $x=-1$ and $x=2$, and also at the end points $x=-2$ and $x=5/2$ Thus

$f(-2)=-3, f(-1)=8\ f(2)=-19\ f(5/2)=-33/2$

Hence the greatest value is $f(-1)=8$ and the least value is $f(2)=-19$

(*b*) Here $f'(x)=x(1+2\log x)=0$ gives $x=0$ or $x=e^{-1/2}$ Since $0<e^{-1/2}<1$, none of these critical points lies in the interval $[1\ e]$ So we only compute the values of $f(x)$ at the end points 1 and e We have

$f(1)=0, f(e)=e^2$

Thus $f(1)=0$ is the least value and $f(e)=e^2$ the greatest value of the function

(*c*) Do yourself The greatest value is $f(1)=1/e$ and the least value is $f(0)=0$

37 We have $\varphi'(t)=5t^4-15t^2-20$

In the interval $]-2, 2[\ \varphi'(t)\neq 0$

We now find $\psi'(t)$ and equate it to zero

Thus $\psi'(t)=12t^2-6t-18=0$ whence $t=-1,\ 3/2$ Both these points lie in the interval $]-2\ 2[$ Again

$\psi''(t)=24t-6,$ $\psi''(-1)=-30<0$

and $\psi''(3/2)=30>0$

Consequently the function $y=f(x)$ has a maximum $y=14$ at $t=-1)$ (*i e* $x=31$) and a min $y=-17\frac{1}{4}$ at $t=3/2$ (*i e* at $x=-1033/32$)

is max or min

$$z=p \log \sin \theta+q \log \cos \theta$$

$$\frac{dz}{d\theta}=p \frac{1}{\sin \theta} \cos \theta+\frac{q}{\cos \theta}(-\sin \theta)=0$$

or $\quad p \cot \theta-q \tan \theta=0$

$$\tan^2 \theta=p/q \quad \text{or} \quad \theta=\tan^{-1} \sqrt{(p/q)}$$

$$\frac{d^2z}{d\theta^2}=-p \operatorname{cosec}^2 \theta-q \sec^2 \theta$$

which is clearly $-$ive when $\tan^2 \theta=p/q$ and hence z is max.

or $\log y$ is max. or y is max

$$\frac{d^2z}{d\theta^2}=-p\,(1+\cot^2 \theta)-q\,(1+\tan^2 \theta)$$

$$=-p\left(1+\frac{q}{p}\right)-q\left(1+\frac{p}{q}\right)$$

$$=-p-q-q-p=-2\,(p+q)$$

We need not calculate the value of $\frac{d^2z}{d\theta^2}$ as we are only concerned with its sign

(k) $y=\frac{a^2}{\cos^2 x}+\frac{b^2}{\sin^2 x}=a^2 \sec^2 x+b^2 \operatorname{cosec}^2 x$

Ist Method

$$\frac{dy}{dx}=2a^2 \sec x \sec x \tan x-2b^2 \operatorname{cosec} x \operatorname{cosec} x \cot x=0$$

$$=2\,[a^2 \sec^2 x \tan x-b^2 \operatorname{cosec}^2 x \cot x]=0$$

$$a^2 \frac{\sin x}{\cos^3 x}-b^2 \frac{\cos x}{\sin^3 x}=0$$

or $\quad \tan^4 x=b^2/a^2 \qquad \tan^2 x=b/a$

$$\frac{d^2y}{dx^2}=2\,[a^2 (\sec^4 x+2 \sec x \sec x \tan x \tan x)$$

$$+b^2 (\operatorname{cosec}^4 x+2 \operatorname{cosec} x \operatorname{cosec} x \cot x \cot x)]$$

Above is clearly $+$ive when $\tan^2 x=b/a$ and hence y is min

Putting $\tan^2 x=b/a$ in the value of y

$$y=a^2 \sec^2 x+b^2 \operatorname{cosec}^2 x=a^2 (1+\tan^2 x)+b^2 (1+\cot^2 x)$$

$$=a^2 (1+b/a)+b^2 (1+a/b)=a^2+b^2+2ab=(a+b)^2$$

Hence the min. value of

$$y=a^2 \sec^2 x+b^2 \operatorname{cosec}^2 x=(a+b)^2$$

2nd Method (without differentiation)

$$y=a^2 \sec^2 x+b^2 \operatorname{cosec}^2 x$$

$$=a^2 (1+\tan^2 x)+b^2 (1+\cot^2 x)$$

(i) Take $x=1$
Now $f'(1-h)=(1-h-1)(1-h-2)^2\{5(1-h)-7\}$
$=-h(1+h)^2(-2-5h)$
$=h(1+h)^2(2+5h)>0$ for small h
and $f(1+h)=(1+h-1)(1+h-2)^2\{5(1+h)-7\}$
$=h(h-1)^2(5h-2)<0$ for small h

$f(x)$ increasing on the left of $x=1$ and decreasing on the right of $x=1$

Hence $f(x)$ is max at $x=1$

(ii) Take $x=2$
$f(2+h)=(2+h-1)(2+h-2)^2\{5(2+h)-7\}$
$=(1+h)h^2(3+5h)>0$ for small h
and $f(2-h)=(2-h-1)(2-h-2)^2\{5(2-h)-7\}$
$=(1-h)h^2(3-5h)>0$ for small h

Hence $f(x)$ is increasing both on left and on right of $x=2$
Therefore there is no maxima or minima at $x=2$

(iii) We now take $x=7/5$ We have
$f''(x)=(x-1)(x-2)^2 5+(5x-7)\,d/dx\,(x-1)(x-2)$
$=(7/5-1)(7/5-1)^2 5+0>0$ at $x=7/5$

Hence $f(x)$ is min at $x=7/5$

Ans Max at $x=1$, neither max nor min at $x=2$ and min at $x=7/5$

Problem Set (B)

1 (a) A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore The ratio of his rate of walking to his rate of rowing is $\sec\alpha$ Prove that he should land at a distance $(b-a\cot\alpha)$ from the place to be reached

(b) A swimmer S is in the sea at a distance d km from the closest point A on a straight shore The house of the swimmer is on the shore at a distance L km from A He can swim at a speed of U km /hrs and walk at a speed of V km /hr At what point on the shore should he land so that he reaches his house in the shortest possible time
(I I T 83)

2 A lane runs at right angles out of a road a feet wide Find how many feet wide the lane must be if it is just possible to

(p) We have $f'(x)=2\cos x-4\cos x\sin x=0$

for max or min This gives $\cos x=0$

or $\sin x=1/2$ The solution in the interval $0 \leqslant x \leqslant \pi/2$

is $\pi/2$ and $\pi/6$ respectively

Now $f''(x)=-2\sin x-4\cos 2x$

> 0 at $x=\pi/2$

and < 0 at $x=\pi/6$

Hence $f(x)$ is min at $x=\pi/2$

and max at $x=\pi/6$

2 (a) $x+y=20$ and $z=xy^3$ is max

$z=y^3(20-y)=20y^3-y^4$

$\frac{dz}{dy}=60y^2-4y^3=0 \quad 4y^2(15-y)=0 \quad y=0, 15$

$\frac{d^2z}{dy^2}=120y-12y^2=12y(10-y)=$ —ive i e max when $y=15$

Hence the two parts are 5 and 15

(b) Ans 9, 1

(c) $x=1, y=1$, min value$=2$

3 $P=2(x+y)=100 \quad x+y=50,$

$A=xy=x(50-x)$ etc

$x=25, y=25$ i e rectangle is a square

4 $y=\frac{ax+b}{(x-1)(x-4)}$

$P(2, -1)$ lies on it $\quad 2a+b=2$

$\frac{dy}{dx}=\frac{(x^2-5x+4)a-(ax+b)(2x-5)}{(x^2-5x+4)^2}$

Since $P(2 \ -1)$ is a turning point i e a point at which max and min occurs

$\frac{dy}{dx}=0$ at $(2, -1)$

$\frac{(4-10+4)a-(2a+b)(4-5)}{(4-10+4)^2}=0$

or $-2a+2a+b=0 \quad b=0$

Hence from (1) $a=1$

$\frac{dy}{dx}=\frac{(x^2-5x+4)\ 1-x(2x-5)}{(x^2-5x+4)^2}=\frac{4-x^2}{(x^2-5x+4)^2}$ $a=1, c=0$

$\frac{dy}{dx}=\frac{(2-x)(2+x)}{(x^2-5x+4)^2}$

2 Find the cylinder of greatest volume which can be inscribed in a cone

3 Show that the radius of the right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone

4 Show that the right circular cylinder of given surface and max volume is such that its height is equal to the diameter of the base

5 (a) An open cylindrical can of given capacity is to be made from a metal sheet of uniform thickness If no allowance is to be made for waste meterial, what will be the most economical ratio of the radius to the height of the can

(b) A cylindrical vessel of volume $25\frac{1}{7}$ cubic meters, which is open at the top is to be manufactured from a sheet of metal Find the dimension of the vessel so that the amount of sheet used in manufacturing it is the least possible (Roorkee 84)

6 (a) A cylindrical gas container is closed at the top and open at the bottom if the iron plate of the top is 5/4 times as thick as the plate forming the cylindrical sides find the ratio of the radius to the height of the cylinder using minimum material for the same capacity

(b) A manufacturer plans to construct a cylindrical can to hold one cubic meter of liquid If the cost of constructing the top and bottom of the can is twice the cost of constructing the sides what are the dimension of the most economical can ? (Roorkee 82)

7 A given quantity of metal is to be cast into a half cylinder, i e with a rectangular base and semi circular ends Show that in order that the total surface area may be minimum the ratio of the height of the cylinder to the diameter of the semi circular ends is $\frac{\pi}{\pi+2}$

8 A thin closed rectangular box is to have one rectangular edge of n times the length of another edge and the volume of the box is given to be V Prove that the least surface S is given by

$$nS^3=54\,(n+1)^2\,V^2$$

9 A tree trunk l feet long is in the shape of a frustum of a cone the radii of its ends being a and b $(a>b)$. It is required to

$$D=\left|\frac{3x-y-3}{\sqrt{(10)}}\right|=\left|\frac{3x-(x^2+7x+2)-3}{\sqrt{(10)}}\right|=\left|\frac{-x^2-4x-5}{\sqrt{(10)}}\right|$$

or $$D=\left|\frac{x^2+4x+5}{\sqrt{(10)}}\right|=\left|\frac{(x+2)^2+1}{\sqrt{(10)}}\right|=\frac{(x+2)^2+1}{\sqrt{(10)}} \text{ as } \frac{N^r}{D^r} \text{ is +ve}$$

$$\frac{dD}{dx}=\frac{2(x+2)}{\sqrt{(10)}}=0, \qquad x=-2$$

and hence y is -8 i e point is $(-2, -8)$

$$\frac{d^2D}{dx^2}=\frac{2}{\sqrt{(10)}}=+\text{ive}$$

and hence min at $(-2, -8)$

8 Let the dimensions of the page be x (length) and y breadth. After leaving the margins these dimensions are (1)

$$x-3 \text{ and } y-2 \quad \text{where } xy=150$$

Area A of printed matter is given by

$$A=(x-3)(y-2)=xy-3y-2x+6$$

or $$A=150-\frac{450}{x}-2x+6 \text{ by (1)}$$

Rest as before, $x=15$ for A to be minimum

$y=10$ by (1)

9 (a) Let P be the pumping station at a distance x from A the point nearest to the town T_1. Also $AB=10$

$PB=10-x$

Let p be the length of the pipe required

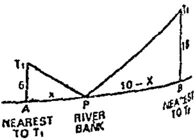


Fig. 5

$$p=T_1P+T_2P$$

$$=\sqrt{(36+x^2)}+\sqrt{\{18^2+(10-x)^2\}} \quad (1)$$

We want p to be minimum

$$\frac{dp}{dx}=\frac{2x}{2\sqrt{(36+x^2)}}-\frac{2(10-x)}{2\sqrt{\{18^2+(10-x)^2\}}}=0$$

$$\frac{x}{\sqrt{(36+x^2)}}=\frac{10-x}{\sqrt{\{18^2+(10-x)^2\}}} \quad \text{square}$$

$$x^2\{18^2+(10-x)^2\}=(10-x)^2(36+x^2)$$

$$18^2x^2=(10-x)^2[36+x^2-x^2]=36(10-x)^2$$

$$18x=\pm 6(10-x)$$

or $$18x=60-6x \text{ or } 18x=-60+6x$$

or $$24x=+60 \text{ or } 12x=-60$$

$$x=5/2 \text{ or } -5$$

28 Prove that the min radius vector of the curve

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=1 \text{ is of length } a+b$$

29 Find the max and min radii vectors of the curve

$$\frac{c^4}{r^2}=\frac{a^2}{\sin^2 t}+\frac{b^2}{\cos^2 t}$$

30 If $xy\,(y-x)=2a^3$, show that y has a minimum when $x=a$. Determine the minimum value. Show that y has a second value at $x=a$ which is less than the minimum. How do you explain this paradox

31 Prove that the minimum value of $\dfrac{(a+x)\,(b+x)}{(c+x)}$, $x > -c$ is

$$\{\sqrt{(a-c)}+\sqrt{(b-c)}\}^2 \qquad \text{(I I T 79)}$$

32 Let $f(x)=\sin^3 x+\lambda \sin^2 x$, $-\pi/2 < x < \pi/2$. Find the intervals in which λ should lie in order that $f(x)$ has exactly one minimum and one maximum (I I T 85)

33 Let $A\,(p^2, -p)$, $B\,(q^2, q)$, $C\,(r^2, -r)$ be the vertices of the triangle ABC. A parallelogram $AFDE$ is drawn with D, E and F on the line segmets BC, CA and AB respectively. Using calculus show that maximum area of such a parallelogram is

$\frac{1}{4}\,(p+q)\,(q+r)\,(p-r)$ (I I T 86)

34 Find all the values of the parameter a for which the point of minimum of the function $f(x)=1+a^2x-x^3$ satisfy the inequality $\dfrac{x^2+x+2}{x^2+5x+6} < 0$

35 Fo what value of a does the function $f(x)=x^3+3\,(a-7)\,x^2+3\,(a^2-9)\,x-1$ have a positive point of maximum ?

Solutions to Problem Set (B)

1 (a) Suppose he lands at D a point at a distance x from B or at distance $b-x$ from the place to be reached i e C. Thus he will have to row a distance $\sqrt{(a^2+x^2)}$ and walk a distance $b-x$ along the shore to reach C. If he rows at the rate of k miles per hour then he will walk at the rate of $k \sec \alpha$ m p h. If T be the time taken then

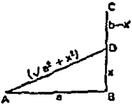


Fig. 13

$T=t_1$ for rowing $+\,t_2$ for walking

(b) $PQ^2 = OP^2 + OQ^2 - 2OP\ OQ\ \cos 60°$
PQ will be minimum when $z = PQ^2$ is minimum

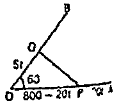


Fig 7

$$PQ^2 = z = (800-20t)^2 + 25t^2 - 2(800-20t)(5t)\tfrac{1}{2}$$

or $z = (800-20t)^2 + 25t^2 - 5(800t - 20t^2)$

$$\frac{dz}{dt} = -40(800-20t) + 50t - 5(800-40t) = 0$$

$$1050t - 36000 = 0$$

$$t = \frac{36000}{1050} = \frac{240}{7}$$ 13

Clearly $\frac{d^2z}{dt^2} = 1050 = +$ive and hence z is min when

$$t = \frac{240}{7} \text{ sec}$$

11 $S = kx^2 \log \frac{1}{x} = -kx^2 \log x$

$\frac{dS}{dx} = -k\left[x^2 \cdot \frac{1}{x} + 2x \log x\right] = 0$ or $-kx(1 + 2\log x) = 0$

$x = 0$ or $\log x = -\frac{1}{2}$ $\quad x = e^{-1/2} = 1/\sqrt{e}$ 14

$$\frac{d^2S}{dx^2} = -k\left(1 + 2\log x + x \cdot \frac{2}{x}\right)$$

$$= -k(3 + 2\log x) = -k(3-1) = -2k = -\text{ive}$$

and hence S is maximum when $x = 1/\sqrt{e}$
Hence the required ratio x is $1 : \sqrt{e}$ for speed to be max

12 Let the speed of the train be v and distance to be covered b ... (1)
so that total time taken is ...
Cost of fuel per hour $= kv^2$ ...)
Also 48 ... by given co... $k = 3/16$
Cost o... ...ur $= \frac{3}{16}$
Other c... is 300

$$\sin\alpha=(1-\cos^2\alpha)^{1/2}=\frac{\sqrt{(b^{2/3}-a^{2/3})}}{b^{1/3}}$$

$$\tan\alpha=\frac{\sqrt{(b^{2/3}-a^{2/3})}}{a^{1/3}}$$

$y=b\sin\alpha-a\tan\alpha$ by (1)

$$=\sqrt{(6^{2/3}-a^{2/3})}\left[b\,\frac{1}{b^{1/3}}-a\,\frac{1}{a^{1/3}}\right]$$

$$=\sqrt{(b^{2/3}-a^{2/3})}\,[b^{2/3}-a^{2/3}]=[b^{2/3}-a^{2/3}]^{3/2}$$ Proved

3 Refer figure of Q 2 Here road is a feet wide and lane is b feet wide *i e*, $CE=a$ and $CD=b$

Let x be the length of the ladder, then

$$x=AC+CB=a\sec\alpha+b\,\mathrm{cosec}\,\alpha$$

For max or min value of x

$$\frac{dx}{d\alpha}=a\sec\alpha\tan\alpha-b\,\mathrm{cosec}\,\alpha\cot\alpha=0$$

$$\tan^3\alpha=b/a$$

$$\frac{d^2x}{d\alpha^2}=a\sec^3\alpha+b\,\mathrm{cosec}^3\alpha+(a\sec\alpha\tan^2\alpha+b\,\mathrm{cosec}\,\alpha\cot^2\alpha)$$

$$=+\text{ive when }\tan^3\alpha=\frac{b}{a}$$

Hence the minimum length of the ladder AB is attained for

$$\tan\alpha=\frac{b^{1/3}}{a^{1/3}}\text{ or }\sin\alpha=\frac{b^{1/3}}{\sqrt{(a^{2/3}+b^{2/3})}},\ \cos\alpha=\frac{a^{1/3}}{\sqrt{(a^{2/3}+b^{2/3})}}$$

Since the ladder is just to pass through the two roads without jamming, its length should not exceed the minimum value of AB

Therefore the length of largest ladder in order to pass it without jamming is

$$x=a\sec\alpha+b\,\mathrm{cosec}\,\alpha\quad\text{where }\tan\alpha=\frac{b^{1/3}}{a^{1/3}}$$

or $$x=a\,\frac{\sqrt{(a^{2/3}+b^{2/3})}}{a^{1/3}}+b\,\frac{\sqrt{(a^{2/3}+b^{2/3})}}{b^{1/3}}$$

$$=\sqrt{(a^{2/3}+b^{2/3})}\,(a^{2/3}+b^{2/3})=(a^{2/3}+b^{2/3})^{3/2}$$

4 Let $y=OP$ and $\angle POQ=\theta$ Then from the figure it is clear that $y=6\,\mathrm{cosec}\,\theta+4\sec\theta$

$$\frac{dy}{d\theta}=-6\,\mathrm{cosec}\,\theta\cot\theta+4\sec\theta\tan\theta=0\text{ for max or min}$$

or $\tan^3\theta=6/4=3/2$ $\qquad\tan\theta=3^{1/3}/2^{1/3}$

15 When the meter of the boat shows a speed of v m.p.h then consumption of petrol is λv^3 per hour where k is a constant. If s be the distance to be covered then it will be covered with effective speed of $v-c$ as it is going against a current of c miles per hour. Hence the total time is $s/(v-c)$ hours.

P = petrol burnt for the entire journey is

$$\frac{s}{v-c}\ (\lambda v^3)$$

$$P=\lambda s\,\frac{v^3}{v-c}$$

Now P will be minimum if $z=1/P$ is maximum

$$z=\frac{1}{ks}\ \frac{v-c}{v^3}=\frac{1}{ks}\left[\frac{1}{v^2}-\frac{c}{v^3}\right]$$

$$\frac{dz}{dv}=\frac{1}{\lambda s}\left[-\frac{2}{v^3}+\frac{3c}{v^4}\right]=0 \qquad v=\frac{3c}{2}$$

$$\frac{d^2z}{dv^2}=\frac{1}{ks}\left[\frac{6}{v^4}-\frac{12c}{v^5}\right]=\frac{6}{\lambda s\,v^4}\left[1-\frac{2c}{v}\right]=-\text{ve for } v=\frac{3c}{2}$$ 18 19

Hence $z=\dfrac{1}{P}$ is maximum so that P is minimum for $v=\dfrac{3c}{2}$

16 (a) Perimeter of the figure = 20

or $2x+2r+\frac{1}{2}(2\pi r)=20$

or $2x=20-\pi r-2r$ (1)

$A=\frac{1}{2}\pi r^2+2r.x$

$=\frac{1}{2}\pi r^2+r(20-\pi r-2r)$ by (1)

or $A=20r-\frac{1}{2}\pi r^2-2r^2$

For max value of A,

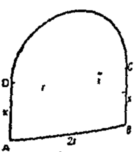


Fig 1

$$\frac{dA}{dr}=20-\pi r-4r=0, \qquad r=\frac{20}{\pi+4}$$

$$\frac{d^2A}{dr^2}=-\pi-4=-\text{ve} \qquad A \text{ is max}$$ 20

$$r=\frac{20}{\pi+4} \text{ and } x=\frac{40}{\pi+4} \qquad \text{by (1)}$$

(b) Proceed as in (a)

Ans length $=\dfrac{20}{\pi+4}$, breadth $=\dfrac{10}{\pi+4}$

$P=2y+3x=16$ given

$A=xy+\frac{1}{2}x\cdot x\sin 60°$

Proceed as above

Refer Q 24 set A, Page 580

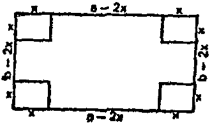


Fig. 21

From the figure it is clear that dimensions of the box are

$L=a-2x,\ B=b-2x,\ H=x$

$V=L\times B\times H$

$=(a-2x)(b-2x)\,x$

or $V=4x^3-2x^2(a+b)+abx$

$a>b$

$$\frac{dV}{dx}=12x^2-4x(a+b)+ab=0$$

$$x=\frac{4(a+b)\pm\sqrt{(16(a+b)^2-48ab)}}{24}=\frac{1}{6}[(a+b)\pm\sqrt{(a^2+b^2-ab)}]$$

Now since $a>b$, the value of x taking +ive sigh with the radical will be greater than

$\frac{1}{6}[(b+b)+\sqrt{(b^2-b^2+b^2)}]$ or $\frac{1}{6}3b$ or $b/2$

or $2x>b$ or $2x-b=$+ive or $b-2x$ is −ive

This is not possible as $b-2x$ is breadth Hence we shall consider the −ive sign with the radical,

$$x=\frac{1}{6}[(a+b)-\sqrt{(a^2+b^2-ab)}] \quad (1)$$

$$\frac{d^2V}{dx^2}=24x-4(a+b)$$

$=4[(a+b)-\sqrt{(a^2+b^2-ab)}-(a+b)]$

$=-4\sqrt{(a^2+b^2-ab)}$

i e −ive and hence V is max

Note $a^2+b^2-ab=(a-b)^2+ab$ i e +ive showing that the value of x given by (1) is real

Square $a=b$, then from (1) $x=\frac{1}{6}(2a-a)=a/6$

8 $S=\pi r(r+l)=$constant (1)

$V=\frac{1}{3}\pi r^2h$

$z=V^2=\frac{1}{9}\pi^2r^4h^2$

or $z=\frac{1}{9}\pi^2r^4(l^2-r^2)$

Here z is a function of two variables r and l and we will eliminate l with the help of (1) in order to make z(i e V^2) a function of a single variable From (1)

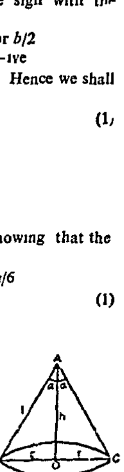


Fig. 22

$$\left(\frac{S}{\pi r}-r\right)=l$$

$$\therefore \quad h=\frac{1000}{x^2}=10$$

Hence the box should be a cube of edge 10 feet

21 $x^2h=250$ Area of base $=x^2$, cost of land $=50x^2$
Cost of digging $=400\ h^2$

$$E=400h^2+50x^2=400\ h^2+50\ \frac{250}{h} \text{ etc,}$$

$$\therefore \quad h=2\ 5 \quad \text{and} \quad x=10$$

22 Dimensions are x breadth, $2x$ length, h height

$$V=x\ 2x\ h=c$$

Area of bottom $=2x^2=$ Area of top
and Area of sides $=2xh+2xh+xh+xh=6xh$
If R rupees be the cost of material for bottom then for the top and sides is $3R$

$$E=R\ (2x^2)+3R\ (2x^2+6xh)=R\ (8x^2+18xh)$$

or
$$E=R\left(8x^2+18x\ \frac{c}{2x^2}\right)=R\left(8x^2+\frac{9c}{x}\right)$$

where R and c are constants

$$\frac{dE}{dx}=R\left(16x-\frac{9c}{x^2}\right) \qquad x=\left(\frac{9c}{16}\right)^{1/3}$$

$$\frac{d^2E}{dx^2}=R\left(16+\frac{18c}{x^3}\right)=+\text{ive and hence minimum}$$

dimensions are

$$\left[\frac{9c}{16}\right]^{1/3},\ 2\left[\frac{9c}{16}\right]^{1/3} \text{ and } \left[\frac{32c}{81}\right]^{1/3}$$

23 $x^2+4hx=40$ $\qquad V=x^2h$ etc

$$x=2\sqrt{\frac{10}{3}},\ h=\sqrt{\frac{10}{3}}$$

24 The dimensions of the box after cutting equal squares of side x on the corner will be

$21-2x$ $16-2x$ and height x

$$V=x\ (21-2x)\ (16-2x)=x\ (336-74x+4x^2)$$

or
$$V=4x^3-74x^2+336x$$

$$\frac{dV}{dx}=12x^2-148x+336=0$$

or
$$3x^2-37x+84=0$$

$$(x-3)\ (3x+28)=0 \qquad x=3$$

$$\frac{d^2V}{dx^2}=6x-37=18-37=-19=-\text{ive for } x=3$$

Hence V is max when $x=3$

or $\qquad z=\pi^2r^2\left[r^2+\dfrac{9V^2}{\pi^2r^4}\right]=\pi^2r^4+\dfrac{9V^2}{r^2}$

$\dfrac{dz}{dr}=4\pi^2r^3-\dfrac{18V^2}{r^3} \qquad r^6=\dfrac{9V^2}{2\pi^2}$

$\dfrac{d^2z}{dr^2}=12\pi^2r^2+\dfrac{54V^2}{r^4}=\dfrac{12\pi^2r^6+54V^2}{r^4}$

$=\dfrac{6\,(9V^2)+54V^2}{r^2}=+\text{ive when } r^6=\dfrac{9V^2}{2\pi^2}$

Hence z i e S^2 is minimum or S is minimum

When $2\pi^2r^6=9V^2$ or $2\pi^2r^6=9\ \frac{1}{9}\pi^2r^4h^2$

or $2r^2=h^2$ or $h=r\sqrt{2}$

Another form

$l=\sqrt{(r^2+h^2)}=\sqrt{(r^2+2r^2)}=r\sqrt{(3)}$

$\sin\alpha=r/l=1/\sqrt{3}$ or $\alpha=\sin^{-1}1/\sqrt{(3)}$

1 $S=\pi rl$ (given) where S is curved surface only

$V=\frac{1}{3}\pi r^2h \qquad z=V^2=\frac{1}{9}\pi^2r^4\,(l^2-r^2)$

$z=\dfrac{1}{9}\pi^2r^4\left[\dfrac{S^2}{\pi^2r^2}-r^2\right]=\dfrac{1}{9}\,[S^2r^2-\pi^2r^6]$

$\dfrac{dz}{dr}=\dfrac{1}{9}\,[2S^2\,r-6\pi^2r^5] \qquad r^4=\dfrac{1}{3}\dfrac{S^2}{\pi^2}$ or $r=0$

$\dfrac{d^2z}{dr^2}=\dfrac{1}{9}\,[2S^2-30\pi^2r^4]=\dfrac{1}{9}\,[2\,S^2-10S^2]=-\text{ive}$ hence max when

$r^4=\dfrac{S^2}{3\pi^2}$ or $3\pi^2r^4=\pi^2r^2l^2 \qquad \dfrac{r}{l}=\dfrac{1}{\sqrt{3}}$ or $\sin\alpha=\dfrac{1}{\sqrt{3}}$

or $\qquad \alpha=\sin^{-1}\dfrac{1}{\sqrt{(3)}}$

2 Let b be the height of the cone and α be its semi vertical angle $LD=x=$radius of the inscribed cylinder and $LM=h$ be its height

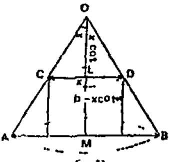


Fig 21

$LM=OM-OL=b-x\cot\alpha$

$V=$volume of cylinder$=\pi r^2h$

$=\pi x^2\,(b-x\cot\alpha)$

$V=\pi\,(bx^2-x^3\cot\alpha)$

$\dfrac{dV}{dx}=\pi\,(2bx-3x^2\cot\alpha)$

$x=0$ or $x=\frac{2}{3}\,b\tan\alpha$.

Clearly $x=0$ is inadmissible and hence we consider the value $x=\frac{2}{3}\,b\tan\alpha$

In that case

$x = a \cos 45° = a/\sqrt{2}$

$y = a \sin 45° = a/\sqrt{2}$

i.e. rectangle is a square

(b) Proceed as above

Ans length $= \sqrt{2}r$ and breadth $= r/\sqrt{2}$

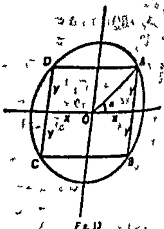


Fig. 13

(c) Area $= \frac{1}{2} PP' \cdot AM$

$= \frac{1}{2}(2b \sin\theta)(a - a\cos\theta)$

or $A = ab\left(\sin\theta - \frac{1}{2}\sin 2\theta\right)$

$$\frac{dA}{d\theta} = ab(\cos\theta - \cos 2\theta) = 0$$

$\cos 2\theta = \cos\theta$ or $2\theta = 2\pi - \theta$

$\theta = 2\pi/3 = 120°$

$P'(a\cos\theta, -b\sin\theta)$

$$\frac{d^2A}{d\theta^2} = ab(-\sin\theta + 2\sin 2\theta)$$

$= ab(-\sin 120° + 2\sin 240°)$

$= ab\left[-\frac{\sqrt{3}}{2} + 2\left\{-\frac{\sqrt{3}}{2}\right\}\right]$

$= -$ive

and hence A is max

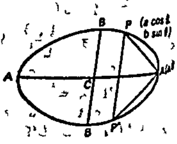


Fig. 14

$A = ab(\sin 120° - \frac{1}{2}\sin 240°)$

$$= ab\left[\frac{\sqrt{3}}{2} - \frac{1}{2}\left\{-\frac{\sqrt{3}}{2}\right\}\right] = \frac{3\sqrt{3}}{4}ab$$

(b) The given equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which represents an ellipse on which any point may be taken as $(a\cos\phi, 2\sin\phi)$. If d be its distance from $(0, -2)$ then let 31

$z = d^2 = a^2\cos^2\phi + 4(1 + \sin\phi)^2$

$$\frac{dz}{d\phi} = -2a^2\cos\phi\sin\phi + 8(1 + \sin\phi)\cos\phi = 0 \quad \ldots(1)$$

$= (4 - a^2)\sin 2\phi + 8\cos\phi$

from (1) we get $\cos\phi = 0$ or $\sin\phi = \frac{4}{a^2 - 4} = \frac{1}{\frac{a^2}{4} - 1} > 1$

by the given condition $4 < a^2 < 8$ and this value is rejected.

We choose $\cos\phi = 0$, $\phi = \pi/2$, so the point becomes $(0, 2)$

Also $\frac{d^2z}{d\phi^2} = (4 - a^2)\, 2\cos 2\phi - 8\sin\phi \ldots$

or $\qquad z=\pi^2r^2\left[r^2+\frac{9V^2}{\pi^2r^4}\right]=\pi^2 r^4+\frac{9V^2}{r^2}$

$\frac{dz}{dr}=4\pi^2r^3-\frac{18V^2}{r^3} \qquad r^6=\frac{9V^2}{2\pi^2}$

$\frac{d^2z}{dr^2}=12\pi^2r^2+\frac{54V^2}{r^4}=\frac{12\pi^2r^6+54V^2}{r^4}$

$=\frac{6\,(9V^2)+54V^2}{r^4}=+\text{ive when } r^6=\frac{9V^2}{2\pi^2}$

Hence z *i e* S^2 is minimum or S is minimum

When $2\pi^2r^6=9V^2$ or $2\pi^2r^6=9\ \frac{1}{9}\pi^2r^4h^2$

or $2r^2=h^2$ or $h=r\sqrt{2}$

Another form

$l=\sqrt{(r^2+h^2)}=\sqrt{(r^2+2r^2)}=r\sqrt{(3)}$

$\sin\alpha=r/l=1/\sqrt{3}$ or $\alpha=\sin^{-1}1/\sqrt{(3)}$

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$V=\frac{1}{3}\pi r^2h \qquad z=V^2=\frac{1}{9}\pi^2r^4\,(l^2-r^2)$

$z=\frac{1}{9}\pi^2r^4\left[\frac{S^2}{\pi^2r^2}-r^2\right]=\frac{1}{9}\,[S^2r^2-\pi^2r^6]$

$\frac{dz}{dr}=\frac{1}{9}\,[2S^2\,r-6\pi^2r^5] \qquad r^4=\frac{1}{3}\frac{S^2}{\pi^2}$ or $r=0$

$\frac{d^2z}{dr^2}=\frac{1}{9}\,[2S^2-30\pi^2r^4]=\frac{1}{9}\,[2\,S^2-10S^2]=-\text{ive}$ hence max when

$r^4=\frac{S^2}{3\pi^2}$ or $3\pi^2r^4=\pi^2r^2l^2 \qquad \frac{r}{l}=\frac{1}{\sqrt{3}}$ or $\sin\alpha=\frac{1}{\sqrt{3}}$

or $\qquad \alpha=\sin^{-1}\frac{1}{\sqrt{(3)}}$

2 Let b be the height of the cone and α be its semi vertical angle $LD=x=$radius of the inscribed cylinder and $LM=h$ be its height

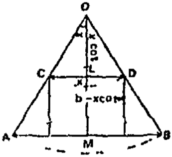


Fig 23

$LM=OM-OL=b-x\cot\alpha$

$V=$volume of cylinder$=\pi r^2h$

$=\pi x^2\,(b-x\cot\alpha)$

$V=\pi\,(bx^2-x^3\cot\alpha)$

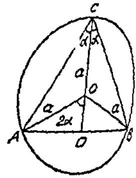
$\frac{dV}{dx}=\pi\,(2bx-3x^2\cot\alpha)$

$x=0$ or $x=\frac{2}{3}\,b\tan\alpha$

Clearly $x=0$ is inadmissible and hence we consider the value $x=\frac{2}{3}\,b\tan\alpha$

when $x=\frac{d}{2}$ i e breadth$=\frac{1}{2}$ diameter

32 Let AB be one of the sides and for the vertex C, it should be in such a position that the altitude CD is max for greatest area. Hence C is as shown in the figure Clearly such a triangle is isosceles Now in order to prove it to be equilateral we must show that



$\angle C=2\alpha=60^\circ \qquad \alpha=30^\circ$

$\triangle ABC=\triangle AOC+\triangle BOC+\triangle AOB$

$=2\,[\frac{1}{2}\,a\,a\sin(\pi-2\alpha)]+\frac{1}{2}\,a\,a\sin 4\alpha$

$A=a^2\,(\sin 2\alpha+\frac{1}{2}\sin 4\alpha)$

$\frac{dA}{d\alpha}=2a^2\,[\cos 2\alpha+\cos 4\alpha]=0$

$\cos 4\alpha=-\cos 2\alpha=\cos(\pi-2\alpha)$

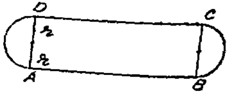
$6\alpha=\pi \quad$ or $\quad \alpha=\pi/6$

Clearly $\frac{d^2A}{d\alpha^2}=2a^2\,(-2\sin 2\alpha-4\sin 4\alpha)=-$ive $\quad A$ is max

Since $\alpha=\pi/6$ or $2\alpha=\angle C=60^\circ$ and triangle being isosceles and hence it is an equilateral triangle

33 Perimeter$=440$ ft

$2x+\pi r+\pi r=440 \quad$ or $\quad 2x+2\pi r=440$ (i)



$A=$ Area of rectangular portion$=x\,2r$

$A=x\,\frac{(440-2x)}{\pi}=\frac{1}{\pi}\,(440x-2x^2)$

$\frac{dA}{dx}=\frac{1}{\pi}\,(440-4x) \qquad x=110$

$\frac{d^2A}{dx^2}=-$ive $\qquad A$ is max when $x=110$

$2r=\frac{440-2x}{\pi}=\frac{440-220}{22/7}=70$

Hence the lengths of the sides are 110 and 70' for max. area

$$\frac{dS}{dr}=-\frac{2V}{r^2}+8r=0 \text{ or } 4r^3=V$$

$$\frac{d^2S}{dr^2}=\frac{4V}{r^3}+8=4\cdot4+8=24=+\text{ive}$$, hence minimum

when $4r^3=V=\pi r^2h$ $\quad \frac{r}{h}=\frac{\pi}{4}$ is the required ratio

(b) Do yourself Ans $r=h=2$ meter

16 $V=\pi r^2h$

If k be the thickness of the sides then that of the top will be $\frac{5}{4}k$

$$S=(2\pi rh)\,k+(\pi r^2)\,\tfrac{5}{4}k$$

or $$S=2\pi rk\,\frac{V}{\pi r^2}+\frac{5}{4}\pi r^2k=k\left(\frac{2V}{r}+\frac{5}{4}\pi r^2\right)$$

Note We have taken into consideration the actual material used and not as in last question in which the base was made from a square sheet thus wasting the material

$$\frac{dS}{dr}=k\left(\frac{-2V}{r^2}+\frac{5}{2}\pi r\right), \qquad r^3=\frac{4V}{5\pi}$$

$$\frac{d^2S}{dr^2}=k\left(\frac{4V}{r^3}+\frac{5}{2}\pi\right)=\pi\left(5\pi+\frac{5}{2}\right)=+\text{ive}$$

When $r^3=\frac{4V}{5\pi}$ or $5\pi r^3=4\pi r^2h \quad \frac{r}{h}=\frac{4}{5}$

(b) *If k be the cost for the side then that of top and bottom* will be $2k$ per square unit.

$$C=(\pi r^2)\,2k+(\pi r^2)\,2k+(2rh)\,k \text{ where } 1=\pi r^2h$$

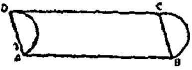
or $$C=2\pi k\left[2r^2+\frac{1}{\pi r}\right] \text{ etc}$$

$$r=\frac{1}{(4\pi)^{1/3}},\quad h=\left(\frac{4}{\pi^2}\right)^{2/3}$$

17 V=Volume of half cylinder

$=\frac{1}{2}\pi r^2h$ (1)

We want the total surface to be minimum It will consist of the following surfaces



S_1=Surface of semi circular ends$=\frac{1}{2}\pi r^2+\frac{1}{2}\pi r^2=\pi r^2$

S_2=Curved surface of half cylinder$=\frac{1}{2}(2\pi rh)=\pi rh$

S_3=Surface area of rectangular base whose length is h and width $2r=2rh$

38 Let 20 be divided into two parts as x and $20-x$ where x is the (circumference of a circle and $20-x$ is the perimeter of a square Then radius of circle is $x/(2\pi)$ and side of the square is $(20-x)/4$ Hence if y denotes the sum of the areas of the circle and square, then

$$y=f(x)=\pi\left(\frac{x}{2\pi}\right)^2+\frac{(20-x)^2}{16}$$

$$=\frac{x^2}{4\pi}+\frac{1}{16}(20-x)^2$$

$$\frac{dy}{dx}=\frac{2x}{4\pi}+\frac{2}{16}(20-x)(-1)=0 \text{ for max, or min}$$

This gives $x=\dfrac{20\pi}{\pi+4}$

and $\dfrac{d^2y}{dx^2}=\dfrac{1}{2\pi}+\dfrac{1}{8}>0$

y is min at $x=\dfrac{20\pi}{\pi+4}$

Min value of $y=\dfrac{100}{\pi+4}$

So there is no maxima in the open interval $0<x<20$ At the end points $x=0$ and $x=20$, we have

$$f(0)=25 \text{ and } f(20)=100/\pi$$

Since $100/\pi$ is the greatest of all the three values $100/\pi$, $100/(\pi+4)$ 25 the greatest value of y is obtained when $x=20$ that is, when the whole wire is bent into a circle to give the maximum area (In our opinion this is not a meaningful question in as much as the wire is not divided into two parts to give a maximum)

39 We have

$$f(x)=\int_1^x [2(t-1)(t-2)^3+3(t-1)^2(t-2)^2]\,dt$$

$$=\int_1^x (t-1)(t-2)^2\{2(t-2)+3(t-1)\}\,dt$$

$$=\int_0^x (t-1)(t-2)^2(5t-7)\,dt$$

$$f'(x)=(x-1)(x-2)^2(5x-7)$$

Now for max or min $f'(x)=0$ This gives

$$x=1,\quad x=2 \text{ and } x=7/5$$

$OM=x$ so that the length of the diagonal of the square cross section is $2x$ and so its area is $2x^2$ [Note that area of a square $=\frac{1}{2}$ (diagonal)^2]

$$OL=b \quad OB=a,$$

If h be the height of the beam then its volume is $2x^2h$

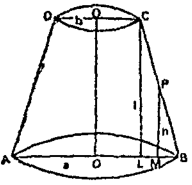


Fig 26

$$V=2x^2h \qquad (1)$$

We have to eliminate the variable h and find its value in terms of x so that V is a function of single variable x

Now from similar triangles CLB and PMB we have

$$\frac{CL}{PM}=\frac{LB}{MB} \quad \text{or} \quad \frac{l}{h}=\frac{OB-OL}{OB-OM}=\frac{a-b}{a-x}$$

$$h=\frac{l}{a-b}(a-x) \qquad (2)$$

Hence from (1) by the help of (2) the volume is

$$V=2x^2\,\frac{l}{a-b}(a-x)=\frac{2l}{a-b}(ax^2-x^3) \qquad (3)$$

$$\frac{dV}{dx}=\frac{2l}{a-b}(2ax-3x^2)=0 \qquad x=0,\ \frac{2a}{3}$$

$$\frac{d^2V}{dx^2}=\frac{2l}{a-b}(2a-6x)=\frac{2l}{a-b}(2a-4a)=-\text{ive and hence}$$

V is max when $x=\frac{2a}{3}$

Hence from (2) the height of the beam of max volume is

$$h=\frac{l}{a-b}\left(a-\frac{2a}{3}\right)=\frac{al}{3(a-b)}$$

) Let the semi vertical angle of the isosceles triangle be α in *which the altitude AD bisects base BC*

$$2s=AB+AC+BC$$
$$=2AC+2DC$$

$AB=AC$ and D is mid point of BC

$s=AC+DC=AF+FC+DC=AF+2CD$ ($CD=CF$)

Now $AF=r\cot\alpha$, $DC=AD\tan\alpha$

$$=(DI+IA)\tan\alpha$$
$$=(r+r\operatorname{cosec}\alpha)\tan\alpha$$

carry a pole b, feet long ($b > a$) from the road into the lane keeping it horizontal

3 A ladder is to be carried in a horizontal position round a corner formed by two streets a ft and b ft wide meeting at right angles Prove that the length of the longest ladder that will pass round the corner without jamming is

$$(a^{2/3}+b^{2/3})^{3/2}$$

4 A tall electric pole is to be kept in vertical position by a stretched straight wire from the pole to the ground The wire has to clear a wall 6 m high and 4 m from the pole What is the least length of the wire that can be used between the pole and the ground

5 One corner of long rectangular sheet of paper of width 1 ft is folded over so as to reach the opposite edge of the sheet Find the minimum length of the crease

6 The lower corner of a leaf in a book is folded over so as to reach the inner edge of the page Show that the fraction of the width folded over when the area of the folded part is min is 2/3

7 A rectangular sheet of metal has four equal square portions removed at the corner and the sides are then turned up so as to form an open rectangular box Show that when the volume contained in the box is max the depth will be

$$\tfrac{1}{6}\{(a+b)-(a^2-ab+b^2)^{1/2}\}$$

where a and b ($a > b$) are the sides of the original rectangle Find the corresponding result if the given metal sheet be a square

8 Show that the semi vertical angle of a right cone of given total surface (including area of base) and max volume is $\sin^{-1}\frac{1}{3}$

9 Show that the semi vertical angle of a cone of max volume and given slant height is $\tan^{-1}\sqrt{2}$

10 Show that a conical tent of given capacity will require the least amount of canvas if its height is $\sqrt{2}$ times its base radius

11 For a given curved surface of a right circular cone, show that the volume is max when semi vertical angle of the cone is

$$\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$-\cos^2\alpha+2\sin^2\alpha+2\sin\alpha=0$$
$$-(1-\sin^2\alpha)+2\sin^2\alpha+2\sin\alpha=0$$

or $$3\sin^2\alpha+2\sin\alpha-1=0$$
$$(\sin\alpha+1)(3\sin\alpha-1)=0$$
$$\sin\beta=-1,\ 1/3 \quad \text{or} \quad \alpha=-\pi/2 \quad \text{or} \quad \sin^{-1}1/3$$

The first value is rejected

$$AD=r\,(1+\operatorname{cosec}\alpha)=r\,(1+3)=4r \qquad \text{Proved}$$

You may show that $\frac{d^2s}{d\alpha^2}=+$ive for $\sin\alpha=1/3$

22 Let h be the height of the cone and r be its radius

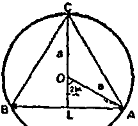


Fig 28

$$h=CL=CO+OL=a+OL$$
$$OL=h-a$$
$$r=LA=\sqrt{(OA^2-OL^2)}$$

or $$r=\sqrt{[a^2-(h-a)^2]}=\sqrt{(2ah-h^2)}$$
$$V=\tfrac{1}{3}\pi r^2h=\tfrac{1}{3}\pi\,(2ah-h^2)\,h$$
$$=\tfrac{1}{3}\pi\,(2ah^2-h^3)$$

$$\frac{dV}{dh}=\frac{\pi}{3}\,(4ah-3h^2)=0 \qquad h=0 \quad \text{or} \quad \frac{4a}{3}$$

$h=0$ is rejected $\qquad h=\frac{4a}{3}=\frac{2}{3}\,(2a)=\frac{2}{3}$ (diameter)

$$\frac{d^2V}{dh^2}=\pi/3\ \ (4a-6h)=\pi/3\ \ (4a-8a)=-\text{ive}$$

and hence V is maximum when $h=\frac{2}{3}$ (diameter)

2nd part

$$S=\pi rl=\pi\sqrt{(2ah-h^2)}\ \sqrt{(h^2+r^2)}$$
$$=\pi\sqrt{(2ah-h^2)}\ \sqrt{(h^2-2ah-h^2)}$$
$$=\pi\sqrt{(2ah-h^2)}\ \sqrt{(2ah)}$$

Let $Z=S^2=\pi^2\,2a\,(2ah^2-h^3)$

S will be maximum when $S^2=Z$ is max

$$\frac{dZ}{dh}=2\pi\,a\,(4ah-3h^2)=0 \qquad h=0\ \ \frac{4a}{3}$$

$h=0$ is rejected and hence $h=\frac{4a}{3}=\frac{2}{3}\,(2a)=\frac{2}{3}$ (diameter)

Also $\frac{d^2Z}{dh^2}=2\pi^2a\,(4a-6h)=2\pi\,a\,(4a-8a)=-$ive and hence S is max when $h=\frac{2}{3}$ of diameter

23 If r be the radius and h the height, then from the figure

$$r^2+\frac{h^2}{4}=a^2 \qquad h^2=4\,(a^2-r^2)$$

cut from it a beam of uniform square section. Prove that the beam of greatest volume that can be cut is $\frac{al}{3(a-b)}$ feet long.

20 Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

21 (a) A cone is circumscribed to a sphere of radius r. Show that volume of the cone is minimum if its altitude is $4r$ and its semi vertical angle is $\sin^{-1}\frac{1}{3}$. (Roorkee 88)

(b) Find the vertical angle of a right circular cone of minimum curved surface that circumscribes a given sphere. (Roorkee 8)

22 The cone of greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$rd the diameter of the sphere. Prove also that the curved surface of the cone is maximum for the same value of altitude.

23. Show that the height of the cylinder of max volume that can be inscribed in a sphere of radius a is $\frac{2a}{\sqrt{3}}$. Also find its radius.

24 Find the max value of the surface of a right circular cylinder which can be inscribed in a sphere of radius a.

25 Tangents are drawn to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

and the auxiliary circle of the ellipse at points where a common ordinate cuts them. Show that if θ be the greatest inclination of these tangents then $\tan\theta=\frac{a-b}{2\sqrt{ab}}$

26 Prove that the min intercept made by the axes on the tangent to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is } a+b$$

Prove further that it is divided at the point of contact into parts which are equal to semi axes respectively.

27 From a variable point of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

normal is drawn to an ellipse. Find the maximum distance of the normal from the centre of ellipse.

$T'_p = \frac{x}{a}\cos t + \frac{y}{a}\sin t = 1$

$$m_2 = -\cot t$$

If θ be the angle between the tangents then

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} = \frac{\cot t\,(1 - b/a)}{1 + b/a \cot^2 t}$$

Now θ will be max when $\tan\theta = y$ is maximum

$$y = \tan\theta = \frac{(a-b)\cot t}{a + b\cot^2 t} = \frac{a \quad b}{a\tan t + b\cot t}$$

Now N' of y is constant and hence it will be maximum when its denominator $Z = a\tan t + b\cot t$ is min

$$\frac{dZ}{dt} = a\sec^2 t - b\,\mathrm{cosec}^2 t = 0 \qquad \tan^2 t = b/a$$

or $\tan t = \sqrt{(b/a)}$ or $\cot t = \sqrt{(a/b)}$

$\frac{d^2Z}{dt^2} = 2a\sec^2 t \tan t + 2b\ \mathrm{cosec}^2 t \cot t = +\text{ive}$ and hence

Z is min when $\tan t = \sqrt{(b/a)}$

or y is Max when $\tan t = \sqrt{(b/a)}$

or $\tan\theta = \frac{a-b}{a\sqrt{(b/a)} + b\sqrt{(a/b)}} = \frac{a-b}{2\sqrt{(a/b)}}$ by (1)

5 Any tangent to the ellipse is

$$\frac{x}{a}\cos t + \frac{y}{b}\sin t = 1$$

or
$$\frac{x}{a\sec t} + \frac{y}{b\,\mathrm{cosec}\, t} = 1$$

It meets the axes at $Q\ (a\sec t, 0)$, $R\ (0, b\,\mathrm{cosec}\, t)$

$y = QR^2 = a^2\sec^2 t + b^2\,\mathrm{cosec}^2 t$

Now refer Q 1 (k) Set 1 page 570

Min value of $y = (a+b)^2$

or $\qquad QR^2 = (a+b)^2$ or $QR = (a+b)$

27 Any normal to the ellipse is

$$ax\sec t - by\,\mathrm{cosec}\, t = a^2 - b^2,$$

If p be the length of perpendicular from (0, 0) then

$$p = \frac{a^2 - b^2}{\sqrt{(a^2\sec^2 t + b^2\,\mathrm{cosec}^2 t)}}$$

Now p will be maximum if its denominator $\sqrt{(a^2\sec^2 t + b^2\,\mathrm{cosec}^2 t)}$ is minimum as its numerator is constant

$$T=\frac{\sqrt{(a^2+x^2)}}{k}+\frac{(b-x)}{k\sec\alpha}$$

$$\frac{dT}{dx}=\frac{1}{k}\left[\frac{2x}{2\sqrt{(a^2+x^2)}}-\frac{1}{\sec\alpha}\right]=0$$

$x^2\sec^2\alpha=a^2+x^2$ or $x^2(\sec^2\alpha-1)=a^2$

or $x^2\tan^2\alpha=a^2$ $\quad x=a\cot\alpha$

$$\frac{d^2T}{dx^2}=\frac{1}{k}\,\frac{\left[\sqrt{(a^2+x^2)}\cdot 1-x\,\frac{2x}{2\sqrt{(a^2+x^2)}}\right]}{(a^2+x^2)}$$

$$=\frac{1}{k}\,\frac{a^2}{(a^2+x^2)^{3/2}}=+\text{ive}$$

for $x=a\cot\alpha$ and hence T is minimum

Hence he should land at a distance $b-x=b-a\cot\alpha$ from the place to be reached i e C

(b) Proceed as in part (a) Ans $L-\dfrac{Ud}{\sqrt{(V^2-U^2)}}$ from A

2 Here AB is the pole of length b feet which is just taken into the lane from the road Its end A will rest on the side of the road and the end B will rest in contact with the side of the lane A point C of its length will be at the turning C as shown If it is just possible to carry the pole from the road into the lane keeping it horizontal then the width CD of the lane should be maximum Let

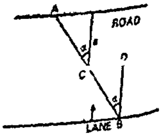


Fig 18

$$CD=y$$

$b=AC+CB$ or $b-a\sec\alpha=CB$

$y=CD=CB\sin\alpha=(b-a\sec\alpha)\sin\alpha=b\sin\alpha-a\tan\alpha$ (i)

For y to be maximum $\dfrac{dy}{d\alpha}=0$

$\dfrac{dy}{d\alpha}=b\cos\alpha-a\sec^2\alpha=0$ $\quad\cos^3\alpha=\dfrac{a}{b}$ or $\cos\alpha=\left(\dfrac{a}{b}\right)^{1/3}$

$\dfrac{d^2y}{d\alpha^2}=-b\sin\alpha-2a\sec\alpha\sec\alpha\tan\alpha=-\text{ive}$

Hence y is max when $\cos\alpha=\dfrac{a^{1/3}}{b^{1/3}}$

For max or min, we have $\frac{dy}{dx}=0$

Then (2) gives

$$y^2-2yx=0 \quad \text{or} \quad y=2x,$$

Since $y=0$ does not satisfy (1), putting $y=2x$ in (1), we get

$$2x^2(2x-x)=2a^3 \quad \text{or} \quad x=a$$

Then $y=2a$

Differentiating (2) again we get

$$2xy\,\frac{d^2y}{dx^2}+2\left(x\frac{dy}{dx}+1\;y\right)\frac{dy}{dx}+2y\,\frac{dy}{dx}-x^2\,\frac{d^2y}{dx^2}$$
$$-2x\,\frac{dy}{dx}-2x\,\frac{dy}{dx}-2y=0$$

In this, putting $x=a$, $y=2a$ and $\frac{dy}{dx}=0$, we get

$$4a^2\,\frac{d^2y}{dx^2}-a^2\,\frac{d^2y}{dx^2}-4a=0 \quad \text{or} \quad \frac{d^2y}{dx^2}=\frac{4}{3a}>0$$

Provided $a>0$ Hence y is minimum when $x=a$ and the minimum value of y is $2a$ To see whether y has another value at $x=a$ less than the minimum, we put $x=a$ in (1)

We then get,

$$ay(y-a)-2a^3=0$$

or $\quad y^2-ay-2a^2=0$

or $\quad (y+a)(y-2a)=0$

Hence $y=-a,\ 2a$

Thus we get a value $-a$ of y which is less than the minimum value $3a$ of y The reason simply is that the minimum value of a function does not mean its least value in an interval [See Greatest and least values of a function before problem Set A on page 563]

31 Put $c+x=y$ Then the expression

$$=\frac{(a-c+y)(b-c+y)}{y}=\frac{(a-c)\,(b-c)}{y}+y+a-c+b-c$$
$$=\left(\frac{\sqrt{\{(a-c)\,(b-c)\}}}{\sqrt{y}}-\sqrt{y}\right)^2$$
$$+a-c+b-c+2\sqrt{\{(a-c)\,(b-c)\}}$$

Hence the given expression is a minimum when the square term is zero, that is when $y=\sqrt{\{(a-c)\,(b-c)\}}$ Thus the minimum value is

$a-c+b-c+2\sqrt{\{(a-c)\,(b-c)\}}$ i e $\quad (\sqrt{\{a-c\}}+\sqrt{\{b-c\}})^2$

$$\frac{d^2y}{d\theta^2}=+6 \operatorname{cosec}\theta \cot\theta+6 \operatorname{cosec}^3\theta+4 \sec\theta \tan^2\theta+4 \sec^3\theta$$

>0 since θ is an acute angle

Hence y is minimum when $\tan\theta=3^{1/3}/2^{1/3}$

Min value of y is given by

$$y=\sqrt{\left[\left(2^{2/3}+3^{2/3}\right)\right]\left[\frac{3\ 2}{3^{1/3}}+\frac{2\ 2}{2^{1/3}}\right]}=2\ (2^{2/3}+3^{2/3})^{3/2}$$

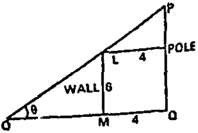


Fig 19

5 The corner C has acquired the position C' when folded over where $CE=x=EC'$ so that $DE=1-x$ Crease is EF and it will be minimum if $EF^2=z$ say is min,

$$z=EF^2=C'E^2+C'F^2$$
$$=x^2+(FM \operatorname{cosec}\theta)^2,$$
$$z=x^2+1 \operatorname{cosec}^2\theta \quad (1)$$

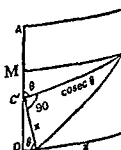


Fig. 20

Again from $\triangle\ C'ED$, $\cos\theta=\dfrac{1-x}{x}$

$$\therefore \quad \sin\theta=\frac{\sqrt{\{x^2-(1-x)^2\}}}{x}=\frac{\sqrt{(2x-1)}}{x}$$

$$z=x^2+\frac{x^2}{2x-1}=\frac{2x^3}{2x-1}=\frac{1}{\dfrac{1}{x^2}-\dfrac{1}{2x^3}}$$

z will be minimum when $t=\dfrac{1}{x^2}-\dfrac{1}{2x^3}$ is maximum

$$\frac{dt}{dx}=-\frac{2}{x^3}+\frac{3}{2x^4}=0 \qquad x=3/4,$$

$$\frac{d^2t}{dx^2}=\frac{6}{x^4}-\frac{6}{x^5}=\frac{6}{x^4}\left(1-\frac{1}{x}\right)=-\text{ive for } x=3/4$$

Hence t is max for $x=3/4$

$z=EF^2$ is min for $x=3/4$

$$z=\frac{2x^3}{2x-1}=2\ \frac{27}{64}\ 2=EF^2 \qquad EF=\frac{3\sqrt{3}}{4}$$

We consider two cases

(i) $-\frac{3}{2} < \lambda < 0$

In this case at $x=0$, we have

$$f'(x)=2\lambda < 0 \text{ since } -\tfrac{3}{2} < \lambda < 0$$

and at $\sin x=-\frac{2\lambda}{3}$, we have

$$f'(x)=-\frac{2\lambda}{9}(9-4\lambda^2) > 0$$

(ii) $0 < \lambda < \frac{3}{2}$

In this case, we have

$f'(x) > 0$ at $x=0$

and $f'(x) < 0$ at $\sin x=-\frac{2\lambda}{3}$

Thus in case (i) we have a max at $x=0$ and a min at

$$\sin x=-\frac{2\lambda}{3},$$

an in case (ii), we have a min at $x=0$ and a max at

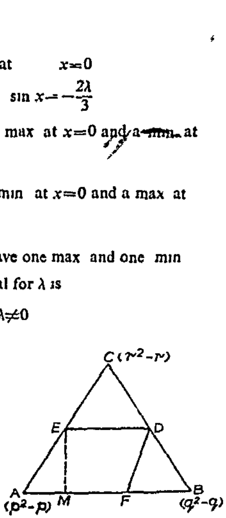
$$\sin x=-\frac{2\lambda}{3}$$

Hence in either case we have one max and one min

Hence the required interval for λ is

$$-\frac{3}{2}< \lambda < \frac{3}{2}, \lambda\neq 0$$

3 Let $AF=x=ED$
and $AE=y=FD$
Then from similar Δ's CED and CAB, we have



$$\frac{CE}{CA}=\frac{ED}{AB}$$

or $\frac{CA-AE}{CA}=\frac{ED}{AB}$

or $\frac{b-y}{b}=\frac{x}{c}$

or $bc-cy=bx$ (1)

$$z=\tfrac{1}{9}\pi^2r^4\left[\left(\frac{S}{\pi r}-r\right)^2-r^2\right]$$

or $$z=\tfrac{1}{9}\pi^2r^4\left[\frac{S^2}{\pi^2r^2}-\frac{2S}{\pi}\right]$$

or $$z=\tfrac{1}{9}\pi^2\left[\frac{S^2}{\pi^2}r^2-\frac{2S}{\pi}r^4\right]=\frac{S}{9}\,[Sr^2-2\pi r^4]$$

Now V is max when V^2 i e z is max

$$\frac{dz}{dr}=\frac{S}{9}[2Sr-8\pi r^3]=0 \qquad r=0 \text{ or } r^2=\frac{S}{4\pi}$$

$r=0$ is obviously rejected

$$\frac{d^2z}{dr^2}=\frac{S}{9}\,[2S-24\pi r^2]=\frac{S}{9}\,[2S-6S]=-\frac{4S^2}{9}=-\text{ive}$$

When $r^2=\dfrac{S}{4\pi}$ Hence z i e V^2 is max or V is max

Now $r^2=\dfrac{S}{4\pi}$ or $4\pi r^2=S=\pi r^2+\pi rl$

$$3\pi r^2=\pi rl \qquad \frac{r}{l}=\frac{1}{3} \text{ or } \sin\alpha=\frac{1}{3}$$

$$\alpha=\sin^{-1}\tfrac{1}{3}$$

9 $V=\frac{1}{3}\pi r^2h$, l is given

$z=V^2=\frac{1}{9}\pi^2r^4(l^2-r^2)=\frac{1}{9}\pi^2(l^2r^4-r^6)$

$dz/dr=\frac{1}{9}\pi^2(4l^2r^3-6r^5)=0$

$r=0$ or $r^2=\frac{2}{3}l^2$ or $r=\sqrt{(2/3)}\,l,\ -\sqrt{(2/3)}\,l$

Clearly $r=0$ and $r=-\sqrt{(2/3)}\,l$ are rejected Hence we consider $r=\sqrt{(2/3)}\,l$

$$\frac{d^2z}{dr^2}=\frac{1}{9}\pi^2(12l^2r^2-30r^4)$$

$$=\frac{1}{9}\pi^2\left[12l^2\,\frac{2}{3}\,l^2-30\,\frac{4}{9}\,l^4\right]=\frac{1}{9}\pi^2l^4\left(-\frac{16}{3}\right)=-\text{ive}$$

Hence z i e V^2 is max when $r=\sqrt{(2/3)}\,l$

$$h^2=l^2-r^2=l^2-\tfrac{2}{3}l^2=\tfrac{1}{3}l^2 \qquad h=\frac{1}{\sqrt{(3)}}\,l$$ Proved

$\tan\alpha=r/h=\sqrt{(2)}$ or $\alpha=\tan^{-1}\sqrt{(2)}$

10 $V=\dfrac{1}{3}\pi r^2h$ (given) $h^2=\dfrac{9V^2}{\pi^2r^4}$

Here because of the tent we are concerned only with the curved surface πrl and not that of the base

$S=\pi rl \qquad S^2=z=\pi^2r^2l^2=\pi^2r^2(r^2+h^2)$

4 We first solve the inequality

$$\frac{x^2+x+2}{x^2+5x+6}<0, \text{ that is, } \frac{x^2+x+2}{(x+3)(x+2)}<0$$

Since $x^2+x+2=(x+\frac{1}{2})+7/4>0$ for $x \in R$, the inequality is satisfied if $(x+3)(x+2) \leqslant 0$ whence $-3<x<-2$ (I)

We now find the extrema of the function $f(x)=1+a^2x-x^3$

For max or min

We have $f'(x)=a^2-3x^2=0$ whence $x=\pm\frac{|a|}{\sqrt{3}}$

Let $x_1=-|a|/\sqrt{3}$ and $x_2=|a|/\sqrt{3}$

For $x<-|a|/\sqrt{3}$ or $x>|a|/\sqrt{3}, f'(x)$ is negative and for $-|a|/\sqrt{3}<x<|a|/\sqrt{3}$, $f(x)$ is positive Thus the function $f(x)$ is decreasing on the left and increasing on the right of x_1 Also $f(x)$ is increasing on the left and decreasing on the right of x_2 Hence x_1 is a point of minimum and x_2 a point of maximum Since minimum point must satisfy the inequality (I)

we must have

$$-3<-|a|/\sqrt{3}<-2$$

or $\quad 2\sqrt{3}<|a|<3\sqrt{3}$

which implies $\quad -3\sqrt{3}<a<-2\sqrt{3}$

or $\quad 2\sqrt{3}<a<3\sqrt{3}$

Thus $\quad a \in (-3\sqrt{3}, -2\sqrt{3}) \cup (2\sqrt{3}, 3\sqrt{3})$ Ans

5 We have $f'(x)=3x^2+6(a-7)x+3(a^2-9)$

Let the equation $f'(x)=0$ have two real roots x_1, x_2 with $x_1<x_2$ Then

$$f'(x) \equiv 3(x-x_1)(x-x_2) \qquad (1)$$

Clearly $f'(x)>0$ for $x<x_1, f'(x)<0$ for $x_1<x<x_2$ and $f'(x)>0$ for $x>x_2$ Hence $x=x_1$ is a point of maximum of $f(x)$ Since by hypothesis $x_1>0$ Then also $x_2>0$

Now both the roots of $f'(x)=0$ are real and positive This occurs if and only if the discriminant

$36(a-7)^2-36(a^2-9)>0$ so that roots are real (1)

$6(a-7)<0$ so that sum is +ve (2)

$$z=\tfrac{1}{9}\pi^2 r^4\left[\left(\frac{S}{\pi r}-r\right)^2-r^2\right]$$

or
$$z=\tfrac{1}{9}\pi^2 r^4\left[\frac{S^2}{\pi^2 r^2}-\frac{2S}{\pi}\right]$$

or
$$z=\tfrac{1}{9}\pi^2\left[\frac{S^2}{\pi^2}r^2-\frac{2S}{\pi}r^4\right]=\frac{S}{9}\,[Sr^2-2\pi r^4]$$

Now V is max when V^2 i e. z is max

$$\frac{dz}{dr}=\frac{S}{9}[2Sr-8\pi r^3]=0 \qquad r=0 \text{ or } r^2=\frac{S}{4\pi}$$

$r=0$ is obviously rejected

$$\frac{d^2z}{dr^2}=\frac{S}{9}\,[2S-24\pi r^2]=\frac{S}{9}\,[2S-6S]=-\frac{4S^2}{9}=-\text{ve}$$

When $r^2=\dfrac{S}{4\pi}$ Hence z i e V^2 is max or V is max

Now $r^2=\dfrac{S}{4\pi}$ or $4\pi r^2=S=\pi r^2+\pi rl$

$$3\pi r^2=\pi rl \qquad \frac{r}{l}=\frac{1}{3} \text{ or } \sin\alpha=\frac{1}{3}$$

$$\alpha=\sin^{-1}\tfrac{1}{3}$$

9 $V=\frac{1}{3}\pi r^2 h$ l is given

$$z=V^2=\tfrac{1}{9}\pi^2 r^4(l^2-r^2)=\tfrac{1}{9}\pi^2(l^2r^4-r^6)$$

$$dz/dr=\tfrac{1}{9}\pi^2(4l^2r^3-6r^5)=0$$

$$r=0 \quad\text{or}\quad r^2=\tfrac{2}{3}l^2 \quad\text{or}\quad r=\sqrt{(2/3)}\,l,\ -\sqrt{(2/3)}\,l$$

Clearly $r=0$ and $r=-\sqrt{(2/3)}\,l$ are rejected Hence we der $r=\sqrt{(2/3)}\,l$

$$\frac{d^2z}{dr^2}=\frac{1}{9}\pi^2(12l^2r^2-30r^4)$$

$$=\frac{1}{9}\pi^2\left[12l^2\,\frac{2}{3}\,l^2-30\,\frac{4}{9}\,l^4\right]=\frac{1}{9}\pi^2 l^4\left(-\frac{16}{3}\right)=-\text{ve}$$

Hence z i e V^2 is max when $r=\sqrt{(2/3)}\,l$

$$h^2=l^2-r^2=l^2-\tfrac{2}{3}l^2=\tfrac{1}{3}l^2 \qquad h=\frac{1}{\sqrt{(3)}}\,l$$

$$\tan\alpha=r/h=\sqrt{(2)} \text{ or } \alpha=\tan^{-1}\sqrt{(2)}$$

Proved

$V=\dfrac{1}{3}\pi r^2 h$ (given) $\qquad h^2=\dfrac{9V^2}{\pi^2 r^4}$

Here because of the tent we are concerned only with curved surface πrl and not that of the base

$S=\pi rl$

$$S^2=z=\pi^2r^2l^2=\pi^2r^2(r^2+h^2)$$

7 If a function $f(x)$ has $f'(a)=0$ and $f''(a)=0$, then
(i) $x=a$ is a maximum for $f(x)$,
(ii) $x=a$ is a minimum for $f(x)$,
(iii) It is difficult to say (i) and (ii),
(iv) $f(x)$ is necessarily a constant function

8 If $y=a \log x+bx^2+x$ has its extremum values at $x=-1$ and $x=2$, then
(a) $a=2, b=-1$, (b) $a=2, b=-\frac{1}{2}$,
(c) $a=-2, b=\frac{1}{2}$, (d) None of these (I I T 81)

9 Let $P(x)=a_0+a_1x^2+a_2x^4+ \quad +a_nx^n$ be a polynomial in a real variable x with $0 < a_0 < a_1 < a_2 < \quad < a_n$ The function $P(x)$ has
(a) Neither a max nor a min ,
(b) Only one maximum,
(c) Only one minimum,
(d) None of these (I I T 86)

10 N characters of information are held on magnetic tape, in batches of x characters each the batch processing time is $\alpha+\beta x^2$ seconds, α and β are constants The optical value of x for fast processing is,
(a) α/β, (b) β/α, (c) $\sqrt{(\alpha/\beta)}$, (d) $\sqrt{(\beta/\alpha)}$,
(M N R 86)

Solutions

1 Ans (a) We have

$\frac{dy}{dx}=x^2-2x$ Integrating, $y=\frac{1}{3}x^3-x^2+c$

Since the curve passes through (2, 0), we get

$$0=\frac{1}{3}\times 2^3-2^2+c \quad \text{or} \quad c=\frac{4}{3}$$

Hence the equation of the curve is

$$y=\frac{1}{3}x^3-x^2+\frac{4}{3}$$

Now for max or min , we have

$$\frac{dy}{dx}=0 \text{ i e } x^2-2x=0, \qquad x=0\ 2$$

and $\frac{d^2y}{dx^2}=2x-2=-2$ at $x=0$

$$\frac{d^2V}{dx^2}=\pi\,(2b-6x\cot\alpha)=\pi\,(2b-4b)=-\text{ive and hence max}$$

when $x=\frac{2}{3}\,b\tan\alpha$ $\quad h=b-x\cot\alpha=b-\frac{2}{3}\,b=\frac{1}{3}\,b$

Max Volume is $\pi r^2h=\pi x^2h=\pi\,\frac{4}{9}\,b^2\tan^2\alpha\,\frac{1}{3}b$

or $V=\frac{4}{27}\,\pi\,b^3\tan^2\alpha$

13 $S=2\pi rh=$ curved surface

$S=2\pi x\,(b-x\cot\alpha)$ as in Q 12

or $S=2\pi\,(bx-x^2\cot\alpha)$

$$\frac{dS}{dx}=2\pi\,(b-2x\cot\alpha)=0 \qquad x=\frac{b}{2}\tan\alpha$$

or $x=\frac{1}{2}\,(b\tan\alpha)=\frac{1}{2}\,(r_1)$

or Radius of cylinder $=(1/2)$ radius of cone

14 $S=2\pi r^2+2\pi rh$ (given) $\qquad \frac{S-2\pi r^2}{2\pi r}=h$

$$V=\pi r^2h=\pi r^2\;\frac{S-2\pi r^2}{2\pi r}=\frac{1}{2}\,(Sr-2\pi r^3)$$

$$\frac{dV}{dr}=\frac{1}{2}\,(S-6\pi r^2)=0 \qquad r=\sqrt{\left(\frac{S}{6\pi}\right)}$$

$$\frac{d^2V}{dr^2}=-6\pi r=-6\pi\sqrt{\left(\frac{S}{6\pi}\right)}=-\sqrt{(6\pi S)}=-\text{ive and}$$

hence max when $r=\sqrt{\left(\frac{S}{6\pi}\right)}$ or $6\pi r^2=S$

or $6\pi r^2=2\pi r^2+2\pi rh$ or $2r=h$

i e its height is equal to diameter of the base

15 The curved surface of the cylindrical can be made by rolling the sheet and there will be no waste of material But in order to have the base of the can, which will be circular of radius r we will have to cut it from a square metal sheet $2r$ by $2r$ and whose area will be $4r^2$ whereas the actual area of the base is only πr^2 But no allowance is to be made for waste of material Hence the … total surface area of the sheet used for making the … cylindrical can is

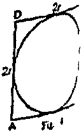


Fig.

$$S=2\pi rh+4r^2$$

Also $V=\pi r^2h$ (given) $\qquad h=V/\pi r^2$

$$S=2\pi r\,\frac{V}{\pi r^2}+4r^2=\frac{2V}{r}+4r^2$$

Maximum value of $A=\frac{1}{2}a^2\left[\sin\frac{2\pi}{3}-\frac{1}{2}\sin\frac{4\pi}{3}\right]$

$$=\tfrac{1}{2}a^2\left[\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{4}\right]=\frac{3\sqrt{3}}{8}a^2$$

Ans (d)

Ans (c)

We have $f'(x)=(3-x)\,2e^{2x}-e^{2x}-4e^x-4xe^x-1$

$=0$ for maxima or minima

or $f'(x)=(5-2x)\,e^{2x}-4\,(1+x)\,e^x-1=0$

This is satisfied for $x=0$

Now $f''(x)=(5-2x)\,2e^{2x}-2e^{2x}-4e^x-4\,(1+x)\,e^x$

$=10-2-4-4=0$ at $x=0$

So we find $f'''(x)$ We have

$f''(x)=(8-4x)\,e^{2x}-4\,(2+x)\,e^x$

$f'''(x)=(8-4x)\,2e^{2x}-4e^{2x}-4e^x-4\,(2+x)\,e^x$

$=16-4-4-8=0$ at $x=0$

So we find the next differential coefficient $f^{iv}(x)$ We have

$f'''(x)=(12-8x)\,e^{2x}-4\,(3+x)\,e^x$

$f^{iv}(x)=(12-8x)\,2e^{2x}-8e^{2x}-4e^x-4\,(3+x)\,e^x$

$=24-8-4-12=0$ at $x=0$

Now $f^{v}(x)=(12-8x)\,4e^{2x}-8\cdot 2e^{2x}-8\cdot 2e^{2x}-4e^x$
$-4e^x-4\,(3+x)\,e^x$

$=48-16-16-4-4-12$ at $x=0$

$=-4\neq 0$

Hence $f(x)$ has neither maximum nor minimum at $x=0$

Ans (d) Max at $x=1$, Min at $x=3$

Ans (c)

We have $f'(x)=\cos x-2\sin 2x=0$

or $\cos x\,(1-4\sin x)=0$

$\cos x=0$ gives $x=n\pi+\frac{\pi}{2}=\frac{(2n+1)\pi}{2}$

$f''(x)=-\sin x-4\cos 2x$

$=-\sin(n\pi+\pi/2)-4\cos(2n\pi+\pi)$ at $x=n\pi+\pi/2$

$=-\sin(n\pi+\pi/2)+4$ [$\cos(2n\pi+\pi)=\cos\pi=-1$]

>0 [$\sin(n\pi+\pi/2)=\pm 1$]

$S=\text{Total surface}=S_1+S_2+S_3$

$S=\pi r^2+\pi rh+2rh=\pi r^2+(\pi+2)\,r\,\frac{2V}{\pi r^2}$ by (1)

$S=\pi r^2+\frac{2V}{\pi r}(\pi+2)$ (1)

$\frac{dS}{dr}=2\pi r-\frac{2V}{\pi r^2}(\pi+2)=0$ $\quad \pi^2r^3=V(\pi+2)$

$\frac{d^2S}{dr^2}=2\pi+\frac{4V(\pi+2)}{\pi r^3}=2\pi+4\pi=6\pi=+\text{ive}$ and hence

S is minimum when $\pi^2r^3=V(\pi+2)$

or $\pi^2r^3=\frac{1}{2}\pi r^2h(\pi+2)$ $\quad \frac{h}{2r}=\frac{\pi}{\pi+2}$

18 Let the edges of the box according to given condition be

x, nx, y

$V=nx^2y$ given [Formula $S=2(xy+yz+zx)$]

$S=2(x\,nx+nxy+yx)$

$S=2\left[nx^2+(n+1)\,x\,\frac{V}{nx^2}\right]$ by (1)

$\frac{dS}{dx}=2\left[2nx-\frac{(n+1)}{n}\frac{V}{x^2}\right]=0,$ $\quad x^3=\frac{(n+1)}{2n^2}V$ ..(2)

$\frac{d^2S}{dx^2}=2\left[2n+2\frac{(n+1)}{n}\frac{V}{x^3}\right]$ which is clearly $+$ive and

hence S is minimum when $x^3=\frac{(n+1)}{2n^2}V$

$S=2\left[nx^2+\frac{(n+1)}{n}\frac{V}{x}\right]=2\left[\frac{n^2x^3+(n+1)V}{nx}\right]$

or $S=\frac{2}{nx}\left[\frac{(n+1)}{2}V+(n+1)V\right]=\frac{2}{nx}\,\frac{3(n+1)V}{2}$

$S^3n^3x^3=27(n+1)^3V^3$

$S^3n^3(n+1)\frac{V}{2n^2}=27(n+1)^3V^3$ by (2)

or $n\,S^3=54(n+1)^2V^2$

19 Let the side of the square base be $2x$ so that its is area is

$(2x)^2=4x^2$

5

Integration

Standard Results

§ 1 Indefinite Integrals

If $\frac{d}{dx}[F(x)+c]=f(x)$, then we say that $F(x)+c$ is an *indefinite integral or anti derivative* of $f(x)$ and we write

$$\int f(x)\,dx=F(x)+c$$

Fundamental Formulae To be committed to memory

Differential Calculus	Integral Calculus
(i) $\frac{d}{dx}(x^{n+1})=(n+1)\,x^n$	$\int x^n\,dx=\frac{x^{n+1}}{n+1}\ (n\neq-1)$

Note The above formula is the most important one and shall be frequently used in the book and we shall henceforth speak it as power formula It should be remembered as follows

Increase the power of x by one and divide by the increased power

e g $\int x^5\,dx=\frac{x^6}{6}$, $\int x^{3/2}\,dx=\frac{x^{5/2}}{5/2}=\frac{2}{5}x^{5/2}$

Deductions from above

(a) $\int\frac{1}{x^n}\,dx=\int x^{-n}\,dx=\frac{x^{-n+1}}{-n+1}=\frac{1}{-(n-1)\,x^{n-1}}\ (n\neq1)$

In other words it means that when x^n is in the denominator then decrease the power of x in the D^r by one and multiply the denominator by the decreased power with sign changed

i e $\int\frac{1}{x^5}\,dx=\frac{1}{-4x^4}$, $\int\frac{1}{x^{7/2}}\,dx=\frac{1}{-5/2\,x^{5/2}}=\frac{-2}{5x^{5/2}}$

(b) $\int\frac{1}{\sqrt{x}}\,dx=\int\frac{1}{x^{1/2}}\,dx=\frac{1}{\frac{1}{2}x^{-1/2}}=2\sqrt{x}$ **(V Imp)**

(c) $\int dx=\int 1\,dx=\int x^0\,dx=\frac{x^{0+1}}{0+1}=x$

Thus $\int 7\,dx=7x$, $\int\frac{1}{2}\,dx=\frac{1}{2}x$

$$s = AF + 2CD = r\cot\alpha + 2\,(r\tan\alpha + r\sec\alpha)$$

or $\quad s = r\,(\cot\alpha + 2\tan\alpha + 2\sec\alpha)$

$$\frac{ds}{d\alpha} = r\,(-\operatorname{cosec}^2\alpha + 2\sec^2\alpha + 2\sec\alpha\tan\alpha) = 0$$

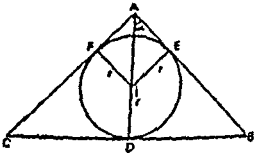


Fig 20

Change in terms of sin and cos we get

$$-\frac{1}{\sin^2\alpha} + \frac{2}{\cos^2\alpha} + \frac{2\sin\alpha}{\cos^2\alpha} = 0$$

$$-(1-\sin^2\alpha) + 2\sin^2\alpha + 2\sin^3\alpha = 0$$

or $\quad 2\sin^3\alpha + 3\sin^2\alpha - 1 = 0$

or $\quad (\sin\alpha + 1)\,(2\sin^2\alpha + \sin\alpha - 1) = 0$

$$(\sin\alpha + 1)\,(\sin\alpha + 1)\,(2\sin\alpha - 1) = 0$$

$\sin\alpha = -1$ or $\frac{1}{2}$ $\qquad \alpha = -\pi/2$ or $\pi/6$

Rejecting first value we choose $\alpha = \pi/6 = 30°$

$$\frac{d^2s}{d\alpha^2} = r\left[2\operatorname{cosec}^2\alpha\cot\alpha + 4\sec^2\alpha\tan\alpha + 2\,(\sec^3\alpha - \sec\alpha\tan^2\alpha)\right]$$

$= +$ive for $\alpha = 30°$ and hence s is minimum

$$s = r\,(\cot 30° + 2\tan 30° + 2\sec 30°)$$

$$= r\left[\sqrt{3} + \frac{2}{\sqrt{3}} + 2\,\frac{2}{\sqrt{3}}\right] = \frac{9r}{\sqrt{3}} = 3\sqrt{3}\,r$$

Perimeter $= 2s = 6r\sqrt{3}$

21 Refer figure of Q 20

$$V = \tfrac{1}{3}\pi r^2 h = \tfrac{1}{3}\pi DC^2\,AD$$

$$= \tfrac{1}{3}\pi r^3\,(\tan\alpha + \sec\alpha)^2\,(1 + \operatorname{cosec}\alpha)$$

$$\frac{dV}{d\alpha} = \tfrac{1}{3}\pi r^3\Big[(\tan\alpha + \sec\alpha)^2\,(-\operatorname{cosec}\alpha\cot\alpha)$$

$$+ 2\,(\tan\alpha + \sec\alpha)\,(\sec^2\alpha + \sec\alpha\tan\alpha)\,(1 + \operatorname{cosec}\alpha)\Big] = 0$$

$$= \tfrac{1}{3}\pi r^3\,(\tan\alpha + \sec\alpha)^2\,[-\operatorname{cosec}\alpha\cot\alpha + 2\sec\alpha\,(1 + \operatorname{cosec}\alpha)] = 0$$

$$-\frac{\cos\alpha}{\sin^2\alpha} + \frac{2}{\cos\alpha} + \frac{2}{\sin\alpha\cos\alpha} = 0$$

$$\frac{d}{dx}\tan^{-1}\frac{x}{a}=\frac{1}{1+\frac{x^2}{a^2}}\text{ d c of }\frac{x}{a}=\frac{1}{a}\quad\frac{a^2}{x^2+a^2}=\frac{a}{x^2+a^2}$$

$$\int\frac{1}{x^2+a^2}\,dx=\frac{1}{a}\tan^{-1}\frac{x}{a}$$

Similarly $\int\frac{1}{\sqrt{(a^2-x^2)}}\,dx=\sin^{-1}\frac{x}{a}$

Important rule In any of the standard results if in place of we have got a linear expression of x of the form ax+b, then the ame formula is applicable but we must divide by the d c of $ax+b$ c by a which is a constant and is coefficient of x

Integrals of squares of trigonometrical functions

(a) $\sin^2 x=\frac{1}{2}(1-\cos 2x)$, $\cos^2 x=\frac{1}{2}(1+\cos 2x)$,

$$\int\sin^2 x\,dx=\tfrac{1}{2}(x-\tfrac{1}{2}\sin 2x),\ \int\cos^2 x\,dx=\tfrac{1}{2}(x+\tfrac{1}{2}\sin 2x)$$

(b) $\tan^2 x=\sec^2 x-1$, $\cot^2 x=\text{cosec}^2\, x-1$

$$\int\tan^2 x\,dx=\tan x-x,\quad\int\cot^2 x\,dx=-\cot x-x$$

(Roorkee 78, 75)

Constant of Integration

$\frac{d}{dx}(x^5)=5x^4$ $\qquad\int 5x^4\,dx=x^5$

Again $\frac{d}{dx}(x^5+c)=5x^4$ $\qquad\int 5x^4\,dx=x^5+c$

From above we observe that

$$\int 5x^4\,dx=x^5\text{ as well as }x^5+c$$

This c is called constant of integration and it should be written by students in all the integrals though we shall not write the same in the answer

§ 2 **Evaluation of Definite Integrals**

We use the following formula, known as the Newton Leibnitz formula

$$\int_a^b f(x)\,dx=\Big[F(x)\Big]_a^b=F(b)-F(a),$$

where $F(x)$ is one of the anti derivatives of the function $f(x)$, i e $F'(x)=f(x)$ $\qquad(a\leqslant x\leqslant b)$

Remark When evaluating integrals with the help of the above formula, the students should keep in mind the condition

$$+\log(x+1)-\frac{1}{x+1}+2\sqrt{x}+2x^{3/2}+\frac{6}{5}x^{5/2}+\frac{2}{7}x^{7/2}$$

Let $f(x)=\dfrac{5\cos x}{2\sin^2 x}+\dfrac{\sin x}{\cos^2 x}+\sqrt{(\cos^2 x+\sin^2 x+2\sin x\cos x)}$

$$\frac{1}{\cos x}+\frac{2\sin x}{\cos^2 x}+\frac{2\sin^2 x}{2\cos^2 x}+\sqrt{\left(\cos^2\frac{x}{4}+\sin^2\frac{x}{4}+2\sin\frac{x}{4}\cos\frac{x}{4}\right)}$$

$$+\frac{(1-\cos x)(1+\cos x)}{(1+\cos x)}+\tan^2 x+\cot^2 x+2$$

$$=\tfrac{5}{2}\operatorname{cosec} x\cot x+3\sec x\tan x+(\cos x+\sin x)+\sec^2 x$$

$$+(\sec^2 x-1)+\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)+\left(\sec^2\frac{x}{2}-1\right)$$

$$+(\sec^2 x+\operatorname{cosec}^2 x)$$

$$=\tfrac{5}{2}\operatorname{cosec} x\cot x+3\sec x\tan x+\cos x+\sin x+3\sec^2 x$$

$$+\left(\cos\frac{x}{4}+\sin\frac{x}{4}\right)+\operatorname{cosec}^2 x+\sec^2\frac{x}{2}-2$$

$$\int f(x)\,dx=-\tfrac{5}{2}\operatorname{cosec} x+3\sec x+(\sin x-\cos x)+3\tan x$$

$$+4(\sin x/4-\cos x/4)-\cot x+2\tan x/2-2x$$

$$\frac{1}{2}\cdot\frac{(2x-3)^{5+1}}{5+1}+\frac{1}{7}\cdot\frac{1}{(-2)(7x-5)^{3-1}}+\frac{1}{5}\cdot 2\sqrt{(5x-4)}$$

$$+(-\tfrac{1}{3})\log(2-3x)+\tfrac{1}{3}\frac{(3x+2)^{(1/2)+1}}{\frac{1}{2}+1}$$

$$=\frac{1}{12}(2x-3)^6-\frac{1}{14}\cdot\frac{1}{(7x-5)^2}+\frac{2}{5}\sqrt{(5x-4)}-\tfrac{1}{3}(2-3x)$$

$$+\tfrac{2}{9}(3x+2)^{3/2}$$

$$\tfrac{1}{2}e^{2x-3}-\tfrac{2}{3}7^{4-3x/2}\frac{1}{\log 7}-\tfrac{1}{3}\cos(3x-\tfrac{1}{2})+\tfrac{5}{2}\sin(\tfrac{2}{5}x-2)+\tfrac{1}{3}\frac{a^{3x+2}}{\log a}$$

$$-\tfrac{1}{3}\tan(2-3x)-\tfrac{1}{4}\{-\cot(3-4x)\}-\tfrac{1}{3}\sec(2-3x)$$

$$-\operatorname{cosec} x-\tfrac{1}{3}\operatorname{cosec}(3x-2)$$

$$\frac{1}{\sqrt{[2^2-(\sqrt{3}x)^2]}}+\frac{1}{\sqrt{[(\sqrt{3})^2-(x/2)^2]}}$$

$$+\frac{1}{((\sqrt{5})^2+(2x)^2)}+\frac{1}{(\sqrt{5})^2+(2-3x)^2}$$

Ans $\dfrac{1}{\sqrt{3}}\sin^{-1}\dfrac{\sqrt{(3x)}}{2}+2\sin^{-1}\dfrac{x}{2\sqrt{3}}+\dfrac{1}{\sqrt{5}}\cdot\tfrac{1}{2}\tan^{-1}\dfrac{2x}{\sqrt{5}}$

$$+\frac{1}{\sqrt{5}}(-\tfrac{1}{3})\tan^{-1}\frac{2-3x}{\sqrt{5}}$$

$$\frac{1}{\sqrt{(3x+4)}-\sqrt{(3x+1)}}=\frac{\sqrt{(3x+4)}+\sqrt{(3x+1)}}{(3x+4)-(3x-1)}$$

$$=\tfrac{1}{3}[\sqrt{(3x+4)}+\sqrt{(3x+1)}]$$

Now as in Q 26, above or Q 1 (k) page 570, the minimum value of $\sqrt{(a^2 \sec^2 t+b^2 \operatorname{cosec}^2 t)}$ is $a+b$

$$\text{Maximum value of } p=\frac{a^2-b^2}{a+b}=a-b$$

28 Since $x=r\cos\theta$, $y=r\sin\theta$, where r is the radius vector

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=1, \quad \text{or} \quad \frac{a^2}{r^2\cos^2\theta}+\frac{b^2}{r^2\sin^2\theta}=1$$

or $\quad r^2=a^2\sec^2\theta+b^2\operatorname{cosec}^2\theta$

The min value of r^2 as shown in Q 1 (k) page 570 is $(a+b)^2$ or min value of r is $a+b$

29 $\frac{c^4}{r^2}=a^2\operatorname{cosec}^2 t+b^2\sec^2 t$

$$r^2=\frac{c^4}{a^2\operatorname{cosec}^2 t+b^2\sec^2 t}$$

Now $a^2\operatorname{cosec}^2 t+b^2\sec^2 t=a^2(1+\cot^2 t)+b^2(1+\tan^2 t)$

$=(a^2+b^2+2ab)+(a^2\cot^2 t+b^2\tan^2 t-2ab)$

$=(a+b)^2+(a\cot t-b\tan t)^2$

The value of $a^2\operatorname{cosec}^2 t+b^2\sec^2 t$ or c^4/r^2 is always greater than $(a+b)^2$ and hence its least value is $(a+b)^2$ when

$a\cot t-b\tan t=0$

i e $\quad \tan^2 t=a/b$

Taking reciprocal the greatest value of $\frac{r^2}{c^4}=\frac{1}{(a+b)^2}$

or greatest value of r is $\frac{c^2}{a+b}$

Again $\frac{c^4}{r}=a^2\operatorname{cosec}^2 t+b^2\sec^2 t$

$=(a^2+b^2)+(a^2\cot^2 t+b^2\tan^2 t)$

$=(a^2+b^2-2ab)+(a^2\cot^2 t+b^2\tan^2 t+2ab)$

$=(a-b)^2+(a\cot t+b\tan t)^2$

Now R H S can be made as great as we please *i e* c^4/r^2 has no maximum value which inturn means that its reciprocal r/c^4 and inturn r has no minimum value

30 The given relation is

$xy(y+x)=2a^3$

Differentiating (1) with respect to x, we get

$x\,2y\frac{dy}{dx}+\ldots y \quad x^2\frac{dy}{dx}-2x\ldots=0$ (?)

$$=\tfrac{1}{64}(1-\cos 4x)^2=\tfrac{1}{64}(1-2\cos 4x+\cos^2 4x)$$

$$=\frac{1}{64}\left(1-2\cos 4x+\frac{1+\cos 8x}{2}\right)$$

$$=\tfrac{1}{128}(3-4\cos 4x+\cos 8x)$$

Ans $(x-\tan x+\sec x)+\frac{1}{4}\left(-\frac{\cos 2x}{2}-\frac{\cos 4x}{4}+\frac{\cos 6x}{6}\right)$

$+\frac{1}{2}(\sin 2x+\tan x-2x)+\frac{1}{128}(3x-\sin 4x+\frac{1}{8}\sin 8x)$

2 Method of Substitution

The *method of substitution* (or *change of variable*) consists in substituting $\varphi(t)$ for x where $\varphi(t)$ is a *continuously differentiable* function of t On substituting, we have

$$\int f(x)\,dx=\int f[\varphi(t)]\,\varphi'(t)\,dt$$

and after integration we return to the old variable by inverse substitution $t=\varphi^{-1}(x)$

Read the following examples in which we shall show that by a proper substitution and changing the variable of integration the question is reduced to a standard form whose integral we know

1 $\int \frac{\sin^{-1} x}{\sqrt{(1-x^2)}}\,dx$

d.c. of $\sin^{-1} x$ is $\frac{1}{\sqrt{(1-x^2)}}$ and hence we put $\sin^{-1} x=t$

$$\frac{1}{\sqrt{(1-x^2)}}\,dx=dt \qquad I=\int t\,dt=\frac{t^2}{2}=\tfrac{1}{2}(\sin^{-1} x)^2$$

2 $\int \frac{(1+\log x)^2}{x}\,dx$ (Roorkee 77)

Put $1+\log x=t$ $\qquad I=\frac{(1+\log x)^3}{3}$

Remark In general $\int [f(x)]^n f'(x)\,dx=\frac{[f(x)]^{n+1}}{n+1}$

If $f(x)=t$ then $f'(x)\,dx=dt$

$$I=\int t^n\,dt=\frac{t^{n+1}}{n+1}=\frac{[f(x)]^{n+1}}{n+1}.$$

Similarly $\int \frac{f'(x)}{f(x)}\,dx=\log f(x)$

Put $f(x)=t$ $\qquad f'(x)\,dx=dt$

$$I=\int \frac{1}{t}\,dt=\log t=\log f(x)$$

3 $\int \frac{\sin 2x}{a^2+b^2\sin^2 x}\,dx$ (Roorkee 77)

Put $a^2+b^2\sin^2 x=t,$ $\qquad b^2\,2\sin x\cos x\,dx=dt$

32 For max or min, we have

$f'(x)=3\sin^2 x\cos x+2\lambda\sin x\cos x=0$

$\therefore\quad \cos x=0$ or $\sin x=0$ or $3\sin x=-2\lambda$

i e $\quad \sin x=-\dfrac{2\lambda}{3}$

$\cos x=0$ gives $x=\pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2}$,

and $\sin x=0$ gives $x=0,\ \pm\pi,\ \pm 2\pi$,

and $\sin x=-\dfrac{2\lambda}{3}$ is meaningful if $-\dfrac{3}{2}<\lambda<\dfrac{3}{2}$

Since $-\dfrac{\pi}{2}<x<\dfrac{\pi}{2}$, the values $x=\pm\dfrac{\pi}{2},\ \pm\dfrac{3\pi}{2},\ \pm\pi,\ \pm$ are ruled out Hence the only values of x for which $f'(x$ are $x=0$ and $x=\sin^{-1}\left(-\dfrac{2\lambda}{3}\right)$ where $-\dfrac{3}{2}<\lambda<\dfrac{3}{2}$

Now $f'(x)=\frac{3}{2}\sin x\sin 2x+\lambda\sin 2x$

$=\sin 2x\,(\frac{3}{2}\sin x+\lambda)$

$f''(x)=\sin 2x\,\frac{3}{2}\cos x+2\cos 2x\,(\frac{3}{2}\sin x+\lambda)$

$=\frac{3}{2}\sin 2x\cos x+3\cos 2x\sin x+2\lambda\cos 2x$

We consider $x=0$ and $x=\sin^{-1}\left(-\dfrac{2\lambda}{3}\right)$

When $x=0,\ f''(x)=2\lambda$

and when $\sin x=-\dfrac{2\lambda}{3}$, we have $\cos x=\sqrt{\left(1-\dfrac{4\lambda^2}{9}\right)}$

$\sin 2x=2\sin x\cos x=2\left(-\dfrac{2\lambda}{3}\right)\sqrt{\left(1-\dfrac{4\lambda^2}{9}\right)}$

$=-\dfrac{4\lambda}{9}\sqrt{(9-4\lambda^2)}$

and $\cos 2x=1-2\sin^2 x=1-2\,\dfrac{4\lambda^2}{9}=1-\dfrac{8\lambda^2}{9}$

Hence in this case

$$f''(x)=\frac{3}{2}\left(-\frac{4\lambda}{9}\right)\sqrt{(9-4\lambda^2)}\sqrt{\left(1-\frac{4\lambda^2}{9}\right)}+3\left(1-\frac{8\lambda^2}{9}\right)\left(-\frac{2\lambda}{3}\right)+2\lambda\left(1-\frac{8\lambda^2}{9}\right)$$

$=-\dfrac{2\lambda}{\ }(9-4\lambda^2)$

Alternative Method

$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx$ D C of $\sin x$ is $\cos x$

Put $\sin x = t$ $\qquad \cos x\,dx = dt$

$$I = \int \frac{1}{t}\,dt = \log t = \log \sin x = -\log \operatorname{cosec} x$$

(iii) $\int \sec x\,dx = \int \frac{\sec x\,(\sec x + \tan x)}{(\sec x + \tan x)}\,dx$

$$\int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx$$

Put $\quad \sec x + \tan x = t \qquad (\sec x \tan x + \sec^2 x)\,dx = dt$

$$I = \frac{1}{t}\,dt = \log t = \log (\sec x + \tan x)$$

$$\int \sec x\,dx = \log (\sec x + \tan x) \qquad (3)$$

(iv) $\int \operatorname{cosec} x\,dx = \int \frac{1}{\sin x}\,dx = \int \frac{1}{2 \sin x/2 \cos x/2}\,dx$

Divide above and below by $\cos^2 x/2$

$$I = \int \frac{\sec^2 x/2\,dx}{2 \tan x/2}$$

Now d c of $\tan x/2 = \frac{1}{2} \sec^2 x/2$

Put $\tan x/2 = t$, $\qquad \frac{1}{2} \sec^2 x/2\,dx = dt$

$$I = \int \frac{1}{t}\,dt = \log t = \log \tan x/2$$

$$\int \operatorname{cosec} x\,dx = \log \tan x/2 \qquad (4)$$

§ 4 Some more examples

1 $\quad \int \frac{e^{\sqrt{x}} \cos e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Put $e^{\sqrt{x}} = t \qquad e^{\sqrt{x}} \frac{1}{2\sqrt{x}}\,dx = dt$

$$I = 2 \int \cos t\,dt = 2 \sin t = 2 \sin e^{\sqrt{x}}$$

2 $\quad \int \frac{x^2}{16 + 25x^6}\,dx = \int \frac{x^2}{4^2 + (5x^3)^2}\,dx$

We put $\quad 5x^3 = t \qquad 15x^2\,dx = dt$

$$I = \frac{1}{15} \int \frac{dt}{4^2 + t^2} = \frac{1}{15}\;\frac{1}{4} \tan^{-1} \frac{t}{4} = \frac{1}{60} \tan^{-1} \frac{5x^3}{4}.$$

Let U denote the area of parallelogram $AFDE$. Then

$U = AF \cdot EM$, where EM is $\perp$ to AB

$= AF \cdot AE \sin A = xy \sin A$

or $U = x\left(\dfrac{bc - bx}{c}\right) \sin A$ from (1)

or $U = \dfrac{b}{c}(cx - x^2) \sin A$

Now $\dfrac{du}{dx} = \dfrac{b}{c}(c - 2x) \sin A = 0$

for max or min of U

This gives $x = \frac{1}{2}c$

And $\dfrac{d^2u}{dx^2} = \dfrac{b}{c}(-2) \sin A < 0$

Hence U is max when $x = \frac{1}{2}c$

Hence max area of parallelogram $AFDE$

$$= \frac{b}{c} \cdot \tfrac{1}{2}c\,(c - \tfrac{1}{2}c) \sin A$$

$$= \tfrac{1}{4}bc \sin A = \tfrac{1}{2}\Delta,$$

where Δ is the area of the $\triangle ABC$

Now $\Delta = \frac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 & q & 1 \\ r^2 & -r & 1 \end{vmatrix}$

$$= \tfrac{1}{2}\begin{vmatrix} p^2 & -p & 1 \\ q^2 - p^2 & q + p & 0 \\ r^2 - p^2 & p - r & 0 \end{vmatrix}$$

$$= \tfrac{1}{2}(q+p)(p-r)\begin{vmatrix} q - p & 1 \\ -(r+p) & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}(q+p)(p-r)(q+r)$$

Thus max area of parallelogram $AFDE$

$$= \tfrac{1}{2}\Delta = \tfrac{1}{4}(p+q)(q+r)(p-r)$$

Problem Set B

Integrate the following

1 $\dfrac{1-\tan x}{1+\tan x}$

2 $\dfrac{\cos x+\sin x}{\cos x-\sin x}$

3 (a) $\operatorname{cosec}^2 x\sqrt{(\cot x)}$

(b) $x^2 \sec x^3$ (M N R 86)

4 $\dfrac{\cos\sqrt{x}}{\sqrt{x}}$ (I.I.T 70)

5 $\dfrac{\sin x \cos x}{a^2 \cos^2 x+b^2 \sin^2 x}$

6 $\dfrac{1}{a^2 \cos^2 x+b^2 \sin^2 x}$

7 $\dfrac{\sin 2x}{1+\sin^2 x}$

8 $\dfrac{1}{\sin^{-1} x\sqrt{(1-x^2)}}$

9 $\dfrac{1}{e^x+1}$

10 $\dfrac{e^x-1}{e^x+1}$

11 $\dfrac{a}{b+ce^x}$

12 $\dfrac{\cot x}{\log \sin x}$

13 $\dfrac{\tan x}{\log \sec x}$

14 $\dfrac{\sec x}{\log(\sec x+\tan x)}$

15 $\dfrac{\operatorname{cosec} x}{\log \tan x/2}$

16 $\dfrac{\sec x \operatorname{cosec} x}{\log \tan x}$

17 $\dfrac{1}{\sqrt{x}\sqrt{(x+1)}}$

18 $\dfrac{1}{x+x \log x}$

19 $\dfrac{1+x}{1+x^2}$

20 $\dfrac{2-x^2}{1+x^2}$

21 $\dfrac{(\sin^{-1} x)^3}{\sqrt{(1-x^2)}}$

22 $\dfrac{1}{\cos^2 x\,(1-\tan x)^2}$

23 (a) $\dfrac{e^{x-1}+x^{e-1}}{e^x+x^e}$ (b) $\displaystyle\int\frac{10x^9+10^x \log_e 10}{10^x+x^{10}}dx$ (M N R 79)

24 $\dfrac{\cos^2 x}{\sin^4 x}$

25 $\tan^m x \sec^2 x$

26 $\dfrac{e^x}{\sqrt{(a+be^x)}}$

27 $\dfrac{1}{(e^x+e^{-x})^2}$

28 $\dfrac{1}{x(\log x)^2}$

29 $\tan x \log \sec x$

30 $\sec x \log(\sec x+\tan x)$

31 $\cot x \log(\sin x)$

32 $\operatorname{cosec} x \log(\tan x/2)$

33 $\dfrac{\cot x}{\sqrt{(\sin x)}}$

34 $\dfrac{\sec^4 x}{\sqrt{(\tan x)}}$

35 $\tan^4 x$

and $3(a^2-9)>0$ so that product is +ve (3)

Now (1) gives $a<\frac{29}{7}$,

(2) gives $a<7$,

and (3) gives $a<-3$ or $a>3$

The value of a satisfying these three inequalities is given by

$$a<-3 \quad \text{or} \quad 3<a<\frac{29}{7}$$

$$a\in(-\infty,-3)\cup(3, 29/7) \quad \text{Ans}$$

Problem Set (C)

[Objective Questions]

1 A curve passes through the point (2, 0), and its gradient at the point (x, y) is x^2-2x for all values of x, then the point of maximum ordinate on the curve is $(0, \frac{4}{3})$

(a) True, (b) False

2 From a fixed point P on the circumference of a circle of radius a the perpendicular PR is drawn to the tangent at Q (a variable, point) then the maximum area of the ΔPQR is

3 At $x=5\pi/6$, $2\sin 3x+3\cos 3x$ is

(a) maximum, (b) minimum, (c) zero, (d) none of these

4 $f(x)=(3-x)\,e^{2x}-4x\,e^{x}-x$ has

(a) a maximum at $x=0$ (b) a minimum at $x=0$,

(c) Neither of two at $x=0$

(d) $f(x)$ is not derivable at $x=0$

5 The function $f(x)=x^5-5x^4+5x^3-1$ has

(a) one minimum and two maxima,

(b) two minima and one maximum,

(c) two minima and two maxima,

(d) one minimum and one maximum

6 $f(x)=\sin x+\cos 2x\ (x>0)$ has minima for =

(a) $\frac{n\pi}{2}$ (b) $\frac{3(n+1)\pi}{2}$, (c) $\frac{(2n+1)\pi}{2}$,

(d) None of these

57 $\tan^3 2x \sec 2x$ (I I T 77)

58 $\cot^3 x \operatorname{cosec}^4 x$

59 $\sqrt{(2+\sin 3x)} \cos 3x$ (I I T 76)

60 $\dfrac{x}{1+x^4}$

61 (a) $\dfrac{1}{x \sin^2 (\log x)}$ (b) $\dfrac{\cot (\log x)}{x}$ (Roorkee 78)

62 $\dfrac{\operatorname{cosec} (\tan^{-1} x)}{1+x^2}$

63 $\dfrac{1}{\sin (x-a) \sin (x-b)}$

64 $\dfrac{1}{\cos (x-a) \cos (x-b)}$

65 $\dfrac{\cos x}{\cos (x-a)}$

66 $\dfrac{\sin x}{\sin (x-a)}$

67 $\dfrac{1}{\sqrt{\{\sin^3 x \sin (x+a)\}}}$

68 $\dfrac{\sin 2x}{\sin^4 x+\cos^4 x}$

69 $\int x^{13/2} (1+x^{5/2})^{1/2}\, dx$ (Roorkee 88)

70 $\int \{1+2 \tan x (\tan x+\sec x)\}^{1/2}\, dx$ (Roorkee 87)

Solution Set (B)

1 We know that $\tan\left(\dfrac{\pi}{4}-x\right)=\dfrac{\tan \pi/4-\tan x}{1+\tan \pi/4 \tan x}=\dfrac{1-\tan x}{1+\tan x}$

$I=\int \tan (\pi/4-x)\, dx=-\log \sec (\pi/4-x)=\log \cos (\pi/4-x)$

2 Divide above and below by $\cos x$ $\quad I=\displaystyle\int \frac{1+\tan x}{1+\tan x}\, dx$

$=\int \tan (\pi/4+x)\, dx=\log \sec (\pi/4+x)$

3 (a) Put $\cot x=t$ $\quad -\operatorname{cosec}^2 x\, dx=dt$

$$I=-\int \sqrt{t}\, dt=-\frac{2}{3} t^{3/2}=-\frac{2}{3} (\cot x)^{3/2}$$

(b) Ans $\frac{1}{3} \log (\sec x^3+\tan x^3)$

4 Put $\sqrt{x}=t$ $\quad \dfrac{1}{2\sqrt{x}}\, dx=dt,$

$I=2 \int \cos t\, dt=2 \sin t=2 \sin \sqrt{x}$

5 d c of $\cos^2 x=-2 \cos x \sin x$, d c of $\sin^2 x=2 \sin x \cos x$

Put $a^2 \cos^2 x+b^2 \sin^2 x=t$ $\quad 2 (b^2-a^2) \sin x \cos x\, dx=dt$

$$I=\frac{1}{2 (b^2-a^2)}\int \frac{dt}{t}=\frac{1}{2 (b^2-a^2)} \log t$$

$$=\frac{1}{2 (b^2-a^2)} \log (a^2 \cos^2 x+b^2 \sin^2 x)$$

Divide N^r and D^r by $\cos^2 x$

$$I=\int \frac{\sec^2 x\, dx}{a^2+b^2 \tan^2 x}$$

Put $b \tan x=t$ $\quad b \sec^2 x\, dx=dt$

$$I=\frac{1}{b}\int \frac{dt}{a^2+t^2}=\frac{1}{b} \cdot \frac{1}{a} \tan^{-1} \frac{t}{a}=\frac{1}{ab} \tan^{-1} \left(\frac{b}{a} \tan x\right)$$

Hence y is max at $x=0$. When $x=0$, we get from (1),

$$y=\tfrac{4}{3}$$

Hence the point of maximum ordinate on the curve is $(0, \tfrac{4}{3})$

2. Ans $\frac{3\sqrt{3}}{8}a^2$

Let the circle be $x^2+y^2=a^2$. We take the point P at $(a, 0)$ and Q at $(a\cos\theta, a\sin\theta)$. Equation of tangent at Q is easily seen to be $x\cos\theta+y\sin\theta=a$ (1)

Let PN be perpendicular from P on (1)

Then $PR=\dfrac{a-a\cos\theta}{\sqrt{(\cos^2\theta+\sin^2\theta)}}=2a\sin^2\dfrac{\theta}{2}$

Also $PQ=\sqrt{[(a-a\cos\theta)^2+a^2\sin^2\theta]}$
$=\sqrt{[a^2(1+\cos^2\theta-2\cos\theta+\sin^2\theta)]}$
$=a\sqrt{[2(1-\cos\theta)]}=2a\sin\dfrac{\theta}{2}$

Hence $QR=\sqrt{(PQ^2-PR^2)}=\sqrt{(4a^2\sin^2\theta/2-4a^2\sin^4\theta/2)}$
$=2a\sin\theta/2\cos\theta/2$

If A denotes the area of $\triangle PQR$, then

$A=\frac{1}{2}PR\,QR=\frac{1}{2}\,2a\sin^2\dfrac{\theta}{2}\;2a\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$
$=\frac{1}{2}a^2(1-\cos\theta)\sin\theta=\frac{1}{2}a^2(\sin\theta-\frac{1}{2}\sin 2\theta)$

For maxima or minima, we have

$\dfrac{dA}{d\theta}=\frac{1}{2}a^2(\cos\theta-\cos 2\theta)=0$

This gives $\cos\theta=\cos 2\theta=\cos(2\pi-2\theta)$

or $\theta=2\pi-2\theta$ i e $\theta=\dfrac{2\pi}{3}$

$\dfrac{d^2A}{d\theta^2}=\frac{1}{2}a^2(-\sin\theta+2\sin 2\theta)$

$=\frac{1}{2}a^2\left(\dfrac{-\sqrt{3}}{2}-\dfrac{2\sqrt{3}}{2}\right)<0$ at $\theta=\dfrac{2\pi}{3}$

Hence A is maximum when $\theta=\dfrac{2\pi}{3}$

19 $I=\int\left[\frac{1}{1+x^2}+\frac{x}{1+x^2}\right]dx=\int\left[\frac{1}{1+x^2}+\frac{1}{2}\quad\frac{2x}{1+x^2}\right]dx$

$=\tan^{-1}x+\frac{1}{2}\log(1+x^2)$

20 $I=\int\frac{-x^2-1+3}{x^2+1}dx=\int\left(-1+\frac{3}{x^2+1}\right)dx$

$=-x+3\tan^{-1}x$

21 Ans $\frac{1}{4}(\sin^{-1}x)^4$ Put $\sin^{-1}x=t$

22 $I=\int\frac{\sec^2 x\,dx}{(1-\tan x)^2}$ Put $1-\tan x=t$ $-\sec^2 x\,dx=dt$

$I=-\int\frac{1}{t^2}dt=-\left(-\frac{1}{t}\right)=\frac{1}{t}=\frac{1}{(1-\tan x)}$

23 (*a*) Multiply above and below by e

$I=\frac{1}{e}\int\frac{ee^{x-1}+ex^{e-1}}{e^x+x^e}dx=\frac{1}{e}\int\frac{e^x+ex^{e-1}}{e^x+x^e}dx$

Put $e^x+x^e=t$ $(e^x+ex^{e-1})\,dx=dt$

$I=\frac{1}{e}\int\frac{dt}{t}=\frac{1}{e}\log t=\frac{1}{e}\log(e^x+x^e)$

(*b*) $\log(10^x+x^{10})$

24 $I=\int\cot^2 x\operatorname{cosec}^2 dx=-\frac{1}{3}\cot^3 x$

25 $\frac{\tan^{m+1}x}{m+1}$ Put $\tan x=t$

26 Put $a+be^x=dt$ $be^x\,dx=dt$

$I=\frac{1}{b}\int\frac{dt}{\sqrt{t}}=\frac{2}{b}\sqrt{t}=\frac{2}{b}\sqrt{(a+be^x)}$

27 $I=\int\frac{1}{e^{-2x}(e^{2x}+1)^2}dx=\int\frac{e^{2x}}{(e^{2x}+1)^2}dx$

Put $e^{2x}+1=t$ $2e^{2x}\,dx=dt$

$I=\frac{1}{2}\int\frac{dt}{t^2}=\frac{1}{2}\left(-\frac{1}{t}\right)=-\frac{1}{2(e^{2x}+1)}$

28 Ans $-\frac{1}{\log x}$ Put $\log x=t$

29 Put $\log\sec x=t$ $\frac{1}{\sec x}\sec x\tan x\,dx=dt$

or $\tan x\,dx=dt$

$I=\int t\,dt=\frac{1}{2}t^2=\frac{1}{2}(\log\sec x)^2$

30 We know that $\int\sec x\,dx=\log(\sec x+\tan x)$

$\frac{d}{dx}\log(\sec x+\tan x)=\sec x$

Put $\log(\sec x+\tan x)=t$ $\sec x\,dx=dt$

$I=\int t\,dt=\frac{1}{2}t^2=\frac{1}{2}[\log(\sec x+\tan x)]^2$

Hence $f(x)$ is minimum at $x=\frac{1}{2}(2n+1)\pi$

7 Ans (iii)

8 Ans (b)

9 Ans (c)

$$P'(x)=2x\,(a_1+2a_2x^2+\quad+na_nx^{2n-2})$$

$=0$ for max or min

This gives $x=0$, since other factor cannot be zero because of the condition

$$0<a_0<a_1<a_2<\quad<a_n$$

Now $P''(x)=2\,(a_1+6a_2x^2+\quad+n(2n-1)\,a_n\,x^{2n-2})$

$$P''(x)=2a_1>0 \quad \text{at} \quad x=0$$

Hence $P(x)$ has only one minimum at $x=0$

10 Ans (c)

$$=\frac{e^{x(\log a+1)}}{\log a+1}=\frac{e^{x\log a}\,e^x}{\log a+\log e}=\frac{a^x e^x}{\log ae}$$

42 $I=\int\frac{1}{2\sqrt{x}(1+x)}\,dx$ Put $\sqrt{x}=t$ $\qquad \frac{1}{2\sqrt{x}}\,dx=dt$

$$I=\frac{dt}{1+t^2}=\tan^{-1} t=\tan^{-1}\sqrt{x}$$

(ii) Ans $\frac{1}{2}\tan^{-1} x^2$ Put $x^2=t$ etc

43 $I=\int x\cos^3 x^2 \sin x^2\,dx$

Put $\cos x^2=t$ $\qquad -(\sin x^2)(2x)\,dx=dt$

$$I=-\frac{1}{2}\int t^3\,dt=-\frac{1}{2}\frac{t^4}{4}=-\frac{1}{8}\cos^4 x^2$$

44 $I=\int\frac{1}{2\sqrt{x}}\tan^4\sqrt{x}\sec^2\sqrt{x}\,dx$

Put $\tan\sqrt{x}=t$ $\qquad (\sec^2\sqrt{x})\frac{1}{2\sqrt{x}}\,dx=dt$

$$I=\int t^4\,dt=\frac{1}{5}t^5=\frac{1}{5}\tan^5\sqrt{x}$$

45 $I=\int\frac{\log x\sin[1+(\log x)^2]}{x}\,dx$

Put $1+(\log x)^2=t$ $\qquad \frac{2\log x}{x}\,dx=dt$

$$I=\frac{1}{2}\int\sin t\,dt=-\frac{1}{2}\cos t=-\frac{1}{2}\cos[1+(\log x)^2]$$

46 $I=\int\frac{x\,dx}{\sqrt{(1-x^2)}\cos^2\sqrt{(1-x^2)}}$

$$=\int\frac{x}{\sqrt{(1-x^2)}}\sec^2\sqrt{(1-x^2)}\,dx$$

Put $\sqrt{(1-x^2)}=t$ $\qquad \frac{1}{2\sqrt{(1-x^2)}}(-2x)\,dx=dt$

$$I=-\int\sec^2 t\,dt=-\tan t=-\tan\sqrt{(1-x^2)}$$

47 $I=\int\tan x\sec^2 x\sqrt{(1-\tan^2 x)}\,dx$

Put $1-\tan^2 x=t$

$$-2\tan x\sec^2 x\,dx=dt$$

$$I=-\frac{1}{2}\int t^{1/2}\,dt$$

$$=-\frac{1}{2}\times\frac{2}{3}t^{3/2}$$

$$=-\frac{1}{3}(1-\tan^2 x)^{3/2}$$

48 $I=\int\sqrt{\left(\frac{x}{(a^3-x^3)}\right)}\,dx=\int\frac{\sqrt{x}}{\sqrt{[(a^{3/2})^2-(x^{3/2})^2]}}\,d$

Put $x^{3/2}=t$ $\qquad \frac{3}{2}\sqrt{x}\,dx=dt$

In other words it means that integral of any constant w.r.t. x is that constant multiplied by x

(ii) $\frac{d}{dx}(\log x)=\frac{1}{x}$ $\quad\quad$ $\int \frac{1}{x} dx=\log x$

(iii) $\frac{d}{dx}(e^x)=e^x$ $\quad\quad$ $\int e^x\, dx=e^x$

(iv) $\frac{d}{dx}(a^x)=a^x \log a$ $\quad\quad$ $\int a^x\, dx=\frac{a^x}{\log a}$

(v) $\frac{d}{dx}(\sin x)=\cos x$ $\quad\quad$ $\int \cos x\, dx=\sin x$

(vi) $\frac{d}{dx}(\cos x)=-\sin x$ $\quad\quad$ $\int \sin x\, dx=-\cos x$

(vii) $\frac{d}{dx}(\tan x)=\sec^2 x$ $\quad\quad$ $\int \sec^2 x\, dx=\tan x$

(viii) $\frac{d}{dx}(\cot x)=-\text{cosec}^2 x$ $\quad\quad$ $\int \text{cosec}^2 x\, dx=-\cot x$

(ix) $\frac{d}{dx}(\sec x)=\sec x \tan x,$ $\quad\quad$ $\int \sec x \tan x\, dx=\sec x$

(x) $\frac{d}{dx}(\text{cosec}\, x)=-\text{cosec}\, x \cot x,$

$\int \text{cosec}\, x \cot x\, dx=-\text{cosec}\, x$

(xi) $\frac{d}{dx}(\sin^{-1} x)=\frac{1}{\sqrt{(1-x^2)}},$ $\quad\quad$ $\int \frac{1}{\sqrt{(1-x^2)}}\, dx=\sin^{-1} x$

(xii) $\frac{d}{dx}(\tan^{-1} x)=\frac{1}{1+x},$ $\quad\quad$ $\int \frac{1}{1+x} dx=\tan^{-1} x$

(xiii) $\frac{d}{dx}\sec^{-1} x=\frac{1}{x\sqrt{(x^2-1)}},$ $\int \frac{1}{x\sqrt{(x^2-1)}}\, dx=\sec^{-1} x$

Extension of the above formulae

$\frac{d}{dx}\tan(2x+3)=\sec^2(2x+3)$ d c of $(2x+3)$

$= 2\sec^2(2x+3)$

$\int \sec^2(2x+3)\, dx=\frac{1}{2}\tan(2x+3)$

$\frac{d}{dx}\sin(q-px)=\cos(q-px)$d c of $(q-px)=-p\cos(q-px)$

$\int \cos(q-px)\, dx=-\frac{1}{p}\sin(q-px)$

(b) $I=\int_0^{\pi/2}\frac{1}{2\cos^2\left(\frac{x}{2}-\frac{\pi}{4}\right)}\,dx=\frac{1}{2}\int\sec^2\left(\frac{x}{2}-\frac{\pi}{4}\right)dx$

$=\frac{1}{2}\,2\left[\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)\right]_0^{\pi/2}=0-(-1)=1$

(c) $I=\int\frac{\sin x+\cos x}{\sqrt{\{(\cos x+\sin x)^2\}}}\,dx=\int dx=x$

(d) (i) $1+\sin x=\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)^2=2\left(\frac{1}{\sqrt{2}}\cos\frac{x}{2}+\frac{1}{\sqrt{2}}\sin\frac{x}{2}\right)^2$

$=2\cos^2\left(\frac{x}{2}-\frac{\pi}{4}\right)$

$I=\frac{1}{2}\int\sec^2\left(\frac{x}{2}-\frac{\pi}{4}\right)dx$

$=\frac{1}{2}\,2\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+\text{constant}$

$=\tan\left(\frac{x}{2}-\frac{\pi}{4}\right)+b$

$a=-\pi/4$ and b is any constant

(ii) $I=\sqrt{2}\int\left(\frac{1}{\sqrt{2}}\sin 2x-\frac{1}{\sqrt{2}}\cos 2x\right)dx$

$=-\sqrt{2}\int\cos\left(2x+\frac{\pi}{4}\right)dx$

$=-\sqrt{2}\,\frac{1}{2}\sin\left(2x+\frac{\pi}{4}\right)+\text{constant}$

$=\frac{1}{\sqrt{2}}\sin\left(\pi+2x+\frac{\pi}{4}\right)+b,\qquad \sin(\pi+\theta)=-\sin\theta$

$=\frac{1}{\sqrt{2}}\sin\left(2x-\left(-\frac{5\pi}{4}\right)\right)+b$

$a=-\frac{5\pi}{4}$, b is any constant

53 $I=\int\frac{\sin x}{\sqrt{(1+\sin x)}}\,dx=\int\frac{(1+\sin x)-1}{\sqrt{(1+\sin x)}}\,dx$

$=\int\sqrt{(1+\sin x)}\,dx-\int\frac{1}{\sqrt{(1+\sin x)}}\,dx$

$=\int\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)dx-\int\frac{dx}{\cos x/2+\sin x/2}$ etc.

54 $\int\frac{1+\cos x}{\sin x\cos x}\,dx=\int\frac{1}{\sin x\cos x}\,dx+\int\frac{\cos x}{\sin x\cos x}\,dx,$

$=2\int\operatorname{cosec}2x\,dx+\int\operatorname{cosec}x\,dx$

$=2.\frac{1}{2}\log\tan(2x/2)+\log\tan x/2$

$\sin(a-b)=\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)$

$$I=\frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\,dx$$

by (1)

$$=\frac{1}{\sin(a-b)}\int[\cot(x-a)-\cot(x-b)]\,dx$$

$$=\frac{1}{\sin(a-b)}[\log\sin(x-a)-\log\sin(x-b)]$$

$$=\frac{1}{\sin(a-b)}\log\frac{\sin(x-a)}{\sin(x-b)}$$

4 Proceed as above

$$I=\frac{1}{\sin(a-b)}\log\frac{\sec(x-b)}{\sec(x-a)}$$

5 $$I=\int\frac{\cos x}{\cos(x-a)}\,dx$$

Put $x-a=t$ $x=a+t$ and $dx=dt$

$$I=\int\frac{\cos(a+t)}{\cos t}\,dt$$

$$=\int\frac{\cos a\cos t-\sin a\sin t}{\cos t}\,dt$$

$$=\int(\cos a-\sin a\tan t)\,dt$$

$$=t\cos a-\sin a\log\sec t$$

$$=(x-a)\cos a-\sin a\log\sec(x-a)$$

66 Proceed as above

$$I=(x-a)\cos a+\sin a\log\sin(x-a)$$

67 $$I=\int\frac{1}{\sqrt{[\sin^3 x\sin(x+\alpha)]}}\,dx$$

$$I=\int\frac{1}{\sqrt{[\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)]}}\,dx$$

$$=\int\frac{1}{\sqrt{\{(\sin^4 x(\cos\alpha+\sin\alpha\cot x)\}}}\,dx$$

$$=\int\frac{\operatorname{cosec}^2 x\,dx}{\sqrt{(\cos\alpha+\sin\alpha\cot x)}}$$

Put $\cos\alpha+\sin\alpha\cot x=t$

$-\sin\alpha\operatorname{cosec}^2 x\,dx=dt$

$$I=-\frac{1}{\sin\alpha}\int\frac{dt}{\sqrt{t}}=-2\sqrt{t}\operatorname{cosec}\alpha$$

$$=-2\operatorname{cosec}\alpha\sqrt{(\cos\alpha+\sin\alpha\cot x)}$$

$$\frac{8x+13}{\sqrt{(4x+7)}}=\frac{2(4x+7)-1}{\sqrt{(4x+7)}}=2\sqrt{(4x+7)}-\frac{1}{\sqrt{(4x+7)}}$$

$$(7x-2)\sqrt{(3x+2)}=\tfrac{7}{3}(3x-\tfrac{6}{7})\sqrt{(3x+2)}$$

$$=\tfrac{7}{3}(3x+2-\tfrac{6}{7}-2)\sqrt{(3x+2)}=\tfrac{7}{3}\{(3x+2)^{3/2}-\tfrac{20}{7}\sqrt{(3x+2)}$$

Ans $\frac{1}{3}[\frac{1}{4}\cdot\frac{2}{3}(2x+4)^{3/2}+\frac{1}{3}\cdot\frac{2}{3}(3x+1)^{3/2}+2\cdot\frac{1}{4}\cdot\frac{2}{3}(4x+7)^{3/2}$

$-\frac{1}{4}\cdot 2\sqrt{(4x+7)}]+\frac{7}{3}[\frac{1}{3}\cdot\frac{2}{5}(3x+2)^{5/2}-\frac{20}{7}\cdot\frac{1}{3}\cdot\frac{2}{3}(3x+2)^{3/2}$

9 We have $\frac{1+x+x^2}{x^2(1+x)}=\frac{1+x}{x^2(1+x)}+\frac{x^2}{x^2(1+x)}=\frac{1}{x^2}+\frac{1}{1+x}$,

$$\frac{2x-1}{(x+1)^2}=\frac{2(x+1)-3}{(x+1)^2}=\frac{2}{x+1}-\frac{3}{(x+1)^2},$$

$$\frac{4-5\sin x}{\cos^2 x}=\frac{4}{\cos^2 x}-\frac{5\sin x}{\cos^2 x}=4\sec^2 x-5\sec x\tan x$$

and $\sin 2x\cos 3x=\frac{1}{2}(\sin 5x-\sin x)$

$$\frac{1}{\sin^2 x\cos^2 x}=4\,\frac{1}{(2\sin x\cos x)^2}=4\ \mathrm{cosec}^2\,2x$$

Alternative

$$\frac{1}{\sin^2 x\cos^2 x}=\frac{\sin^2 x+\cos^2 x}{\sin^2 x\cos^2 x}=\frac{1}{\cos^2 x}+\frac{1}{\sin^2 x}$$

$$=\sec^2 x+\mathrm{cosec}^2\,x \text{ etc}$$

Ans $-\frac{1}{x}+\log(1+x)+2\log(x+1)+\frac{3}{(x+1)}+4\tan x-5x$ …

$-4\cdot\frac{1}{2}\cot 2x+\frac{1}{10}\cos 5x$ …

10 We have $\frac{\sin x}{1+\sin x}=\frac{1+\sin x-1}{1+\sin x}=1-\frac{1}{1+\sin x}$ …

$$=1-\frac{1-\sin x}{\cos^2 x}=1-(\sec^2 x-\sec x \ldots$$

$\sin x\sin 2x\sin 3x=\frac{1}{2}(2\sin x\sin 3x)\sin 2x$

$=\frac{1}{2}(\cos 2x-\cos 4x)\sin 2x$

$=\frac{1}{4}(2\sin 2x\cos 2x-2\sin 2x\cos 4x)$

$=\frac{1}{4}[\sin 4x-(\sin 6x-\sin 2x)]$

$=\frac{1}{4}(\sin 2x+\sin 4x-\sin 6x)$,

$$\sec^2 x\cos^2 2x=\left(\frac{\cos 2x}{\cos x}\right)^2=\left(\frac{2\cos^2 x-1}{\cos x}\right)^2$$

$$=\left(2\cos x-\frac{1}{\cos x}\right)^2=4\cos^2 x+\frac{1}{\cos^2 x}-4$$

$=2(1+\cos 2x)+\sec^2 x-4$

$=2\cos 2x+\sec^2 x-2,$

and $\sin^4 x\cos^4 x=\frac{1}{16}(2\sin x\cos x)^4=\frac{1}{16}\sin^4 2x=\frac{1}{64}(2\sin^2 2x)^2$ …

$$\sin(a-b)=\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)$$

$$I=\frac{1}{\sin(a-b)}\int\frac{\sin(x-b)\cos(x-a)-\cos(x-b)\sin(x-a)}{\sin(x-a)\sin(x-b)}\,dx$$

by (1)

$$=\frac{1}{\sin(a-b)}\int[\cot(x-a)-\cot(x-b)]\,dx$$

$$=\frac{1}{\sin(a-b)}[\log\sin(x-a)-\log\sin(x-b)]$$

$$=\frac{1}{\sin(a-b)}\log\frac{\sin(x-a)}{\sin(x-b)}$$

64 Proceed as above

$$I=\frac{1}{\sin(a-b)}\log\frac{\sec(x-b)}{\sec(x-a)}$$

65 $$I=\int\frac{\cos x}{\cos(x-a)}\,dx$$

Put $x-a=t$ $\quad x=a+t$ and $dx=dt$

$$I=\int\frac{\cos(a+t)}{\cos t}\,dt$$

$$=\int\frac{\cos a\cos t-\sin a\sin t}{\cos t}\,dt$$

$$=\int(\cos a-\sin a\tan t)\,dt$$

$$=t\cos a-\sin a\log\sec t$$

$$=(x-a)\cos a-\sin a\log\sec(x-a)$$

66 Proceed as above

$$I=(x-a)\cos a+\sin a\log\sin(x-a)$$

67 $$I=\int\frac{1}{\sqrt{[\sin^3 x\sin(x+\alpha)]}}\,dx$$

$$I=\int\frac{1}{\sqrt{[\sin^3 x(\sin x\cos\alpha+\cos x\sin\alpha)]}}\,dx$$

$$=\int\frac{1}{\sqrt{((\sin^4 x(\cos\alpha+\sin\alpha\cot x))}}\,dx$$

$$=\int\frac{\operatorname{cosec}^2 x\,dx}{\sqrt{(\cos\alpha+\sin\alpha\cot x)}}$$

Put $\cos\alpha+\sin\alpha\cot x=t$

$$-\sin\alpha\operatorname{cosec}^2 x\,dx=dt$$

$$I=-\frac{1}{\sin\alpha}\int\frac{dt}{\sqrt{t}}=-2\sqrt{t}\operatorname{cosec}\alpha$$

$$=-2\operatorname{cosec}\alpha\sqrt{(\cos\alpha+\sin\alpha\cot x)}$$

$$I=\frac{1}{b^2}\int\frac{dt}{t}=\frac{1}{b^2}\log t=\frac{1}{b^2}\log(a^2+b^2\sin^2 x)$$

Remark Note that $\int \frac{f'(x)}{\sqrt{[f(x)]}}\,dx=2\sqrt{[f(x)]}$

Put $f(x)=t$ $\quad f'(x)\,dx=dt$

$$I=\int\frac{1}{\sqrt{t}}\,dt=2\sqrt{t}=2\sqrt{[f(x)]}$$

4 (a) $\int\frac{\sec^2 x\,dx}{\sqrt{(\tan x)}}$ $\quad$ (b) $\int\frac{1}{x\sqrt{(1+\log x)}}\,dx$ (Roorkee 77)

Put $\tan x=t$

Here d c of $\tan x$ is $\sec^2 x$

$$\sec^2 x\,dx=dt$$

$$I=\int\frac{dt}{\sqrt{t}}=2\sqrt{t}=2\sqrt{(\tan x)}$$

Another form $I=\int\frac{\sqrt{(\tan x)}}{\sin x\cos x}\,dx$

Divide above and below by $\cos^2 x$

$$I=\int\frac{\sec^2 x}{\tan x}\sqrt{(\tan x)}\,dx=\int\frac{\sec^2 x}{\sqrt{(\tan x)}}\,dx=2\sqrt{(\tan x)}$$

§ 3 Four Important Formulae

Integration of tan x, cot x, sec x, cosec x

(i) $\int\tan x\,dx=\int\frac{\sec x\tan x}{\sec x}\,dx$ $\quad$ D C of $\sec x$ is $\sec x\tan x$

Put $\sec x=t$ $\quad \sec x\tan x\,dx=dt$

$$I=\int\frac{1}{t}\,dt=\log t=\log\sec x=-\log\cos x \qquad (1)$$

$$I=\int\tan x\,dx=\log\sec x=-\log\cos x$$

Alternative Method

$\int\tan x\,dx=\int\frac{\sin x}{\cos x}\,dx$ $\quad$ Put $\cos x=t$ $\quad -\sin x\,dx=dt$

$$I=-\int\frac{dt}{t}=-\log t=-\log\cos x=\log\sec x$$

(ii) $\int\cot x\,dx=\int\frac{\mathrm{cosec}\,x\cot x}{\mathrm{cosec}\,x}\,dx$

Now D C of $\mathrm{cosec}\,x$ is $-\mathrm{cosec}\,x\cot x$

Put $\mathrm{cosec}\,x=t$ $\quad -\mathrm{cosec}\,x\cot x\,dx=dt$

$$I=-\int\frac{dt}{t}=-\log t=-\log\mathrm{cosec}\,x=\log\sin x$$

$$\int\cot x\,dx=\log\sin x=-\log(\mathrm{cosec}\,x)$$

$$\int \frac{1}{\sqrt{(a^2-x^2)}} dx = \sin^{-1} x/a$$

$$\int \sqrt{(x^2+a^2)}\, dx = \frac{x}{2}\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2 \log\,[x+\sqrt{(x^2+a^2)}]$$

or $$= \frac{x}{2}\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2 \sinh^{-1}\frac{x}{a}$$

$$\int \sqrt{(x^2-a^2)}\, dx = \frac{x}{2}\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2 \log\,[x+\sqrt{(x^2-a^2)}]$$

or $$= \frac{x}{2}\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2 \cosh^{-1}\frac{x}{a}$$

$$\int \sqrt{(a^2-x^2)}\, dx = \frac{x}{2}\sqrt{(a^2-x^2)} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}$$

7 Application of above formulae to the Integrals

$$\int \frac{dx}{ax^2+bx+c},\ \int \frac{dx}{\sqrt{(ax^2+bx+c)}},\ \int \sqrt{(ax^2+bx+c)}\, dx$$

Express ax^2+bx+c as sum or difference of two squares and pply formulae no (1), o (9) as the case may be

Rule to express a quadratic as sum or difference of two squares

$$3x^2+4x-7 = 3\left[x^2+\frac{4}{3}x - \frac{7}{3}\right]$$

$$= 3\left[\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)-\frac{7}{3}-\frac{4}{9}\right]$$

$$= 3\left[\left(x+\frac{2}{3}\right)^2-\left(\frac{5}{3}\right)^2\right]$$

$$4-3x-2x^2 = -2\left[x^2+\frac{3}{2}x-2\right]$$

$$= -2\left[\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-2-\frac{9}{16}\right]$$

$$= -2\left[\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{(41)}}{4}\right)^2\right]$$

$$= 2\left[\left(\frac{\sqrt{(41)}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2\right]$$ **Note**

Rule Make the coefficient of x^2 equal to $+1$ by taking out the constant and then add and subtract square of half the coefficient of x

Ex 1 $\int \frac{1}{4x^2+4x+5}\, dx$

$$4x^2+4x+5 = 4\,(x^2+x+5/4) = 4\,(x^2+x+\tfrac{1}{4}+5/4-\tfrac{1}{4})$$
$$= 4[(x+\tfrac{1}{2})^2+1^2]$$

3 $\int \frac{e^x(1+x)}{\sin^2(xe^x)}dx = \int e^x(1+x)\,\text{cosec}^2(xe^x)\,dx$

Here we think of $\int \text{cosec}^2 x\,dx = -\cot x$

Put $xe^x = t$ $\quad (xe^x + e^x\,1)\,dx = dt$

$I = \int \text{cosec}^2 t\,dt = -\cot t = -\cot(xe^x)$

Similarly $\int \frac{e^x(x+1)\,dx}{\cos^2(xe^x)} = \tan(xe^x)$ (Roorkee 79)

4 $\int \frac{x^3\tan^{-1}x^4}{1+x^8}dx$

Put $\tan^{-1}x^4 = t$, $\quad \frac{1}{1+(x^4)^2}(4x^3)\,dx = dt$

$I = \frac{1}{4}\int t\,dt = \frac{1}{4}\frac{t^2}{2} = \frac{1}{8}(\tan^{-1}x^4)^2$ (Roorkee 1982)

5 $\int \frac{x\tan^{-1}x^2}{1+x^4}dx$ Ans $\frac{1}{4}(\tan^{-1}x^2)^2$

(Roorkee 75)

Put $\tan^{-1}x^2 = t$ etc

6 $I = \int x^2 \sec x^3\,dx$

Put $x^3 = t$ $\quad 3x^2\,dx = dt$

$I = \frac{1}{3}\int \sec t\,dt$

$= \frac{1}{3}\log(\sec x^3 + \tan x^3)$

§ 5 Integral of $\int \frac{1}{a\sin x + b\cos x}dx$

Here we put $\quad a = r\cos\alpha \quad b = r\sin\alpha$

$I = \frac{1}{r}\int \frac{dx}{\sin(x+\alpha)} = \frac{1}{r}\int \text{cosec}(x+\alpha)\,dx$

$= \frac{1}{r}\log\tan\frac{x+\alpha}{2}$

where $r = \sqrt{(a^2+b^2)}$ and $\tan\alpha = b/a$

Note If we put $b = r\cos\alpha$, $a = r\sin\alpha$.

Then $I = \frac{1}{r}\int \frac{dx}{\cos(x-\alpha)} = \frac{1}{r}\int \sec(x-\alpha)\,dx$

$= \frac{1}{r}\left[\log[\sec(x-\alpha) + \tan(x-\alpha)]\right]$

where $r = \sqrt{(a^2+b^2)}$ and $\tan\alpha = a/b$

6 $\int \frac{1}{\sqrt{(a^2-x^2)}} dx = \sin^{-1} x/a$

7 $\int \sqrt{(x^2+a^2)}\, dx = \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{1}{2}a^2 \log [x+\sqrt{(x^2+a^2)}]$

or $= \frac{x}{2}\sqrt{(x^2+a^2)} + \frac{1}{2}a^2 \sinh^{-1} \frac{x}{a}$

8 $\int \sqrt{(x^2-a^2)}\, dx = \frac{x}{2}\sqrt{(x^2-a^2)} - \frac{1}{2}a^2 \log [x+\sqrt{(x^2-a^2)}]$

or $= \frac{x}{2}\sqrt{(x^2-a^2)} - \frac{1}{2}a^2 \cosh^{-1} \frac{x}{a}$

9 $\int \sqrt{(a^2-x^2)}\, dx = \frac{x}{2}\sqrt{(a^2-x^2)} + \frac{1}{2}a^2 \sin^{-1} \frac{x}{a}$

§ 7 Application of above formulae to the Integrals

$$\int \frac{dx}{ax^2+bx+c},\ \int \frac{dx}{\sqrt{(ax^2+bx+c)}},\ \int \sqrt{(ax^2+bx+c)}\, dx$$

Express ax^2+bx+c as sum or difference of two squares and apply formulae no (1), to (9) as the case may be

Rule to express a quadratic as sum or difference of two squares

$$3x^2+4x-7 = 3\left[x^2+\frac{4}{3}x-\frac{7}{3}\right]$$
$$= 3\left[\left(x^2+\frac{4}{3}x+\frac{4}{9}\right)-\frac{7}{3}-\frac{4}{9}\right]$$
$$= 3\left[\left(x+\frac{2}{3}\right)^2-\left(\frac{5}{3}\right)^2\right]$$
$$4-3x-2x^2 = -2\left[x^2+\frac{3}{2}x-2\right]$$
$$= -2\left[\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-2-\frac{9}{16}\right]$$
$$= -2\left[\left(x+\frac{3}{4}\right)^2-\left(\frac{\sqrt{(41)}}{4}\right)^2\right]$$
$$= 2\left[\left(\frac{\sqrt{(41)}}{4}\right)^2-\left(x+\frac{3}{4}\right)^2\right]$$

Note

Rule Make the coefficient of x^2 equal to $+1$ by taking out the constant and then add and subtract square of half the coefficient of x

Ex 1 $\int \frac{1}{4x^2+4x+5} dx$

$$4x^2+4x+5 = 4\,(x^2+x+5/4) = 4\,(x^2+x+\tfrac{1}{4}+5/4-\tfrac{1}{4})$$
$$= 4[(x+\tfrac{1}{2})^2+1^2]$$

37 $\frac{\sin \sqrt{x}}{\sqrt{x}}$

$\frac{1}{x^2 \cos^3 1/x}$

39 $\sec^7 x \tan x$

$\left(1-\frac{1}{x^2}\right) e^{x+1/x}$

41 $a^x e^x$

40 $\frac{1}{x \log x \log (\log x)}$

42 (a) $\frac{1}{2\sqrt{x}(1+x)}$ (b) $\frac{x}{1+x^4}$ (IIT 78)

43 $x \cos^3 x^2 \sin x^2$

44 $\frac{1}{2\sqrt{x}} \tan \sqrt{x} \sec^2 \sqrt{x}$

45 $\frac{\log x \sin [1+(\log x)^2]}{x}$

46 (a) $\frac{x}{\sqrt{(1-x^2)} \cos^2 \sqrt{(1-x^2)}}$

47 $\tan x \sec^2 x \sqrt{(1-\tan^2 x)}$

48 (a) $\sqrt{\left(\frac{x}{a^3-x^3}\right)}$

(b) $\frac{x}{\sqrt{(4-x^4)}}$ (Roorkee 76) (c) $\int \frac{a^x \, dx}{\sqrt{(1-a^{-x})}}$ (MNR 8

(d) $\frac{1}{x^2 (x^4+1)^{3/4}}$ (IIT 84)

49 $\frac{1}{\sin x+\cos x}$

50 $\frac{\sec x}{a \tan x+b}$

51 $\frac{dx}{\sin x+\sqrt{3} \cos x}$

52 (a) $\frac{1}{\sqrt{(1+\sin x)}}$ (MNR 8

(b) $\int_0^{\pi/2} \frac{dx}{1+\sin x}$ is

(a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\frac{3}{2}$

(c) $\int \frac{\sin x+\cos x}{\sqrt{(1+\sin 2x)}} \, dx$

(d) For what values of a and b the following equations are correct (Roorkee 79)

(i) $\int \frac{dx}{1+\sin x}=\tan \left(\frac{x}{2}+a\right)+b$

(ii) $\int (\sin 2x-\cos 2x) \, dx=\frac{1}{\sqrt{2}} \sin (2x-a)+b$ (Roorkee 7

53 $\frac{\sin x}{\sqrt{(1+\sin x)}}$

54 $\frac{1+\cos x}{\sin x \cos x}$

$\frac{1}{3 \sin x-4 \sin^3 x}$

55 $\frac{1}{4 \cos^3 x-3 \cos x}$

Ex 8 $\int \sqrt{[(x^2+4x-5)]}\,dx$

$=\int \sqrt{[(x+2)^2-3^2]}\,dx$ As in Ex (3)

$=\frac{1}{2}(x+2)\sqrt{[(x+2)^2-3^2]}-\frac{1}{2}3^2\log\{(x+2)+\sqrt{[(x+2)^2-3^2]}\}$
formula (8), § 6

Ex 9 $\int \sqrt{[(1+3x-x^2]}\,dx$

$$I=\int\sqrt{\left[\left(\frac{\sqrt{(13)}}{2}\right)^2 \quad (x-3/2)^2\right]}dx$$

$$=\frac{x-3/2}{2}\sqrt{\left[\left(\frac{\sqrt{(13)}}{2}\right)^2-(x-3/2)^2\right]}$$

$$+\frac{1}{2}\left(\frac{\sqrt{(13)}}{2}\right)^2\sin^{-1}\frac{(x-3/2)}{\sqrt{(13)}/2} \quad \text{formula (9), § 6}$$

§ 8 Application to the integrals

$$\int\frac{px+q}{ax^2+bx+c}dx, \int\frac{px+q}{\sqrt{(ax^2+bx+c)}}dx,$$

$$\int(px+q)\sqrt{(ax^2+bx+c)}\,dx$$

Here we should write

$px+q=l\,[\text{d c of }(ax^2+bx+c)]+m$ (1)

Find the values of l, m by comparing the cofficients of x and constant term on both sides of (1) In this way the question will reduce to sum of two integrals one of which will be done as explained in the last nine examples and other will be done as under

$\int\frac{f'(x)}{f(x)}dx=\log f(x)$ by putting $f(x)=t$ for Ist

$\int\frac{f'(x)}{\sqrt{\{f(x)\}}}dx=2\sqrt{\{f(x)\}}$ Putting $f(x)=t$ for 2nd

$\int f'(x)\sqrt{\{f(x)\}}\,dx=\frac{2}{3}[f(x)]^{3/2}$ by putting $f(x)=t$ for 3rd

Ex 10 $\int\frac{3x+2}{4x^2+4x+5}dx$

$3x+2=l\,(\text{d c of }4x^2+4x+5)+m$

or $3x+2=l\,(8x+4)+m$

Compare x and constant $8l=3$ and $4l+m=2$

$l=3/8$, $m=2-4l=2-\frac{3}{2}=\frac{1}{2}$,

$$I=\frac{3}{8}\int\frac{(8x+4)\,dx}{4x^2+4x+5}+\frac{1}{2}\int\frac{dx}{4x^2+4x+5}$$

For first integral put $4x^2+4x+5=t$ $(8x+4)\,dx=dt$

$$I=\frac{3}{8}\int\frac{dt}{t}+\frac{1}{2}\int\frac{dx}{4x^2+4x+5}$$

7 As in Q 5 d c of $\sin^2 x = 2 \sin x \cos x = \sin 2x$

Put $1+\sin^2 x = t$ $\qquad 2 \sin x \cos x\, dx = dt$

$$I = \int \frac{dt}{t} = \log t = \log (1+\sin^2 x)$$

8 Ans $\log \sin^{-1} x$ $\qquad$ Put $\sin^{-1} x = t$

Note In all the questions involving e^x multiply above and below by e^x and then put $e^x = t$

9 $I = \int \frac{e^x\, dx}{e^x (e^x+1)}$ $\qquad$ Put $e^x = t$ $\qquad e^x\, dx = dt$

$$I = \int \frac{dt}{t(t+1)} = \int \left(\frac{1}{t} - \frac{1}{t+1}\right) dt \quad \text{By partial fractions.}$$

$$= \log t - \log (t+1) = \log e^x - \log (e^x+1) = x - \log (e^x+1)$$

10 $I = \int \frac{e^x (e^x-1)}{e^x (e^x+1)}\, dx$ $\qquad$ Put $e^x = t$ $\qquad e^x\, dx = dt$

$$= \int \frac{t-1}{t(t+1)}\, dt = \int \left(\frac{2}{t+1} - \frac{1}{t}\right) dt$$

$$= 2 \log (t+1) - \log t = 2 \log (e^x+1) - \log e^x$$

$$= 2 \log (e^x+1) - x$$

11 Ans $\frac{a}{b} [x - \log (b+ce^x)]$

12 We know $\int \cot x\, dx = \log \sin x$ $\qquad \frac{d}{dx} (\log \sin x) = \cot x$

Put $\log \sin x = t$ $\qquad \cot x\, dx = dt$

Put $I = \int \frac{dt}{t} = \log t = \log (\log \sin x)$

13 Ans $\log (\log \sec x)$

14 Ans $\log [\log (\sec x + \tan x)]$

15 Ans $\log (\log \tan x/2)$

16 Put $\log \tan x = t$ $\qquad \frac{1}{\tan x} \sec^2 x\, dx = dt$ or $\sec x \operatorname{cosec} dx = dt$

or $\frac{\cos x}{\sin x} \frac{1}{\cos^2 x}\, dx = dt$

$$I = \int \frac{dt}{t} = \log t = \log (\log \tan x)$$

17 Put $\sqrt{x}+1 = t$ $\qquad \frac{1}{2\sqrt{x}}\, dx = dt$

$$I = \int \frac{2dt}{t} = 2 \log t = 2 \log (1+\sqrt{x})$$

18 $I = \int \frac{1}{x(1+\log x)}\, dx = \log (1+\log x)$

3 $\int \frac{dx}{a+b\sin x}$, 4 $\int \frac{p\cos x+q\sin x+r}{a\cos x+b\sin x+c}$

5 $\int \frac{p\cos x+q\sin x}{a\cos x+b\sin x}dx$

Rule for 1, 2 and 3

Write $\cos x=\cos^2 x/2-\sin^2 x/2$, $\sin x=2\sin x/2\cos x/2$ Then divide above and below by $\cos^2 x/2$ The numerator shall become $\sec^2 x/2$ and the denominator will be a quadratic in $\tan x/2$ (change $\sec^2 x/2$ into $1+\tan^2 x/2$) Putting $\tan x/2=t$ the question shall reduce to the form

$$\int \frac{dt}{at^2+bt+c}$$

Rule for 4

Express numerator as $l\,(D^r)+m$ (d c of D^r)$+n$

Find l, m, n by comparing the coefficients of $\sin x$, $\cos x$ and constant term and split the integral into sum of three integrals as

$$l\int dx+m\int \frac{(\text{d c of } D^r)}{D^r}\,dx+n\int \frac{dx}{a\cos x+b\sin x+c}$$

$=lx+m\log(D^r)+n$ (as explained in rule above)

Rule for 5

Here express numerator as $l(D^r)+m$ (d c of D^r) and find l and m by comparing the coefficients of $\sin x$ and $\cos x$ Now split the integral into sum of two intergrals

$$l\int dx+m\int \frac{(\text{d c of } D^r)}{D^r}\,dx$$

$$=lx+m\log(D^r)$$

shall illustrate the above by giving suitable examples

Ex 13 $\int \frac{dx}{4\sin^2 x+4\sin x\cos x+5\cos^2 x}$

Divide above and below by $\cos^2 x$

$I=\int \frac{\sec^2 x\,dx}{4\tan^2 x+4\tan x+5}$, Put $\tan x=t$

$I=\int \frac{dt}{4t^2+4t+5}=\frac{1}{4}\int \frac{dt}{t^2+t+5/4}$

$=\frac{1}{4}\int \frac{dt}{(t+\frac{1}{2})^2+1^2}=\frac{1}{4}\tan^{-1}(t+\frac{1}{2})$ as in Ex 1

$=\frac{1}{4}\tan^{-1}(\tan x+\frac{1}{2})$

31 Ans $\frac{1}{2}(\log \sin x)^2$

32 $\frac{1}{2}(\log \tan x/2)^2$

33 $I=\int \frac{\cos x}{\sin x\sqrt{(\sin x)}}\,dx$ Put $\sin x=t$

$I=\int \frac{dt}{t^{3/2}}=\int t^{-3/2}\,dt=\frac{-2}{\sqrt{t}}=\frac{-2}{\sqrt{(\sin x)}}$

34 $I=\int \frac{\sec^2 x \sec^2 x\,dx}{\sqrt{(\tan x)}}$ Put $\tan x=t$ $\quad \sec^2 x\,dx=dt,$

$I=\int \frac{1+t^2}{\sqrt{t}}\,dt=\int \left(\frac{1}{\sqrt{t}}+t^{3/2}\right)dt$

$=2\sqrt{t}+\frac{2}{5}t^{5/2}=\frac{2}{5}\sqrt{t}\,(5+t^2)$

$=\frac{2}{5}\sqrt{(\tan x)}\,(5+\tan^2 x)$

35 $\int \tan^4 x\,dx=\int \tan^2 x\,(\sec^2 x-1)\,dx$

$=\int[\tan^2 x \sec^2 x-(\sec^2 x-1)]\,dx$

$=\frac{1}{3}\tan^3 x-\tan x+x$

Similarly $\int \tan^3 x\,dx=\frac{1}{2}\tan^2 x-\log \sec x$

36 $I=\int \frac{1}{x^2}\sec^2\frac{1}{x}\,dx$ Put $\frac{1}{x}=t$ $\quad -\frac{1}{x^2}\,dx=dt$

$I=-\int \sec^2 t\,dt, =-\tan t=-\tan(1/x)$

37 $\int \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx$ Put $\sqrt{x}=t$ $\quad \frac{1}{2\sqrt{x}}\,dx=dt$

$I=2\int \sin t\,dt=-2\cos t=-2\cos\sqrt{x}$

38 $I=\int\left(1-\frac{1}{x^2}\right)e^{(x+1/x)}\,dx$ Put $x+\frac{1}{x}=t$

$I=\int e^t\,dt=e^t=e^{(x+1/x)}$

39 $I=\int \sec^p x\tan x\,dx=\int \sec^{p-1}x\sec x\tan x\,dx$

Put $\sec x=t$ $\quad \sec x\tan x\,dx=dt$

$I=\int t^{p-1}\,dt=t^p/p=\frac{1}{p}\sec^p x$

40 $I=\int \frac{1}{x\log x\log(\log x)}\,dx$

Put $\log(\log x)=t$ $\quad \frac{1}{\log x}\,\frac{1}{x}\,dx=dt$

$I=\int \frac{1}{t}\,dt=\log t=\log[\log(\log x)]$

41 We know that $a^x=e^{x\log a}$

$I=\int e^{x\log a}\,e^x\,dx=\int e^{x(\log a+1)}\,dx$

Ex. 17 $\int \frac{dx}{5+4 \sin x}$ or $\int \frac{dx}{4+5 \sin x}$

$$I=\int \frac{dx}{5+4\,(2 \sin x/2 \cos x/2)} \text{ or } \int \frac{dx}{4+5\,(2 \sin x/2 \cos x/2)}$$

Divide above and below by $\cos^2 x/2$

$$=\int \frac{\sec^2 x/2\, dx}{5\,(1+\tan^2 x/2)+8 \tan x/2} \text{ or } \int \frac{\sec^2 x/2\, dx}{4\,(1+\tan^2 x/2)+10 \tan x/2}$$

Put $\tan x/2=t$ $\quad \frac{1}{2} \sec^2 x/2\, dx=dt$

$$I=\int \frac{2dt}{5t^2+8t+5} \text{ or } \int \frac{2dt}{4t^2+10t+4}$$

$$=\frac{2}{5} \int \frac{dt}{t^2+\frac{8}{5}t+1} \text{ or } \frac{2}{4} \int \frac{dt}{t^2+\frac{5}{2}t+1}$$

$$=\frac{2}{5} \int \frac{dt}{(t+\frac{4}{5})^2+(\frac{3}{5})^2} \text{ or } \frac{1}{2} \int \frac{dt}{(t+\frac{5}{4})^2-(\frac{3}{4})^2}$$

$$=\frac{2}{5}\, \frac{1}{3/5} \tan^{-1} \frac{t+\frac{4}{5}}{3/5} \text{ or } \frac{1}{2}\, \frac{1}{2(3/4)} \log \frac{(t+5/4)-3/4}{(t+5/4)+3/4}$$

formula 1 and 2

$$=\frac{2}{3} \tan^{-1} \frac{5 \tan x/2+4}{3} \text{ or } \frac{1}{3} \log \frac{2 \tan x/2+1}{2\,(\tan x/2+2)}$$

Ex 18 $\int \frac{2+3 \cos \theta}{\sin \theta+2 \cos \theta+3}\, d\theta$

Write $N^r=l\,(D^r)+m$ (d c of D^r) $+n$

Let $2+3 \cos \theta=l\,(\sin \theta+2 \cos \theta+3)+m\,(\cos \theta-2 \sin \theta)+n$

Comparing coefficients of constant, $\sin \theta$, and $\cos \theta$, we get

$3l+n=2, \quad 2l+m=3, \quad l-2m=0$

$l=6/5, \quad m=3/5, \quad n=-8/5$

$$I=\int l\, d\theta+m \int \frac{\cos \theta-2 \sin \theta}{\sin \theta+2 \cos \theta+3}\, d\theta + n \int \frac{d\theta}{\sin \theta+2 \cos \theta+3}$$

$$=l\theta+m \log (\sin \theta+2 \cos \theta+3)+n\, I_1$$

Now $I_1=\int \frac{d\theta}{\sin \theta+2 \cos \theta+3}$

$$=\int \frac{d\theta}{2 \sin \theta/2 \cos \theta/2+2\,(\cos^2 \theta/2-\sin^2 \theta/2)+3}$$

Divide above and below by $\cos^2 \theta/2$

$$I_1=\int \frac{\sec^2 \theta/2\, d\theta}{2 \tan \theta/2+2\,(1-\tan^2 \theta/2)+3\,(1+\tan^2 \theta/2)}$$

Put $\tan \theta/2=t$ so that $\frac{1}{2} \sec^2 \theta/2\, d\theta=dt$

$$I=\frac{2}{3}\int\frac{dt}{\sqrt{[(a^{3/2})^2-t^2]}}=\frac{2}{3}\sin^{-1}\frac{t}{a^{3/2}}=\frac{2}{3}\sin^{-1}\frac{x^{3/2}}{a^{3/2}}$$

(a) $\frac{1}{2}\sin^{-1}\frac{x^2}{2}$ (c) $\frac{1}{\log a}\sin^{-1}(a^x)$

(d) $I=\int\frac{dx}{x^2x^3\left(1+\frac{1}{x^4}\right)^{3/2}}$ Put $1+\frac{1}{x^4}=t$

$$-\frac{4}{x^5}dx=dt$$

$$I=-\frac{1}{4}\int\frac{dt}{t^{3/4}}=-\frac{1}{4}4t^{1/4}=-t^{1/4}=-\left(1+\frac{1}{x^4}\right)^{1/4}$$

$$=-\frac{1}{x}(1+x^4)^{1/4}$$

49 $I=\int\frac{1}{\sin x+\cos x}dx=\int\frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin x+\frac{1}{\sqrt{2}}\cos x\right)}dx$

$$=\frac{1}{\sqrt{2}}\int\frac{1}{\sin(x+\pi/4)}dx=\frac{1}{\sqrt{2}}\int\operatorname{cosec}\left(x+\frac{\pi}{4}\right)dx$$

$$=\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\frac{\pi}{8}\right)$$

50 $I=\int\frac{\sec x\,dx}{a\tan x+b}=\int\frac{dx}{a\sin x+b\cos x}$

Now see § 5 page 626

51 $\int\frac{dx}{\sin x+\sqrt{3}\cos x}$

Put $1=r\cos\alpha$, $\sqrt{3}=r\sin\alpha$ $\alpha=\pi/3$.

$r^2=1+3=4$ or $r=2$ and $\tan\alpha=\sqrt{3}$

$$I=\int\frac{dx}{r\sin(x+\alpha)}=\int\frac{1}{2}\operatorname{cosec}(x+\pi/3)\,dx$$

$$=\frac{1}{2}\log\tan\left(\frac{x}{2}+\frac{\pi}{6}\right)$$

52 $\int\frac{dx}{\sqrt{(1+\sin x)}}=\int\frac{dx}{\sqrt{(\sin^2 x/2+\cos^2 x/2+2\sin x/2\cos x/2)}}$

$$=\int\frac{dx}{\sin\frac{x}{2}+\cos\frac{x}{2}}=\int\frac{1}{\sqrt{2}\left(\frac{1}{\sqrt{2}}\sin\frac{x}{2}+\frac{1}{\sqrt{2}}\cos\frac{x}{2}\right)}dx$$

$$=\int\frac{1}{\sqrt{2}\sin(x/2+\pi/4)}dx=\frac{1}{\sqrt{2}}\int\operatorname{cosec}(x/2+\pi/4)\,dx$$

$$=1/\sqrt{2}\;2\log\tan(x/4+\pi/8)=\sqrt{2}\log\tan(x/4+\pi/8)$$

3 $\dfrac{x^2}{25x^6-16}$

4 $\dfrac{\cos x}{4-\sin^2 x}$

5 $\dfrac{3x^2}{\sqrt{(9-16x^6)}}$

6 $\dfrac{1}{(1+x^2)\sqrt{\{p^2+q^2(\tan^{-1}x)^2\}}}$

7 $\dfrac{x^2}{\sqrt{(x^6+1)}}$

8 $\dfrac{1}{[(1-x^2)\{(2\sin^{-1}x)^2-9\}]^{1/2}}$

9 $\sec x \tan x \sqrt{(\sec^2 x+1)}$

10 $\cos x \sqrt{(4-\sin^2 x)}$

11 $x^2\sqrt{(q^2x^6-p^2)}$

12 $\dfrac{\cos x}{\sin^2 x+4\sin x+5}$

13 $\dfrac{1}{x\{(\log x)^2+4\log x-1\}}$

14 $\dfrac{x}{x^4+x^2+1}$

15 $\dfrac{3x+1}{2x^2+x+1}$

16 $\dfrac{2x^2+3x+4}{x^2+6x+10}$

17 $\dfrac{1-x}{4x^2-4x-3}$

18 $\dfrac{3x+4}{x^2+4x+2}$

19 $\dfrac{1}{\sqrt{(1-4x+x^2)}}$

20 $\dfrac{1}{\sqrt{(x^2+3x+1)}}$

21 $\dfrac{1}{\sqrt{(x^2+2x+4)}}$

22 $\dfrac{x}{\sqrt{(9+8x-x^2)}}$

23 $\dfrac{x+3}{\sqrt{(x^2+2x+2)}}$

24 $\dfrac{x+2}{\sqrt{(4x-x^2)}}$

25 $\sqrt{\left(\dfrac{a+x}{a-x}\right)}$

26 $\sqrt{\left(\dfrac{1-x}{1+x}\right)}$ **(I I T 1971)**

27 $x\sqrt{\left(\dfrac{a^2-x^2}{a^2+x^2}\right)}$

28 $\sqrt{(4+8x-5x^2)}$

29 $\sqrt{(x^2-x+1)}$

30 $(x-2)\sqrt{(2x^2-6x+5)}$

31 $\dfrac{1}{3-2\cos^2 x}$

32 $\dfrac{1}{4\sin^2 x+9\cos^2 x}$

33 $\dfrac{1}{2-3\cos 2x}$

34 $\dfrac{1}{\sin^2 x+\sin 2x}$

35 $\dfrac{1}{(2\sin x+3\cos x)^2}$

36 $\dfrac{1}{\cos x(\sin x+2\cos x)}$

37 $\dfrac{1}{2+\sin x+\cos x}$

38 $\dfrac{2-\sin x}{2\cos x+3}$

39 $\dfrac{\cos x}{5-3\cos x}$

40 $\dfrac{\sin x}{\sin x-\cos x}$ or $\dfrac{1}{1-\cot x}$ **(I I T 78)**

41 $\dfrac{\sin x+2\cos x}{2\sin x+\cos x}$

$$I=\frac{2}{3}\int\frac{}{\sqrt{[(a^{3}}}$$

(a) $\frac{1}{2}\sin^{-1}\frac{x^{-}}{2}$

(d) $I=\int\frac{dx}{x^2\, x^3\left(1\right.}$

$$-\frac{4}{x^5}\,dx=$$

$$I=-\frac{1}{4}\int\frac{dt}{t^{3/4}}=$$

$$=-\frac{1}{x}(1+x^4)^{1/4}$$

49 $I=\int\frac{1}{\sin x+\cos x}$

$$=\frac{1}{\sqrt{2}}\int\frac{1}{\sin(x+\pi/4)}$$

$$=\frac{1}{\sqrt{2}}\log\tan\left(\frac{x}{2}+\right.$$

50 $I=\int\frac{\sec x\,dx}{a\tan x+b}=\int$

Now see § 5 page 626

51 $\int\frac{dx}{\sin x+\sqrt{3}\cos x}$

Put $1=r\cos\alpha$, $\sqrt{3}=r$

$r^2=1+3=4$ or

$$I=\int\frac{dx}{r\sin(x+\alpha)}$$

$$=\frac{1}{2}\log\tan\left(\frac{x}{2}\right.$$

52. $\int\frac{dx}{\sqrt{(1+\sin x)}}=\int\frac{}{\sqrt{(}}$

$$=\int\frac{dx}{\sin\frac{x}{2}+\cos\frac{x}{2}}=$$

$$=\int\frac{1}{\sqrt{2}\sin(x/2+\pi/4)}$$

$=1/\sqrt{2}\;2\log\tan(x/4$

$$I=\frac{1}{4q}\int\sqrt{(t^2-p^2)}\,dt=\frac{1}{4q}\left[\frac{t}{2}\sqrt{(t^2-p^2)}\right.$$
$$\left.-\tfrac{1}{2}p^2\log\left(t+\sqrt{(t^2-p^2)}\right)\right]$$

12 Put $\sin x=t$ $\quad\cos x\,dx=dt$

$$I=\int\frac{dt}{t^2+4t+5}=\int\frac{dt}{(t+2)^2+1}=\tan^{-1}\frac{(t+2)}{1}$$
$$=\tan^{-1}(\sin x+2)$$

13 Put $\log x=t$ $\quad\frac{1}{x}dx=dt$

$$I=\int\frac{1}{t^2+4t-1}\,dt=\int\frac{dt}{(t+2)^2-(\sqrt{5})^2}$$
$$=\frac{1}{2\sqrt{5}}\log\frac{(t+2)-\sqrt{5}}{(t+2)+\sqrt{5})}\text{ where } t=\log x$$

14 Put $x^2=t$ $\quad 2x\,dx=dt$

$$I=\tfrac{1}{2}\int\frac{dt}{t^2+t+1}=\tfrac{1}{2}\int\frac{dt}{t^2+t+\frac{1}{4}+(1-\frac{1}{4})}$$
$$=\tfrac{1}{2}\int\frac{dt}{\left(t+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}=\frac{1}{2\;\frac{\sqrt{3}}{2}}\tan^{-1}\frac{t+\frac{1}{2}}{\sqrt{3}/2}$$
$$=\frac{1}{\sqrt{3}}\tan^{-1}\frac{2x^2+1}{\sqrt{3}}$$

15 $I=\int\frac{3x+1}{2x^2+x+1}\,dx$

$N^r=l\,(\text{d c of } D^r)+m$

$3x+1=l\,(4x+1)+m$ Compare

$4l=3,\ l+m=1$ $\quad l=3/4\quad m=\frac{1}{4}$

$$I=l\int\frac{4x+1}{2x^2+x+1}dx+m\int\frac{dx}{2x^2+x+1}$$

Put $2x^2+x+1=t$ for (1) $\quad(4x+1)\,dx=dt$

$$I=l\int\frac{dt}{t}+\frac{m}{2}\int\frac{dx}{x^2+\frac{1}{2}x+\frac{1}{2}}$$
$$=l\log t+\frac{m}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2+\frac{1}{2}-\frac{1}{16}}$$
$$=l\log(2x^2+x+1)+\frac{m}{2}\int\frac{dx}{\left(x+\frac{1}{4}\right)^2+\left(\frac{\sqrt{7}}{4}\right)^2}$$

$=\log\tan x+\log\tan x/2=\log(\tan x\tan x/2)$

55 $I=\int\frac{1}{4\cos^3 x-3\cos x}\,dx=\int\frac{1}{\cos 3x}\,dx=\int\sec 3x\,dx$

$=\frac{1}{3}\log(\sec 3x+\tan 3x)$

56 $I=\int\operatorname{cosec} 3x\,dx=\frac{1}{3}\log\tan\frac{3x}{2}$

57 $I=\int\tan^3 2x\sec 2x\,dx=\int\tan^2 2x\tan 2x\sec 2x\,dx$

$=\int(\sec^2 2x-1)\sec 2x\tan 2x\,dx$

Put $\sec 2x=t$ $\quad 2\sec 2x\tan 2x\,dx=dt$

$$I=\frac{1}{2}\int(t^2-1)\,dt=\frac{1}{2}\left(\frac{t^3}{3}-t\right)$$

$=\frac{1}{6}(\sec^3 2x-3\sec 2x)$

58 $I=\int\cot^3 x\operatorname{cosec}^4 x\,dx=\int\cot^3 x\operatorname{cosec}^2 x\operatorname{cosec}^2 x\,dx$

$=\int\cot^3 x(1+\cot^2 x)\operatorname{cosec}^2 x\,dx$

Put $\cot x=t$ $\quad -\operatorname{cosec}^2 x\,dx=dt$

$$I=-\int(t^3+t^5)\,dt=-\left(\frac{t^4}{4}+\frac{t^6}{6}\right)$$

$$=-\left(\frac{1}{4}\cot^4 x+\frac{1}{6}\cot^6 x\right)$$

59 $I=\int\sqrt{(2+\sin 3x)}\cos 3x\,dx$

Put $2+\sin 3x=t$ $\quad 3\cos 3x\,dx=dt$

$$I=\frac{1}{3}\int\sqrt{t}\,dt=\frac{1}{3}\cdot\frac{2}{3}t^{3/2}=\frac{2}{9}(2+\sin 3x)^{3/2}$$

60 Put $x^2=t$ $\quad 2x\,dx=dt$

$$I=\frac{1}{2}\int\frac{dt}{1+t^2}=\frac{1}{2}\tan^{-1}t=\frac{1}{2}\tan^{-1}x^2,$$

61 $I=\int\frac{1}{x}\operatorname{cosec}^2\log x\,dx$

Put $\log x=t$ $\quad \frac{1}{x}\,dx=dt$

$I=\int\operatorname{cosec}^2 t\,dt=-\cot t=-\cot(\log x)$

(b) Ans $\log(\sin\log x)$

62 $I=\int\frac{\operatorname{cosec}(\tan^{-1}x)}{1+x^2}\,dx$

Put $\tan^{-1}x=t$ $\quad \frac{1}{1+x^2}\,dx=dt$

$I=\int\operatorname{cosec} t\,dt=\log\tan t/2=\log\tan(\frac{1}{2}\tan^{-1}x)$

63 $I=\int\frac{1}{\sin(x-a)\sin(x-b)}\,dx$

Put $a-b=(x-b)-(x-a)$

23 Proceed as in Q 22

$I=\sqrt{(x^2+2x+2)}+2\log\{(x+1)+[(x+1)^2+1)\}$

24 $I=-\sqrt{(4x-x^2)}+4\sin^{-1}\frac{x-2}{2}$

25
$$I=\int\sqrt{\left(\frac{a+x}{a-x}\right)}dx=\int\frac{a+x}{\sqrt{(a^2-x^2)}}\,dx$$
$$=\int\frac{a}{\sqrt{(a^2-x^2)}}\,dx-\tfrac{1}{2}\int\left(\frac{-2x}{\sqrt{(a^2-x^2)}}\right)dx$$
$$=a\sin^{-1}\frac{x}{a}-\frac{1}{2}\int\frac{dt}{\sqrt{t}},\text{ where } a^2-x^2=t$$
$$=a\sin^{-1}\frac{x}{a}-\frac{1}{2}\;2\sqrt{t}$$
$$=a\sin^{-1}\frac{x}{a}-\sqrt{(a^2-x^2)}$$

26
$$I=\int\sqrt{\left(\frac{1-x}{1+x}\right)}dx=\int\frac{1-x}{\sqrt{(1-x^2)}}\,dx$$
$$=\int\frac{1}{\sqrt{(1-x^2)}}dx+\frac{1}{2}\int\frac{-2x}{\sqrt{(1-x^2)}}\,dx$$
$$=\sin^{-1}x+\sqrt{(1-x^2)}$$

27
$$I=\int x\sqrt{\left(\frac{a^2-x^2}{a^2+x^2}\right)}dx$$

Put $x^2=t$, $\quad 2x\,dx=dt$
$$I=\tfrac{1}{2}\int\sqrt{\left(\frac{a^2-t}{a^2+t}\right)}\,dt=\frac{1}{2}\int\frac{a^2-t}{\sqrt{(a^4-t^2)}}\,dt$$
$$=\frac{1}{2}\int\frac{a^2}{\sqrt{(a^4-t^2)}}\,dt+\tfrac{1}{4}\int\frac{-2t}{\sqrt{(a^4-t^2)}}\,dt$$
$$=\frac{a^2}{2}\sin^{-1}\frac{t}{a^2}+\frac{1}{4}\;2\sqrt{(a^4-t^2)}$$
$$=\frac{a^2}{2}\sin^{-1}\frac{x^2}{a^2}+\frac{1}{2}\sqrt{(a^4-x^4)}$$

28
$$I=\int\sqrt{(4+8x-5x^2)}dx=\sqrt{5}\sqrt{\left[-\left(x^2-\frac{8}{5}x-\frac{4}{5}\right)\right]}\,dx$$
$$=\sqrt{5}\int\sqrt{\left[-\left(x^2-\frac{8}{5}x+\frac{16}{25}-\frac{4}{5}-\frac{16}{25}\right)\right]}\,dx$$
$$=\sqrt{5}\int\sqrt{\left[\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2\right]}\,dx$$
$$=\sqrt{5}\left[\frac{x-4/5}{2}\sqrt{\left[\left(\frac{6}{5}\right)^2-\left(x-\frac{4}{5}\right)^2\right]}\right.$$
$$\left.+\frac{1}{2}\left(\frac{6}{5}\right)^2\sin^{-1}\frac{x-4/5}{6/5}\right]$$

$$= -2 \operatorname{cosec} \alpha \sqrt{\left(\frac{\cos \alpha \sin x + \sin \alpha \cos x}{\sin x}\right)}$$

$$= -2 \operatorname{cosec} \alpha \sqrt{\left(\frac{\sin (x+\alpha)}{\sin x}\right)}$$

68 $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Divide N^r and D^r by $\cos^4 x$

$$I = \int \frac{2 \tan x \sec^2 x \, dx}{1 + \tan^4 x}$$

Put $\tan^2 x = t$ $\qquad 2 \tan x \sec^2 x \, dx = dt$

$$I = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} (\tan^2 x)$$

69 Put $1 + x^{5/2} = t$

Then $\frac{5}{2} x^{3/2} \, dx = dt$

$$\int x^{13/2} \left(1 + x^{5/2}\right)^{1/2} dx$$

$$= \int \left(1 + x^{5/2}\right)^{1/2} x^5 \; x^{3/2} \, dx$$

$$= \int t^{1/2} (t-1)^2 \tfrac{2}{5} \, dt$$

$$= \tfrac{2}{5} \int t^{1/2} (t^2 - 2t + 1) \, dt$$

$$= \tfrac{2}{5} \int (t^{5/2} - 2t^{3/2} + t^{1/2}) \, dt$$

$$= \tfrac{2}{5} \left[\tfrac{2}{7} t^{7/2} - \tfrac{4}{5} t^{5/2} + \tfrac{2}{3} t^{3/2}\right] + c$$

$$= \tfrac{2}{5} t^{3/2} \left[\tfrac{2}{7} t^2 - \tfrac{4}{5} t + \tfrac{2}{3}\right] + c$$

$$= \tfrac{2}{5} (1 + x^{5/2})^{3/2} \left[\tfrac{2}{7} (1 + x^{5/2})^2 - \tfrac{4}{5} (1 + x^{5/2}) + \tfrac{2}{3}\right] + c$$

70 The given expression is $(1 + 2 \tan^2 x + 2 \tan x \sec x)^{1/2}$

$$(\sec^2 x + \tan^2 x + 2 \tan x \sec x)^{1/2} = \sec x + \tan x$$

$$I = \int (\sec x \tan x) \, dx$$

$$= \log (\sec x + \tan x) + \log \sec x + c$$

$$= \log \sec x (\sec x + \tan x) + c$$

§ 6 Nine Standard Results

1 $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

2. $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{(x-a)}{(x+a)}$ when $x > a$.

3 $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{a+x}{a-x}$ when $x < a$

4 $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log [x + \sqrt{(x^2 + a^2)}]$ or $\sinh^{-1} \left(\frac{x}{a}\right)$

5 $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log [x + \sqrt{(x^2 - a^2)}]$ or $\cosh^{-1} \frac{x}{a}$

$$= \frac{1}{5} \cdot \frac{1}{2 \cdot \frac{1}{\sqrt{5}}} \log \frac{t - \frac{1}{\sqrt{5}}}{t + \frac{1}{\sqrt{5}}} = \frac{1}{2\sqrt{5}} \log \frac{\sqrt{5}\tan x - 1}{\sqrt{5}\tan x + 1}$$

34 $I = \int \frac{1}{\sin^2 x + \sin 2x}\, dx = \int \frac{dx}{\sin^2 x + 2\sin x \cos x}$

Divide above and below by $\cos^2 x$

$$I = \int \frac{\sec^2 x\, dx}{\tan x + 2\tan x} \quad \text{Put } \tan x = t$$

$$I = \int \frac{dt}{t^2 + 2t} = \int \frac{dt}{t(t+2)}$$

$$= \frac{1}{2} \int \left(\frac{1}{t} - \frac{1}{t+2} \right) dt \quad \text{Partial fractions}$$

$$= \frac{1}{2} [\log t - \log (t+2)] = \frac{1}{2} \log \frac{\tan x}{\tan x + 2}$$

35 $I = \int \frac{1}{(2\sin x + 3\cos x)^2}\, dx$

Here denominator consists of terms of the form $\cos^2 x$, $\sin^2 x$ and $\sin x \cos x$ but we need not open the square. Divide above and below by $\cos^2 x$

$$I = \int \frac{\sec^2 x\, dx}{(2\tan x + 3)^2}$$

Put $2\tan x + 3 = t$, $\quad 2\sec^2 x\, dx = dt$

$$I = \frac{1}{2} \int \frac{dt}{t^2} = \frac{1}{2}\left(-\frac{1}{t}\right) = -\frac{1}{2(2\tan x + 3)}$$

36 Proceed as above.

$$I = \log (\tan x + 2)$$

37 $I = \int \frac{1}{2 + \sin x + \cos x}\, dx$

$$= \int \frac{1}{2 + 2\sin x/2 \cos x/2 + \cos^2 x/2 - \sin^2 x/2}$$

Divide above and below by $\cos^2 x/2$

$$I = \int \frac{\sec^2 x/2\, dx}{2(1 + \tan^2 x/2) + 2\tan x/2 + 1 - \tan^2 x/2}$$

Put $\tan \frac{x}{2} = t$

$$I = \int \frac{2\, dt}{t^2 + 2t + 3} = 2 \int \frac{dt}{(t+1)^2 + (\sqrt{2})^2}$$

$$= 2 \frac{1}{\sqrt{2}} \tan^{-1} \frac{t+1}{\sqrt{2}} = \sqrt{2} \tan^{-1} \frac{(\tan x/2 + 1)}{\sqrt{2}}$$

$$= -2 \operatorname{cosec} \alpha \sqrt{\left(\frac{\cos \alpha \sin x + \sin \alpha \cos x}{\sin x}\right)}$$

$$= -2 \operatorname{cosec} \alpha \sqrt{\left(\frac{\sin (x+\alpha)}{\sin x}\right)}$$

68 $I = \int \frac{\sin 2x}{\sin^4 x + \cos^4 x} dx = \int \frac{2 \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

Divide N^r and D^r by $\cos^4 x$

$$I = \int \frac{2 \tan x \sec^2 x \, dx}{1 + \tan^4 x}$$

Put $\tan^2 x = t$ $\quad\quad 2 \tan x \sec^2 x \, dx = dt$

$$I = \int \frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} (\tan^2 x)$$

69 Put $1 + x^{3/2} = t$

Then $\frac{3}{2} x^{1/2} dx = dt$

$$\int x^{7/2} \left(1 + x^{3/2}\right)^{1/2} dx$$

$$= \int \left(1 + x^{3/2}\right)^{1/2} x^3 \cdot x^{1/2} dx$$

$$= \int t^{1/2} (t-1)^2 \tfrac{2}{3} dt$$

$$= \tfrac{2}{3} \int t^{1/2} (t^2 - 2t + 1) \, dt$$

$$= \tfrac{2}{3} \int (t^{5/2} - 2t^{3/2} + t^{1/2}) \, dt$$

$$= \tfrac{2}{3} [\tfrac{2}{7} t^{7/2} - \tfrac{4}{5} t^{5/2} + \tfrac{2}{3} t^{3/2}] + c$$

$$= \tfrac{4}{3} t^{3/2} [\tfrac{1}{7} t^2 - \tfrac{2}{5} t + \tfrac{1}{3}] + c$$

$$= \tfrac{4}{3} (1 + x^{3/2})^{3/2} [\tfrac{1}{7} (1+x^{3/2})^2 - \tfrac{2}{5}(1 + x^{3/2}) + \tfrac{1}{3}] + c.$$

70 The given expression is $(1 + 2 \tan^2 x + 2 \tan x \sec x)^{1/2}$

$$(\sec^2 x + \tan^2 x + 2 \tan x \sec x)^{1/2} = \sec x + \tan x$$

$$I = \int (\sec x + \tan x) \, dx$$

$$= \log (\sec x + \tan x) + \log \sec x + c$$

$$= \log \sec x (\sec x + \tan x) + c$$

§ 6 Nine Standard Results

1 $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

2 $\int \frac{1}{x^2 - a^2} dx = \frac{1}{2a} \log \frac{(x-a)}{(x+a)}$ when $x > a$.

3 $\int \frac{1}{a^2 - x^2} dx = \frac{1}{2a} \log \frac{a+x}{a-x}$ when $x < a$

4 $\int \frac{1}{\sqrt{(x^2 + a^2)}} dx = \log [x + \sqrt{(x^2 + a^2)}]$ or $\sinh^{-1} \left(\frac{x}{a}\right)$

5 $\int \frac{1}{\sqrt{(x^2 - a^2)}} dx = \log [x + \sqrt{(x^2 - a^2)}]$ or $\cosh^{-1} \frac{x}{a}$

$$I = -\tfrac{1}{3}\, x + \tfrac{5}{3} \int \frac{2\, dt}{8t^2+2}$$

$$= -\tfrac{1}{3}\, x + \tfrac{5}{3} \int \frac{dt}{4t^2+1}$$

$$= -\tfrac{1}{3}\, x + \frac{5}{3 \cdot 4} \int \frac{dt}{t^2+(\frac{1}{2})^2}$$

$$= -\tfrac{1}{3}\, x + \frac{5}{12} \frac{1}{\frac{1}{2}} \tan^{-1} \frac{t}{1/2}$$

$$= -\tfrac{1}{3}\, x + \tfrac{5}{6} \tan^{-1} (2 \tan x/2)$$

40 $\int \frac{\sin x}{\sin x - \cos x}\, dx$

$N^r = l\,(D^r) + m$ (d c of D^r)

Here you need not write n Even if you write n its value will be zero

$\sin x = l\,(\sin x - \cos x) + m\,(\cos x + \sin x)$

Campare the coefficients of $\sin x$ and $\cos x$

$1 = l + m,\ -l + m = 0 \qquad l = m = \tfrac{1}{2}$

$$I = l \int \frac{(\sin x - \cos x)}{(\sin x + \cos x)}\, dx + m \int \frac{\cos x + \sin x}{\sin x - \cos x}\, dx$$

$$= lx + m \log (\sin x - \cos x)$$

$$= \tfrac{1}{2}\, x + \tfrac{1}{2} \log (\sin x - \cos x)$$

41 Proceed as above

$I = \tfrac{3}{5}\, x + \tfrac{2}{5} \log (2 \sin x + \cos x)$

§ 4 Integration by Parts

Rule If u and v be two functions of x then

$$\int uv\, dx = u \int v\, dx - \int \left[\frac{du}{dx} \int v\, dx \right] dx$$

The above formula can be stated as under

The integral of the Product of two functions

First function u multiplied by integral of 2nd function v − integral of [d c of Ist function multiplied by integral of 2nd already written before]

Note In applying the above rule care has to be taken in choosing the first and 2nd function properly Whenever x^n where n is a+ive integer is one of the two functions then it should be chosen as first function because after differentiation in the second part its power will be reduced by unity But this is possible if we know the integral of the second function If we choose x^n as 2nd function then it will be integrated and its power will increase e g

$$\int x^3 \sin 2x\, dx$$

$$I=\frac{1}{4}\int\frac{dx}{(x+\frac{1}{2})^2+1^2}=\frac{1}{4}\tan^{-1}(x+\tfrac{1}{2})$$ formula (1)

Ex 2 $\int\frac{1}{x^2+4x-5}dx=\int\frac{1}{(x+5)(x-1)}dx$

$$I=\int\frac{1}{x^2+4x+4-5-4}dx=\int\frac{1}{(x+2)^2-3^2}dx$$

$$=\frac{1}{2\cdot 3}\log\frac{(x+2)-3}{(x+2)+3}=\frac{1}{6}\log\frac{x-1}{x+5}$$ formula (2)

Note You could do it by partial fractions also

Ex 3 $\int\frac{dx}{1+3x-x^2}$

$$1+3x-x^2=-(x^2-3x-1)=-(x^2-3x+9/4-1-9/4)$$

$$=[(x-3/2)^2-(\sqrt{(13/2)^2}]=(\sqrt{(13)}/2)^2-(x-3/2)^2$$

$$I=\int\frac{dx}{(\sqrt{(13)}/2)^2-(x-3/2)^2}$$

$$=\frac{1}{2\sqrt{(13)}/2}\log\frac{(\sqrt{(13)}/2+(x-3/2)}{(\sqrt{(13)}/2-[x-3/2)}$$ formula (3) § 6

$$=\frac{1}{\sqrt{(13)}}\log\frac{[\sqrt{(13)}-3]+2x}{[\sqrt{(13)}+3]-2x}$$

Ex 4 $\int\frac{dx}{\sqrt{(4x^2+4x+5)}}$

$$I=\frac{1}{2}\int\frac{dx}{[(x+\frac{1}{2})^2+1^2]}$$ as in Ex (1)

$$=\frac{1}{2}\log[(x+\tfrac{1}{2})+\sqrt{\{(x+\tfrac{1}{2})^2+1^2\}}]$$ formula (4), § 6

Ex 5 $\int\frac{dx}{\sqrt{(x^2+4x-5)}}$

$$I=\int\frac{dx}{\sqrt{[(x+2)^2-3^2]}}$$ as in Ex (2)

$$=\log[(x+2)+\sqrt{\{(x+2)^2-3^2\}}]$$ formula (5), § 6

Ex 6 $\int\frac{dx}{\sqrt{(1+3x-x^2)}}$

$$I=\int\frac{dx}{\sqrt{\left[\left(\frac{\sqrt{(13)}}{2}\right)^2-(x-3/2)\right]}}$$

$$=\sin^{-1}\frac{x-3/2}{\sqrt{(13)}/2}=\sin^{-1}\frac{2x-3}{\sqrt{(13)}}$$ formula (6), § 6

Ex 7 $\int\sqrt{(4x^2+4x+5)}\,dx$

$$I=2\int\sqrt{[(x+\tfrac{1}{2})^2+1^2]}\,dx$$ as in Ex (1)

$$=2\left[\tfrac{1}{2}(x+\tfrac{1}{2})\sqrt{\{(x+\tfrac{1}{2})^2+1^2\}}+\tfrac{1}{2}\log[(x+\tfrac{1}{2})+\sqrt{\{(x+\tfrac{1}{2})^2+1^2\}}]\right]$$
formula (7), § 6

Ex 3 $\int x^2 \log x\, dx$

$$=\int (\log x)\, x^2\, dx$$

$$=(\log x)\int x^2\, dx - \int\left[\frac{d}{dx}(\log x)\int x^2\, dx\right]dx$$

$$=\frac{x^3}{3}\log x - \int \frac{1}{x}\frac{x^3}{3}\,dx$$

$$=\frac{x^3}{3}\log x - \frac{1}{3}\frac{x^3}{3} - \frac{x^3}{9}(3\log x - 1)$$

Note from Ex 2, 3 we conclude that

$$\int x^n \log x\, dx = \frac{x^{n+1}}{(n+1)^2}[(n+1)\log x - 1]$$

Ex 4 (a) $\int \log x\, dx$ **(M N R 79)**

(*b*) $\int \log(x+1)\, dx$ **(Roorkee 74)**

Here there is only one function whose integral we do not know and we choose the other function as unity which will be taken as 2nd function

$$\int \log x\, dx = \int (\log x)\, 1\, dx$$

$$=\log x \int 1\, dx - \int\left[\frac{d}{dx}\log x\int 1\, dx\right]dx$$

$$=x\log x - \int \frac{1}{x}.x\, dx$$

$$=x(\log x - 1)$$

(*b*) $I = x\log(x+1) - \int \frac{x}{x+1}dx$

$$=x\log(x+1) - \int\left(1 - \frac{1}{x+1}\right)dx$$

$$=x\log(x+1) - x + \log(x+1) - (x+1)\log(x+1) - x$$

Ex 4 (c) $\int x\log(1+x)\, dx$ **(Roorkee 75)**

$$=\frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\frac{x^2}{x+1}dx$$

$$=\frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\frac{x^2-1+1}{x+1}dx$$

$$=\frac{x^2}{2}\log(1+x) - \frac{1}{2}\int\left\{(x-1) + \frac{1}{x+1}\right\}dx$$

$$=\frac{x^2}{2}\log(1+x) - \frac{1}{2}\left[\frac{x^2}{2} - x + \log(x+1)\right]$$

$$=\frac{x^2-1}{2}\log(1+x) - \frac{x^2}{4} + \frac{x}{2}$$

Ex 5 $\int \tan^{-1} x\, dx$ **(Roorkee 77)**

$=\frac{3}{8}\log(4x^2+4x+5)+\frac{1}{2}\cdot\frac{1}{2}\tan^{-1}(x+\frac{1}{2})$ as in Ex 1 P 639

Ex 11 $\int\frac{(3x+2)}{\sqrt{(4x^2+4x+5)}}\,dx$

As in Ex 10 $\quad 3x+2=l\,(8x+4)+m$ etc

$$I=\frac{3}{8}\int\frac{8x+4}{\sqrt{(4x^2+4x+5)}}dx+\frac{1}{2}\int\frac{dx}{\sqrt{(4x^2+4x+5)}}$$

Put $x^2+4x+5=t \qquad (8x+4)\,dx=dt$

$$I=\frac{3}{8}\int\frac{dt}{\sqrt{t}}+\frac{1}{2}\int\frac{dx}{\sqrt{(4x^2+4x+5)}}$$

$=\frac{3}{8}\,2\sqrt{t}+\frac{1}{2}\cdot\frac{1}{2}\log[(x+\frac{1}{2})+\sqrt{\{(x+\frac{1}{2})^2+1\}}]$ as in Ex 4 P 640

$=\frac{3}{4}\sqrt{(4x^2+4x+5)}+\frac{1}{4}\log[(x+\frac{1}{2})+\sqrt{\{(x+\frac{1}{2})^2+1\}}]$

Ex 12 $\int(3x+2)\sqrt{(4x^2+4x+5)}\,dx$

$I=\frac{3}{8}\int(8x+4)\sqrt{(4x^2+4x+5)}\,dx+\frac{1}{2}\int\sqrt{(4x^2+4x+5)}\,dx$

As in Ex 10

Put $4x^2+4x+5=t \qquad (8x+4)\,dx=dt$

$$I=\frac{3}{8}\int\sqrt{t}\,dt+\frac{1}{2}\int\sqrt{(4x^2+4x+5)}\,dx$$

$=\frac{3}{8}\cdot\frac{2}{3}t^{3/2}+\frac{1}{2}\int\sqrt{(4x^2+4x+5)}\,dx$

$=\frac{1}{4}(4x^2+4x+5)^{3/2}+\frac{1}{2}\,2\Big[\frac{1}{2}(x+\frac{1}{2})\sqrt{\{(x+\frac{1}{2})^2+1^2\}}$

$+\frac{1^2}{2}\log\{(x+\frac{1}{2})+\sqrt{(x+\frac{1}{2})^2+1^2)}\}\Big]$ As in Ex 7 P 640

§ 9 Application to the integrals

$$\int\frac{dx}{a\cos^2x+2b\sin x\cos x+c\sin^2x}$$

$$\int\frac{dx}{a\cos^2x+b}\qquad\int\frac{dx}{a+b\sin^2x}$$

In above type of questions divide above and below by $\cos^2x$ The numerator shall become $\sec^2x$ and in the denominator we will have a quadratic in $\tan x$ (change $\sec^2x$ into $1+\tan^2x$) Putting $\tan x=t$ the question shall reduce to the form

$$\int\frac{dt}{at^2+bt+c}$$

The integral of above has been explained in application (1)

§ 10 Application to the integrals

1 $\quad\int\frac{1}{a\cos x+b\sin x+c}\,dx,\quad$ 2 $\quad\int\frac{dx}{a+b\cos x}$

or $I = x\sqrt{(x^2-a^2)} - I \quad a^2 \log [x+\sqrt{(x^2-a^2)}]$

$2I = x\sqrt{(x^2-a^2)} - a^2 \log [x+\sqrt{(x^2-a^2)}]$

or $I = \frac{x}{2}\sqrt{(x^2-a^2)} - \frac{a^2}{2} \log [x+\sqrt{(x^2-a^2)}]$

Ex 8 $\int \sqrt{(a^2-x^2)}\, dx = \int \sqrt{(a^2-x^2)}\, 1\, dx$ **(I I T 74)**

$= \sqrt{(a^2-x^2)} \int 1 \; dx - \int \left[\frac{d}{dx} [\sqrt{(a^2-x^2)}] \int 1\, dx \right] dx$

$= x\sqrt{(a^2-x^2)} - \int \frac{1}{2\sqrt{(a^2-x^2)}} (-2x).x\, dx$

$= x\sqrt{(a^2-x^2)} - \int \frac{-x^2}{\sqrt{(a^2-x^2)}}\, dx$

$= x\sqrt{(a^2-x^2)} - \int \frac{a^2 \quad x^2-a^2}{\sqrt{(a^2-x^2)}}\, dx$ Split into two

$= x\sqrt{(a^2-x^2)} - \int \sqrt{(a^2-x^2)}\, dx + \int \frac{a^2}{\sqrt{(a^2-x^2)}}\, dx$

or $I = x\sqrt{(a^2-x^2)} - I + a^2 \sin^{-1} x/a$

$2I = x\sqrt{(a^2-x^2)} + a^2 \sin^{-1} x/a$

or $I = \frac{x}{2}\sqrt{(a^2-x^2)} + \frac{a^2}{2} \sin^{-1} \frac{x}{a}$

Note We may similarly prove that

$I = \int \sqrt{(x^2+a^2)}\, dx = \frac{x}{2} \sqrt{(x^2+a^2)} + \frac{a^2}{2} \log [x+\sqrt{(x^2+a^2)}]$

Successive integration by parts

Ex 9 $\int x^3 \sin 2x\, dx$

$= x^3 \int \sin 2x\, dx - \int \left[\frac{d}{dx} (x^3) \int \sin 2x\, dx \right] dx$

$= -\frac{x^3 \cos 2x}{2} - \int 3x^2 \left(\frac{-\cos 2x}{2}\right) dx$

$= -\frac{x^3}{2} \cos 2x + \frac{3}{2} \int x^2 \cos 2x\, dx$ (1)

Again $\int x^2 \cos 2x\, dx = x^2 \frac{\sin 2x}{2} - \int 2x \; \frac{\sin 2x}{2}\, dx$

$= \frac{x^2}{2} \sin 2x - \int x \sin 2x\, dx$

Putting in (1)

$I = -\frac{x^3}{2} \cos 2x + \frac{3}{4} x^2 \sin 2x - \frac{3}{2} \int x \sin 2x$ (2)

Ex 14 $\int \frac{dx}{1+3\sin^2 x}$

Divide above and below by $\cos^2 x$

$$I = \int \frac{\sec^2 x\, dx}{\sec^2 x + 3\tan^2 x} = \int \frac{\sec^2 x\, dx}{1+\tan^2 x + 3\tan^2 x}$$

$$= \int \frac{\sec^2 x\, dx}{1+4\tan^2 x}$$

Put $2\tan x = t$ $\qquad 2\sec^2 x\, dx = dt$

$$I = \frac{1}{2}\int \frac{dt}{1+t^2} = \frac{1}{2}\tan^{-1} t = \frac{1}{2}\tan^{-1}(2\tan x)$$

Ex 15 $\int \frac{d\theta}{(\sin\theta - 2\cos\theta)(2\sin\theta + \cos\theta)}$

If we multiply the factors in D^r it becomes of the form of Ex 13,, but we need not multiply Divide above and below by $\cos^2\theta$

$$I = \int \frac{\sec^2\theta\, d\theta}{(\tan\theta - 2)(2\tan\theta + 1)} \qquad \text{Put } \tan\theta = t$$

$$I = \int \frac{dt}{(t-2)(2t+1)} \qquad \text{Split into partial fractions}$$

$$I = \int \left[\frac{1}{5(t-2)} - \frac{2}{5(2t+1)}\right] dt$$

$$= \frac{1}{5}\log(t-2) - \frac{1}{5}\log(2t+1)$$

$$= \frac{1}{5}\log\frac{t-2}{2t+1} = \frac{1}{5}\log\frac{\tan\theta - 2}{2\tan\theta + 1}$$

Ex. 16 $\int \frac{dx}{4+5\cos x}$ **(Roorkee 83)**

$$I = \int \frac{dx}{4+5(\cos^2 x/2 - \sin^2 x/2)}$$

Divide above and below by $\cos^2 x/2$

$$I = \int \frac{\sec^2 x/2\, dx}{4\sec^2 x/2 + 5(1 - \tan^2 x/2)}$$

$$I = \int \frac{\sec^2 x/2\, dx}{9 - \tan^2 x/2} \qquad \sec^2 x/2 = 1 + \tan^2 x/2$$

Put $\tan x/2 = t$ $\qquad \frac{1}{2}\sec^2 x/2\, dx = dt$

$$I = \int \frac{2\,dt}{9-t^2} = 2\,\frac{1}{2.3}\log\frac{3+t}{3}$$

$= \frac{1}{3}$ lo [illegible] $x/2$

$$=x^3\left(\frac{-\cos 2x}{2}\right)-(3x^2)\left(\frac{-\sin 2x}{4}\right)$$
$$+(6x)\left(\frac{\cos 2x}{8}\right)-(6)\left(\frac{\sin 2x}{16}\right)+0$$

We have stopped here because differentiation of 6 will be zero

$$\int x^3 \sin 2x\,dx=\frac{-x^3}{2}\cos 2x+\tfrac{3}{4}x^2\sin 2x+\tfrac{3}{4}x\cos 2x$$
$$-\tfrac{3}{8}\sin 2x$$

Above is same as result (3) of Ex 10

$$\int x^3 e^{4x}\,dx$$

Here e^{4x} will be successively integrated and x^3 will be successively differentiated

$$\int x^3 e^{4x}\,dx=x^3\left(\frac{e^{4x}}{4}\right)-(3x^2)\left(\frac{e^{4x}}{16}\right)+(6x)\left(\frac{e^{4x}}{64}\right)$$
$$-6\left(\frac{e^{4x}}{256}\right)+0$$
$$=\frac{1}{4}x^3e^{4x}-\frac{3}{16}x^2e^{4x}+\frac{3}{32}xe^{4x}-\frac{3}{128}e^{4x}$$

Above is same as result (3) of Ex 10

Similarly $\int x^2\cos x\,dx=x^2(\sin x)-(2x)(-\cos x)+2(-\sin x)$
$$=x^2\sin x+2x\cos x-2\sin x$$

$$\int x^2 e^{2x}\,dx=e^{2x}\left(\frac{x^2}{2}-\frac{x}{2}+\frac{1}{4}\right)$$ **(Roorkee 78)**

Ex 11 (*a*) $\int x\sec^2 x\,dx$

$$=x\int\sec^2 x\,dx-\int\left[\frac{d}{dx}(x)\int\sec^2 x\,dx\right]dx$$
$$=x\tan x-\int 1\cdot\tan x\,dx$$
$$=x\tan x-\log\sec x$$

Note $\int x\tan^2 x\,dx=\int x(\sec^2 x-1)\,dx$
$$=\int x\sec^2 x\,dx-\int x\,dx$$
$$=x\tan x-\log\sec x-\frac{x^2}{2}\text{ by Ex 11}$$

(*b*) $$\int x\frac{(\sec 2x-1)}{(\sec 2x+1)}\,dx=\int x\frac{(1-\cos 2x)}{(1+\cos 2x)}\,dx=\int x\frac{(2\sin^2 x)}{(2\cos^2 x)}\,dx$$
$$=\int x\tan^2 x\,dx$$
$$=\int x(\sec^2 x-1)\,dx$$
$$=\int x\sec^2 x\,dx-\int x\,dx$$
$$=x\tan x-\log\sec x-\frac{x^2}{2}\text{ by Ex 11}$$

$$I_1=\int \frac{2dt}{t^2+2t+5}=2\int \frac{dt}{(t+1)^2+2^2}$$

$$=2\cdot\frac{1}{2}\tan^{-1}\frac{t+1}{2}=\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$$

$$I=l\theta+m\log(\sin\theta+2\cos\theta+3)+n\tan^{-1}\left(\frac{\tan\theta/2+1}{2}\right)$$

where $l=6/5,\quad m=3/5,\quad n=-8/5$

19 $\int \frac{3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$

Write $N^r=l\,(D^r)+m$ (d c. of D^r)

Let

$3\sin x+2\cos x=l\,(3\cos x+2\sin x)+m\,(-3\sin x+2\cos x)$

Comparing coefficients of $\sin x$ and $\cos x$ on both sides

$$3=2l-3m,\quad 2=3l+2m$$

Solving we get $l=12/13,\quad m=-5/13$

$$I=l\int dx+m\int \frac{-3\sin x+2\cos x}{3\cos x+2\sin x}\,dx$$

$$=lx+m\log(3\cos x+2\sin x)$$

$$=\tfrac{12}{13}x-\tfrac{5}{13}\log(3\cos x+2\sin x)$$

20 $\int \frac{dx}{p+q\tan x}=\int \frac{\cos x\,dx}{p\cos x+q\sin x}$

Write $N^r=l\,(D^r)+m$ (d c of D^r)

$$\cos x=l\,(p\cos x+q\sin x)+m\,(-p\sin x+q\cos x)$$

Compare $\cos x$ and $\sin x$

$$1=lp+mq,\quad 0=lq-pm$$

Solving $l=\frac{p}{p^2+q^2},\ m=\frac{q}{p^2+q^2}$

$$I=l\int dx+m\int \frac{-p\sin x+q\cos x}{p\cos x+q\sin x}\,dx$$

$$\frac{p}{p^2+q^2}x+\frac{q}{p^2+q^2}\log(p\cos x+q\sin x)$$

Note Putting $p=q=1$ we get

$$\int \frac{dx}{1+\tan x}=\tfrac{1}{2}x+\tfrac{1}{2}\log(\cos x+\sin x)$$

Problem Set (C).

Integrate the following

1 $\frac{3x^2}{4+5x^2},\ \frac{3x^3}{x^4+1}$ (M N R 81) 2. $\frac{1}{e^x+e^{-x}}$.

$$=\frac{e^{ax}}{r}\cos(bx-\alpha)$$

$$\int e^{4x}\sin 3x\,dx=\frac{e^{4x}}{4^2+3^2}(4\sin 3x-3\cos 3x)$$

$$=\frac{e^{4x}}{r}\sin(3x-\alpha)$$

where $r=\sqrt{(4^2+3^2)}=5,\ \tan\alpha=b/a=3/4$

Cancellation of integrals

Ex 14 $\int e^x[f(x)+f'(x)]\,dx=e^x f(x)$

Note the above form

$I=\int e^x f(x)\,dx+\int e^x f'(x)\,dx$

Integrate 1st by parts

$$I=f(x)\int e^x\,dx-\int\left[\frac{d}{dx}f(x)\int e^x\,dx\right]dx+\int e^x f'(x)\,dx$$

$=f(x)\,e^x-\int f'(x)\,e^x\,dx+\int e^x f'(x)\,dx$

$=e^x f(x)$ The last two integrals cancel

Hence $\int e^x(\sin x+\cos x)\,dx=e^x\sin x$

$f(x)=\sin x,\ f'(x)=\cos x$

$$\int e^{x/2}\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)dx=\frac{1}{\sqrt{2}}\int e^t(\sin t+\cos t)\,(2dt)$$

(Roorkee 82)

$=\sqrt{2}\int e^t(\sin t+\cos t)\,dt=\sqrt{2}\,e^t\sin t$ where $\frac{x}{2}=t$

$=\sqrt{2}e^{x/2}\sin x/2$

$\int e^x(\tan x+\sec^2 x)\,dx=e^x\tan x$

$f(x)=\tan x,\ f'(x)=\sec^2 x$

$$\int e^x\left(\log x+\frac{1}{x}\right)dx=e^x\log x$$

$f(x)=\log x,\ f'(x)=\frac{1}{x}$

$\int e^x(\cot x-\operatorname{cosec}^2 x)\,dx=e^x\cot x,$

$f(x)=\cot x,\ f'(x)=-\operatorname{cosec}^2 x$

Ex 15 $\int e^x\frac{1+\sin x}{1+\cos x}\,dx=e^x\tan\frac{x}{2}$

$$I=\int\frac{e^x}{1+\cos x}\,dx+\int e^x\frac{\sin x}{1+\cos x}\,dx$$

$$=\int\frac{e^x}{2\cos^2 x/2}\,dx+\int e^x\frac{2\sin x/2\cos x/2}{2\cos^2 x/2}\,dx$$

$=\int e^x(\tan x/2+\frac{1}{2}\sec^2 x/2)\,dx=e^x\tan x/2$

Solution Set (C)

1 Put $\sqrt{5}x^2=t$ $\quad I=\dfrac{1}{\sqrt{5}}\displaystyle\int\dfrac{dt}{4+t^2}=\dfrac{1}{2\sqrt{5}}\tan^{-1} t/2$

$$=\frac{1}{2\sqrt{5}}\tan^{-1}\frac{\sqrt{5}x^2}{2}$$

(b) $\tan^{-1} x^3$

2 Change e^{-x} to $\dfrac{1}{e^x}$ $\quad I=\displaystyle\int\dfrac{e^x\,dx}{1+e^{2x}}$ $\quad$ Put $e^x=t$ etc

$\tan^{-1} e^x$

3 Put $5x^4=t$ $\quad 20x^3\,dx=dt$ $\quad I=\dfrac{1}{20}\displaystyle\int\dfrac{dt}{t^2-4^2}$

$$=\frac{1}{20}\,\frac{1}{2\cdot 4}\log\frac{t-4}{t+4}=\frac{1}{160}\log\frac{5x^4-4}{5x^4+4}$$

4 Put $\sin x=t$ $\quad \cos x\,dx=dt$

$$I=\int\frac{dt}{2^2-t^2}=\frac{1}{2\cdot 2}\log\frac{2+t}{2-t}=\tfrac{1}{4}\log\frac{2+\sin x}{2-\sin x}$$

5 Put $4x^3=t$ $\quad 12x^2\,dx=dt$

$$I=\tfrac{1}{4}\int\frac{dt}{\sqrt{(3^2-t^2)}}=\tfrac{1}{4}\sin^{-1}\frac{t}{3}=\tfrac{1}{4}\sin^{-1}\frac{4}{3}x^3$$

6 Put $q\tan^{-1}x=t$ $\quad q\,\dfrac{1}{1+x^2}\,dx=dt$

$$I=\frac{1}{q}\int\frac{dt}{\sqrt{(p^2+t^2)}}=\frac{1}{q}\log\{t+\sqrt{(p^2+t^2)}\}\text{ where } t=q\tan^{-1}x$$

7 Put $x^4=t$ $\quad I=\tfrac{1}{4}\displaystyle\int\dfrac{dt}{\sqrt{(t^2+1)}}=\tfrac{1}{4}\log\,(t+\sqrt{(t^2+1)})$

$$=\tfrac{1}{4}\log\,[(x^4+\sqrt{(x^8+1)}]$$

8 Put $2\sin^{-1}x=t$ $\quad \dfrac{2}{\sqrt{(1-x^2)}}\,dx=dt$

$$I=\tfrac{1}{2}\int\frac{dt}{\sqrt{(t^2-9)}}=\tfrac{1}{2}\log\{t+\sqrt{(t^2-9)}\}$$

$$\tfrac{1}{2}\log\{2\sin^{-1}x+\sqrt{((2\sin^{-1}x)^2-9)}\}$$

9 Put $\sec x=t$ $\quad \sec x\tan x\,dx=dt$

$$I=\int\sqrt{(t^2+1)}\,dt=\tfrac{1}{2}t\sqrt{(t^2+1)}+\tfrac{1}{2}\log\{t+\sqrt{(t^2+1)}\}$$

10 Put $\sin x=t$ $\quad \cos x\,dx=dt$

$$I=\int\sqrt{(4-t^2)}\,dt=\frac{t}{2}\sqrt{(4-t^2)}+\tfrac{1}{2}\,4\sin^{-1}\frac{t}{2}$$

11 Put $qx^4=t$ $\quad 4qx^3\,dx=dt$

$$I=(\sin^{-1} x)\frac{x^2}{2}-\int\frac{1}{\sqrt{(1-x^2)}}\,\frac{x^2}{2}\,dx$$

$$=\tfrac{1}{2}x^2\sin^{-1}x+\tfrac{1}{2}\int\frac{1-x^2-1}{\sqrt{(1-x^2)}}\quad \text{Split into two}$$

$$=\tfrac{1}{2}x^2\sin^{-1}x+\tfrac{1}{2}\left[\int\sqrt{(1-x^2)}\,dx-\int\frac{1}{\sqrt{(1-x^2)}}\,dx\right]$$

$$=\tfrac{1}{2}x^2\sin^{-1}x+\tfrac{1}{2}\left[\frac{x}{2}\sqrt{(1-x^2)}+\tfrac{1}{2}\sin^{-1}x-\sin^{-1}x\right]$$

$$=\tfrac{1}{2}x^2\sin^{-1}x+\frac{x}{4}\sqrt{(1-x^2)}-\tfrac{1}{4}\sin^{-1}x$$

Problem Set (D)

Integrate the following

1 (*a*) $x^2\sin 2x$ (I I T 74) (*b*) $\int_0^{\pi/2} x^2\sin x\,dx$ (Roorkee 79)

2 $x\sin x\cos x$ 3 $x\tan^{-1}x$ (Roorkee 79)

4 $x\sec^2 2x$ (I I T 75) 5 $(\log x)^2$ (I I T 71)

6 $\sin(\log x)$ (I I T 73) 7 $e^x\sin x$ 8 $x^2 a^x$

9 $\frac{1}{x^2}\log(x^2+a^2)$ (M N R 80) 10 $\frac{x\tan^{-1}x}{1+x^2}$

11 $\frac{\sin^{-1}x}{(1-x^2)^{3/2}}$ 12 $\sin^{-1}\frac{2x}{1+x^2}$, $\cos^{-1}\frac{1-x^2}{1+x^2}$, $\tan^{-1}\frac{2x}{1-x^2}$

13 $\frac{x}{1+\cos x}$ 14 (i) $\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$, (ii) $\int_0^{1/2}\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}$ (I I T 84)

15 $x^3\cos x^2$ 16 $\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}$

17 $x\sin^2 x$, $x\cos^2 x$ (I I T 72) 18 $\cos\sqrt{x}$

19 (*a*) $x^2(\log x)^2$ (Roorkee 76) (*b*) $\sqrt{x}(\log x)^2$ (Roorkee 81)

20 $\sin x\log(\sec x+\tan x)$ 21 $\cos x\log\tan x/2$

22 $\log(1+x^2)$ 23 $\log\{x+\sqrt{(x^2+a^2)}\}$

24 $e^x\frac{1-\sin x}{1-\cos x}$ 25 $e^x\left[\frac{1+\sqrt{(1-x^2)}\sin^{-1}x}{\sqrt{(1-x^2)}}\right]$

$$=\frac{3}{4}\log(2x^2+x+1)+\frac{1}{8}\frac{1}{\sqrt{7/4}}\tan^{-1}\frac{x+\frac{1}{4}}{\sqrt{7/4}}$$

$$=\frac{3}{4}\log(2x^2+x+1)+\frac{1}{2\sqrt{7}}\tan^{-1}\frac{4x+1}{\sqrt{7}}$$

16 $I=\int\frac{2x^2+3x+4}{x^2+6x+10}\,dx$

Write $N^r=l\,(D^r)+m$ (d c of D^r)$+n$

$$2x^2+3x+4=l\,(x^2+6x+10)+m\,(2x+6)+n$$

Compare $l=2,\ 6l+2m=3,\ 10l+6m+n=4$

$$l=2\quad m=-9/2,\ n=11$$

$$I=l\int dx+m\int\frac{2x+6}{x^2+6x+10}\,dx+n\int\frac{dx}{(x+3)^2+1^2}$$

$$=lx+m\log(x^2+6x+10)+n\,\frac{1}{1}\tan^{-1}\frac{(x+3)}{1}$$

$$=2x-\tfrac{9}{2}\log(x^2+6x+10)+11\tan^{-1}(x+3)$$

17 $-\frac{1}{8}\log(4x^2-4x-3)+\frac{1}{16}\log\frac{2x-3}{2x+1}$

18 $\frac{3}{2}\log(x^2+4x+2)-\frac{1}{\sqrt{2}}\log\frac{x+2-\sqrt{2}}{x+2+\sqrt{2}}$

19 $I=\int\frac{dx}{\sqrt{[5-(x+2)^2]}}=\sin^{-1}\frac{x+2}{\sqrt{5}}$

20 $I=\int\frac{dx}{\sqrt{\left[\left(x+\frac{3}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2\right]}}$

$$=\log\left[(x+3/2)+\sqrt{[(x+3/2)^2-(\sqrt{5}/2)^2]}\right]$$

21 $I=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x^2+x+4/3)}}=\frac{1}{\sqrt{3}}\int\frac{dx}{\sqrt{(x+\frac{1}{2})^2+[\sqrt{(13/2\sqrt{3})^2}]}}$

$$=(1/\sqrt{3})\log\left[(x+\tfrac{1}{2})+\sqrt{\{(x+\tfrac{1}{2})^2+13/12\}}\right]$$

22 $\int\frac{x}{\sqrt{(9+8x-x^2)}}\,dx$

We write $x=l\,(8-2x)+m$ Compare

$$-2l=1,\ 8l+m=0,\qquad l=-\tfrac{1}{2},\ m=4$$

$$I=l\int\frac{8-2x}{\sqrt{(9+8x-x^2)}}\,dx+m\int\frac{dx}{\sqrt{[25-(x^2-8x+16)]}}$$

Put $9+8x-x^2=t$ $\quad(8-2x)\,dx=dt$

$$I=l\int\frac{1}{\sqrt{t}}\,dt+m\int\frac{dx}{\sqrt{[5^2-(x-4)^2]}}$$

$$=l\,2\sqrt{t}+m\sin^{-1}\frac{x-4}{5}=-\sqrt{(9+8x-x^2)}+4\sin^{-1}\frac{x-4}{5}$$

$= x(\log x)^2 - 2\int 1 \cdot \log x\, dx$

$= x(\log x)^2 - 2\left[x \log x - \int \frac{1}{x} x\, dx\right]$

$= x(\log x)^2 - 2x \log x + 2x$

6 $I = \int \sin(\log x)$

Put $\log x = t$ $\quad x = e^t$ and $dx = e^t\, dt$

$I = \int e^t \sin t$

$= e^t(-\cos t) - \int e^t(-\cos t)\, dt$

$= -e^t \cos t + \int e^t \cos t\, dt$

$= -e^t \cos t + [e^t \sin t - \int e^t \sin t\, dt]$ Transpose

$2\int e^t \sin t\, dt = -e^t \cos t + e^t \sin t$

$\int e^t \sin t\, dt$

$= \frac{1}{2}e^t(\sin t - \cos t) - \frac{1}{2} x[\sin(\log x) - \cos(\log x)]$

7 Do yourself $\frac{e^x}{2}(\sin x \quad \cos x)$

8 $\int x^2 a^x\, dx$

We know that $\int a^x\, dx = \frac{a^x}{\log a}$

$I = x^2 \frac{a^x}{\log a} - \int 2x \frac{a^x}{\log a} dx$

$= \frac{x^2 a^x}{\log a} - \frac{2}{\log a}\left[x \frac{a^x}{\log a} - \int 1 \frac{a^x}{\log a} dx\right]$

$= x^2 \frac{a^x}{\log a} \quad \frac{2x\, a^x}{(\log a)^2} + \frac{2}{(\log a)^2}\int a^x\, dx$

$= \frac{x^2 a^x}{\log a} \quad \frac{2x\, a^x}{(\log a)^2} + \frac{2a^x}{(\log a)^3}$

$= \frac{a^x}{(\log a)^3}\left[x^2 (\log a)^2 - 2x \log a + 2\right]$

Note We could have straight away written the answer by applying successive integration

$\int x^2 a^x\, dx = x^2\left(\frac{a^x}{\log a}\right) - (2x)\left(\frac{a^x}{(\log a)^2}\right) + 2\left(\frac{a^x}{(\log a)^3}\right) - 0$

$= \frac{a^x}{(\log a)^3}\left[x^2 (\log a)^2 - 2x(\log a) + 2\right]$

9 $I = \frac{-1}{x}\log(x^2+a^2) + \int \frac{1}{x} \frac{1}{x^2+a^2} 2x\, dx$

$= \frac{-1}{x}\log(x^2+a^2) + 2\int \frac{1}{x^2+a^2} dx$

$$=\sqrt{5}\left[\frac{5x-4}{10\sqrt{5}}\sqrt{(4+8x-5x^2)}+\frac{18}{25}\sin^{-1}\frac{5x-4}{6}\right]$$

29 $I=\int\sqrt{(x^2-x+1)}\,dx=\int\sqrt{\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}\,dx$

$$=\frac{x-\frac{1}{2}}{4}\sqrt{\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}+\frac{1}{2}\left(\frac{\sqrt{3}}{2}\right)^2\log\left\{\left(x-\frac{1}{2}\right)+\sqrt{\left[\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2\right]}\right\}$$

$$=\frac{2x-1}{4}\sqrt{(x^2-x+1)}+\frac{3}{8}\log\left\{\frac{2x-1}{2}+\sqrt{(x^2-x+1)}\right\}$$

30 $I=\int(x-2)\sqrt{(2x^2-6x+5)}\,dx$

Put $x-2=l(4x-6)+m$ Compare

$4l=1$ $-6l+m=-2,$ $l=\frac{1}{4}, m=-\frac{1}{2}$

$$I=l\int(4x-6)\sqrt{(2x^2-6x+5)}\,dx+m\int\sqrt{(2x^2-6x+5)}\,dx$$

Put $2x^2-6x+5=t$ $(4x-6)\,dx=dt$

$$I=l\int\sqrt{t}\,dt+m\sqrt{2}\int\sqrt{(x^2-3x+5/2)}\,dx$$

$$=l\,\tfrac{2}{3}t^{3/2}+m\sqrt{2}\int\sqrt{[(x-3/2)^2+(\tfrac{1}{2})^2]}\,dx$$

$$=\tfrac{1}{4}\cdot\tfrac{2}{3}(2x^2-6x+5)^{3/2}-\tfrac{1}{2}\sqrt{2}I_1$$

$$=\tfrac{1}{6}(2x^2-6x+5)^{3/2}-\frac{1}{\sqrt{2}}I_1$$

$$I_1=\frac{x-\frac{3}{2}}{2}\sqrt{[(x-\tfrac{3}{2})^2+(\tfrac{1}{2})^2]}+\tfrac{1}{2}(\tfrac{1}{2})^2\log\{(x-\tfrac{3}{2})+\sqrt{[(x-\tfrac{3}{2})^2+(\tfrac{1}{2})^2]}\}$$

31 Divide above and below by $\cos^2 x$ etc

$$I=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$$

32 As above $I=\frac{1}{6}\tan^{-1}(\frac{2}{3}\tan x)$

33 $I=\int\frac{dx}{2-3\cos}$ ... $\frac{dx}{2\cos^2 x}$

$=\int\frac{dx}{5-6\cos^2}$ ... $\frac{x\,dx}{{}^2\,x)-}$

Put $\tan x=t,$ se

$I=\int$ ... t $=\frac{1}{5}\int$

$$=\left[\frac{1}{2}-\frac{\sqrt{3}}{2}\cdot\frac{\pi}{6}\right]=\left[\frac{1}{2}-\frac{\sqrt{3}}{12}\pi\right]$$

15 $I=\int x^3\cos x^2\,dx$ Put $x^2=t$, $2x\,dx=dt$

$I=\int x^2\cos x^2\cdot x\,dx=\frac{1}{2}\int t\cos t\,dt$

$=\frac{1}{2}\left[t\sin t-\int 1\cdot\sin t\,dt\right]$

$=\frac{1}{2}\left[t\sin t+\cos t\right]=\frac{1}{2}\left[x^2\sin x^2+\cos x^2\right]$

16 $I=\int\frac{x\tan^{-1}x}{(1+x^2)^{3/2}}\,dx$ Put $x=\tan\theta$, $dx=\sec^2\theta\,d\theta$

$I=\int\frac{\theta\tan\theta}{\sec^3\theta}\sec^2\theta\,d\theta=\int\theta\sin\theta\,d\theta$

$=\theta(-\cos\theta)-\int 1\cdot(-\cos\theta)\,d\theta$

$=-\theta\cos\theta+\sin\theta$

Now $\tan\theta=x$ $\sin\theta=\frac{x}{\sqrt{(1+x^2)}}$, $\cos\theta=\frac{1}{\sqrt{(1+x^2)}}$

$I=\frac{x-\tan^{-1}x}{\sqrt{(1+x^2)}}$

17 $\sin^2 x=\frac{1-\cos 2x}{2}$, $\cos^2 x=\frac{1+\cos 2x}{2}$

$I=\int x\sin^2 x\,dx=\int\frac{x(1-\cos 2x)}{2}\,dx$

$=\int\frac{x}{2}dx-\frac{1}{2}\int x\cos 2x\,dx$

$=\frac{x^2}{4}-\frac{1}{2}\left[\frac{x\sin 2x}{2}-\int 1\cdot\frac{\sin 2x}{2}\,dx\right]$

$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x+\frac{1}{2}\left[-\frac{\cos 2x}{4}\right]$

$=\frac{x^2}{4}-\frac{1}{4}x\sin 2x-\frac{1}{8}\cos 2x$

Similarly $\int x\cos^2 x\,dx=\frac{x^2}{4}+\frac{1}{4}x\sin 2x+\frac{1}{8}\cos 2x$

18 $I=\int\cos\sqrt{x}\,dx$

Put $\sqrt{x}=t$ so that $\frac{1}{2\sqrt{x}}\,dx=dt$

$I=\int(\cos t)\,2t\,dt=2\int t\cos t\,dt$

Integrating by parts we get

$I=2\left[t\sin t-\int 1\cdot\sin t\,dt\right]$

$=2\left[t\sin t+\cos t\right]$

$=2\left[\sqrt{x}\sin\sqrt{x}+\cos\sqrt{x}\right]$

38 $I=\int\frac{2-\sin x}{2\cos x+3}\,dx$

$$I=\int\frac{2}{2(\cos^2 x/2-\sin^2 x/2)+3}\,dx-\int\frac{\sin x\,dx}{(2\cos x+3)}$$

$$=\int\frac{2\sec^2 x/2\,dx}{2(1-\tan^2 x/2)+3(1+\tan^2 x/2)}$$

$$+\frac{1}{2}\int\frac{-2\sin x}{(2\cos x+3)}\,dx$$

Put $\tan x/2=t$ in Ist and apply

$$\int\frac{f'(x)}{f(x)}\,dx=\log f(x) \text{ in 2nd}$$

$$I=\int\frac{4\,dt}{5+t^2}+\tfrac{1}{2}\log(2\cos x+3)$$

$$=4\,\frac{1}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}}+\frac{1}{2}\log(2\cos x+3)$$

$$=\frac{4}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan x/2\right)+\frac{1}{2}\log(2\cos x+3)$$

Note We may write $N^r=l\,(D^r)+m$ (d c of D^r)+3

$$2-\sin x=l\,(2\cos x+3)+m\,(-2\sin x)+n$$

Compare coefficients of $\cos x$, $\sin x$ and constant

$2l=0,\ -2m=-1,\ n=2 \qquad l=0,\ m=\tfrac{1}{2},\ n=2$

$$I=0+\frac{1}{2}\int\frac{-2\sin x\,dx}{2\cos x+3}+2\int\frac{dx}{2\cos x+3}$$

$=\tfrac{1}{2}\log_e(2\cos x+3)+$as above

39 $\int\frac{\cos x}{5-3\cos x}\,dx$

$$\cos x=l\,(5-3\cos x)+m\,(3\sin x)+n$$

Compare the coefficients of $\cos x$ $\sin x$ and constant

$-3l=1,\ 3m=0\ \ 5l+n=0$

$l=-1/3,\ m=0\ \ n=5/3$

$$I=l\int\frac{5-3\cos x}{5-3\cos x}\,dx+0+n\int\frac{dx}{5-3\cos x}$$

$$=-\frac{1}{3}\int dx+\frac{5}{3}\int\frac{dx}{5-3(\cos^2 x/2-\sin^2 x/2)}$$

$$=-\frac{x}{3}+\frac{5}{3}\int\frac{\sec^2 x/2\,dx}{5(1+\tan^2 x/2)-3(1-\tan^2 x/2)}$$

Put $\tan\frac{x}{2}=t \qquad \tfrac{1}{2}\sec^2 x/2\,dx=dt$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\,dx=\log\{x+\sqrt{(x^2+a^2)}\}$$

$$\frac{d}{dx}\log\{x+\sqrt{(x^2+a^2)}\}=\frac{1}{\sqrt{(x^2+a^2)}}$$

Choosing 1 is the other function

$$I=x\log\{x+\sqrt{(x^2+a^2)}\}-\int x\,\frac{d}{dx}\log\{x+\sqrt{(x^2+a^2)}\}\,dx$$

$$=x\log\{x+\sqrt{(x^2+a^2)}\}-\int x\,\frac{1}{\sqrt{(x^2+a^2)}}\,dx$$

$$=x\log\{x+\sqrt{(x^2+a^2)}\}-\frac{1}{2}\int\frac{2x}{\sqrt{(x^2+a^2)}}\,dx$$

$$=x\log[x+\sqrt{(x^2+a^2)}]-\tfrac{1}{2}\,2\sqrt{(x^2+a^2)}$$

$$=x\log[x+\sqrt{(x^2+a^2)}]-\sqrt{(x^2+a^2)}$$

$$\int\frac{f'(x)}{\sqrt{(f(x))}}\,dx=2\sqrt{(f(x))}$$

24 $$I=\int e^x\,\frac{(1-\sin x)}{(1-\cos x)}\,dx$$

$$=\int\left[\frac{e^x}{2\sin^2 x/2}-e^x\,\frac{2\sin x/2\cos x/2}{2\sin^2 x/2}\right]dx$$

$$=\int e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx-\int e^x\cot\frac{x}{2}\,dx$$

Integrate first by parts

$$I=e^x\left(-\cot\frac{x}{2}\right)-\int e^x\left(-\cot\frac{x}{2}\right)dx-\int e^x\cot\frac{x}{2}\,dx$$

$= -e^x\cot x/2$ The last two integrals cancel

25 $$I=\int e^x\left[\frac{1+\sqrt{(1-x^2)}\sin^{-1}x}{\sqrt{(1-x^2)}}\right]dx$$

$$=\int e^x\,\frac{1}{\sqrt{(1-x^2)}}\,dx+\int e^x\sin^{-1}x\,dx$$

Integrate 1st by parts

$I=e^x\sin^{-1}x-\int e^x\sin^{-1}x\,dx+\int e^x\sin^{-1}x\,dx$

$I=e^x\sin^{-1}x$ The last two integrals cancel

Note Above is also of the form

$$\int e^x[f(x)+f'(x)]\,dx=e^x f(x)$$

$$f(x)=\sin^{-1}x \text{ and } f'(x)=\frac{1}{\sqrt{(1-x)^2}}$$

26 $$I=\int e^x\,\frac{(2+\sin 2x)}{(1+\cos 2x)}\,dx$$

we will choose x^3 as Ist and choose $\sin 2x$ whose integral we know as 2nd

$$\int x^3 \log x \, dx$$

Here we would like to choose x^3 as first but we do not know the integral of $\log x$ and hence here we shall per force choose x^3 as 2nd function and $\log x$ as first function

$$\int \log x \, dx, \int \sqrt{(x^2-a^2)}\, dx, \int \sqrt{(a^2-x^2)}\, dx, \int \sin^{-1} x \, dx$$

Here in the above integrals there is only one function and as such we shall choose unity as the other function This unity will be treated as 2nd function whose integral is to be taken

We shall make all the above ideas clear by giving the following examples

Ex 1 (a) $\int x \cos x \, dx$ (Roorkee 77)

Here x will be chosen as first function and $\cos x$ whose integral we know as 2nd function

$$\int x \cos x \, dx = x \int \cos x \, dx - \int \left[\frac{d}{dx}(x) \int \cos x \, dx\right] dx$$

$$= x \sin x - \int 1 \sin x \, dx$$

$$= x \sin x + \cos x$$

Ex 1 (b) Prove $\int x^3 e^{x^2} dx = \frac{1}{2} e^{x^2} (x^2-1)$ (M.N.R 80)

Put $x^2 = t$ $\quad 2x\, dx = dt \quad I = \frac{1}{2} \int t e^t \, dt$

$$= \frac{1}{2}\left[t \int e^t \, dt - \int \left[\frac{d}{dt}(t) \int e^t \, dt\right] dt\right]$$

$$= \frac{1}{2}\left[t e^t - \int 1 \, e^t \, dt\right] = \frac{1}{2} [t e^t - e^t] = \frac{1}{2} e^{x^2} (x^2-1)$$

Ex 2 $\int x \log x \, dx$

Here if we choose x as first function then $\log x$ will be second function whose integral we do not know and hence we will choose $\log x$ as Ist and x as 2nd

$$\int x \log x \, dx = \int (\log x) . x \, dx$$

$$= \log x \int x \, dx - \int \left[\frac{d}{dx}(\log x) \int x \, dx\right] dx$$

$$= \frac{x^2}{2} (\log x) - \int \frac{1}{x} \frac{x^2}{2} dx$$

$$= \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx$$

$$= \frac{x^2}{2} \log x - \frac{x^2}{4} = \frac{x^2}{4} (2 \log x - 1)$$

$$\int \frac{1}{\sqrt{(x^2+a^2)}}\, dx = \log\{x+\sqrt{(x^2+a^2)}\}$$

$$\frac{d}{dx}\log\{x+\sqrt{(x^2+a^2)}\} = \frac{1}{\sqrt{(x^2+a^2)}}$$

Choosing 1 as the other function

$$I = x\log\{x+\sqrt{(x^2+a^2)}\} - \int x\,\frac{d}{dx}\log\{x+\sqrt{(x^2+a^2)}\}\,dx$$

$$= x\log\{x+\sqrt{(x^2+a^2)}\} - \int x\,\frac{1}{\sqrt{(x^2+a^2)}}\,dx$$

$$= x\log\{x+\sqrt{(x^2+a^2)}\} - \frac{1}{2}\int\frac{2x}{\sqrt{(x^2+a^2)}}\,dx$$

$$= x\log[x+\sqrt{(x^2+a^2)}] - \tfrac{1}{2}\,2\sqrt{(x^2+a^2)}$$

$$= x\log[x+\sqrt{(x^2+a^2)}] - \sqrt{(x^2+a^2)}$$

$$\int\frac{f'(x)}{\sqrt{(f(x))}}\,dx = 2\sqrt{(f(x))}$$

24 $I = \int e^x\,\dfrac{(1-\sin x)}{(1-\cos x)}\,dx$

$$= \int\left[\frac{e^x}{2\sin^2 x/2} - e^x\,\frac{2\sin x/2\cos x/2}{2\sin^2 x/2}\right]dx$$

$$= \int e^x\left(\frac{1}{2}\operatorname{cosec}^2\frac{x}{2}\right)dx - \int e^x\cot\frac{x}{2}\,dx$$

Integrate first by parts

$$I = e^x\left(-\cot\frac{x}{2}\right) - \int e^x\left(-\cot\frac{x}{2}\right)dx - \int e^x\cot\frac{x}{2}\,dx$$

$= -e^x\cot x/2$ The last two integrals cancel

25 $I = \int e^x\left[\dfrac{1+\sqrt{(1-x^2)}\,\sin^{-1}x}{\sqrt{(1-x^2)}}\right]dx$

$$= \int e^x\,\frac{1}{\sqrt{(1-x^2)}}\,dx + \int e^x\sin^{-1}x\,dx$$

Integrate 1st by parts

$I = e^x\sin^{-1}x - \int e^x\sin^{-1}x\,dx + \int e^x\sin^{-1}x\,dx$

$I = e^x\sin^{-1}x$ The last two integrals cancel

Note Above is also of the form

$$\int e^x[f(x)+f'(x)]\,dx = e^x f(x)$$

$$f(x) = \sin^{-1}x \text{ and } f'(x) = \frac{1}{\sqrt{(1-x)^2}}$$

26 $I = \int e^x\,\dfrac{(2+\sin 2x)}{(1+\cos 2x)}\,dx$

Here also there is one function and the other be chosen as unity

$$\int \tan^{-1} x\, dx = \int (\tan^{-1} x)\, 1\, dx$$

$$= (\tan^{-1} x) \int 1\, dx - \int \left[\frac{d}{dx} (\tan^{-1} x) \int 1\, dx\right] dx$$

$$= x \tan^{-1} x - \int \frac{1}{1+x^2}\, x\, dx$$

$$= x \tan^{-1} x - \tfrac{1}{2} \int \frac{2x}{(1+x^2)}\, dx$$

[$2x$ is d c of $1+x^2$]

$$= x \tan^{-1} x - \tfrac{1}{2} \log (1+x^2)$$

$$\int \frac{f'(x)}{f(x)}\, dx = \log f(x)$$

Ex 6 $\int \sin^{-1} x\, dx = \int (\sin^{-1} x)\, 1\, dx$

$$= (\sin^{-1} x) \int 1\, dx - \int \left[\frac{d}{dx} (\sin^{-1} x) \int 1\, dx\right] dx$$

$$= x \sin^{-1} x - \int \frac{1}{\sqrt{(1-x^2)}}\, x\, dx$$

$$= x \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{(1-x^2)}}\, dx$$

[$-2x$ is d c of $1-x^2$]

$$= x \sin^{-1} x + \tfrac{1}{2}\, 2\sqrt{(1-x^2)} = x \sin^{-1} x + \sqrt{(1-x^2)}$$

$$\int \frac{f'(x)}{\sqrt{[f(x)]}}\, dx = 2\sqrt{[f(x)]}$$

Reappearance of original integral

Ex 7 $\int \sqrt{(x^2-a^2)}\, dx$

Here also there is one function and as such we choose the other as unity

$$I = \int \sqrt{(x^2-a^2)}\, dx = \sqrt{(x^2-a^2)}\, 1\, dx$$

$$= \sqrt{(x^2-a^2)} \int 1\, dx - \int \left[\frac{d}{dx} [\sqrt{(x^2-a^2)}] \int 1\, dx\right] dx$$

$$= \sqrt{(x^2-a^2)} - \int \frac{1}{2\sqrt{(x^2-a^2)}}\, 2x\, x\, dx$$

$$= x\sqrt{(x^2-a^2)} - \int \frac{x^2}{\sqrt{(x^2-a^2)}}\, dx$$

$$= x\sqrt{(x^2-a^2)} - \int \frac{x^2-a^2+a^2}{\sqrt{(x^2-a^2)}}\, dx \qquad \text{Split into two}$$

$$= x\sqrt{(x^2-a^2)} - \int \sqrt{(x^2-a^2)}\, dx - \int \frac{a^2}{\sqrt{(x^2-a^2)}}\, dx$$

$\sin x = t$ $\quad \cos x\,dx = dt$, Also when $x = \pi/2$, $t = 1$ and when $x = 0$, $t = 0$

$$I = \int_0^1 \frac{dt}{(1+t)(2+t)}$$

Now split into partial fractions by the method of supression

$$I = \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t} \right) dt$$

$$= \Big[\log(1+t) - \log(2+t) \Big]_0^1 = \left[\log \frac{1+t}{2+t} \right]_0^1$$

$$= \log \frac{2}{3} - \log \frac{1}{2} = \log \frac{2}{3} - \frac{1}{2} = \log \frac{4}{3}$$

(b) $\displaystyle\int_0^1 \frac{2e^x\,dx}{e^{3x} - 6e^{2x} + 11e^x - 6} = \log \frac{(e^x-1)(e^x-3)^3}{(e^x-2)^4}$

(M N R 82)

Put $e^x = t$ $\quad e^x\,dx = dt$ $\quad I = \displaystyle\int \frac{2t\,dt}{(t-1)(t-2)(t-3)}$

Now split into partial fractions and integrate

Ex 2 $\quad \displaystyle\int \frac{1 - \cos x}{\cos x\,(1 + \cos x)}\,dx$

Here we have $\cos x$ but its d c i e $-\sin x$ is not present in the numerator and as such we cannot make the substitution of $\cos x = t$ But we simply put $\cos x = t$ to split the integrand into partial fractions

$$\frac{1 - \cos x}{\cos x\,(1 + \cos x)} = \frac{1-t}{t(1+t)} = \left(\frac{1}{t} - \frac{2}{1+t} \right) = \left(\frac{1}{\cos x} - \frac{2}{1+\cos x} \right)$$

$$I = \int \left(\frac{1}{\cos x} - \frac{2}{1+\cos x} \right) dx = \int \left(\sec x - \sec^2 \frac{x}{2} \right) dx$$

$$= \log(\sec x + \tan x) - 2\tan x/2$$

Ex 3 $\quad \displaystyle\int \frac{dx}{x\,(x^n+1)}$

(Roorkee 79) (Important)

Here we multiply above and below by x^{n-1}

$$I = \int \frac{x^{n-1}\,dx}{x^n\,(x^n+1)}$$

Now d c of x^n is nx^{n-1} which is present in the numerator Therefore we make the substitution of

$x^n = t$ $\quad nx^{n-1}\,dx = dt$

$$I = \frac{1}{n} \int \frac{dt}{t(t+1)} = \frac{1}{n} \int \left(\frac{1}{t} - \frac{1}{t+1} \right) dt$$

Again $\int x \sin 2x \, dx = x\left(-\frac{\cos 2x}{2}\right) - \int 1 \left(-\frac{\cos 2x}{2}\right) dx$

$$= -\frac{x \cos 2x}{2} + \frac{1}{2}\int \cos 2x \, dx$$

$$= -\frac{x \cos 2x}{2} + \frac{\sin 2x}{4}$$

Putting in (2)

$$I = -\frac{x^3}{2}\cos 2x + \frac{3}{4} x^2 \sin 2x + \frac{3}{4} x \cos 2x - \frac{3}{8} \sin 2x \qquad (3)$$

Ex 10 $\int x^3 e^{4x} \, dx = x^3 \frac{e^{4x}}{4} - \int 3x^2 \frac{e^{4x}}{4} \, dx$

$$= \frac{x^3}{4} e^{4x} - \frac{3}{4}\int x^2 e^{4x} \, dx \qquad (1)$$

$$\int x^2 e^{4x} = x^2 \frac{e^{4x}}{4} - \int 2x \frac{e^{4x}}{4} \, dx$$

$$= \frac{x^2}{4} e^{4x} - \frac{1}{2}\int x e^{4x} \, dx$$

Putting in (1)

$$I = \frac{x^3}{4} e^{4x} - \frac{3}{16} x^2 e^{4x} + \frac{3}{8}\int x e^{4x} \, dx \qquad (2)$$

$$\int x e^{4x} \, dx = x \frac{e^{4x}}{4} - \int 1 \frac{e^{4x}}{4} \, dx$$

$$= \frac{x}{4} e^{4x} - \frac{e^{4x}}{4} dx$$

Putting in (2) we get

$$I = \frac{x^3}{4} e^{4x} - \frac{3}{16} x^2 e^{4x} + \frac{3}{32} x e^{4x} - \frac{3}{128} e^{4x} \qquad (3)$$

Note You have seen that in examples 9 and 10 we had to integrate successively thrice by parts Below we give a method with the help of which we can straight away write down the answer

Rule Start as usual and then go on successively integrating one function and successively differentiating the other and attach alternately + and − signs The successive integration and differentiation be written within brackets

$$\int x^3 \sin 2x \, dx$$

Here $\sin 2x \, dx$ will be successively integrated and x^3 will be successively differentiated

Ex 5 $\int \frac{x^2\,dx}{(a+bx)^3}$ (I I T 78)

Put $a+bx=t$ $\quad b\,dx=dt$

Also $x=\frac{t-a}{b}$

$$I=\int \frac{1}{t^3}\,\frac{(t-a)^2}{b^2}\,\frac{dt}{b}=\frac{1}{b^3}\int \frac{t^2-2at+a^2}{t^2}\,dt=\frac{1}{b^3}\int\left(1-\frac{2a}{t}+\frac{a^2}{t^2}\right)dt$$

$$=\frac{1}{b^3}\left[t-2a\log t-\frac{a^2}{t}\right]$$ Now put $t=a+bx$

Ex 6 $\int \frac{x^3}{(1+x^2)^2}\,dx$

Now $x^3=x^2\cdot x$ and d c of x^2 is $2x$

Hence we put $1+x^2=t$ $\quad 2x\,dx=dt$ and $x^2=t-1$

$$I=\int \frac{x^2}{(1+x^2)^2}\,x\,dx=\int \frac{t-1}{t^2}\,\tfrac{1}{2}\,dt$$

$$=\tfrac{1}{2}\int\left(\frac{1}{t}-\frac{1}{t^2}\right)dt=\tfrac{1}{2}\left[\log t+\frac{1}{t}\right]$$

$$=\tfrac{1}{2}\left[\log(x^2+1)+\frac{1}{x^2+1}\right]$$

Ex 7 $\int \frac{1}{x\sqrt{(1+x^n)}}\,dx$ See also Ex 3

Here we multiply above and below by x^{n-1}

$$I=\int \frac{x^{n-1}\,dx}{x^n\sqrt{(1+x^n)}}$$

Now d c of x^n is nx^{n-1} Also in order to remove the fractional powers we put $1+x^n=t^2$

$\therefore$ $nx^{n-1}\,dx=2t\,dt$ $\quad$ Also $x^n=t^2-1$

$$I=\frac{1}{n}\int \frac{2t\,dt}{(t^2-1)\,t}=\frac{2}{n}\int \frac{dt}{t^2-1}$$

$$=\frac{2}{n}\,\frac{1}{2}\,\log\frac{t-1}{t+1}$$

$$=\frac{1}{n}\log\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}$$

$$\int \frac{1}{x\sqrt{(1+x^3)}}\,dx=\frac{1}{3}\log\frac{\sqrt{(1+x^3)}-1}{\sqrt{(1+x^3)}+1}$$

Ex 8 $\int \frac{x^2\,dx}{\sqrt{(1-x)}}$ (I I T 80)

Put $1-x=t^2$ $\quad dx=-2t\,dt$ Also $x=1-t^2$

Ex 12 $\int \tan^{-1} x\, dx$

We have already done this question in Ex 5 Here we shall do it by alternative method

Put $x=\tan\theta$ $dx=\sec^2\theta\, d\theta$ and $\tan^{-1}\tan\theta=\theta$

$$I=\int \tan^{-1} x\, dx=\int \tan^{-1}(\tan\theta)\sec^2\theta\, d\theta$$

$$=\int \theta \sec^2\theta \quad \text{integrate by parts}$$

$$=\theta\tan\theta-\log\sec\theta \quad \text{by Ex 11}$$

$$=x\tan^{-1}x-\log\sqrt{(1+\tan^2\theta)}$$

$$=x\tan^{-1}x-\tfrac{1}{2}\log(1+x^2) \quad \text{Same as in Ex 5}$$

Ex 13 (a) $\int e^{ax}\sin bx\, dx$

$$I=e^{ax}\left(\frac{-\cos bx}{b}\right)-\int ae^{ax}\left(\frac{-\cos bx}{b}\right)dx$$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\int e^{ax}\cos bx\, dx$$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b}\left[e^{ax}\left(\frac{\sin bx}{b}\right)-\int ae^{ax}\left(\frac{\sin bx}{b}\right)dx\right]$$

$$=-\frac{e^{ax}\cos bx}{b}+\frac{a}{b^2}e^{ax}\sin bx-\frac{a^2}{b^2}I$$

Transpose I to the other side

$$I\left(1+\frac{a^2}{b^2}\right)=e^{ax}\frac{(a\sin bx-b\cos bx)}{b^2}$$

or $I(a^2+b^2)=e^{ax}(a\sin bx-b\cos bx)$

or $$I=\frac{e^{ax}}{(a^2+b^2)}(a\sin bx-b\cos bx)$$

(b) $\int_0^{\pi/2} e^x \sin x\, dx$ (Roorkee 78)

Put $a=b=1$ in the above

Another form If we put $a=r\cos\alpha$, $b=r\sin\alpha$, then

$$a^2+b^2=r^2$$

and $$\tan\alpha=\frac{b}{a} \quad \text{or} \quad \alpha=\tan^{-1}\frac{b}{a}$$

$$\therefore \quad I=\int e^{ax}\sin bx\, dx=\frac{e^{ax}}{r^2}(r\cos\alpha\sin bx-r\sin\alpha\cos bx)$$

$$=\frac{e^{ax}}{r}\sin(bx-\alpha)$$

Similarly

$$I=\int e^{ax}\cos bx\, dx=\frac{e^{ax}}{a^2+b^2}(a\cos bx+b\sin bx)$$

$=\frac{1}{15}\sqrt{(1+x)}\,[3x^4-4x^2+8]$

Ex 11 $\int\frac{1}{(x+1)\sqrt{(x-1)}}\,dx$

Put $x-1=t^2$ $\quad dx=2t\,dt$, Also $x=t^2+1$

$$I=\int\frac{2t\,dt}{(t^2+2)\,t}=2\int\frac{1}{t^2+(\sqrt{2})^2}dt$$

$$=2\;\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}=\sqrt{2}\tan^{-1}\frac{t}{\sqrt{2}}=\sqrt{2}\tan^{-1}\sqrt{\left(\frac{x-1}{2}\right)}$$

Ex 12 $\int\frac{1+x^2}{\sqrt{(1-x^2)}}dx$

Here we write $1+x^2=-[-x^2-1]=-(1-x^2-2)=2-(1-x^2)$

$$I=\int\frac{2-(1-x^2)}{\sqrt{(1-x^2)}}dx=\int\frac{2}{\sqrt{(1-x^2)}}dx-\int\sqrt{(1-x^2)}\,dx$$

$$=2\sin^{-1}x-\left\{\frac{x}{2}\sqrt{(1-x^2)}+\tfrac{1}{2}\sin^{-1}x\right\}$$

$$=\frac{3}{2}\sin^{-1}x-\frac{x}{2}\sqrt{(1-x^2)}$$

Ex 13 $\int x\sqrt{\left(\frac{1-x}{1+x}\right)}dx$

$$I=\int\frac{x(1-x)}{\sqrt{(1-x^2)}}dx=\int\frac{x}{\sqrt{(1-x^2)}}dx+\int\frac{-x^2}{\sqrt{(1-x^2)}}$$

$$=-\tfrac{1}{2}\int\frac{-2x}{\sqrt{(1-x^2)}}dx+\int\frac{1-x^2-1}{\sqrt{(1-x^2)}}dx$$

$$=-\tfrac{1}{2}\,2\sqrt{(1-x^2)}+\int\sqrt{(1-x^2)}\,dx-\int\frac{1}{\sqrt{(1-x^2)}}dx$$

$$\int\frac{f'(x)}{\sqrt{f(x)}}dx=2\sqrt{f(x)}$$

$$=-\sqrt{(1-x^2)}+\left[\frac{x}{2}\sqrt{(1-x^2)}+\tfrac{1}{2}\sin^{-1}x\right]-\sin^{-1}x$$

$$=\left(\frac{x}{2}-1\right)\sqrt{(1-x^2)}-\tfrac{1}{2}\sin^{-1}x$$

Ex 14 $\int\frac{dx}{(2x+3)\sqrt{(4x+5)}}$

and hence deduce the integral of $\int\frac{8x^2+22x+17}{(2x+3)\sqrt{(4x+5)}}dx$

Put $4x+5=t^2$ $\quad 4dx=2t\,dt$ and $x=\frac{t^2-5}{4}$

$$2x+3=\frac{t^2-5}{2}+3=\frac{t^2+1}{2}$$

$f(x) = \tan x/2, f'(x) = \frac{1}{2}\sec^2 x/2$

Proceeding directly

$$I = \int e^x \tan x/2 \; dx + \int \tfrac{1}{2} e^x \sec^2 x/2 \; dx$$

$$= \tan x/2 \int e^x dx - \int\left[\frac{d}{dx}(\tan x/2) \int e^x dx\right] dx$$

$$+ \int \tfrac{1}{2} e^x \sec^2 x/2 \; dx$$

$$= \tan\frac{x}{2} e^x - \int (\tfrac{1}{2}\sec^2 x/2 \, e^x dx + \int \tfrac{1}{2} e^x \sec^2 x/2 \; dx$$

$= e^x \tan x/2$ The last two cancel

(b) $\int \frac{(x-1)e^x}{(x+1)^3} dx$ **(I I T 83)**

$$I = \int \frac{(x+1-2)}{(x+1)^3} e^x dx = \int e^x \left[\frac{1}{(x+1)^2} + \frac{-2}{(x+1)^3}\right] dx = e^x \frac{1}{(x+1)^2}$$

by Ex 14

Here $f(x) = \frac{1}{(x+1)^2}$ and $f'(x) = \frac{-2}{(x+1)^3}$

16 $\int [\sin(\log x) + \cos(\log x)] \, dx$

$I = \int 1 \sin(\log x) \, dx + \int \cos(\log) \, dx$

$= x \sin(\log x) - \int x \cos(\log x) \frac{1}{x} dx + \int \cos(\log x) \, dx$

$x \sin(\log x)$ as the last two integrals cancel

17 $\int \sec^3 x \, dx$

$I = \int \sec^3 x \, dx = \int \sec x \; \sec^2 x \, dx$

$= \sec x \tan x \; \int\left[\frac{d}{dx}(\sec x) \int \sec^2 x \, dx\right] dx$

$= \sec x \tan x - \int \sec x \tan x \; \tan x \, dx$

$= \sec x \tan x - \int \sec x (\sec^2 x - 1) \, dx$

$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$

$2 \int \sec^3 x \, dx = \sec x \tan + \log(\sec x + \tan x)$

$\int \sec^3 x \, dx = \frac{1}{2} \sec x \tan x + \frac{1}{2} \log(\sec x + \tan x)$

18 $\int \sin\sqrt{x} \, dx$

Put $x = t^2$ $dx = 2t \, dt$

$I = \int 2t \sin t \, dt$ Integrate by parts

$= 2t(-\cos t) - \int 2(\cos t) \, dt$

$= -2t \cos t + 2 \sin t$ Now put $t = \sqrt{x}$

$= 2[\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}]$

Similarly $\int \cos\sqrt{x} \, dx = 2(\cos\sqrt{x} + \sqrt{x}\sin\sqrt{x})$ **(I I T 77)**

19 $\int x \sin^{-1} x \, dx$

$$I=\int\frac{6t^5\,dt}{t^3-t^2}=6\int\frac{t^3}{t-1}\,dt$$

$$=6\int\frac{t^3-1+1}{t-1}\,dt=6\int\left\{(t^2+t+1)+\frac{1}{t-1}\right\}dt$$

$$=6\left[\frac{t^3}{3}+\frac{t^2}{2}+t+\log(t-1)\right] \text{ where } t=(1+x)^{1/6}$$

Ex 17 (a) $\int_0^{\pi/4}\frac{\sin x+\cos x}{9+16\sin 2x}\,dx$ **(I I T 83)**

Put $\sin x-\cos x=t$ $\quad(\cos x+\sin x)\,dx=dt$

Also squaring we get $1-2\sin x\cos x=t^2$ or $1-t^2=\sin 2x$

Again when $x=\pi/4$, $t=0$, when $x=0$, $t=-1$

$$I=\int_{-1}^{0}\frac{dt}{9+16(1-t^2)}=\int_{-1}^{0}\frac{dt}{5^2-(4t)^2}$$

$$=\frac{1}{4}\cdot\frac{1}{2\cdot 5}\left[\log\frac{5+4t}{5-4t}\right]_{-1}^{0} \qquad \int\frac{1}{a^2-x^2}\,dx=\frac{1}{2a}\log\frac{a+x}{a-x}$$

$$=\frac{1}{40}\left[\log 1-\log\frac{1}{9}\right]=\frac{1}{40}\log 3^2=\frac{1}{20}\log 3$$

(b) $\int\frac{(\sin\theta-\cos\theta)}{\sqrt{(\sin 2\theta)}}$

Put $\sin\theta+\cos\theta=t$ $\quad(\cos\theta-\sin\theta)\,d\theta=dt$

Also $1+\sin 2\theta=t^2$ $\quad\sin 2\theta=t^2-1$

$$I=\int\frac{-dt}{\sqrt{(t^2-1)}}=-\log\{t+\sqrt{(t^2-1)}\}$$

$$=-\log\{\sin\theta+\cos\theta+\sqrt{(\sin 2\theta)}\}$$

18 $\int\frac{dx}{(x+1)^2(x^2+1)}$ **(M N R 79)**

Let $\frac{1}{(x+1)^2(x^2+1)}=\frac{A}{(x+1)^2}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$

$$1=A(x^2+1)+B(x+1)(x^2+1)+(Cx+D)(x+1)^2$$

Put $x=-1$ $\quad 1=2A$ or $A=1/2$

Put $x^2=-1$ $\quad 1=(Cx+D)(x^2+2x+1)=(Cx+D)\,2x$

or $1=2Cx^2+2Dx=-2C+2Dx$ $\quad x^2=-1$

$C=-\frac{1}{2},\ D=0$

Comparing coefficient of x^3 we get

$0=B+C$ $\quad B=-C=1/2$

26 $e^x \frac{2+\sin 2x}{1+\cos 2x}$ 27 $\frac{x e^x}{(x+1)^2}$

28 $\frac{\log x}{(1+\log x)^2}$ 29 $\frac{x-\sin x}{1-\cos x}$

30 $\frac{2x+\sin 2x}{1+\cos 2x}$ or $\frac{x+\sin x}{1+\cos x}$ (Roorkee 80)

31 $\int (e^{\log x}+\sin x) \cos x \, dx$ (I I T 81)

32 $\int \frac{\log_e x}{x^3} \, dx$ (Roorkee 86)

33 Evalute $\int_0^1 \log\left[\sqrt{(1-x)}+\sqrt{(1+x)}\right] dx$ (I I T 88)

Solution Set (D)

1 Proceed as in Ex 9 any of the two methods

$$\int x^2 \sin 2x \, dx = x^2\left(-\frac{\cos 2x}{2}\right)-(2x)\left(-\frac{\sin 2x}{4}\right)$$

$$+(2)\left(\frac{\cos 2x}{8}\right) = -\tfrac{1}{2}x^2 \cos 2x+\tfrac{1}{2}x \sin 2x+\tfrac{1}{4}\cos 2x$$

(b) Do yourself

2 $I=\int x \sin x \cos x \, dx=\frac{1}{2}\int x \sin 2x \, dx$

$$=\tfrac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right)-\int 1 \cdot \left(-\frac{\cos 2x}{2}\right) dx\right]$$

$$=-\tfrac{1}{4}x \cos 2x+\tfrac{1}{4}\frac{\sin 2x}{2}$$

$$=-\tfrac{1}{4}x \cos 2x+\tfrac{1}{8}\sin 2x$$

3 $I=\int x \tan^{-1} x \, dx=(\tan^{-1} x)\left(\frac{x^2}{2}\right)-\int \frac{1}{1+x^2} \cdot \frac{x^2}{2} dx$

$$=\tfrac{1}{2}x^2 \tan^{-1} x-\tfrac{1}{2}\int \frac{x^2+1-1}{x^2+1} \, dx$$

$$=\tfrac{1}{2}x^2 \tan^{-1} x-\tfrac{1}{2}\int\left(1-\frac{1}{x^2+1}\right) dx$$

$$=\tfrac{1}{2}x^2 \tan^{-1} x-\tfrac{1}{2}[x-\tan^{-1} x]$$

$$=\tfrac{1}{2}(x^2+1)\tan^{-1} x-\tfrac{1}{2}x$$

4 $I=\int x \sec^2 2x \, dx=x \frac{\tan 2x}{2}-\int 1 \cdot \frac{\tan 2x}{2} dx$

$$=\tfrac{1}{2}x \tan 2x-\tfrac{1}{2}\cdot\tfrac{1}{2}\log \sec 2x$$

$$=\tfrac{1}{2}x \tan 2x-\tfrac{1}{4}\log \sec 2x$$

5 $I=\int (\log x)^2 dx=\int 1 \cdot (\log x)^2 dx$

$$=(\log x)^2 . x-\int 2(\log x) \cdot \frac{1}{x} \cdot x \, dx$$

Now $1-\frac{1}{x^2}$ is differentiation of $x+\frac{1}{x}$ and hence we write

$$x^2+\frac{1}{x^2}=x^2+\frac{1}{x^2}+2-2=\left(x+\frac{1}{x}\right)^2-2$$

$$I=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-(\sqrt{2})^2}\quad \text{Put } x+\frac{1}{x}=t \qquad \left(1-\frac{1}{x^2}\right)dx=dt$$

$$I=\int\frac{dt}{t^2-(\sqrt{2})^2}=\frac{1}{2\sqrt{2}}\log\frac{t-\sqrt{2}}{t+\sqrt{2}}$$

or $$I=\frac{1}{2\sqrt{2}}\log\frac{\left(x+\frac{1}{x}\right)-\sqrt{2}}{\left(x+\frac{1}{x}\right)+\sqrt{2}}$$

$$=\frac{1}{2\sqrt{2}}\log\frac{x^2-x\sqrt{2}+1}{x^2+x\sqrt{2}+1}$$

Proceeding as above

$$\int\frac{x^2-1}{x^4+x^2+1}\,dx=\int\frac{dt}{t^2-1}=\tfrac{1}{2}\log\frac{x^2-x+1}{x^2+x+1}$$

$$\int\frac{x^2-1}{x^4-x^2+1}\,dx=\int\frac{dt}{t^2-3}=\frac{1}{2\sqrt{3}}\log\frac{x^2-x\sqrt{3}+1}{x^2+x\sqrt{3}+1}$$

Deduction I.

$$\int\frac{x^2}{x^4+1}\,dx,\ \int\frac{x^2\,dx}{x^4+x^2+1},\ \int\frac{x^2\,dx}{x^4-x^2+1}$$

Write $x^2=\frac{1}{2}[(x^2+1)+(x^2-1)]$

Hence each integral is

$\frac{1}{2}$ [Ans of Q 19 − Ans of Q 20]

Deduction II

$$\int\frac{1}{x^4+1}\,dx,\ \int\frac{1}{x^4+x^2+1}\,dx,\ \int\frac{1}{x^4-x^2+1}\,dx$$

write $1=\frac{1}{2}[(x^2+1)-(x^2-1)]$

Hence each integral is

$\frac{1}{2}$ [Ans of Q 19 - Ans of Q 20]

Problem Set (E)

1 (a) Prove $\int\frac{1}{(e^x-1)}\,dx=\log(e^x-1)-x$

$$=\frac{-1}{x}\log(x^2+a^2)+2\ \frac{1}{a}\ \tan^{-1}\frac{x}{a}$$

10 $I=\int \frac{x^2\tan^{-1}x}{1+x^2}\,dx$ Put $x=\tan\theta$ $\quad dx=\sec^2\theta\,d\theta$

$$I=\int\frac{(\tan^2\theta)\,\theta}{\sec^2\theta}\sec^2\theta\,d\theta=\int\theta\tan^2\theta\,d\theta$$

$=\theta\tan\theta-\log\sec\theta-\frac{\theta^2}{2}$ by note to Ex 11 P 661

$=x\tan^{-1}x-\log\sqrt{(1+\tan^2\theta)}\ \ \frac{1}{2}(\tan^{-1}x)^2$

$=x\tan^{-1}x-\frac{1}{2}\log(1+x^2)-\frac{1}{2}(\tan^{-1}x)^2$

11 $I=\int\frac{\sin^{-1}x}{(1\ \ x^2)^{3/2}}\,dx$ Put $x=\sin\theta$ $\quad dx=\cos\theta$

$$I=\int\frac{\theta\cos\theta\,d\theta}{\cos^3\theta}=\int\theta\sec^2\theta\,d\theta$$

$=\theta\tan\theta-\log\sec\theta$ by Ex 11

$=\theta\tan\theta+\log\cos\theta$

Now since $\sin\theta=x$, $\quad\cos\theta=\sqrt{(1-x^2)}$ and $\tan\theta=\frac{x}{\sqrt{(1-x^2)}}$

$$I=\frac{x}{\sqrt{(1-x^2)}}\sin^{-1}x+\frac{1}{2}\log(1-x^2)$$

12 By definition each is equal to $2\tan^{-1}x$

$I=2\int\tan^{-1}x\,dx$

$=2\,[x\tan^{-1}x-\frac{1}{2}\log(1+x^2)]$ by Ex 5 Page 657

13 $I=\int\frac{x}{1+\cos x}\,dx=\int\frac{x}{\cos\ x/2}\,dx=\int x\,(\frac{1}{2}\sec^2 x/2)\,dx$

$=x\tan\frac{x}{2}-\int 1\ \tan\frac{x}{2}\,dx$

$=x\tan x/2-2\log\sec x/2$

14 (i) $I=\int\frac{x\sin^{-1}x}{\sqrt{(1-x^2)}}\,dx$ $\quad$ Put $x=\sin\theta$ $\quad dx=\cos\theta\,d\theta$

(M N R 78)

$$I=\int\frac{\theta\sin\theta\cos\theta\,d\theta}{\cos\theta}=\int\theta\sin\theta\,d\theta$$

$=\theta\,(-\cos\theta)-\int 1\,(-\cos\theta)\,d\theta$

$=-\theta\cos\theta+\sin\theta$

$=-\sqrt{(1\ \ x^2)}\sin^{-1}x+x$

(ii) $I=\Big[x-\sqrt{(1-x^2)}\sin^{-1}x\Big]_0^{1/2}$

11 $\int \frac{dx}{x+\sqrt{(x-1)}} = \log[x+\sqrt{(x-1)}] - \frac{2}{\sqrt{3}} \tan^{-1} \frac{2\sqrt{(x-1)}+1}{\sqrt{3}}$

12 $\int \sqrt{\left(\frac{x+1}{x-1}\right)}\, dx = \sqrt{(x^2-1)} + \log[x+\sqrt{(x^2-1)}]$

13 $\int \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx = \sin^{-1} x - \sqrt{(1-x^2)}$

14 $\int x \sqrt{\left(\frac{1-x^2}{1+x^2}\right)}\, dx = \frac{1}{2}[\sin^{-1} x^2 + \sqrt{(1-x^4)}]$.

15 $\int \frac{x\, dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}$

$$= \frac{1}{a-b}\left\{\frac{2}{5}(x+a)^{5/2} - \frac{2a}{3}(x+a)^{3/2} - \frac{2}{5}(x+b)^{5/2} + \frac{2b}{3}(x+b)^{3/2}\right\}$$

16 $\int \frac{x\, dx}{(px+q)^{3/2}} = \frac{2}{p^2}\left\{\sqrt{(px+q)} + \frac{q}{\sqrt{(px+q)}}\right\}$

17 $\int \frac{1+x^{1/2}-x^{2/3}}{1+x^{1/3}}\, dx = -\frac{3}{4}t^8 + \frac{6}{7}t^7 + t^6 - \frac{6}{5}t^5 - \frac{3}{2}t^4$

$$+2t^3+3t^2-6t-3\log(t^2+1)+6\tan^{-1} t$$

where $t = x^{1/6}$

18 $\int x^3 \sqrt{(a^2+x^2)}\, dx$

$$= \frac{2}{3}\left\{\frac{1}{5}(a^2+x^2)^{5/2} - \frac{a^2}{3}(a^2+x^2)^{3/2}\right\}$$

19 $\int \frac{x\sqrt{(a^2-x^2)}}{a^2+x^2}\, dx = \sqrt{(a^2-x^2)} - \frac{a}{\sqrt{2}} \log \frac{a\sqrt{2}+\sqrt{(a^2-x^2)}}{a\sqrt{2}-\sqrt{(a^2-x^2)}}$

20 $\int \frac{dx}{(x+2)\sqrt{(x-1)}} = \frac{2}{\sqrt{3}} \tan^{-1} \sqrt{\left(\frac{x-1}{3}\right)}$

21 $\int \frac{dx}{(x+2)\sqrt{(3x+4)}} = 2\tan^{-1} \sqrt{\left(\frac{3x+4}{2}\right)}$

and hence prove $\int \frac{3x^2+13x+9}{(x+2)\sqrt{(3x+4)}}\, dx$

$$= \frac{2}{9}(3x+4)^{3/2} + \sqrt{2}\tan^{-1}\sqrt{\left(\frac{3x+4}{2}\right)}$$

22 $\int_0^1 \frac{dx}{(1+x^2)\sqrt{(2+x^2)}} = \frac{\pi}{6}$

23 $\int_0^{\pi/4} \sqrt{(\tan\theta)}\, d\theta = \frac{1}{\sqrt{2}} \log(\sqrt{2}-1) + \frac{\pi}{2\sqrt{2}}$

24 $\int_0^{\pi/2} \frac{dx}{\cos^4 x - \cos^2 x \sin^2 x + \sin^4 x} = \frac{\pi}{2}$

11 Put $x-1=t^2$ $dx=2t\,dt$

$$I=\int\frac{2t\,dt}{t^2+1+t}=\frac{(2t+1)-1}{t^2+t+1}\,dt$$

$$\int\frac{2t+1}{t^2+t+1}\,dt \quad \int\frac{1}{(t+1/2)^2+(\sqrt{3}/2)^2}\,dt \text{ etc}$$

12 $$I=\int\frac{x+1}{\sqrt{(x^2-1)}}\,dx=\int\left[\frac{x}{\sqrt{(x^2-1)}}+\frac{1}{\sqrt{(x^2-1)}}\right]dx$$

13 $$I=\int\frac{1+x}{\sqrt{(1-x^2)}}\,dx=\int\left[\frac{1}{\sqrt{(1-x^2)}}+\frac{x}{\sqrt{(1-x^4)}}\right]dx$$

14 Put $x^2=t$ first and then as in Q 12 and 13

15 Multiply above and below by $\sqrt{(x+a)}-\sqrt{(x+b)}$

$$I=\int\frac{x\,[\sqrt{(x+a)}-\sqrt{(x+b)}]}{(x+a)-(x+b)}\,dx$$

$$=\frac{1}{(a-b)}\int(x+a-a)\sqrt{(x+a)}-(x+b-b)\sqrt{(x+b)}]\,dx$$

$$I=\frac{1}{a-b}\int\left[(x+a)^{3/2}-a\,(x+a)^{1/2}-(x+b)^{3/2}+b\,(x+b)^{1/2}\right]dx$$

16 Put $px+q=t^2$

17 Put $x=t^6$

18 Write $x^5=x^3\,x^2$ and put $a^3+x^3=t^2$

19 Put $a^2-x^2=t^2$ $a^2-t^2=x^2$

and $-2x\,dx=dt$

$$I=-\int\frac{t\,t\,dt}{2a^2-t^2}=\int\frac{2a^2-t^2-2a^2}{2a^2-t^2}\,dt$$

$$=\int\left[1-\frac{2a^2}{(a\sqrt{2})^2-t^2}\right]dt$$

20 Put $x-1=t^2$

21 Put $3x+4=t^2$

and $3x^2+10x+9=(3x+4)(x+2)+1$

22 Proceed as in solved *Ex 15 Page 678*

23 Put $\tan\theta=t^2$ $\sec^2\theta\,d\theta=2t\,dt$

or $d\theta=\dfrac{2t\,dt}{1+\tan^2\theta}=\dfrac{2t\,dt}{1+t^4}$ and limits are adjusted as 0 to 1

$$I=\int_0^1\left[\frac{t\;2t\,dt}{t^4+1}=\int\frac{2\,t^2\,dt}{t^4+1}\right.$$

$$=\int_0^1\left[\frac{(t^2+1)+(t^2-1)}{t^4+1}\right]dt$$

Both the above have been calculated in Q 19 20 Page 680

24 Divide above and below by $\cos^4 x$

19 (a) $I=\int x^3 (\log x)^2\, dx$

$$= \frac{x^4}{4} (\log x)^2 - \int 2 (\log x) \cdot \frac{1}{x} \cdot \frac{x^4}{4}\, dx$$

$$= \frac{x^4}{4} (\log x)^2 - \frac{1}{2} \int x^3 \log x\, dx$$

$$= \frac{x^4}{4} (\log x)^2 - \frac{1}{2}\left[\frac{x^4}{4} \log x - \int \frac{1}{x} \cdot \frac{x^4}{4}\, dx\right]$$

$$= \frac{x^4}{4} (\log x)^2 - \frac{1}{8} x^4 \log x + \frac{1}{8}\int x^3\, dx$$

$$= \frac{x^4}{4} (\log x)^2 - \frac{1}{8} x^4 \log x + \frac{1}{32} x^4$$

(b) $\frac{2}{3} x^{3/2} \left[(\log x)^2 - \frac{4}{3} \log x + \frac{8}{9}\right]$

20 $I=\int \sin x \log (\sec x + \tan x)\, dx$

$$I = (-\cos x) \log (\sec x + \tan x) - \int \left[\frac{d}{dx} \log (\sec x + \tan x)\, (-\cos x)\right] dx$$

Now $\int \sec x\, dx = \log (\sec x + \tan x)$,

$$\frac{d}{dx} \log (\sec x + \tan x) = \sec x$$

$$I = -\cos x \log (\sec x + \tan x) - \int \sec x\, (-\cos x)\, dx$$

$$= -\cos x \log (\sec x + \tan x) + \int 1\, dx$$

$$= x - \cos x \log (\sec x + \tan x)$$

$I = \int \cos x \log \tan \frac{x}{2}\, dx$

$\int \operatorname{cosec} x\, dx = \log \tan \frac{x}{2}$, $\quad \frac{d}{dx}\left(\log \tan \frac{x}{2}\right) = \operatorname{cosec} x$

Hence as in Q 20 $I = \sin x \log \tan x/2 - x$

22 $I=\int \log (1+x^2)\, dx = \int 1 \cdot \log (1+x^2)\, dx$

$$= x \log (1+x^2) - \int x \cdot \frac{1}{1+x^2} \cdot 2x\, dx$$

$$= x \log (1+x^2) - 2 \int \frac{x^2+1-1}{x^2+1}\, dx$$

$$= x \log (1+x^2) - 2\int \left[1 - \frac{1}{1+x^2}\right] dx$$

$$= x \log (1+x^2) - 2x + 2 \tan^{-1} x$$

23 $I=\int \log [x + \sqrt{(x^2+a^2)}]\, dx$

We know that

$$=2\tan^{-1} t = 2\tan^{-1}\sqrt{\left(x+1+\frac{1}{x}\right)}$$

28 Dividing both N^r and D^r by x^2, we get

$$I=\int\frac{(1-1/x^2)\,dx}{(x+1/x)\sqrt{(1/x^2+x^2)}}$$

$$I=\int\frac{dt}{t\sqrt{(t^2-2)}},\ \text{Put } x+\frac{1}{x}=t,\left(1-\frac{1}{x^2}\right)dx=dt$$

$$=\int\frac{t\,dt}{t^2\sqrt{(t^2-2)}},\ \text{now put } t^2-2=z^2,\quad 2t\,dt=2z\,dz$$

$$=\int\frac{z\,dz}{(z^2+2)\,z}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}$$

$$=\int\frac{1}{\sqrt{2}}\cos^{-1}\frac{\sqrt{2}}{\sqrt{(z^2+2)}}=\frac{1}{\sqrt{2}}\cos^{-1}\frac{\sqrt{2}}{t}$$

$$=\frac{1}{\sqrt{2}}\cos^{-1}\frac{\sqrt{2}}{x+1/x}=\frac{1}{\sqrt{2}}\cos^{-1}\frac{x\sqrt{2}}{(x^2+1)}+c$$

29 Put $x=t^3$ $\quad I=3\int\frac{t^2+1}{(t^3+1)}\,dt$

$$I=3\int\frac{(t^2+1)}{(t+1)(t^2-t+1)}\ \text{P F's}$$

$$=3\int\left\{\frac{2}{3}\frac{1}{(t+1)}+\frac{1}{3}\frac{t-1}{t^2-t+1}\right\}dt$$

$$=2\log(t+1)+\tfrac{1}{2}\int\frac{2t-1+3}{(t^2+t+1)}\,dt$$

$$=2\log(t+1)+\tfrac{1}{2}\log(t^2-t+1)+\frac{3}{2}\int\frac{dt}{\left(t-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$=\tfrac{1}{2}\left[4\log(t+1)-\log(t^2-t+1)\right]+\frac{3}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1}\frac{t-\frac{1}{2}}{\sqrt{3}/2}$$

$$\tfrac{1}{2}\left[\log(t+1)^4(t^2-t+1)\right]-\sqrt{3}\tan^{-1}\frac{2t-1}{\sqrt{3}}$$

where $t=x^{1/3}$

Miscellaneous Problem Set (F)

1 $\int\frac{\tan x}{a^2+b^2\tan^2 x}\,dx$

2. $\int\frac{\sqrt{\tan x}\,dx}{\sin x\cos x}$

3 $\int\frac{e^x(1+x)\,dx}{\sin^2(xe^x)}$

4 $\int\frac{e^{\sqrt{x}}\cos e^{\sqrt{x}}}{\sqrt{x}}\,dx$

5 $\int\frac{dx}{(x^2-2x\cos\alpha+1)}$

6 $\int\frac{3+4\sin x+2\cos x}{2\sin x\ [illegible]\cos x}\,dx$

7. $\int\frac{dx}{\sqrt{[(x-a)(x-b)]}}$

8 $\int e^{-3x}\cos^3 x\,dx$

$$= \int \left[2\,\frac{e^x\,2}{\cos^2 x} + e^x \frac{2\sin x \cos x}{2\cos^2 x} \right] dx$$

$\int e^x \sec^2 x\,dx + \int e^x \tan x\,dx$

Integrate first by parts

$I = e^x \tan x - \int e^x \tan x\,dx + \int e^x \tan x\,dx$

$= e^x \tan x$ The last two integrals cancel

Above is of the form

$$\int e^x [f(x) + f'(x)]\,dx = e^x f(x),$$

where $f(x) = \tan x$ and $f'(x) = \sec^2 x$

27 $\int \frac{xe^x}{(1+x)^2}\,dx = \int e^x \frac{(x+1-1)}{(x+1)^2}\,dx$

$$= \int e^x \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2} \right] dx$$

$$= \int e^x [f(x) + f'(x)]\,dx$$

where $f(x) = \frac{1}{x+1}$ and $f'(x) = \frac{1}{(x+1)^2}$

$$= e^x f(x) = e^x \frac{1}{x+1}$$

28 Put $\log x = t$ $x = e^t$ and $dx = e^t\,dt$

$$I = \int \frac{\log x}{(1+\log x)^2}\,dx = \int \frac{t\,e^t\,dt}{(1+t)^2} = e^t \frac{1}{t+1} \text{ by Q 27}$$

$$= \frac{x}{1+\log x}$$

29 Split into two as in Q 24 $I = -x \cot x/2$

30 Split into two as in Q 26 $I = x \tan x$

31 $I = \int x \cos x\,dx + \int \sin x \cos x\,dx = x \sin x + \cos x + \frac{1}{2} \sin^2 x$

32 Ans $-\frac{1}{4x^2}(2 \log_e x + 1) + c$ [Hints Integrate by parts]

§ 5 Integration of Rational and Irrational fractions.

Students are fully conversant with the process of splitting a given fraction into partial fractions. In the case of irrational fraction we make such substitution so that the fractional powers are removed. The proce.s will be the following examples

Ex 1 (a) $\int_0^{\pi/2}$ $x\,dx$

Here we know th

40. If m and n are integers, show that

$$\int_0^{\pi} \sin mx \sin nx \, dx = 0 \text{ if } m \neq n \text{ and } \frac{\pi}{2} \text{ if } m = n$$

41 $\int_a^b \frac{dx}{\sqrt{[(x-a)(b-x)]}} = \pi \quad (b > a)$

42 $\int_a^b \sqrt{\left(\frac{x-a}{b-x}\right)} dx = \frac{\pi}{2}(b-a)$

43 $\int_a^b \sqrt{[(x-a)(b-x)]} \, dx = \frac{\pi}{8}(b-a)^2$

44 $\int_0^{\infty} \frac{dx}{\{x+\sqrt{(1+x^2)}\}^n}$, where n is an integer > 1

45 $\int \left[\frac{(\cos 2x)^{1/2}}{\sin x}\right] dx$ **(I I T 87)**

Solution to Miscellaneous Problem Set (F)

1 Change to $\sin x$ and $\cos x$ and it reduces to Q 5 P 627

3 Divide above and below by $\cos^2 x$

$$I = \int \frac{\sqrt{(\tan x)} \sec^2 x \, dx}{\tan x} = \int \frac{\sec^2 x}{\sqrt{(\tan x)}} dx = 2\sqrt{(\tan x)}$$

3 Put $xe^x = t$ $\quad (1 \cdot e^x + x e^x)\, dx = dt$

$I = \int \operatorname{cosec}^2 t \, dt = -\cot t = -\cot (xe^x)$

4 Put $e^{\sqrt{x}} = t$ $\quad e^{\sqrt{x}} \frac{1}{2\sqrt{x}} dx = dt$

$I = 2 \int \cos t \, dt = 2 \sin t = 2 \sin e^{\sqrt{x}}$

5 $x^2 - 2x \cos\theta + 1 = (x - \cos\theta)^2 + (1 - \cos^2\theta)$

$$I = \int \frac{dx}{(x-\cos\theta)^2 + \sin^2\theta} = \frac{1}{\sin\theta} \tan^{-1} \frac{x - \cos\theta}{\sin\theta}$$

6 $I = \int_0^{\pi} \left[2 - \frac{3}{3 + 2\sin x + \cos x}\right] dx$

Now see I_2 of Ex 18 P 645

7 $I = \int \frac{dx}{\sqrt{[(x-a)(x-b)]}} = \int \frac{dx}{\sqrt{(x-a)}\sqrt{[x-a+(a-b)]}}$,

Put $x - a = z^2$ $\quad dx = 2z \, dz$

$$I = \int \frac{2z \, dz}{z\sqrt{(z^2 + a - b)}} = 2 \int \frac{dz}{\sqrt{[z^2 + (a-b)]}}$$

$= 2 \log [z + \sqrt{[z^2 + (a-b)]}] = 2 \log [\sqrt{(x-a)} + \sqrt{(x-b)}]$

$$=\frac{1}{n}\left[\log t-\log(t+1)\right]=\frac{1}{n}\log\frac{x^n}{x^n+1}$$

Similarly $\int\frac{dx}{x(x^3+1)}=\frac{1}{3}\log\frac{x^3}{x^3+1}$ etc (Roorkee 80)

and $\int\frac{dx}{x(x^4+1)}=\frac{1}{4}\log\frac{x^4}{x^4+1}$ etc (Roorkee 80)

3 (b) $\int\frac{dx}{(x+1)(2+2x+x^2)}$ (Roorkee 74)

$$I=\int\frac{dx}{(x+1)\{(x+1)^2+1\}}=\int\frac{dt}{t(t^2+1)}=\tfrac{1}{2}\log\frac{t^2}{t^2+1}$$

where $t=x+1$

Ex 4 $\int\frac{dx}{\sin x+\sin 2x}$

$$I=\int\frac{dx}{\sin x(1+2\cos x)}=\int\frac{\sin x\,dx}{\sin^2 x(1+2\cos x)}$$

$$=\int\frac{\sin x\,dx}{(1-\cos x)(1+\cos x)(1+2\cos x)}$$

Now d c of $\cos x$ is $-\sin x$ which is given in numerator and hence we make the substitution $\cos x=t$ $\quad -\sin x\,dx=dt$

$$I=-\int\frac{dt}{(1-t)(1+t)(1+2t)}$$

We split the integrand into partial fractions by the method of supression

$$I=-\int\left[\frac{1}{6(1-t)}-\frac{1}{2(1+t)}+\frac{4}{3(1+2t)}\right]dt$$

$$=\int\left[\frac{1}{6}\left(\frac{-1}{1-t}\right)+\frac{1}{2}\frac{1}{(1+t)}-\frac{2}{3}\frac{2}{(1+2t)}\right]dt$$

$$=\frac{1}{6}\log(1-t)+\frac{1}{2}\log(1+t)-\frac{2}{3}\log(1+2t)$$

$$=\frac{1}{6}\log(1-\cos x)+\frac{1}{2}(1+\cos x)-\frac{2}{3}\log(1+2\cos x)$$

(b) $\int_0^{\pi/2}\frac{\sin x\cos x}{\cos^2 x-3\cos x+2}dx=\log(9/8)$ (M N R 81)

Put $\cos x=t$ $\quad I=-\int_0^1\frac{t\,dt}{(t+1)(t+2)}=\int_0^1\left(\frac{2}{t+2}-\frac{1}{t+1}\right)dt$

$$=\left[\log\frac{(t+2)^2}{(t+1)}\right]_0^1=\log\frac{9}{2}-\log 4=\log(9/8)$$

$$=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int\left[(x^2-1)+\frac{1}{1+x^2}\right]dx$$

$$=\frac{1}{4}\left[(x^4-1)\tan^{-1}x-\frac{x^3}{3}+x\right]$$

15 Integrate by parts

$$I=\frac{x^2}{2}\sin^{-1}x-\frac{1}{2}\int\frac{x^2}{\sqrt{(1-x^2)}}\,dx$$

$$=\frac{1}{2}x^2\sin^{-1}x+\frac{1}{2}\int\frac{1-x^2-1}{\sqrt{(1-x^2)}} \qquad \text{(Note)}$$

$$=\frac{1}{2}x^2\sin^{-1}x+\frac{1}{2}\int\left(\sqrt{(1-x^2)}-\frac{1}{\sqrt{(1-x^2)}}\right)dx$$

$$=\frac{1}{2}x^2\sin^{-1}x+\frac{1}{2}\left[\frac{x}{2}\sqrt{(1-x^2)}+\frac{1}{2}\sin^{-1}x-\sin^{-1}x\right]$$

$$=\tfrac{1}{2}x^2\sin^{-1}x+\tfrac{1}{4}x\sqrt{(1-x^2)}-\tfrac{1}{4}\sin^{-1}x$$

16 $$I=\frac{x^2}{2}\sec^{-1}x-\frac{1}{2}\int\frac{x^2}{x\sqrt{(x^2-1)}}\,dx$$

$$=\frac{x^2}{2}\sec^{-1}x-\frac{1}{4}\int\frac{2x}{\sqrt{(x^2-1)}}\,dx \qquad \text{Put } x^2-1=t \text{ etc}$$

$$I=\tfrac{1}{2}\left[x^2\sec^{-1}x-\sqrt{(x^2-1)}\right]$$

17 Put $x=a\tan^2\theta \qquad dx=2a\tan\theta\sec^2\theta\,d\theta$

$$I=\int\sin^{-1}\sqrt{\left(\frac{a\tan^2\theta}{a\sec^2\theta}\right)}\,2a\tan\theta\sec^2\theta\,d\theta$$

$$=\int\sin^{-1}(\sin\theta)\,2a\tan\theta\sec^2\theta\,d\theta$$

$$=a\int\theta\,(2\tan\theta\sec^2\theta)\,d\theta \qquad \text{Integrate by parts}$$

$$I=a\left[\theta\sec^2\theta-\int\sec^2\theta\cdot 1\,d\theta\right]$$

$$\int 2\tan\theta\sec^2\theta\,d\theta=\int 2\sec\theta(\sec\theta\tan\theta\,d\theta)=2\,\frac{\sec^2\theta}{2}$$

$$=a\left[\theta(1+\tan^2\theta)-\tan\theta\right]$$

$$a\left[\left(1+\frac{x}{a}\right)\tan^{-1}\sqrt{\left(\frac{x}{a}\right)}-\sqrt{\left(\frac{x}{a}\right)}\right]$$

$$=(a+x)\tan^{-1}\sqrt{\left(\frac{x}{a}\right)}-\sqrt{(ax)}$$

18 Put $x=2a\cos\theta \qquad \dfrac{2a-x}{a}=\dfrac{2a(1-\cos\theta)}{a}$

$$\sqrt{\left(\frac{2a-x}{a}\right)}=\sqrt{(4\sin^2\theta/2)}=2\sin\theta/2$$

$$I=\int(2a\cos\theta)\sin^{-1}(\tfrac{1}{2}\,2\sin\theta/2)\,(-2a\sin\theta\,d\theta)$$

$$=\int\tfrac{1}{2}\theta\,(-4a^2\sin\theta\cos\theta)\,d\theta=-a^2\int\theta\sin 2\theta\,d\theta$$

$$I=\int \frac{(1-t^2)^2\,(-2t\,dt)}{t}=-2\int(1-2t^2+t^4)\,dt$$

$$=-2\left[t-\frac{2}{3}t^3+\frac{1}{5}t^5\right]$$

$$=-2t\left[1-\frac{2}{3}t^2+\frac{1}{5}t^4\right] \text{ Now put } t=\sqrt{(1-x)}$$

$$=-\frac{2\sqrt{(1-x)}}{15}\left[15-10(1-x)+3(1-2x+x^2)\right]$$

$$=-\frac{2\sqrt{(1-x)}}{15}\left[8+4x+3x^2\right]$$

Similarly $\int \frac{x^2}{\sqrt{(x+1)}}dx=\frac{1}{15}\sqrt{(1+x)}\left[8-4x+3x^2\right]$

Ex 9 $\int \frac{x^5}{\sqrt{(1+x^3)}}dx$ (I I T 1976)

Here $x^5=x^3\cdot x^2$ and d c of x^3 is $3x^2$ (I I T 75)

In order to remove fractional powers we put $1+x^3=t^2$

$3x^2\,dx=2t$, Also $x^3=t^2-1$

$$I=\int \frac{x^5}{\sqrt{(1+x^3)}}dx=\int \frac{x^3\,x^2\,dx}{\sqrt{(1+x^3)}}$$

$$=\int \frac{(t^2-1)}{t}\left(\frac{2}{3}\,t\,dt\right)=\frac{2}{3}\int(t^2-1)\,dt$$

$$=\frac{2}{3}\left(\frac{t^3}{3}-t\right)=\frac{2}{9}t\,(t^2-3)$$

$$=\frac{2}{9}\sqrt{(1+x^3)}\,(1+x^3-3)=\frac{2}{9}\sqrt{(1+x^3)}\,(x^3-2)$$

Ex 10 $\int \frac{x^5}{\sqrt{(1+x^2)}}\,dx$

As above $x^5=x^4\cdot x$ and d c of x^2 is $2x$

Put $1+x^2=t^2$ $2x\,dx=2t\,dt$ and $x^2=t^2-1$

$$I=\int \frac{x^4\,x\,dx}{\sqrt{(1+x^2)}}=\int\frac{(t^2-1)}{t}\,t\,dt$$

$$=\int(t^4-2t^2+1)\,dt=\left(\frac{t^5}{5}-2\frac{t^3}{3}+t\right)$$

$$=\frac{1}{15}t\,(3t^4-10t^2+15)$$

$$=\frac{1}{15}\sqrt{(1+x^2)}\,[3(1+x^4+2x^2)-10(1+x^2)+15)]$$

$$I_1+I_2=x\log(\log x)-\frac{x}{\log x}$$

22 $I=x(\log x)^2-2\int x\log x\,\frac{1}{x}\,dx$

$=x(\log x)^2-2\int\log x\,dx$

$=x(\log x)^2-2[x\log x-x]$ Ex 4 (a) Page 657

$=x[(\log x)^2-2\log x+2]$

23 $\frac{x^2+1}{(x+1)^2}=\frac{x^2-1+2}{(x+1)^2}=\frac{x-1}{x+1}+\frac{2}{(x+1)^2}$

$=f(x)+f'(x)$

$I=\int e^x[f(x)+f'(x)]\,dx=e^x f(x)$

$=e^x\,\frac{x-1}{x+1}$

Alternative Method

$I=\frac{e^x(x^2+1)}{(x+1)^2}$ Integrate by parts

$I=e^x(x^2+1)\left[-\frac{1}{x+1}\right]+\int\frac{1}{x+1}\,e^x(x^2+1+2x)\,dx$

$=-e^x\,\frac{x^2+1}{x+1}+\int e^x(x+1)\,dx$

$=-e^x\,\frac{x^2+1}{x+1}+e^x x$

$=e^x\left[\frac{-x^2-1+x^2+x}{x+1}\right]=e^x\,\frac{x-1}{x+1}$

24 $\left(\frac{1-x}{1+x^2}\right)^2=\frac{1+x^2-2x}{(1+x^2)^2}$

$=\frac{1}{1+x^2}+\frac{-2x}{(1+x^2)^2}=f(x)+f'(x)$

$I=\int e^x[f(x)+f'(x)]\,dx=e^x f(x)$

$=e^x\,\frac{1}{1+x^2}$

25 $\frac{x^3-x+2}{(x^2+1)^2}=\frac{(x^2+1)(x+1)+(1-2x-x^2)}{(x^2+1)^2}$

$=\frac{x+1}{x^2+1}+\frac{1-2x-x^2}{(x^2+1)^2}=f(x)+f'(x)$

$I=\int e^x[f(x)+f'(x)]\,dx=e^x f(x)$

$=e^x\,\frac{x+1}{x^2+1}$

26 $I=I_1-I_2$

$$I=\frac{1}{2}\int\frac{t\,dt}{\frac{t^2+1}{2}\,t}=\int\frac{dt}{t^2+1}=\tan^{-1}t=\tan^{-1}\sqrt{(4x+5)}$$

Now $8x^2+22x+17=(2x+3)(4x+5)+2$

$$I=\int\frac{8x^2+22x+17}{(2x+3)\sqrt{(4x+5)}}\,dx=\int\frac{(2x+3)(4x+5)+2}{(2x+3)\sqrt{(4x+5)}}\,dx$$

$$=\int\sqrt{(4x+5)}+2\int\frac{dx}{(2x+3)\sqrt{(4x+5)}}$$

$$=\frac{1}{4}\cdot\frac{2}{3}(4x+5)^{3/2}+2\tan^{-1}\sqrt{(4x+5)} \text{ as proved above}$$

$$=\frac{1}{6}(4x+5)^{3/2}+2\tan^{-1}\sqrt{(4x+5)}$$

Ex 15 $\displaystyle\int_0^{1/\sqrt{3}}\frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$

Put $x=\frac{1}{t}$ $\quad dx=-\frac{1}{t^2}dt$

Also when $x=\frac{1}{\sqrt{3}}$, $t=\sqrt{3}$ and when $x=0$, $t=\infty$

$$I=\int_{\infty}^{\sqrt{3}}\frac{-\frac{1}{t^2}\,dt}{\frac{1+t^2}{t^2}\cdot\frac{\sqrt{(t^2-1)}}{t}}=\int_{\infty}^{\sqrt{3}}\frac{-t\,dt}{(1+t^2)\sqrt{(t^2-1)}}$$

Now differentiation of t^2 is $2t$ which is present and in order to remove fractional power we put $t^2-1=z^2$

$2t\,dt=2z\,dz$ and $t^2=1+z^2$

Also when $t=\sqrt{3}$ then $z=\sqrt{2}$ and when $t=\infty$, $z=\infty$

$$I=-\int_{\infty}^{\sqrt{2}}\frac{z\,dz}{(z^2+2)z}=-\int_{\infty}^{\sqrt{2}}\frac{dz}{z^2+2}$$

$$=-\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{z}{\sqrt{2}}\right]_{\infty}^{\sqrt{2}}=-\frac{1}{\sqrt{2}}\left[\tan^{-1}1-\tan^{-1}\infty\right]$$

$$=-\frac{1}{\sqrt{2}}\left[\frac{\pi}{4}-\frac{\pi}{2}\right]=-\frac{1}{\sqrt{2}}\left[-\frac{\pi}{4}\right]=\frac{\pi}{4\sqrt{2}}$$

Ex 16 (a) $\displaystyle\int\frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}}$

(b) $\displaystyle\int\frac{x^{1/2}}{x^{1/2}-x^{1/3}}\,dx$ (Roorkee 81)

In order to remove the fractional powers $\frac{1}{2}$ and $\frac{1}{3}$ we should put $(1+x)=t^6$ where 6 is the L C M of 2 and 3

$$=\frac{\sin x\cos x-x(1-\sin^2 x)}{(x\sin x+\cos x)\cos x}=\frac{\sin x-x\cos x}{x\sin x+\cos x}$$

30 $\sin\theta+\cos\theta$ is differential coefficient of $\sin\theta-\cos\theta$ Hence we put $\sin\theta-\cos\theta=t$ $(\cos\theta+\sin\theta)\,d\theta=dt$ Also on squaring $\sin^2\theta+\cos^2\theta-2\sin\theta\cos\theta=t^2$

or $1-t^2=\sin 2\theta$

$$I=\int\frac{dt}{\sqrt{(1-t^2)}}=\sin^{-1}t=\sin^{-1}(\sin\theta-\cos\theta)$$

31 $$I=\int\sqrt{\left(\frac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)}dx=\int\frac{\sin(x-\alpha)}{\sqrt{[\sin(x+\alpha)\sin(x-\alpha)]}}dx$$

$$=\int\left\{\frac{\sin x\cos\alpha}{\sqrt{(\sin^2 x-\sin^2\alpha)}}-\frac{\cos x\sin\alpha}{\sqrt{(\sin^2 x-\sin^2\alpha)}}\right\}dx$$

$$=\cos\alpha\int\frac{\sin x\,dx}{\sqrt{(\cos^2\alpha-\cos^2 x)}}-\sin\alpha\int\frac{\cos x\,dx}{\sqrt{(\sin^2 x-\sin^2\alpha)}}$$

$\cos x=t$ for Ist and $\sin x=z$ for 2nd

$$=\cos\alpha\cos^{-1}(\cos x\sec\alpha)-\sin\alpha\cosh^{-1}(\sin x\operatorname{cosec}\alpha)$$

32 $$\frac{1-\cos x}{\cos x(1+\cos x)}=\frac{1}{\cos x}-\frac{2}{1+\cos x}\quad \text{P F 's}$$

$$I=\int\left(\sec x-2\,\frac{1}{2\cos^2\frac{x}{2}}\right)dx$$

$$=\log(\sec x+\tan x)-2\tan\frac{x}{2}$$

33 $$I=\int\frac{1}{x(x^2+1)^3}\,dx=\int\frac{x\,dx}{x^2(x^2+1)^3}$$

Put $x^2+1=t$ $\qquad 2x\,dx=dt$

$$I=\tfrac{1}{2}\int\frac{dt}{t^3(t-1)}=-\tfrac{1}{2}\int\frac{1}{t^3}\,\frac{t^3-1-t^3}{t-1}\,dt$$

$$=-\tfrac{1}{2}\int\frac{1}{t^3}\left[t^2+t+1-\frac{t^3}{t-1}\right]dt$$

$$=-\tfrac{1}{2}\int\left[\frac{1}{t}+\frac{1}{t^2}+\frac{1}{t^3}-\frac{1}{t-1}\right]dt$$

$$=-\tfrac{1}{2}\left[\log t-\frac{1}{t}-\frac{1}{2t^2}-\log(t-1)\right]$$

where $t=x^2+1$

34, 35, 36 In all these multiply above and below by e^x and put $e^x=t$ and split into partial fractions

34 Ans $2\log(e^x-1)-x$

35 Ans $\tfrac{1}{4}\log(e^x-1)(e^x+3)^3$

$$\int \frac{1}{(x+1)^2(x^2+1)}dx=\int\left[\frac{1}{2}\frac{1}{(x+1)^2}+\frac{1}{2}\frac{1}{x+1}-\frac{1}{2}\frac{x}{x^2+1}\right]dx$$

$$=\frac{1}{2}\left[-\frac{1}{x+1}+\log(x+1)-\tfrac{1}{2}\log(x^2+1)\right]$$

Special Type

19 $\int\frac{x^2+1}{x^4+1}dx, \int\frac{x^2+1}{x^4+x^2+1}dx, \int\frac{x^2+1}{x^4-x^2+1}dx$

Divide above and below by x^2

$$I=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}}$$

Now $1+\frac{1}{x^2}$ is differentiation of $x-\frac{1}{x}$ and hence we write

$x^2+\frac{1}{x^2}$ as $x^2+\frac{1}{x^2}-2+2=\left(x-\frac{1}{x}\right)^2+2$

$$I=\int\frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+2}\quad \text{Put } x-\frac{1}{x}=t \qquad \left(1+\frac{1}{x^2}\right)dx=dt$$

$$I=\int\frac{dt}{t^2+2}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}$$

or $I=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x-\frac{1}{x}}{\sqrt{2}}=\frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{x\sqrt{2}}$

Proceeding as above

$$\int\frac{x^2+1}{x^4+x^2+1}dx=\int\frac{dt}{t^2+3}=\frac{1}{\sqrt{3}}\tan^{-1}\frac{x^2-1}{x\sqrt{3}}$$

$$\int\frac{x^2+1}{x^4-x^2+1}dx=\int\frac{dt}{t^2+1}=\tan^{-1}\frac{x^2-1}{x}$$

20 $\int\frac{x^2-1}{x^4+1}dx, \int\frac{x^2-1}{x^4+x^2+1}, \int\frac{x^2+1}{x^4-x^2+1}dx$

Divide above and below by x^2

$$I=\int\frac{\left(1-\frac{1}{x^2}\right)dx}{x^2+\frac{1}{x^2}}$$

$$I=\int_0^{\pi}\sin^2 mx\,dx=\tfrac{1}{2}\int_0^{\pi}(1-\cos 2mx)\,dx$$

$$=\tfrac{1}{2}\left[x-\frac{\sin 2mx}{2m}\right]_0^{\pi}=\frac{\pi}{2}-0=\frac{\pi}{2}$$

41, 42, 43 Put $x=a\cos^2 t+b\sin^2 t$

$x-a=a\cos^2 t+b\sin^2 t-a=b\sin^2 t-a(1-\cos^2 t)$

$=(b-a)\sin^2 t$

$b-x=b-a\cos^2 t-b\sin^2 t=(b-a)\cos^2 t$

$dx=2(b-a)\sin t\cos t\,dt$

Also when $x=a$ then $\sin t=0$ $\quad t=0$

$x=b$ then $\cos t=0$ $\quad t=\pi/2$

For Q 41

$$I=\int_0^{\pi/2}\frac{2(b-a)\sin t\cos t\,dt}{(b-a)\sin t\cos t}=2\left[t\right]_0^{\pi/2}=\pi$$

For Q 42

$$I=\int_0^{\pi/2}\frac{\sin t}{\cos t}\,2(b-a)\sin t\cos t\,dt$$

$$=(b-a)\int_0^{\pi/2}2\sin^2 t\,dt=(b-a)\int_0^{\pi/2}(1-\cos 2t)\,dt$$

$$=(b-a)\left[t-\frac{\sin 2t}{2}\right]_0^{\pi/2}=(b-a)\frac{\pi}{2}$$

For Q 43

$$I=\int_0^{\pi/2}(b-a)\sin t\cos t\;2(b-a)\sin t\cos t\,dt$$

$$=2(b-a)^2\int_0^{\pi/2}\sin^2 t\cos^2 t\,dt$$

$$=2(b-a)^2\int_0^{\pi/2}\frac{1-\cos 2t}{2}\cdot\frac{1+\cos 2t}{2}\,dt$$

$$=\frac{(b-a)^2}{2}\int(1-\cos^2 2t)\,dt=\frac{(b-a)^2}{4}\int_0^{\pi/2}2\sin^2 2t\,dt$$

$$=\frac{(b-a)^2}{4}\int_0^{\pi/2}(1-\cos 4t)\,dt=\frac{(b-a)^2}{4}\left[t-\frac{\sin 4t}{4}\right]_0^{\pi/2}=\frac{\pi}{8}(b-a)^2$$

44 Put $x+\sqrt{(1+x^2)}=u$ so that

$(u-x)^2=1+x^2$ or $u^2-2ux=1$

or $$x=\frac{1}{2}\left(u-\frac{1}{u}\right)$$

$$dx=\frac{1}{2}\left(1+\frac{1}{u^2}\right)du$$

Also $u=1$ when $x=0$, and $u=\infty$ when $x=\infty$

(*b*) Evaluate $\int \frac{x-1}{(x-3)(x-2)}\,dx$ (Roorkee 78)

2 (*a*) Prove $\int \frac{1}{(e^x-1)^2}\,dx = x - \log(e^x-1) - \frac{1}{e^x-1}$

(*b*) Evaluate $\int_1^2 \frac{dx}{x(x+1)^2}$ (Roorkee 77)

(*c*) Evaluate $\int \frac{e^x}{(1+e^x)}\,dx$ (Roorkee 76)

Prove the following

3 (*a*) $\int \frac{dx}{x\{6(\log x)^2+7\log x+2\}} = \log \frac{1+\log x^2}{2+\log x^3}$

(*b*) $\int \frac{x\,dx}{(x^2-a^2)(x^2-b^2)} = \frac{1}{2(a^2-b^2)} \log \frac{x^2-a^2}{x^2-b^2}$ (Roorkee 76)

4 $\int \frac{x^2+1}{(x^2+2)(2x^2+1)} = \frac{1}{3\sqrt{2}} \left\{ \tan^{-1} \frac{x}{\sqrt{2}} + \tan^{-1} x\sqrt{2} \right\}$

5 $\int_0^{\pi/2} \frac{\cos^2\theta\, d\theta}{\cos^2\theta + 4\sin^2\theta} = \frac{\pi}{6}$

6 $\int \frac{dx}{\sin x\,(5+4\cos x)} = \frac{1}{18}\log(1-\cos x) - \frac{1}{2}\log(1+\cos x) + \frac{4}{9}\log(5+4\cos x)$

7 $\int \frac{dx}{x\log x\,[(\log x)^2-3\log x-10]}$
$= -\frac{1}{10}\log(\log x) + \frac{1}{35}\log(\log x-5) + \frac{1}{14}\log(\log x+2)$

8 $\int \frac{x-1}{(x+1)(x^2+1)} = -\log(x+1) + \frac{1}{2}\log(x^2+1)$

9 (*a*) $\int \frac{x\,dx}{(1+x)(1+x^2)} = -\frac{1}{2}\log(1+x) + \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1} x$

(*b*) Evaluate $\int \frac{x^4}{(x-1)(x^2+1)}\,dx$ (Roorkee 86)

10 (*a*) $\int \frac{dx}{(1+x)(1+x^2)} = \frac{1}{2}\log(1+x) - \frac{1}{4}\log(1+x^2) + \frac{1}{2}\tan^{-1} x$

(*b*) $\int \frac{x^2+x+3}{(x-2)(x+1)}\,dx = x + 3\log(x-2) - \log(x+1)$ (Roorkee 75)

(*c*) Evaluate $\int \frac{1}{(x-1)(x^2+1)}\,dx$ (Roorkee 84)

(*d*) $\int \frac{x^3-1}{x^3+x}\,dx$ (Roorkee 88)

$$h \to 0,\ n \to \infty$$

is defined as the integral of $f(x)$ w r t x between the limits a to b and is written as

$$\int_a^b f(x)\,dx = \Big(x\Big)\Big]_a^b = F(b) - F(a)$$

where $F(x)$ is a function whose differentiation is $f(x)$ or $F(x)$ is the integral of $f(x)$ w r t

We shall illustrate the above by giving few examples

Note The above sum can also be written as under

$$\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(a+rh)$$

where $$h = \frac{b-a}{n}$$

Recollect the following From summation of series

$$\sum_{r=1}^{n} r = \frac{n(n+1)}{2},\quad \sum_{r=1}^{n} r^2 = \frac{n(n+1)(2n+1)}{6}$$

$$\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2 = \frac{n^2(n+1)^2}{4},\quad \Sigma 1 = n$$

$$\sum_{r=1}^{n} x^r = \text{sum of a G P} = \frac{x(x^n-1)}{x-1}$$

$$\sum_{r=1}^{n} ax^{r-1} = \frac{a(x^n-1)}{x-2} \text{ as first term will be } a$$

$\sin A + \sin(A+B) + \sin(A+2B) + \quad + \sin(A+\overline{n-1}B)$ (n terms)

$$= \sin\left(\frac{A+A+\overline{n-1}\,B}{2}\right)\frac{\sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

$\cos A + \cos(A+B) + \cos(A+2B) + \quad + \cos(A+\overline{n-1}B)$ (n terms)

$$= \cos\left(\frac{A+A+\overline{n-1}\,B}{2}\right)\frac{\sin\frac{nB}{2}}{\sin\frac{B}{2}}$$

Here the angles are in A P whose first term is A and common difference is B

25 $\int_0^{\pi/2} \frac{\cos x\, dx}{1-\sin^2 x+\sin^4 x} = \frac{\pi}{2} - \frac{1}{2\sqrt{3}} \log (2-\sqrt{3})$

26 $\int \frac{\sin^{-1} \sqrt{x} - \cos^{-1} \sqrt{x}}{\sin^{-1} \sqrt{x} + \cos^{-1} \sqrt{x}}\, dx$ (*IIT 83*)

27 Evaluate $\int \frac{(x-1)\, dx}{(x+1) \sqrt{(x^3+x^2+x)}}$

28 Prove that

$$\int \frac{x^2-1}{x^2+1} \frac{dx}{\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \cos^{-1} \frac{x\sqrt{2}}{x^2+1} + c$$

29 $\int \frac{1+x^{-2/3}}{1+x}\, dx$ (Roorkee 87)

Solutions to Problem Set (E)

1&2 Multiply above and below by e^x and then put $e^x = t$ and now split into partial farctions

3 Put $\log x = t$ and then split into partial fractions

4 Write $x^2 = t$ and split $\frac{t+1}{(t+2)(2t+1)}$ into partial fractions and

5 Divide above and below by $\cos^2 \theta$

$$I = \int \frac{d\theta}{1+4 \tan^2 \theta} = \int \frac{\sec^2 \theta\, d\theta}{(1+\tan^2 \theta)(1+4 \tan^2 \theta)}$$

Now put $\tan \theta = t$ $\qquad \sec^2 d\theta = dt$

$$I = \int \frac{dt}{(1+t^2)(1+4t^2)}$$

Now split into partial fractions as in Ex 4 above

6 Proceed as in Solved Ex 4, 674

7 Write $\log x^x = x \log x$ and then put $\log x = t$ $\qquad 1/x\, dx = dt$

$$I = \int \frac{dt}{t(t^2-3t-10)} = \int \frac{dt}{t(t-5)(t+2)}$$

Now split into partial fractions

8, 9, 10 Split into partial fractions In 9 (b) and 10 ( ) First Divide *For 9 (b), we have, after resolving into partial fractions*

$$\int \frac{x^4}{(x-1)(x^2+1)}\, dx = \int \left[x+1+ \text{[illegible]} \frac{x+1}{x^2+1)}\right] dx$$

$= \frac{1}{2} x^2 + x - \frac{1}{2} \log$ [illegible] $(x^2+$ [illegible] c

For 10 (d), We have [illegible] $+1) dx$ [illegible] $x+$ [illegible]

$$=\frac{b-a}{3}\left[\,3a^2+3\,(ab-a^2)+(b^2-2ab+a^2)\,\right]$$

$$=\frac{b-a}{3}\,[b^2+ba+a^2]=\tfrac{1}{3}\,(b^3-a^3)$$

Note Had we started with the result

$$\int_a^b f(x)\,dx=\underset{n\to\infty}{\mathrm{Lt}}\ \sum_{r=1}^{n} f(a+rh),\ \text{where } nh=b-a,$$

then also we would have obtained the same result

2 $\int_1^3 x^3\,dx=20$

Here $a=1$, $b=3$, $b-a=3-1=2$, $f(x)=x^3$ and $h=\dfrac{b-a}{n}=\dfrac{2}{n}$

By definition

$$\int_1^3 x^3\,dx=\underset{n\to\infty}{\mathrm{Lt}}\ h\sum_{r=1}^{n} f(a+rh)$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{2}{n}\sum_{r=1}^{n} f\left(1+\frac{2r}{n}\right)\qquad h=\frac{2}{n}\tag{1}$$

Now $f(x)=x^3$ $\qquad f\left(1+\dfrac{2r}{n}\right)=\left(1+\dfrac{2r}{n}\right)^3$

or $f\left(1+\dfrac{2r}{n}\right)=1+3\left(\dfrac{2r}{n}\right)+3\left(\dfrac{2r}{n}\right)^2+\left(\dfrac{2r}{n}\right)^3$

$$=1+\frac{6}{n}\,r+\frac{12}{n^2}\,r^2+\frac{8}{n^3}\,r^3)\tag{2}$$

Hence from (1) by the help of (2)

$$\int_1^3 x^3\,dx=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{2}{n}\sum_{r=1}^{n}\left(1+\frac{6}{n}\,r+\frac{12}{n^2}\,r^2+\frac{8}{n^3}\,r^3\right)$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{2}{n}\left(\Sigma 1+\frac{6}{n}\,\Sigma r+\frac{12}{n^2}\,\Sigma r^2+\frac{8}{n^3}\,\Sigma r^3\right)$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ \frac{2}{n}\left[n+\frac{6}{n}\ \frac{n\,(n+1)}{2}+\frac{12}{n^2}\ \frac{n\,(n+1)\,(2n+1)}{6}\right.$$

$$\left.+\frac{8}{n^3}\ \frac{n^2\,(n+1)^2}{4}\right]$$

$$=\underset{n\to\infty}{\mathrm{Lt}}\ 2\left\{1+\frac{3}{n^2}\,n^2\left(1+\frac{1}{n}\right)+\frac{2}{n^3}\,n^3\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)\right.$$

$$\left.+\frac{2}{n^4}\,n^4\left(1+\frac{1}{n}\right)^2\right]$$

$$I=\int_0^{\pi/4}\frac{\sec^2 x\,\sec^2 x\,dx}{1-\tan^2 x+\tan^4 x}$$

Put $\tan x=t$ $\quad\sec^2 x\,dx=dt$ and adjust the limits

$$I=\int_0^1\frac{(1+t^2)}{t^4-t^2+1}\,dt \text{ etc as in Q 19 P 680}$$

$$=\left[\tan^{-1}\frac{t^2-1}{t}\right]_0^1=\tan^{-1}0-\tan^{-1}(-\infty)$$

$$=0-(-\pi/2)=\pi/2$$

25 Put $\sin x=t$ $\quad\cos x\,dx=dt$ and adjust the limits

$$I=\int_0^1\frac{dt}{t^4-t^2+1}=\tfrac{1}{2}\int_0^1\frac{(t^2+1)-(t^2-1)}{t^4-t^2+1}\,dt$$

Both the above have been calculated in Q 19, 20 P 680

26 $$I=\int\frac{\sin^{-1}\sqrt{x}-\cos^{-1}\sqrt{x}}{\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}}\,dx$$

$$=\int\frac{\sin^{-1}\sqrt{x}-(\pi/2-\sin^{-1}\sqrt{x})}{\pi/2}\,dx$$

$$[\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x}=\pi/2]$$

$$=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-\int dx=\frac{4}{\pi}\int\sin^{-1}\sqrt{x}\,dx-x+c$$

Now put $x=\sin^2\theta$ Then $dx=2\sin\theta\cos\theta\,d\theta$

Then $\int\sin^{-1}\sqrt{x}\,dx=\int\theta\,2\sin\theta\cos\theta\,d\theta=\int\theta\sin 2\theta\,d\theta$

$$=\theta\left(-\tfrac{1}{2}\cos 2\theta\right)-\int 1\left(-\tfrac{1}{2}\cos 2\theta\right)d\theta$$

$$=-\tfrac{1}{2}\theta\cos 2\theta+\tfrac{1}{2}\cdot\tfrac{1}{2}\sin 2\theta$$

$$=-\tfrac{1}{2}\theta(1-2\sin^2\theta)+\tfrac{1}{2}\sin\theta\cos\theta$$

$$=-\tfrac{1}{2}\sin^{-1}\sqrt{x}\,(1-2x)+\tfrac{1}{2}\sqrt{x}\,\sqrt{(1-x)}$$

Hence $$I=\frac{4}{\pi}\left[-\tfrac{1}{2}(1-2x)\sin^{-1}\sqrt{x}+\tfrac{1}{2}\sqrt{(x-x^2)}\right]-x+c$$

$$=\frac{2}{\pi}[\sqrt{(x-x^2)}-(1-2x)\sin^{-1}\sqrt{x}]-x+c$$

27 We write the integral as

$$I=\int\frac{x^2-1}{(x+1)^2}\,\frac{dx}{\sqrt{(x^3+x^2+x)}}=\int\frac{(x^2-1)}{(x^2+2x+1)}\,\frac{dx}{\sqrt{(x^3+x^2+x)}}$$

$$=\int\frac{(1-1/x^2)}{(x+2+1/x)}\,\frac{dx}{\sqrt{(x+1+1/x)}}$$ dividing both N^r and D^r by x^2

$$I=\int\frac{2t\,dt}{(t^2+1)\,t}=\int\frac{2dt}{(t^2+1)},$$ Putting $x+1+\dfrac{1}{x}=t^2$

$$\left(1-\frac{1}{x^2}\right)dx=2t\,dt$$

Note $\int_0^{\pi/2} \cos x \, dx = 1$ Here $a=0$, $b=\pi/2$

$$b-a=\pi/2=nh$$

Hence from (1)

$$\int_0^{\pi/2} \cos x \, dx = \underset{n\to\infty}{\text{Lt}} \; 2 \cdot 1 \cos\left(0+\frac{1}{2} \cdot \frac{\pi}{2}\right) \sin \frac{1}{2} \cdot \frac{\pi}{2}$$

$= 2 \cos \pi/4 \sin \pi/4 = \sin 2 \cdot \pi/4 = \sin \pi/2 = 1$

$$\int_a^b \sin x \, dx = \cos a - \cos b$$

The above can be similarly proved as in Ex 3

4 $\int_a^b e^x \, dx = e^b - e^a$

$$\int_a^b f(x) \, dx = \underset{n\to\infty}{\text{Lt}} \; h \sum_{r=1}^{n} f(a+rh) \quad \text{where } nh=b-a$$

$$f(x)=e^x, \qquad f(a+rh)=e^{a+rh}=e^a \, e^{rh}$$

$$\int_a^b f(x) \, dx = \underset{n\to\infty}{\text{Lt}} \; h \sum_{r=1}^{n} e^a \, e^{rh}$$

$$= \underset{n\to\infty}{\text{Lt}} \; h e^a \, [e^h + e^{2h} + e^{3h} + \dots + e^{nh}]$$

$$= \underset{n\to\infty}{\text{Lt}} \; h \, e^a \, e^h \, \frac{[e^{nh}-1]}{e^h-1}$$

$$= \underset{n\to\infty}{\text{Lt}} \; \frac{h}{e^h-1} \, e^h \, e^a \, (e^{nh}-1) \tag{1}$$

$$\text{Lt} \; \frac{h}{e^h-1} = \text{Lt} \; \frac{h}{(1+h+h^2/2!+\dots)-1} = \text{Lt} \; \frac{h}{h+h^2/2!+\dots}$$

$$= \text{Lt} \; \frac{1}{1+h/2!+\dots} = 1, \qquad \text{when } n\to\infty, \; h\to 0$$

$\text{Lt} \; e^h = e^0 = 1$

Hence from (1), we have

$$\int_a^b e^x \, dx = 1 \cdot 1 \, e^a \, [e^{b-a}-1] = e^b - e^a$$

Note $\int_0^1 e^x \, dx = e-1$

Here $a=0$ $b=1$, $b-a=nh$ or $1-0=nh$

9 If $P=\int e^{ax}\cos bx\,dx$, $Q=\int e^{ax}\sin bx\,dx$, then prove that

$$(P^2+Q^2)(a^2+b^2)=e^{2ax}$$

and $\tan^{-1}\dfrac{Q}{P}+\tan^{-1} b/a=bx$

10 $\int \dfrac{x}{1+\sin x}\,dx$

11 $\int \dfrac{x}{1+\sec x}\,dx$

12 $\int \tan^{-1}\sqrt{\left(\dfrac{1-x}{1+x}\right)}dx$

13 $\int x^2\tan^{-1} x\,dx$

14 $\int x^3\tan^{-1} x\,dx$

15 $\int x\sin^{-1} x\,dx$

16 $\int x\sec^{-1} x\,dx$

17 $\int \sin^{-1}\sqrt{\left(\dfrac{x}{a+x}\right)}dx$

18 $\int x\sin^{-1}\tfrac{1}{2}\sqrt{\left(\dfrac{2a-x}{a}\right)}dx$

19 $\int_a^b \dfrac{\log x}{x}\,dx$

20 $\int \sqrt{(x^2+1)}\,\dfrac{[\log(x^2+1)-2\log x]}{x^4}\,dx$

21 $\int\left[\log(\log x)+\dfrac{1}{(\log x)^2}\right]dx$

22 $\int(\log x)^2\,dx$

23 $\int \dfrac{e^x(x^2+1)}{(x+1)^2}dx$

24 $\int e^x\left(\dfrac{1-x}{1+x^2}\right)^2 dx$

25 $\int \dfrac{e^x(x^3-x+2)}{(1+x^2)^2}dx$

26 $\int\left[\dfrac{1}{\log x}-\dfrac{1}{(\log x)^2}\right]dx$

27 $\int \dfrac{dx}{\sqrt[4]{[(x-1)^3(x+2)^5]}}$

28 $\int \dfrac{e^{2x}\,dx}{\sqrt[4]{(e^x+1)}}$

29 $\int \dfrac{x^2}{(x\sin x+\cos x)^2}\,dx$

30 $\int \dfrac{\sin\theta+\cos\theta}{\sqrt{(\sin 2\theta)}}\,d\theta$

31 $\int\sqrt{\left(\dfrac{\sin(x-\alpha)}{\sin(x+\alpha)}\right)}dx$

32 $\int \dfrac{1-\cos x}{\cos x(1+\cos x)}\,dx$

33 $\int \dfrac{1}{x(x^2+2)^2}\,dx$

34 $\int \dfrac{(e^x-1)}{(e^x+1)}\,dx$

35 $\int \dfrac{e^x\,dx}{(e^x-3e^{-x}+2)}$

36 $\int \dfrac{dx}{1+3e^x+2e^{2x}}$

37 $\int \dfrac{1-x\sin x+\cos x}{x(1+\cos x)}\,dx$

38 $\int \dfrac{\cos x+x\sin x}{x(x+\cos x)}\,dx$

39 If $s_n=\int_0^{\pi/2}\dfrac{\sin(2n-1)x}{\sin x}\,dx$ $\quad t_n=\int_0^{\pi/2}\left(\dfrac{\sin nx}{\sin x}\right)^2 dx$

and n is an integer show that

$$s_{n+1}-s_n=0,\; t_{n+1}-t_n=s_{n+1}$$

or $$S < \left[\frac{1}{a} - \frac{1}{a+nh}\right] \quad (3)$$

$$\left(\frac{1}{a+h} - \frac{1}{b+h}\right) < S < \left(\frac{1}{a} - \frac{1}{a+nh}\right) \text{ by (2) and (3)}$$

Hence $\lim_{h\to 0}\left[\frac{1}{a+h} - \frac{1}{b+n}\right] \leqslant \lim_{n\to\infty} S \leqslant \lim_{h\to 0}\left[\frac{1}{a} + \frac{1}{a+nh}\right]$

or $\left[\frac{1}{a} - \frac{1}{b}\right] \leqslant \lim S \leqslant \frac{1}{a} - \frac{1}{b}$ (4)

Hence from (4) we get $\text{Lt } S = \frac{1}{a} - \frac{1}{b}$

$$\int_a^b \frac{1}{x^2}\,dx = \frac{1}{a} - \frac{1}{b}$$

6 $\int_a^b 1/\sqrt{x}\,dx = 2\,(\sqrt{b} - \sqrt{a})$

$\int_a^b f(x)\,dx = \underset{n\to\infty}{\text{Lt}}\; h \sum_{r=1}^{n} f(a+rh)$ when $nh = b-a$

Here $f(x) = \frac{1}{\sqrt{x}}$ $\quad f(a+rh) = \frac{1}{\sqrt{(a+rh)}}$

$$\int_a^b \frac{1}{\sqrt{x}}\,dx = \underset{n\to\infty}{\text{Lt}}\; h\left(\frac{1}{\sqrt{(a+h)}} + \frac{1}{\sqrt{(a+2h)}} + \frac{1}{\sqrt{(a+nh)}}\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\left(\frac{h}{\sqrt{(a+h)}} + \frac{h}{\sqrt{(a+2h)}} + \frac{h}{\sqrt{(a+rh)}}\right)$$

$$= \underset{n\to\infty}{\text{Lt}}\; S \quad (1)$$

Now $h = [\sqrt{a+2h} - \sqrt{(a+h)}]\,[\sqrt{(a+2h)} + \sqrt{(a+h)}]$

$> [\sqrt{(a+2h)} - \sqrt{(a+h)}]\,[\sqrt{(a+h)} + \sqrt{(a+h)}]$

or $> [\sqrt{(a+2h)} - \sqrt{(a+h)}]2\sqrt{(a+h)}$

$$\frac{h}{\sqrt{(a+h)}} > 2\,[\sqrt{(a+2h)} - \sqrt{(a+h)}] \quad (2)$$

Hence from (1) by the help of (2)

$S > 2\,[\{\sqrt{(a+2h)} - \sqrt{(a+h)}\} + \{\sqrt{(a+3h)} - \sqrt{(a+2h)}\}$

$+ \{\sqrt{(a+\overline{n+1}\,h)} - \sqrt{(a+nh)}\}]$

or $S > 2\,[\sqrt{(a+b-a+h)} - \sqrt{(a+h)}] = 2\,[\sqrt{(b+h)} - \sqrt{(a+h)}]$

(3)

8 Proceed as in Q 13 (*a*) P 662 $I = -\frac{e^{-ax}}{r}\cos(bx+\alpha)$

where $r=\sqrt{(a^2+b^2)}$ and $\alpha=\tan^{-1} b/a$

9 Refer page 662 Another form

$$P=\frac{e^{ax}}{r}\cos(bx-\alpha) \quad Q=\frac{e^{ax}}{r}\sin(bx-\alpha)$$

Squaring and adding we get

$$P^2+Q^2=\frac{e^{2ax}}{r^2}\cdot 1 \quad \text{or} \quad (P^2+Q^2)(a^2+b^2)=e^{2ax}$$

Dividing we get

$$\frac{Q}{P}=\tan(bx-\alpha) \quad \text{or} \quad \tan^{-1}\frac{Q}{P}+\alpha=bx$$

or $\tan^{-1}\frac{Q}{P}+\tan^{-1}\frac{b}{a}=bx$

10 $I=\int\frac{x}{1+\sin x}dx=\int\frac{x(1-\sin x)}{1-\sin^2 x}dx$

$=\int x\sec^2 x\,dx-\int x\sec x\tan x\,dx$

Integrate each by parts

$I=[x\tan x-\log\sec x]-[x\sec x-\log(\sec x+\tan x)]$

11 $I=\int\frac{x}{\sec x+1}dx=\int\frac{x\cos x}{1+\cos x}dx$

$=\int x\frac{1+\cos x-1}{1+\cos x}dx=\int\left(x-\frac{x}{2\cos^2 x/2}\right)dx$

$I=\frac{x^2}{2}-x\tan\frac{x}{2}+2\log\sec\frac{x}{2}$ as in Q 13 P 668

12 Put $x=\cos\theta$ $\quad\frac{1-\cos\theta}{1+\cos\theta}=\tan^2\theta/2$ and $dx=-\sin\theta\,d\theta$

$I=\int\tan^{-1}(\tan\theta/2)(-\sin\theta\,d\theta)=-\frac{1}{2}\int\theta\sin\theta\,d\theta$

$=-\frac{1}{2}[-\theta\cos\theta+\sin\theta]=\frac{1}{2}[x\cos^{-1}x-\sqrt{(1-x^2)}]$

13 Integrating by parts

$I=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x^3}{x^2+1}dx$

$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\left(x-\frac{x}{x^2+1}\right)dx$ (Note)

$=\frac{x^3}{3}\tan^{-1}x-\frac{1}{6}x^2+\frac{1}{6}\log(x^2+1)$

14 $I=\frac{x^4}{4}\tan^{-1}x-\frac{1}{4}\int\frac{x^4-1+1}{1+x^2}dx$ (Note)

or $$S<\left[\frac{1}{a}-\frac{1}{a+nh}\right] \tag{3}$$

$$\left(\frac{1}{a+h}-\frac{1}{b+h}\right)<S<\left(\frac{1}{a}-\frac{1}{a+nh}\right) \text{ by (2) and (3)}$$

Hence $\lim\limits_{h\to 0}\left[\frac{1}{a+h}-\frac{1}{b+h}\right]\leqslant \lim\limits_{n\to\infty} S\leqslant \lim\limits_{h\to 0}\left[\frac{1}{a}+\frac{1}{a+nh}\right]$

or $$\left[\frac{1}{a}-\frac{1}{b}\right]\leqslant \lim S\leqslant \frac{1}{a}-\frac{1}{b} \tag{4}$$

Hence from (4) we get Lt $S=\frac{1}{a}-\frac{1}{b}$

$$\int_a^b \frac{1}{x^2}\,dx=\frac{1}{a}-\frac{1}{b}$$

6 $$\int_a^b 1/\sqrt{x}\,dx=2\,(\sqrt{b}-\sqrt{a})$$

$$\int_a^b f(x)\,dx=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(a+rh) \text{ when } nh=b-a$$

Here $f(x)=\dfrac{1}{\sqrt{x}}$ $\qquad f(a+rh)=\dfrac{1}{\sqrt{(a+rh)}}$

$$\int_a^b \frac{1}{\sqrt{x}}\,dx=\underset{n\to\infty}{\text{Lt}}\ h\left(\frac{1}{\sqrt{(a+h)}}+\frac{1}{\sqrt{(a+2h)}}+\frac{1}{\sqrt{(a+nh)}}\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\left(\frac{h}{\sqrt{(a+h)}}+\frac{h}{\sqrt{(a+2h)}}+\frac{h}{\sqrt{(a+rh)}}\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\ S \tag{1}$$

Now $h=[\sqrt{a+2h})-\sqrt{(a+h)}]\ [\sqrt{(a+2h)}+\sqrt{(a+h)}]$

$>[\sqrt{(a+2h)}-\sqrt{(a+h)}]\ [\sqrt{(a+h)}+\sqrt{(a+h)}]$

or $>[\sqrt{(a+2h)}-\sqrt{(a+h)}]2\sqrt{(a+h)}$

$$\frac{h}{\sqrt{(a+h)}}>2\,[\sqrt{(a+2h)}-\sqrt{(a+h)}] \tag{2}$$

Hence from (1) by the help of (2)

$S>2\,[\{\sqrt{(a+2h)}-\sqrt{(a+h)}\}+\{\sqrt{(a+3h)}-\sqrt{(a+2h)}\}$

$+\ \{\sqrt{(a+\overline{n+1}\,h)}-\sqrt{(a+nh)}\}]$

or $S>2\,[\sqrt{(a+b-a+h)}-\sqrt{(a+h)}]=2\,[\sqrt{(b+h)}-\sqrt{(a+h)}]$ (3)

$$=a^2\left[\tfrac{1}{2}\theta\cos 2\theta-\frac{\sin 2\theta}{4}\right], \text{ integrating by parts}$$

$$=\frac{a^2}{2}\left[\theta(2\cos\theta-1)-\cos\theta\sqrt{(1-\cos^2\theta)}\right]$$

Now put $\cos\theta=\dfrac{x}{2a}$

$$I=\left[\frac{a^2}{8}\,2\sqrt{(x^2-2a)}\cos^{-1}\frac{x}{2a}-x\sqrt{(4a^2-x)}\right]$$

19 Integrating by parts

$$I=\Big[\log x\log x\Big]_a^b-\int\log x\,\frac{1}{x}\,dx=\Big[(\log x)^2\Big]_a^b-I$$

$$2I=(\log b)^2-(\log a)^2=(\log b+\log a)(\log b-\log a)$$

$$I=\frac{1}{2}\log(ab)\log\frac{b}{a}$$

20 $$I=\int\frac{x\sqrt{\left(1+\frac{1}{x^2}\right)}}{x^4}\log\frac{(x^2+1)}{x^2}\,dx$$

$$=\int\frac{1}{x^3}\sqrt{\left(1-\frac{1}{x^2}\right)}\log\left(1+\frac{1}{x}\right)dx$$

Put $1+\dfrac{1}{x^2}=t \qquad \dfrac{-2}{x^3}\,dx=dt$

$I=-\frac{1}{2}\int\sqrt{t}\log t\,dt$ Integrate by parts

$$=-\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log t-\frac{2}{3}\int t^{3/2}\frac{1}{t}\,dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log t-\frac{2}{3}\int\sqrt{t}\,dt\right]$$

$$=-\frac{1}{2}\left[\frac{2}{3}t^{3/2}\log t-\frac{2}{3}\cdot\frac{2}{3}t^{3/2}\right]$$

$$=\frac{t^{3/2}}{9}\left[2-3\log t\right] \text{ where } t=1+\frac{1}{x^2}=\frac{x^2+1}{x^2}$$

21 $I=I_1+I_2$

$$I_1=\int\log(\log x)\,dx=x\log(\log x)-\int x\frac{1}{\log x}\frac{1}{x}\,dx$$

$$=x\log(\log x)-\int\frac{1}{\log x}\,dx$$

$$=x\log(\log x)-\left[x\frac{1}{\log x}-\int x\left(-\frac{1}{\log x}\right)^2\frac{1}{x}\,dx\right]$$

$$=x\log(\log x)-\frac{x}{\log x}-\int\frac{1}{(\log x)^2}\,dx$$

$$I_2=\int\frac{1}{(\log x)^2}\,dx$$

Ex 2 $\int_1^4 (3x^2+2x)\,dx=10$

Proceed as above

Ex 3 $\int_1^2 (x^2+5)\,dx=\frac{22}{3}$

Ex 4 $\int_0^1 x^2\,dx=\frac{1}{4}$

Ex 5 $\int_a^b \sin^2 x\,dx=\frac{1}{2}\,[(b-a)-\cos(b+a)\sin(b-a)]$

$$\sin^2 x=\frac{1}{2}\left[1-\cos 2x\right]=\frac{1}{2}-\frac{1}{2}\cos 2x$$

Now proceed as in Ex 2

Ex 6 $\int_a^b \sqrt{x}\,dx=\frac{2}{3}\,(b^{3/2}-a^{3/2})$

§ 7 Summation of Series

We know that

$$\int_a^b f(x)\,dx=\operatorname*{Lt}_{n\to\infty} h\sum_{r=1}^{n} f(a+rh) \text{ where } nh=b-a$$

Now put $a=0,\ b=1$ $\quad nh=1-0=1$ or $h=\frac{1}{n}$

$$\int_0^1 f(x)\,dx=\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\,\Sigma f(r/n)$$

Rule Express the given series in the form $\Sigma\,\frac{1}{n}f(r/n)$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and the limit of the sum is $\int_0^1 f(x)\,dx$

Ex 1 $\operatorname*{Lt}_{n\to\infty}\left[\frac{1}{n+1}+\frac{1}{n+2}+\quad+\frac{1}{2n}\right]$

$$=\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\left[\frac{1}{1+\frac{1}{n}}+\frac{1}{1+\frac{2}{n}}+\quad+\frac{1}{1+\frac{n}{n}}\right]$$

$$=\operatorname*{Lt}_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{1}{\left(1+\frac{r}{n}\right)} \quad \text{Standard form}$$

$$I_1=\int\frac{1}{\log x}\,dx=x\,\frac{1}{\log x}-\int x\left(-\frac{1}{\log x}\right)^2\frac{1}{x}\,dx$$

$$I_1=x\,\frac{1}{\log x}+\int\left(\frac{1}{\log x}\right)^2 dx=x\,\frac{1}{\log x}+I_2$$

or $I=I_1-I_2=x\,(\log x)^{-1}$

27 $$I=\int\frac{1}{\left[\left(\frac{x-1}{x+2}\right)^3(x+2)^8\right]^{1/4}}\,dx$$

$$=\int\frac{1}{(x+2)^2\left(\frac{x-1}{x+2}\right)^{3/4}}\,dx$$

Put $\frac{x-1}{x+2}=t$ or $1-\frac{3}{x+2}=t$ $\quad\frac{3}{(x+2)^2}\,dx=dt$

$$I=\frac{1}{3}\int t^{-3/4}\,dt=\frac{1}{3}\,4t^{1/4}=\frac{4}{3}\left(\frac{x-1}{x+2}\right)^{1/4}$$

28 $$I=\int\frac{e^x\,e^x\,dx}{(e^x+1)^{1/4}}$$

Put $e^x+1=t^4$ $\qquad e^x\,dx=4t^3\,dt$

$$I=\int\frac{(t^4-1)\,4t^3}{t}\,dt=4\int(t^6-t^2)\,dt$$

$$=4\left[\frac{t^7}{7}-\frac{t^3}{3}\right]=\frac{4t^3}{21}\,[3t^4-7]$$

$$=\frac{4}{21}\,(e^x+1)^{3/4}\,[3e^x-4]$$

29 Differentiation of $x\sin x+\cos x$ is $x\cos x$

Then $$I=\int\frac{x^2\,dx}{(x\sin x+\cos x)^2}=\int\frac{x\cos x}{(x\sin x+\cos x)^2}\,\frac{x}{\cos x}\,dx$$

Integrate by parts $\left[\int\frac{1}{t^2}\,dt=-\frac{1}{t}\right]$

$$I=\frac{-1}{(x\sin x+\cos x)}\,\frac{x}{\cos x}+\int\frac{1}{(x\sin x+\cos x)}\cdot\frac{\cos x\cdot 1-x\,(-\sin x)}{\cos^2 x}\,dx$$

$$=-\frac{1}{x\sin x+\cos x}\,\frac{x}{\cos x}+\int\sec^2 x\,dx$$

$$=-\frac{1}{x\sin x+\cos x}\,\frac{x}{x}$$

$$=\frac{-x+x\sin^2 x+\sin}{(x\sin x+\cos x)}$$

When $r=0$, $x=0$, and when $r=n-1$ $x=\frac{n-1}{n}=1-\frac{1}{n}\to 1$ as $n\to\infty$

$$\text{Reqd Lt} = \int_0^1 \frac{dx}{1+x^2} = \Big[\tan^{-1}x\Big]_0^1 = \tan^{-1}1-\tan^{-1}0$$

$$=\pi/4-0=\pi/4$$

Ex 4 $\underset{n\to\infty}{\text{Lt}}\left(\frac{1}{1+n^3}+\frac{4}{8+n^3}+\quad\frac{r^2}{r^3+n^3}\quad+\quad+\frac{1}{2n}\right)$

Now $\frac{1}{2n}=\frac{n^2}{n^3+n^3}$

$$\text{The given expression}=\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{r^2}{r^3+n^3}=A\text{, say}$$

We should put it in the form $\frac{1}{n}\Sigma f\left(\frac{r}{n}\right)$

$$A=\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{r^2}{n^3(1+(r/n)^3)}=\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{1}{n}\frac{(r/n)^2}{1+(r/n)^3}$$

Now put $r/n=x$ and $1/n=dx$ and limits for x are 0 to 1

$$\text{Lt}=\int_0^1\frac{x^2\,dx}{1+x^3}=\frac{1}{3}\int_0^1\frac{3x^2\,dx}{1+x^3}=\frac{1}{3}\Big[\log(1+x^3)\Big]_0^1=\tfrac{1}{3}\log 2$$

$$\text{Similarly}\ \underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{r^3}{r^4+n^4}=\tfrac{1}{4}\log 2$$

Ex 5 $\underset{n\to\infty}{\text{Lt}}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\quad+\frac{1}{n}\right]$

The last term $\frac{1}{n}=\frac{n+n}{n^2+n^2}$

$$\text{The given expression}=\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{n+r}{n^2+r^2}$$

Express in the form $\frac{1}{n}\Sigma f\left(\frac{r}{n}\right)$

$$\therefore\ \text{Limit}=\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{n(1+r/n)}{n^2(1+r^2/n^2)}=\text{Lt}\sum_{r=1}^{n}\frac{1}{n}\left(\frac{1+r/n}{1+r^2/n^2}\right)$$

Put $\frac{r}{n}=x$ and $\frac{1}{n}=dx$ and as usual limits are adjusted as 0 to 1

$$\text{Lt}=\int_0^1\frac{1+x}{1+x^2}dx=\int_0^1\frac{1}{1+x^2}dx+\tfrac{1}{2}\int_0^1\frac{2x\,dx}{1+x^2}\,dx$$

36 Ans $\log \frac{e^x(1+e^x)}{(1+2e^x)^2}$

37 $\int \frac{1+x\sin x+\cos x}{x(1+\cos x)}\,dx$

$$=\int\left[\frac{1+\cos x}{x(1+\cos x)}+\frac{x\sin x}{x(1+\cos x)}\right]dx$$

$$=\int\left(\frac{1}{x}+\frac{\sin x}{1+\cos x}\right)dx=\int\left(\frac{1}{x}+\tan\frac{x}{2}\right)dx$$

$$=\log x+2\log\sec x/2$$

38 $\int \frac{\cos x+x\sin x}{x(x+\cos x)}\,dx$

$$I=\int\frac{(x+\cos x)-x+x\sin x}{x(x+\cos x)}\,dx$$

$$=\int\left[\frac{1}{x}-\frac{(1-\sin x)}{(x+\cos x)}\right]dx$$

$$=\log x-\log(x+\cos x)=\log\frac{x}{x+\cos x}$$

39 $s_{n+1}-s_n=\int_0^{\pi/2}\frac{\sin(2n+1)x-\sin(2n-1)x}{\sin x}\,dx$

$$=\int_0^{\pi/2}\frac{2\sin x\cos 2nx}{\sin x}\,dx=\int_0^{\pi/2}2\cos 2nx\,dx$$

$$=2\left[\frac{\sin 2nx}{2n}\right]_0^{\pi/2}=\frac{1}{n}\sin n\pi=0$$

where n is an integer

$$t_{n+1}-t_n=\int_0^{\pi/2}\frac{\sin^2(n+1)x-\sin^2 nx}{\sin^2 x}\,dx$$

$$=\int_0^{\pi/2}\frac{\sin(2n+1)x\sin x}{\sin^2 x}\,dx=\int_0^{\pi/2}\frac{\sin(2n+1)x}{\sin x}\,dx=s_{n+1}$$

[$\because$ $\sin^2 A-\sin^2 B=\sin(A+B)\sin(A-B)$]

40 $I=\int_0^{\pi}\sin mx\sin nx\,dx$

$$=\tfrac{1}{2}\int_0^{\pi}[\cos(m-n)x-\cos(m+n)x]\,dx;\ m\neq n$$

$$=\tfrac{1}{2}\left[\frac{\sin(m-n)x}{m-n}-\frac{\sin(m+n)x}{m+n}\right]_0^{\pi}=0$$

$\because$ $m-n$ and $m+n$ are both integers and $\sin r\pi=0$,

where r is an integer

When $m=n$ then

Put $\frac{r}{n}=x$ and $\frac{1}{n}=dx$ and the limits are adjusted as from 0 to 1

$$\log A=\int_0^1 \log(1+x^2)\,dx$$

$$=\left[x\log(1+x^2)\right]_0^1-\int_0^1 x\,\frac{1}{(1+x^2)}\,2x\,dx$$

$$=\left[x\log(1+x^2)\right]_0^1-2\int_0^1 \frac{x^2+1-1}{x^2+1}\,dx$$

$$=\log 2\quad 2\int_0^1\left(1-\frac{1}{1+x^2}\right)dx=\log 2-2\left[x-\tan^{-1}x\right]_0^1$$

or $\log A=\log 2-2+\frac{2\pi}{4}$ or $\log A-\log 2=\frac{\pi-4}{2}$

or $\log\frac{A}{2}=\frac{\pi-4}{2}\qquad \frac{A}{2}=e^{(\pi-4)/2}$

or $A=2e^{(\pi-4)/2}$

Ex 8 $\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left[\sin^2\frac{\pi}{2n}+\sin^2\frac{2\pi}{2n}+\sin^2\frac{3\pi}{2n}+\quad+\sin^2\frac{n\pi}{2n}\right]$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\sum_{r=1}^{n}\sin^2\frac{r\pi}{2n}$$

Put $\frac{r}{n}=x$, $\frac{1}{n}=dx$ and limits are 0 to 1

$$\text{Lt}=\int_0^1 \sin^2\frac{\pi x}{2}\,dx$$

Put $\frac{\pi x}{2}=t \qquad \frac{\pi}{2}dx=dt$ or $dx=\frac{2}{\pi}\,dt$

Also when $x=1$, $t=\pi/2$ and when $x=0$, $t=0$

$$Lt=\int_0^{\pi/2}\sin^2 t\,\frac{2}{\pi}dt=\frac{1}{\pi}\int_0^{\pi/2}\left(1-\cos 2t\right)dt$$

$$=\frac{1}{\pi}\left[t-\frac{\sin 2t}{2}\right]_0^{\pi/2}=\frac{1}{\pi}\left[\frac{\pi}{2}-0\right]=\frac{1}{2}$$

Problem Set (G)

1 $\underset{n\to\infty}{\text{Lt}}\left[\frac{\sqrt{(n+1)}+\sqrt{(n+2)}+\quad+\sqrt{(2n)}}{n\sqrt{n}}\right]=\frac{2}{3}(2\sqrt{2}-1)$

2 $\underset{n\to\infty}{\text{Lt}}\sum_{r=1}^{n}\frac{1}{n}\sqrt{\left(\frac{n+r}{n-r}\right)}=\frac{\pi}{2}+1$

Above reduces to

$$I=\int_1^{\infty}\frac{\frac{1}{2}(1+1/u)}{u^n}\,du=\frac{1}{2}\int_0^{\infty}\left(\frac{1}{u^n}+\frac{1}{u^{n+1}}\right)du$$

$$=\frac{1}{2}\left[-\frac{1}{(n-1)}\frac{1}{u^{n-1}}-\frac{1}{(n+1)}\frac{1}{u^{n+1}}\right]_1^{\infty}$$

$$=\frac{1}{2}\left\{\frac{1}{n-1}+\frac{1}{n+1}\right\}=\frac{n}{n^2-1}$$

45 $I=\int\frac{\sqrt{(2\cos^2 x-1)}}{\sin^2 x}\sin x\,dx$ Put $\cos x=t$

$$I=\int\frac{\sqrt{(2t^2-1)}}{(1-t^2)}(-dt)=\int\frac{(2t^2-1)\,dt}{(t^2-1)\sqrt{(2t^2-1)}}$$

$$=\int\frac{2(t^2-1)+1}{(t^2-1)\sqrt{(2t^2-1)}}\,dt$$

$$=\int\frac{2}{\sqrt{(2t^2-1)}}\,dt+\int\frac{1}{(t^2-1)\sqrt{(2t^2-1)}}\,dt=I_1+I_2$$

$$I_1=\sqrt{2}\int\frac{1}{\sqrt{(t^2-\frac{1}{2})}}\,dt=\sqrt{2}\log\{t+\sqrt{(t^2-\tfrac{1}{2})}\}$$

$=\sqrt{2}\log[\sqrt{2}t+\sqrt{(2t^2-1)}]$ rejected the constant

$=\sqrt{2}\log[\sqrt{2}\cos x+\sqrt{(\cos 2x)}]$

for I_2 Proceed as is Q 15 P 678 by first putting $t=1/z$ and then $2-z^2=u^2$

$$I_2=\int\frac{du}{u^2-1}=\frac{1}{2}\log\frac{u-1}{u+1}=\frac{1}{2}\log\frac{\sqrt{(2-z^2)}-1}{\sqrt{(2-z^2)}+1}$$

$$=\frac{1}{2}\log\frac{\sqrt{(2t^2-1)}-t}{\sqrt{(2t^2-1)}+t}=\frac{1}{2}\log\frac{\sqrt{(\cos 2x)}-\cos x}{\sqrt{(\cos 2x)}+\cos x}$$

$$I=I_1+I_2$$

§ 6 Definite Integral as the limit of sum

Definition *If $f(x)$ be a single valued continuous function in* the interval (a, b) where $b>a$ and if the interval (a, b) be divided into n equal parts of length h, by the points

$a+h,\ a+2h,\ a+3h\ \ a+\overline{n-1}\,h$ so that
$a+nh=b$ or $nh=b-a$ then

$$\underset{h\to 0}{\mathrm{Lt}}\ h\,[f(a)+f(a+h)+f(a+2h)+\ \ f(a+\overline{n-1}\,h)]$$

written as $\underset{h\to 0}{\mathrm{Lt}}\ h\sum_{r=0}^{n-1}f(a+rh)$

or $\underset{n\to\infty}{\mathrm{Lt}}\sum_{r=1}^{n-1}f(a+rh)$ $\qquad h=\dfrac{b-a}{n}$ and when

Lower limit must be zero and in that case we can replace x by $a-x$

Application

$$I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}\,dx$$

Here the lower limit is zero and hence we can replace x by $a-x$ i e x by $\pi/2-x$

$$I=\int_0^{\pi/2}\frac{\sin(\pi/2-x)}{\sin(\pi/2-x)+\cos(\pi/2-x)}\,dx=\int_0^{\pi/2}\frac{\cos x}{\cos x+\sin x}\,dx$$

Adding $2I=\int_0^{\pi/2}\frac{\sin x+\cos x}{\sin x+\cos x}\,dx=\int_0^{\pi/2}1\,dx=\left[x\right]_0^{\pi/2}=\frac{\pi}{2}$

$I=\pi/4$

5 $\int_{-a}^{a} f(x)\,dx=2\int_0^a f(x)\,dx$ if $f(-x)=f(x)$

$=0$ if $f(-x)=-f(x)$

Proof $I=\int_{-a}^{a} f(x)\,dx=\int_{-a}^{0} f(x)\,dx+\int_0^a f(x)\,dx$

Putting $x=-t$ in first integral

$$I=-\int_a^0 f(-t)\,dt+\int_0^a f(x)\,dx,$$

$$\int_0^a f(-x)\,dx+\int_0^a f(x)\,dx \text{ by prop 1 and 2}$$

$$=2\int_0^a f(x)\,dx \text{ if } f(-x)=f(x)$$

$$=0 \text{ if } f(-x)=-f(x)$$

Application

$$\int_{-a}^{a} x^3\,dx=0$$

$$f(x)=x^3,\ f(-x)=(-x)^3=-x^3=-f(x)$$

$$\int_{-a}^{a} x^4\,dx=2\int_0^a x^4\,dx$$

$$f(x)=x^4,\ f(-x)=(-x)^4=x^4=f(x)$$

In other words when $f(x)$ is an odd function of x the result is zero and when $f(x)$ is an even function of x the result is

$$2\int_0^a f(x)\,dx$$

$$\int_{-\pi/2}^{\pi/2}\cos^2 x\,dx=2\int_0^{\pi/2}\cos^2 x\,dx$$

Formula $\quad \sin$ or $\cos\left\{\frac{\text{Ist angle}+\text{last angle}}{2}\right\}\frac{\sin\frac{n\,\text{Diff}}{2}}{\sin\frac{\text{Diff}}{2}}$

Ex 1 $\int_a^b x^2 = \frac{1}{3}(b^3-a^3)$

Here $f(x)=x^2 \qquad f(a)=a^2, f(a+h)=(a+h)^2,$

$f(a+2h)=(a+2h)^2$ etc

$$\int_a^b f(x)\,dx = \underset{h\to 0}{\text{Lt}} \sum_{r=0}^{n-1} h\, f(a+rh) = \underset{n\to\infty}{\text{Lt}} \sum_{r=0}^{n-1} f(a+rh)$$

where $nh=b-a$

$$\int_a^b x^2\,dx = \underset{n\to\infty}{\text{Lt}}\ h\,[f(a)+f(a+h)+f(a+2h)+\ \ +f(a+\overline{n-1}\,h)]$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\,[a^2-(a+h)^2+(a+2h)^2+\ \ +(a+\overline{n-1}\,h)^2]$$

Grouping the terms of a^2, $2ah$ and h^2 in the above we get

$$I=\underset{n\to\infty}{\text{Lt}}\ h\,[a^2\Sigma 1+2ah\ \{(1+2+3\ \ (n-1)\}$$
$$+h^2\{1^2+2^2+3^2+\ \ (n-1)^2\}]$$

Now $\Sigma 1 = n \quad \sum_{r=1}^{n} r=\frac{n(n+1)}{2}$ and $\sum_{r=1}^{n} r^2=\frac{n(n+1)(2n+1)}{6}$

Replacing n by $n-1$ in the above

$$\sum_{r=1}^{n-1} r=\frac{(n-1)\,n}{2},\ \sum_{r=1}^{r-1} r^2=\frac{(n-1)\,n\,(2n-1)}{6}$$

$$I=\underset{n\to\infty}{\text{Lt}}\ h\left\{a\ \ n+2ah\,\frac{(n-1)\,n}{2}+h^2\,\frac{(n-1)\,(n)\,(2n-1)}{6}\right\}$$

We have to take the limit of the above sum when $n\to\infty$ or $h\to 0$ and $nh=b-a$

Therefore we write the above as below

$$I=\underset{n\to\infty}{\text{Lt}}\left\{a^2\,(nh)+a\,(nh)^2\left(1-\frac{1}{n}\right)+\frac{1}{6}\,(nh)^3\left(1-\frac{1}{n}\right)\left(2-\frac{1}{n}\right)\right\}$$

Now when $n\to\infty$, $\frac{1}{n}\to 0$ and $nh=b-a$

$$I=\left\{a^2\,(b-a)+a\,(b-a)^2\,1+\frac{1}{6}\,(b-a)^3\,1\,2\right\}$$
$$=(b-a)\,[a^2+a(b-a)+\tfrac{1}{3}(b-a)^2]$$

7 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Proof Put $x = a+b-t$ Then $dx = -dt$

$$\int_a^b f(x)\,dx = \int_b^a f(a+b-t)\,dt$$

$$= \int (a+b-x)\,dx$$

by Prop 1 and 2,

Application

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{(\tan x)}} = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\cos x)}}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}dx$$

Then by prop 7

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx, \quad a+b=\pi/2$$

Hence $2I = \int_{\pi/6}^{\pi/3} dx = \frac{\pi}{3} - \frac{\pi}{6}$ or $I = \frac{\pi}{12}$

Removal of x

(i) Consider $\int_0^\pi \frac{x \sin x}{1+\cos^2 x}$ **(Roorkee 80))**

We know how to integrate $I = \int_0^\pi \frac{\sin x}{1+\cos^2 x}\,dx$ by putting

$\cos x = t$, *i e* $-\sin x\,dx = dt$

Also when $x=\pi$, $t=-1$, when $x=0$, $t=1$

$$I = \int_{+1}^{-1} \frac{-dt}{1+t^2} = -\left[\tan^{-1} t\right]_{+1}^{-1} = -[\tan^{-1}(-1) - \tan^{-1}$$

$$= -[-\pi/4 - (\pi/4)] = \pi/2$$

Now $I = \int_0^\pi \frac{x \sin x}{1+\cos^2 x}\,dx$ (1)

In order to remove x apply property no 4 *i e*

$$I = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}dx = \int_0^\pi \frac{(\pi-x)\sin x}{1+\cos^2 x}dx \quad (2)$$

Adding (1) and (2), we get

$$2I = \int_0^\pi (x+\pi-x)\,\frac{\sin x}{1+\cos^2 x}\,dx$$

or $$I = \frac{\pi}{2}\int_0^\pi \frac{\sin x}{1+\cos^2 x}dx = \frac{\pi}{2}\left(\frac{\pi}{2}\right) = \frac{\pi^2}{4}$$

as calculated above

$$=\underset{n\to\infty}{\text{Lt}}\ 2\left[1+3\left(1+\frac{1}{n}\right)+2\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)+2\left(1+\frac{1}{n}\right)^2\right]$$

$=2\,[1+3+2\cdot 1\cdot 2+2\cdot 1]=2\cdot 10=20$

Note $\int_a^b x^3\,dx=\frac{1}{4}(b^4-a^4)$

This will be evaluated as in Ex 1 It will be convenient if we take the definition as

$$\int_a^b f(x)\,dx=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(a+rh) \text{ where } nh=b-a, \text{ instead of}$$

$$\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=0}^{n-1} f(a+rh) \text{ when } nh=b-a$$

3 $\int_a^b \cos x\,dx=\sin b-\sin a$

$$\int_a^b f(x)\,dx=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n} f(a+rh) \text{ where } nh=b-a,$$

$$f(x)=\cos x,\ f(a+rh)=\cos(a+rh)$$

$$\int_a^b \cos x\,dx=\underset{n\to\infty}{\text{Lt}}\ h\sum_{r=1}^{n}\cos(a+rh)$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\,[\cos(a+h)+\cos(a+2h)+\ \ldots\ +\cos(a+nh)]$$

$$=\underset{n\to\infty}{\text{Lt}}\ h\cos\frac{a+h+a+nh}{2}\cdot\frac{\sin nh/2}{\sin h/2}$$ See P 700

$$=\underset{n\to\infty}{\text{Lt}}\ h\cos\left(a+(n+1)\frac{h}{2}\right)\frac{\sin nh/2}{\sin h/2}$$

$$=\underset{n\to\infty}{\text{Lt}}\ 2\cdot\frac{h/2}{\sin h/2}\cos\left\{a+\tfrac{1}{2}nh\left(1+\frac{1}{n}\right)\right\}\sin\frac{nh}{2}$$

Now we know that

$$\underset{\theta\to 0}{\text{Lt}}\ \frac{\sin\theta}{\theta}=1 \quad\text{and}\quad \underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}=0$$

$$\int_a^b f(x)\,dx=2\cdot 1\cos\left[a+\tfrac{1}{2}(b-a)\right]\sin\frac{b-a}{2}$$

$$=2\cos\frac{b+a}{2}\sin\frac{b-a}{2}$$

$$=\sin b-\sin a$$

$$2I=\int_0^{\pi/2} \log(\sin x\cos x)\, dx$$

$$=\int_0^{\pi/2} \log \frac{\sin 2x}{2}\, dx$$

$$=\int_0^{\pi/2} \log \sin 2x\, dx-\int_0^{\pi/2} \log 2\, dx$$

Put $2x=t$ in the Ist, $2\,dx=dt$ and limits become 0 to π

$$2I=\frac{1}{2}\int_0^{\pi} \log \sin t\, dt \quad \left[x\log 2\right]_0^{\pi/2}$$

Now apply Prop 6 in Ist $f(2a-x)=f(x)$

$$2I=\frac{1}{2}\, 2\int_0^{\pi/2} \log \sin t\, dt-\frac{\pi}{2}\log 2$$

or $2I=\int_0^{\pi/2} \log \sin x\, dx-(\pi/2)\log 2$ by Prop I

or $2I=I+\frac{\pi}{2}\log \frac{1}{2}$

$$I=\int_0^{\pi/2} \log \sin x\, dx=\int_0^{\pi/2} \log \cos x\, dx=\frac{\pi}{2}\log(1/2)$$

The above may be taken as a standard result

Ex 2 (*a*) $\int_0^a \frac{dx}{x+\sqrt{(a^2-x^2)}}$ or $\int_0^{\pi/2} \frac{1}{1+\tan\theta}\, d\theta$

(Roorkee 80)

Put $x=a\sin\theta$ $\quad dx=a\cos\theta\, d\theta$
When $x=a$, $\sin\theta=1$ $\quad \theta=\pi/2$
When $x=0$, $\sin\theta=0$ $\quad \theta=0$

$$I=\int_0^{\pi/2} \frac{a\cos\theta\, d\theta}{a\sin\theta+a\cos\theta}=\int_0^{\pi/2} \frac{\cos\theta\, d\theta}{\sin\theta+\cos\theta} \qquad (1)$$

Now as in Ex on page 713 we apply Prop 4

$$I=\int_0^{\pi/2} \frac{\cos(\pi/2-\theta)\, d\theta}{\sin(\pi/2-\theta)+\cos(\pi/2-\theta)}=\int_0^{\pi/2} \frac{\sin\theta\, d\theta}{\cos\theta+\sin\theta} \qquad (2)$$

Adding (1) and (2) we get

$$2I=\int_0^{\pi/2} \frac{\cos\theta+\sin\theta}{\cos\theta+\sin\theta}\, d\theta=\int_0^{\pi/2} 1\, d\theta$$

or $2I=\left[\theta\right]_0^{\pi/2}=\pi/2$

$I=\pi/4$

$$I=\int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta}\sec^2\theta\, d\theta$$

or $$I=\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$$

Apply prop 4 *i e* $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$I=\int_0^{\pi/4} \log\{1+\tan(\pi/4-\theta)\}d\theta$$

$$=\int_0^{\pi/4} \log\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right)dx \qquad \tan(\pi/4)=1$$

$$=\int_0^{\pi/4} \log\frac{2}{1+\tan\theta}\,d\theta$$

$$=\int_0^{\pi/4} \log 2\, d\theta-\int_0^{\pi/4} \log(1+\tan\theta)\,d\theta$$

or $$I=\Big[\theta\log 2\Big]_0^{\pi/4}-I$$

or $2I=\frac{\pi}{4}\log 2 \qquad I=\frac{\pi}{8}\log 2$

Ex 5 $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx=\frac{\pi}{2}(\pi-2)$

Change to $\sin x$ and $\cos x$

$$I=\int_0^{\pi} \frac{x\sin x}{1+\sin x}\,dx \tag{1}$$

In order to remove x, we apply prop 4

$$I=\int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\sin(\pi-x)}\,dx=\int_0^{\pi} \frac{(\pi-x)\sin x}{1+\sin x}\,dx \tag{2}$$

Adding (1) and (2) we get

$$2I=\int_0^{\pi}(x+\pi-x)\frac{\sin x}{1+\sin x}\,dx=\pi\int_0^{\pi}\frac{\sin x}{1+\sin x}\,dx$$

$$I=\frac{\pi}{2}\int_0^{\pi}\frac{\sin x+1-1}{1+\sin x}\,dx=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1}{1+\sin x}\right)dx$$

$$=\int_0^{\pi}\left(1-\frac{1-\sin x}{1-\sin^2 x}\right)dx=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1-\sin x}{\cos^2 x}\right)dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right)dx$$

Hence proceeding as above we have from (1)

$$\int_0^1 e^x\,dx = 1 \cdot 1 \cdot e^0\,[e^1-1] = e-1$$

5 $\int_a^b \frac{1}{x^2}\,dx = \frac{1}{a}-\frac{1}{b}$

$$\int_a^b f(x)\,dx = \underset{n\to\infty}{\mathrm{Lt}}\; h \sum_{r=1}^{n} f(a+rh), \text{ where } nh=b-a$$

Here $f(x)=\frac{1}{x^2}$, $\quad f(a+rh)=\frac{1}{(a+rh)^2}$

$$\int_a^b \frac{1}{x^2}\,dx = \underset{n\to\infty}{\mathrm{Lt}}\; h\left[\sum_{r=1}^{n} \frac{1}{(a+h)^2}+\frac{1}{(a+2h)^2}+\dots+\frac{1}{(a+nh)^2}\right] \quad (1)$$

$$= \underset{n\to\infty}{\mathrm{Lt}}\; S \text{ say}$$

We know that

$$(a+5h)^2=(a+5h)\,(a+5h) < (a+5h)\,(a+6h)$$
$$(a+5h)^2=(a+5h)\,(a+5h) > (a+5h)\,(a+4h)$$

Taking reciprocal and changing the sign of inequality

$$\frac{h}{(a+5h)^2} > \frac{h}{(a+5h)\,(a+6h)} = \left(\frac{1}{a+5h}-\frac{1}{a+6h}\right)$$

$$< \frac{h}{(a+5h)\,(a+4h)} = \left(\frac{1}{a+4h}-\frac{1}{a+5h}\right)$$

Hence from (1) by the help of above relations

$$S > \left[\left(\frac{1}{a+h}-\frac{1}{a+2h}\right)+\left(\frac{1}{a+2h}-\frac{1}{a+3h}\right)+\dots+\left(\frac{1}{a+nh}-\frac{1}{a+\overline{n+1}\,h}\right)\right]$$

or $\quad S > \left[\frac{1}{a+h}-\frac{1}{a+\overline{n+1}\,h}\right]$ Put $nh=b-a$

or $\quad S > \left[\frac{1}{a+h}-\frac{1}{b+h}\right] \quad (2)$

Also $\quad S < \left[\left(\frac{1}{a}-\frac{1}{a+h}\right)+\left(\frac{1}{a+h}-\frac{1}{a+2h}\right)+\dots+\left(\frac{1}{a+\overline{n-1}\,h}-\frac{1}{a+nh}\right)\right]$

$$I=\int_0^{\pi/4} \frac{\log (1+\tan \theta)}{\sec^2 \theta} \sec^2 \theta \, d\theta$$

or $$I=\int_0^{\pi/4} \log (1+\tan \theta) \, d\theta$$

Apply prop 4 *i e* $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$I=\int_0^{\pi/4} \log \{1+\tan (\pi/4-\theta)\} d\theta$$

$$=\int_0^{\pi/4} \log \left(1+\frac{1-\tan \theta}{1+\tan \theta}\right) dx \qquad \tan (\pi/4)=1$$

$$=\int_0^{\pi/4} \log \frac{2}{1+\tan \theta} \, d\theta$$

$$=\int_0^{\pi/4} \log 2 \, d\theta-\int_0^{\pi/4} \log (1+\tan \theta) \, d\theta$$

or $$I=\Big[\theta \log 2\Big]_0^{\pi/4}-I$$

or $2I=\frac{\pi}{4}\log 2 \qquad I=\frac{\pi}{8} \log 2$

Ex 5 $\int_0^{\pi} \frac{x \tan x}{\sec x+\tan x} \, dx=\frac{\pi}{2}(\pi-2)$

Change to $\sin x$ and $\cos x$

$$I=\int_0^{\pi}\frac{x \sin x}{1+\sin x} \, dx \tag{1}$$

In order to remove x, we apply prop 4

$$I=\int_0^{\pi}\frac{(\pi-x) \sin (\pi-x)}{1+\sin (\pi-x)} \, dx=\int_0^{\pi}\frac{(\pi-x) \sin x}{1+\sin x} \, dx \tag{2}$$

Adding (1) and (2) we get

$$2I=\int_0^{\pi}(x+\pi-x) \frac{\sin x}{1+\sin x} \, dx=\pi \int_0^{\pi} \frac{\sin x}{1+\sin x} \, dx$$

$$I=\frac{\pi}{2} \int_0^{\pi}\frac{\sin x+1-1}{1+\sin x} \, dx=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1}{1+\sin x}\right) dx$$

$$=\int_0^{\pi}\left(1-\frac{1-\sin x}{1-\sin^2 x}\right) dx=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1-\sin x}{\cos^2 x}\right) dx$$

$$=\frac{\pi}{2}\int_0^{\pi}\left(1-\frac{1}{\cos^2 x}+\frac{\sin x}{\cos^2 x}\right) dx$$

Again writing $h=[\sqrt{(a+h)}-\sqrt{a}]\,[\sqrt{(a+h)}+\sqrt{a}]$

$$<[\sqrt{(a+h)}-\sqrt{a}]\,[\sqrt{(a+h)}+\sqrt{(a+h)}]$$

or $<[\sqrt{(a+h)}-\sqrt{a}]\,2\sqrt{(a+h)}$

$$\frac{h}{\sqrt{(a+h)}}<2\,[\sqrt{(a+h)}-\sqrt{a}] \quad (4)$$

Hence from (1) by the help of (4)

$$S<2\,[\{\sqrt{(a+h)}-\sqrt{a}\}+\{\sqrt{a+2h)}-\sqrt{(a+h)}\}+$$

$$+\{\sqrt{(a+nh)}-\sqrt{a+\overline{n-1}h}\}) \quad \text{Put } nh=b-a$$

or $S<2\,[\sqrt{b}-\sqrt{a}]$ (5)

Hecne from (3) and (5) we get that

$S>2[\sqrt{(b+h)}-\sqrt{(a+h)}$ and $S<2\,[\sqrt{b}-\sqrt{a}]$

$$\int_a^b \frac{1}{\sqrt{x}}\,dx=\underset{n\to\infty}{\text{Lt}}\,S$$

When $n\to\infty$, $h\to 0$

$$S=2\,(\sqrt{b}-\sqrt{a})$$

Exercise

Regarding integral as the limit of a sum, prove the following

Ex 1 $\int_0^1 (3x-2)\,dx=-\frac{1}{2}\,a$

Here $a=0$, $b=1$, $nh=b-a=1$, $a+rh=0+r\,\frac{1}{n}=\frac{r}{n}$

$$f(x)=3x-2=f\,(a+rh)=f\left(\frac{r}{n}\right)=3\,\frac{r}{n}-2$$

$$\int_0^1 (3x-2)\,dx=\underset{n\to\infty}{\text{Lt}}\;h\sum_{r=1}^{n} f\,(a+rh)$$

$$=\underset{n\to\infty}{\text{Lt}}\;h\sum_{r=1}^{n}\left(3\,\frac{r}{n}-2\right)$$

$$=\underset{n\to\infty}{\text{Lt}}\;h\left[\frac{3}{n}\,\Sigma\,r-2\Sigma\,1\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\;h\left[\frac{3}{n}\;\frac{n\,(n+1)}{2}-2n\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\left[\frac{3}{2}\,nh\left(1+\frac{1}{n}\right)-2nh\right] \quad \text{Put } nh=1$$

$$=\frac{3}{2}\;1\;1-2\;1=\frac{3}{2}-2=-\frac{1}{2}$$

Put $x=\tan\theta$ $\quad dx=\sec^2\theta\, d\theta$

The limits are adjusted as 0 to $\pi/2$

$$I=\int_0^{\pi/2}\frac{\log\sec^2\theta}{\sec^2\theta}\sec^2\theta\,d\theta=\int_0^{\pi/2}\log\sec^2\theta\,d\theta$$

or $I=2\int_0^{\pi/2}\log\sec\theta\,d\theta=-2\int_0^{\pi/2}\log\cos\theta\,d\theta$

$$=-2\ \frac{\pi}{2}\log\tfrac{1}{2}\text{ by solved Ex 1 P 716}=-\pi\log\tfrac{1}{2}=\pi\log 2$$

Ex 8 $\int_0^{\pi}\log(1+\cos x)\,dx=\pi\log\frac{1}{2}$

$$I=\int_0^{\pi}\log[1+\cos(\pi-x)]\,dx\text{ by prop 4}$$

or $\quad I=\int_0^{\pi}\log(1-\cos x)\,dx$

$$2I=\int_0^{\pi}[\log(1+\cos x)+\log(1-\cos x)]\,dx$$

$$=\int_0^{\pi}\log(1-\cos^2 x)\,dx=\int_0^{\pi}\log\sin^2 x\,dx$$

$$=2\int_0^{\pi}\log\sin x\,dx$$

$$=2\cdot 2\int_0^{\pi/2}\log\sin x\,dx\text{ (by Prop 6)}=4\,\pi/2\,\log\tfrac{1}{2}$$

$$2I=2\pi\log\tfrac{1}{2}\quad\text{or}\quad I=\pi\log\tfrac{1}{2}$$

Ex 9 $\int_0^{na}f(x)\,dx=n\int_0^{a}f(x)\,dx$ if $f(x)=f(a+x)$

and hence deduce that $\int_a^{na}f(x)=(n-1)\int_0^{a}f(x)\,dx$
when $f(x)=f(a+x)$

$$\int_0^{na}f(x)\,dx=\int_0^{a}f(x)\,dx+\int_a^{2a}f(x)\,dx+\cdots\int_{(n-1)a}^{na}f(x)\,dx$$

$\int_a^{2a}f(x)\,dx$ Put $x=a+t$ and adjust the limits

$$I=\int_0^{a}f(a+t)\,dt=\int_0^{a}f(a+x)\,dx=\int_0^{a}f(x)\,dx$$

Similarly in $I=\int_{2a}^{3a}f(x)\,dx$, Put $x=a+t$

Put $\frac{r}{n}=x$ and $\frac{1}{n}=dx$ When $r=1$, $x=\frac{1}{n}\to 0$, $r=n$, then $x=\frac{n}{n}\to 1$ as $n\to\infty$

$$I=\int_0^1 \frac{1}{1+x}dx=\Big[\log(1+x)\Big]_0^1=\log 2-\log 1=\log 2$$

Similarly $\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{n+1}+\frac{1}{n+2}+\quad+\frac{1}{6n}\right]$ [I I T 81]

$$=\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\sum_{r=1}^{5n}\frac{1}{(1+r/n)}$$

Put $r/n=x$, $1/n=dx$ Also when $r=1$, $x=1/n=0$ and when $r=5n$, $x=5n/n=5$

$$\text{Lt}=\int_0^5\frac{dx}{1+x}=\Big[\log(1+x)\Big]_0^5=\log 6-\log 1=\log 6$$

Ex 2 (a) $\underset{n\to\infty}{\text{Lt}}=\left[\frac{1}{\sqrt{(n^2-1^2)}}+\frac{1}{\sqrt{(n^2-2^2)}}+\quad\frac{1}{\sqrt{\{n^2-(n-1)^2\}}}\right]$

$$=\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\left[\frac{1}{\sqrt{\left(1-\frac{1}{n^2}\right)}}+\frac{1}{\sqrt{\left[1-\left(\frac{2}{n}\right)^2\right]}}+\quad\frac{1}{\sqrt{\left[1-\left(\frac{n-1}{n}\right)^2\right]}}\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\sum_{r=1}^{n-1}\frac{1}{\sqrt{\left[1-\left(\frac{r}{n}\right)^2\right]}}$$

Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx

Also when $r=1$, $x=\frac{1}{n}=0$, when $r=n-1$, $x=\frac{n-1}{n}=1-\frac{1}{n}=1$

$$\text{Lt}=\int_0^1\frac{1}{\sqrt{(1-x^2)}}dx=\Big[\sin^{-1}x\Big]_0^1=\sin^{-1}1-\sin^{-1}0=\pi/2$$

Ex 3 $\underset{n\to\infty}{\text{Lt}}\left[\frac{n}{n^2}+\frac{n}{n^2+1^2}+\frac{n}{n^2+2^2}+\quad\frac{n}{n^2+(n-1)^2}\right]=\frac{\pi}{4}$

Now $\underset{n\to\infty}{\text{Lt}}\frac{n}{n^2}\left[\frac{1}{1+0^2}+\frac{1}{1+\left(\frac{1}{n}\right)^2}+\frac{1}{1+\left(\frac{2}{n}\right)^2}+\quad\frac{1}{1+\left(\frac{n-1}{n}\right)^2}\right]$

$$=\underset{n\to\infty}{\text{Lt}}\frac{1}{n}\sum_{r=0}^{n-1}\frac{1}{1+(r/n)^2}$$

Put $\frac{r}{n}=x$ $\frac{1}{n}=dx$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\tan\left(\tfrac{1}{2}\pi-\tfrac{1}{2}\alpha\right)-\tan^{-1}\tan\left(\tfrac{1}{2}\pi-\alpha\right)\right]$$

$$=\frac{2\pi}{\sin\alpha}\cdot\tfrac{1}{2}\alpha=\frac{\pi\alpha}{\sin\alpha}$$

Ex 11 Evaluate $\int_0^{2\pi}\frac{\sin^2\theta}{a-b\cos\theta}\,d\theta,\ a>b>0$

(Roorkee 88)

$$I=\int_0^{2\pi}\frac{\sin^2\theta\,d\theta}{a-b\cos\theta}=2\int_0^{\pi}\frac{\sin^2\theta\,d\theta}{a-b\cos\theta}\quad\text{by prop VI}$$

Also $I=2\int_0^{\pi}\frac{\sin^2\theta\,d\theta}{a+b\cos\theta}$ by prop IV

$$2I=2\int_0^{\pi}\frac{2a\sin^2\theta}{a^2-b^2\cos^2\theta}\,d\theta$$

$$=8a\int_0^{\pi/2}\frac{\sin^2\theta\,d\theta}{a^2-b^2\cos^2\theta},\ \text{by prop VI}$$

$$=8a\int_0^{\pi/2}\frac{-\cos^2\theta+1}{a^2-b^2\cos^2\theta}\,d\theta$$

$$=8a\int_0^{\pi/2}\frac{\frac{1}{b^2}\left(a^2-b^2\cos^2\theta\right)-\frac{a^2}{b^2}+1}{a^2-b^2\cos^2\theta}$$

$$=8a\left[\frac{1}{b^2}\theta\right]_0^{\pi/2}-8a\,\frac{a^2-b^2}{b^2}\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{a^2(1+\tan^2\theta)-b^2}$$

$$=8a\left[\frac{1}{b^2}\cdot\frac{\pi}{2}\right]-8a\,\frac{(a^2-b^2)}{b^2}\int_0^{\infty}\frac{dt}{a^2t^2+a^2-b^2}$$

putting $\tan\theta=t$

$$=\frac{4a}{b^2}\pi-\frac{8a(a^2-b^2)}{b^2}\cdot\frac{1}{a^2}\int_0^{\infty}\frac{dt}{t^2+\left(\frac{\sqrt{(a^2-b^2)}}{a}\right)^2}$$

$$=\frac{4a}{b^2}\pi-\frac{8(a^2-b^2)}{ab^2}\cdot\frac{a}{\sqrt{(a^2-b^2)}}\left[\tan^{-1}\frac{ta}{\sqrt{(a^2-b^2)}}\right]_0^{\infty}$$

$$=\frac{4a}{b^2}\pi-\frac{8\sqrt{(a^2-b^2)}}{b^2}\left(\frac{\pi}{2}-0\right)$$

$$I=\frac{2\pi}{b^2}\left[a-\sqrt{(a^2-b^2)}\right]$$

$$=\left[\tan^{-1} x\right]_0^1+\frac{1}{2}\left[\log(1+x^2)\right]_0^1$$

$$=(\tan^{-1} 1-\tan^{-1} 0)+\tfrac{1}{2}[\log 2-\log 1]=\frac{\pi}{4}+\frac{1}{2}\log 2$$

Ex. 6 *Evaluate* $\underset{n\to\infty}{\text{Lt}}\left[\frac{n!}{n^n}\right]^{1/n}$

$$\frac{n!}{n^n}=\frac{1}{n}\ \frac{2}{n}\ \frac{3}{n}\ \ \frac{n}{n}$$

Let $A=\underset{n\to\infty}{\text{Lt}}\left[\frac{n!}{n^n}\right]^{1/n}=\underset{n\to\infty}{\text{Lt}}\left[\frac{1}{n}\ \frac{2}{n}\ \frac{3}{n}\ \ \frac{n}{n}\right]^{1/n}$

$$\log A=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left[\log\frac{1}{n}+\log\frac{2}{n}+\log\frac{3}{n}+\ \ \log\frac{n}{n}\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\sum_{r=1}^{n}\log r/n$$

Put $\frac{r}{n}=x$ and $\frac{1}{n}=dx$ and limits are adjusted as 0 to 1

$$\log A=\int_0^1 \log x\,dx$$

$$=\left[x\log x\right]_0^1-\int_0^1 x\,\frac{1}{x}\,dx=\left[x\log x-x\right]_0^1$$

$$=[1\log 1-1]-\left[\underset{x\to 0}{\text{Lt}}\ x\log x-0\right]=-1$$

$$A=e^{-1}=\frac{1}{e}$$

Note $\quad\underset{x\to 0}{\text{Lt}}\ x\log x=\underset{x\to 0}{\text{Lt}}\ \frac{\log(x)}{1/x}=\frac{1/x}{-1/x^2}=-x=0$

We have used the rule for limits in the case of indeterminate forms

Ex 7 $\quad\underset{n\to\infty}{\text{Lt}}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

Let $A=\underset{n\to\infty}{\text{Lt}}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\ \ \left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$

$$\log A=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right)+\log\left(1+\frac{2^2}{n^2}\right)+\ \ \log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$=\underset{n\to\infty}{\text{Lt}}\ \frac{1}{n}\sum_{r=1}^{n}\log(1+r^2/n^2)$$

First remove x and then divide above and below by $\cos^2 x$

7 $\int_0^{\pi} x \log \sin x = \frac{\pi^2}{2} \log \frac{1}{2}$

Remove x Ist and apply prop 6 to make the limits 0 to $\pi/2$ and it becomes solved example 1 P 716

8 $\int_0^{\pi/2} \frac{\cos x - \sin x}{1+\sin x \cos x}\, dx = 0$

use Prop 4 $I = -I$ or $2I = 0$ $I = 0$

9 $\int_0^{\pi/2} \sin 2x \log \tan x \, dx = 0$

use Prop 4 i e $\int_0^a f(a-x)\, dx = \int_0^a f(x)\, dx$

$$I = \int_0^{\pi/2} \sin 2\,(\pi/2 - x) \log \tan (\pi/2 - x)\, dx$$

$$= \int_0^{\pi/2} \sin 2x \log \cot x = -\int_0^{\pi/2} \sin 2x \log \tan x \, dx$$

$I = -I$ or $2I = 0$ $I = 0$

10 (a) $\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx = 2$

Here $f(-x) = \frac{(-x)\sin^{-1}(-x)}{\sqrt{[1-(-x)^2]}} = \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} = f(x)$

Therefore by prop 5

$I = 2\int_0^{\pi/2} \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx$ Put $x = \sin\theta$

$$I = 2\int_0^{\pi/2} \frac{\theta \sin\theta \cos\theta \, d\theta}{\cos\theta} = 2\int_0^{\pi/2} \theta \sin d\theta$$

Integrate by parts

$$I = 2\left\{\theta(-\cos\theta) + \int_0^1 \cos\theta \cdot 1 \, d\theta\right\}$$

$$= 2\left\{-\theta\cos\theta + \sin\theta\right\}_0^{\pi/2}$$

$$= 2\,[(0+1)-(0)] = 2$$

Note $\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}\, dx = 0$

Here $f(-x) = -f(x)$ and hence by Prop 5 $I = 0$

(b) Evaluate $\int_{-1}^{1} \log \frac{2-x}{2+x}\, dx$ **(Roorkee 86)**

$\int_0^1 \sqrt{\left(\frac{1+x}{1-x}\right)} = \int \frac{1+x}{\sqrt{(1-x^2)}}\,dx$ See Q 13 p 683

3 $\underset{n\to\infty}{\text{Lt}} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \ldots + \frac{1}{8n}\right] = \frac{3}{8}$

$\frac{1}{n} = \frac{n^2}{(n+0)^3},\ \frac{1}{8n} = \frac{n^2}{(n+n)^3}$ (Note)

4 $\underset{n\to\infty}{\text{Lt}} \sum_{r=1}^{\infty} \frac{1}{\sqrt{(n^2+r^2)}} = \log(1+\sqrt{2})$

5 $\underset{n\to\infty}{\text{Lt}} \left[\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots+\sqrt{n}}{n\sqrt{n}}\right] = \frac{2}{3}$

6 $\underset{n\to\infty}{\text{Lt}} \frac{1}{n}\left[\sec^2\frac{\pi}{4n} + \sec^2\frac{2\pi}{4n} + \ldots + \sec^2\frac{n\pi}{4n}\right] = \frac{4}{\pi}$

7 Compute $\underset{n\to\infty}{\text{Lim}} \left(\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{an}\right)$ where a is a positive integer. Calculate approximately

$$\frac{1}{100} + \frac{1}{101} + \frac{1}{102} + \ldots + \frac{1}{300}.$$

Ans $\log_e a$, $\log_e 3 = 1.1$ approx

§ 7 General Properties of Definite Integrals

1 $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$

i.e. change of variable does not make any difference

2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Interchanging the limits amounts to change of sign

3. $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

If $\int f(x)\,dx = F(x) + c$

Then L H S $= \Big[F(x)+c\Big]_a^b = [F(b)+c] - [F(a)+c] = F(b) - F(a)$

R H S $= [F(c) - F(a)] + [F(b) - F(c)] = F(b) - F(a) = $ L.H S

4 $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$ V Imp

Proof Put $x = a - t$, $dx = -dt$

Hence $\int_0^a f(x)\,dx = -\int_a^0 f(a-t)\,dt = \int_0^a f(a-x)\,dx$ by prop 1 and 2

$$=\left[x-\frac{x^2}{2}\right]_0^1+\left[\frac{x^2}{2}-x\right]_1^2$$

$$=\left(1-\frac{1}{2}\right)+\frac{4-1}{2}-1=\frac{1}{2}+\frac{3}{2}-1=1$$

(c) If $f(x)=x$ for $0 \leqslant x < 1$

$=\sqrt{x}$ for $1 \leqslant x \leqslant 2$

evaluate $\int_0^2 f(x)\,dx$

$$I=\int_0^1 x^2\,dx+\int_1^2 \sqrt{x}\,dx=\frac{1}{3}(4\sqrt{2}-1)$$

(d) $\int_{-\pi/2}^{\pi/2} \sqrt{(\cos x-\cos^3 x)}\,dx=4/5$

$$I=\int \sqrt{(\cos x)}\sqrt{(\sin^2 x)}\,dx=\int_{-\pi/2}^{\pi/2} \sqrt{(\cos x)}\,|\sin x|\,dx$$

$$=\int_{-\pi/2}^{0} \sqrt{(\cos x)}\,(-\sin x)\,dx+\int_0^{\pi/2} \sqrt{(\cos x)}\,(\sin x)\,dx$$

Now put $\cos x=t$ etc

13 Evaluate $\int_a^b \frac{|x|}{x}\,dx$ where $a > b$

Solution if $0 \leqslant a < b$, then $\frac{|x|}{x}=1$, therefore

$$\int_a^b f(x)\,dx=\int_a^b dx=b-a$$

If $a < b \leqslant 0$, then $f(x)=-1$ and

$$\int_a^b f(x)\,dx=\int_a^b (-1)\,dx=(-b)-(-a)=a-b$$

Finally if $a < 0 < b$, then we divide the given integral into two integrals as follows,

$$\int_a^b f(x)\,dx=\int_a^0 f(x)\,dx+\int_0^b f(x)\,dx$$

$$=\int_a^0 (-1)\,dx+\int_0^b 1\,dx$$

$$=0-(-a)+b-0=b+a$$

The above three cases may be represented by a single formula

$$\int_a^b \frac{|x|}{x}\,dx=|b|-|a|$$

$$f(x)=\cos^3 x, f(-x)=\cos^3(-x)=\cos^3 x=f(x)$$

$$\int_{-\pi/2}^{\pi/2} \sin^3 dx=0$$

$$f(x)=\sin^3 x, f(-x)=\sin^3(-x)=(-\sin x)^3$$
$$=-\sin^3 x=-f(x)$$

6 $\int_0^{2a} f(x)\,dx=2\int_0^a f(x)\,dx$ if $f(2a-x)=f(x)$

$=0$ if $f(2a-x)=-f(x)$

Proof $I=\int_0^{2a} f(x)\,dx=\int_0^a f(x)\,dx+\int_a^{2a} f(x)\,dx,$

putting $x=2-t$ in 2nd so that $dx=-dt$

$$I=\int_0^a f(x)\,dx-\int_a^0 f(2a-t)\,dt,$$

$I=\int_0^a f(x)\,dx+\int_0^a f(2a-x)\,dx$ by prop 1 and 2

Hence $I=2\int_0^a f(x)\,dx$ if $f(2a-x)=f(x)$

and $=0$ if $f(2a-x)=-f(x)$

$$\int_0^{\pi} \cos^3 x\,dx=0$$

$$f(x)=\cos^3 x \quad f(2a-x)=\cos^3(\pi-x)=(-\cos x)^3$$
$$=-\cos^3 x=-f(x)$$

But $\int_0^{\pi} \cos^4 x\,dx=2\int_0^{\pi/2} \cos^4 x\,dx$

Similarly $\int_0^{\pi} \sin^3 x\,dx=2\int_0^{\pi/2} \sin^3 x\,dx$

$$f(2a-x)=\sin^3(\pi-x)=\sin^3 x=f(x)$$

Also $\int_0^{\pi} \sin^4 x\,dx=2\int_0^{\pi/2} \sin^4 x$ as above

$I=\int_0^{\pi} \frac{\sin 2kx}{\sin x}\,dx=0$ where k is an integer

$$f(2a-x)=\frac{\sin 2k(\pi-x)}{\sin(\pi-x)}=\frac{\sin(2k\pi-2kx)}{\sin x}$$

$$=\frac{\sin(-2kx)}{\sin x}=-\frac{\sin 2kx}{\sin x}=-f(x)$$

Since $f(2a-x)=-f(x)$ therefore by prop VI

$$I=\int_0^{\pi} \frac{\sin 2kx}{\sin x}\,dx=0$$

16 Consider the integral $I=\int_0^{\pi} \frac{dx}{5-2\cos x}$

Making the substitution $\tan \frac{1}{2} x=t$, we have

$$\int_0^{2\pi} \frac{dx}{5-2\cos x}=\int_0^0 \frac{2dt}{(1+t)\left[5-\frac{2(1-t^2)}{(1+t^2)}\right]}=0$$

The result is obviously wrong since the integrand is positive and consequently the integral of this function cannot be equal to zero. Find the mistake.

Solution The mistake lies in the substitution $\tan \frac{1}{2} x=t$ since the function $\tan \frac{1}{2} x$ is discontinuous at $x=\pi$ a point in the since interval $(0, 2\pi)$

17 Is it possible to make the substitution $x=\sec t$ in the integral $I=\int_0^1 \sqrt{(1+x^2)}\,dx$

Solution No, since $\sec t \geqslant 1$ whereas the interval of integration is $[0, 1]$

18 Find the mistake in the following evaluation of the integral

$$\int_0^{\pi} \frac{dx}{1+2\sin^2 x}=\int_0^{\pi} \frac{dx}{\cos^2 x+3\sin^2 x}=\int_0^{\pi} \frac{\sec^2 x\,dx}{1+3\tan^2 x}$$

$$=\frac{1}{\sqrt{3}}\left[\tan^{-1}(\sqrt{3}\tan x)\right]_0^{\pi}=0$$

Solution The Newton Leibniz formula is not applicable here since the anti derivative $F(x)=\frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}\tan x)$ has a discontinuity at the point $x=\pi/2$

Indeed, $F(\frac{1}{2}\pi-0)=\lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}\{\sqrt{3}\tan(\pi/2-h)\}$

$$=\lim_{h\to 0} \frac{1}{\sqrt{3}} \tan^{-1}(\sqrt{3}\cot h)=\frac{1}{\sqrt{3}}\tan^{-1}\infty=\frac{\pi}{2\sqrt{3}}$$

and similarly $F(\pi/2+0)=-\frac{\pi}{2\sqrt{3}}$. Thus the limit of $F(x)$ at $x=\pi/2$ does not exist and hence it is discontinuous at $x=\pi/2$

19 Let f and g be functions satisfying the following conditions

(a) $f(0)=1$, (b) $f'(x)=g(x)$, $g'(x)=f(x)$,

(c) $g(0)=0$

Find $f(1)$ accurate to three decimal places

Solution We have $f''(x)=g'(x)=f(x)$,

Prove

(ii) $I=\int_0^{\pi} x f(\sin x)\ dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\ dx$

$$=\pi\int_0^{\pi/2} f(\sin x)\ dx$$

$I=\int_0^{\pi} x f(\sin x)\ dx$

We apply property 4 *i e*

$$\int_0^a f(x)\ dx=\int_0^a f(a-x)\ dx$$

$$I=\int_0^{\pi} (\pi-x) f \sin(\pi-x)\ dx=\int_0^{\pi} (\pi-x) f(\sin x)\ dx$$

Adding (1) and (2) we get

$$2I=\int_0^{\pi} (x+\pi-x) f(\sin x)\ dx=\pi\int_0^{\pi} f(\sin x)\ dx$$

$$I=\frac{\pi}{2}\int_0^{\pi} f(\sin x)\ dx$$

Now for the second result we shall use property 6 *i e*

$$\int_0^{2a} f(x)\ dx=2\int_0^{a} f(x)\ dx$$

if $f(2a-x)=f(x)$

Here $2a=\pi$ and $f(x)=f(\sin x)$

$$f(2a-x)=f(\sin(\pi-x))=f(\sin x)=f(x)$$

$$I=\frac{\pi}{2}\ 2\int_0^{\pi/2} f(\sin x)\ dx=\pi\int_0^{\pi/2} f(\sin x)\ dx$$

In general $\int_0^a x\phi(x)\ dx=\frac{1}{2}\ a\int_0^a \phi(x)\ dx$ provided

$$\phi(a-x)=\phi(x)$$

Solved Examples

Ex 1 $\quad I=\int_0^{\pi/2} \log\sin x\ dx=\frac{\pi}{2}\log\frac{1}{2}$

By prop (4) $\int_0^a f(x)\ dx=\int_0^a f(a-x)\ dx$

$$I=\int_0^{\pi/2} \log\sin(\pi/2-x)\ dx=\int_0^{\pi/2} \log\cos x\ dx$$

$$2I=\int_0^{\pi/2} (\log\sin x+\log\cos x)\ dx$$

$$\int_0^1 \frac{n!}{(n-k)!} x^k (1-x)^{n-k}\, dx = \frac{k!}{n+1}$$

or $\int_0^1 x^k (1-x)^{n-k}\, dx = \frac{k!\,(n-k)!}{n!} \cdot \frac{1}{n+1} = [{}^nc_k\,(n+1)]^{-1}$

21 Compute the integral

$$I = \int_0^{\pi} \sqrt{\left\{\frac{1}{2}(1+\cos 2x)\right\}}\, dx$$

Solution We have

$$I = \int_0^{\pi} \sqrt{\left\{\frac{1}{2}(1+\cos 2x)\right\}}\, dx = \int_0^{\pi} \sqrt{(\cos^2 x)}\, dx = \int_0^{\pi} |\cos x|\, dx$$

$$= 2\int_0^{\pi/2} |\cos x|\, dx, \qquad |\cos(\pi - x)| = |-\cos x| = |\cos x|$$

(see Prop 6 of § 7)

$$= 2\int_0^{\pi/2} \cos x\, dx = 2\Big[\sin x\Big]_0^{\pi/2} = 2\,[1-0] = 2$$

Note If we ignore the fact that $\cos x$ is negative in $]\frac{1}{2}\pi, \pi[$, and put $\sqrt{\left(\frac{1+\cos 2x}{2}\right)} = \cos x$, we get the wrong result

$$\int_0^{\pi} \cos x\, dx = \Big[\sin x\Big]_0^{\pi} = 0$$

22 (a) Show that if $f(t)$ is an odd function, then

$\int_a^x f(t)\, dt$ is an even function i e

$$\int_a^x f(t)\, dt = \int_a^{-x} f(t)\, dt$$

(b) Can $\int_a^x f(t)\, dt$ be an odd function if $f(t)$ is an even function ?

Solution (a) Let $F(x) = \int_a^x f(t)\, dt$

Then $F(-x) = \int_a^{-x} f(t)\, dt = \int_{-a}^{x} f(-u)\,(-du)$ where $t = -u$

$$= \int_{-a}^{x} f(u)\, du \quad [\;f(-u) = -f(u)]$$

$$= \int_{-a}^{a} f(u)\, du + \int_a^x f(u)\, du$$

$$= 0 + \int_a^x f(x)\, dx, \qquad f(-u) = -f(u)$$

$$= F(x)$$

Prove

(ii) $I=\int_0^\pi x f(\sin x)\,dx=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$ (I I T 82

$$=\pi\int_0^{\pi/2} f(\sin x)\,dx$$

$I=\int_0^\pi x f(\sin x)\,dx$ (1)

We apply property 4 i e

$$\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$$

$$I=\int_0^\pi (\pi-x)\,f\sin(\pi-x)\,dx=\int_0^\pi (\pi-x)\,f(\sin x)\,dx \quad (2)$$

Adding (1) and (2) we get

$$2I=\int_0^\pi (x+\pi-x)f(\sin x)\,dx=\pi\int_0^\pi f(\sin x)\,dx$$

$$I=\frac{\pi}{2}\int_0^\pi f(\sin x)\,dx$$

Now for the second result we shall use property 6 i e

$$\int_0^{2a} f(x)\,dx=2\int_0^a f(x)\,dx$$

$$\text{if } f(2a-x)=f(x)$$

Here $2a=\pi$ and $f(x)=f(\sin x)$

$$f(2a-x)=f(\sin(\pi-x))=f(\sin x)=f(x)$$

$$I=\frac{\pi}{2}\,2\int_0^{\pi/2} f(\sin x)\,dx=\pi\int_0^{\pi/2} f(\sin x)\,dx$$

In general $\int_0^a x\phi(x)\,dx=\frac{1}{2}a\int_0^a \phi(x)\,dx$ provided

$$\phi(a-x)=\phi(x)$$

Solved Examples

Ex 1 $I=\int_0^{\pi/2} \log\sin x\,dx=\frac{\pi}{2}\log\frac{1}{2}$

By prop (4) $\int_0^a f(x)\,dx=\int_0^a f(a-x)\,dx$

$$I=\int_0^{\pi/2} \log\sin(\pi/2-x)\,dx=\int_0^{\pi/2}\log\cos x\,dx$$

$$2I=\int_0^{\pi/2}(\log\sin x+\log\cos x)\,dx$$

or $-\int_x^a f(t)\,dt - \int_a^{x+T} f(t)\,dt = 0$

or $-\int_x^{x+T} f(t)\,dt = 0$ (1)

To prove (1) we write

$$\int_x^{x+T} f(t)\,dt = \int_x^a f(t)\,dt + \int_a^T f(t)\,dt + \int_T^{x+T} f(t)\,dt \quad (2)$$

Putting $t=T+z$ in the last integral in (2) we have

$$\int_T^{x+T} f(t)\,dt = \int_0^x f(T+z)\,dz = \int_0^x f(z)\,dz = \int_0^x f(t)\,dt$$

From (2), we have

$$\int_x^{x+T} f(t)\,dt = \int_x^a f(t)\,dt + \int_a^T f(t)\,dt + \int_0^x f(t)\,dt$$

$$= \int_0^T f(t)\,dt \quad (3)$$

Now (3) shows that $\int_x^{x+T} f(t)\,dt$ is independent of x. Putting $x=-\frac{1}{2}T$ in (3), we get

$\int_x^{x+T} f(t)\,dt = \int_{-T/2}^{T/2} f(t)\,dt = 0$ since $f(t)$ is an odd function

Hence (1) is proved

§ 8 Area

1 The area between the curve $y=f(x)$, x axis and two ordinates at the points $x=a$ and $x=b$ $(b > a)$ is given by formula

$$A = \int_a^b f(x)\,dx = \int_a^b y\,dx$$

It represents the shaded area in the figure below

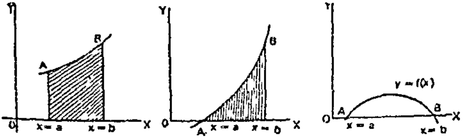


Fig 1

2 Similarly if the area between the curve and y axis and two abscissas drawn at the points $y=a$ and $y=b$ $(b > a)$ then the corresponding formula will be

$$A = \int_a^b x\,dy$$

The following integrals are each equal to $\pi/4$ and may be calculated as above

$$\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx,\quad \int_0^{\pi/2} \frac{1}{1+\tan x}\,dx$$

Change to sin and cos

$$\int_0^{\pi/2} \frac{1}{1+\cot x}\,dx,\quad \int_0^{\pi/2} \frac{\tan x}{1+\tan x}\,dx$$

$$\int_0^{\pi/2} \frac{\cot x}{1+\cot x}\,dx,\quad \int_0^{\pi/2} \frac{1}{1+\sqrt{(\tan x)}}\,dx,\quad \int_0^{\pi/2} \frac{\sqrt{\cot x}}{\sqrt{\cot x}+\sqrt{\tan x}}\,dx$$

(*IIT* 83)

Ex 2 (*b*) $I=\int_{\pi/6}^{\pi/3} \frac{1}{\sqrt{1+(\cot x)}}\,dx=\pi/12$

$$I=\int_{\pi/6}^{\pi/3} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\,dx \qquad (1) \quad \text{Put } x=\pi/2-t$$

$$I=\int_{\pi/3}^{\pi/6} \frac{\sqrt{(\cos t)}}{\sqrt{(\cos t)}+\sqrt{(\sin t)}}\,(-dt)$$

$$=\int_{\pi/6}^{\pi/3} \frac{\sqrt{(\cos t)}}{\sqrt{(\cos t)}+\sqrt{(\sin t)}}\,dt$$

or $$I=\int_{\pi/6}^{\pi/3} \frac{\sqrt{(\cos x)}\,dx}{\sqrt{(\cos x)}+\sqrt{(\sin x)}} \quad \text{Prop } I \qquad (2)$$

Adding (1) and (2) we get

$$2I=\int_{\pi/6}^{\pi/3} 1\,dx=\Big[\,x\,\Big]_{\pi/6}^{\pi/3}=\pi/6 \qquad I=\frac{\pi}{12}$$

Ex 3 $\int_0^{\infty} \frac{x\,dx}{(1+x)(1+x^2)}=\pi/4$

Put $x=\tan\theta$ $\quad dx=\sec^2\theta\,d\theta$

When $x=\infty$, $\tan\theta=\infty$, $\quad \theta=\pi/2$

$$I=\int_0^{\pi/2} \frac{\tan\theta\sec^2\theta\,d\theta}{(1+\tan\theta)(\sec^2\theta)}\quad d\theta$$

Now change to $\sin\theta$ and $\cos\theta$

$$I=\int_0^{\pi/2} \frac{\sin\theta\,d\theta}{\cos\theta+\sin\theta}=\pi/4 \text{ as in Prop IV page 713}$$

Ex 4 $\int_0^{1} \frac{\log(1+x)}{1+x^2}\,dx=\frac{\pi}{8}\log 2$

Put $x=\tan\theta$ $\quad dx=\sec^2\theta\,d\theta$

When $x=1$, $\tan\theta=1$ $\quad \theta=\pi/4$

$x=0$, $\tan\theta=0$ $\quad \theta=0$

Ex 2 Find the area enclosed by the parabola

$$ay=3(a^2-x^2)$$

and the axis of x

The parabola cuts the axis of x i e $y=0$ in points given by

$$0=3(a^2-x^2) \qquad x=a, -a$$

$$\text{Area}=\int_{-a}^{a} y\,dx=\frac{3}{a}\int_{-a}^{a}(a^2-x^2)\,dx$$

Here $f(-x)=a^2-(-x)^2=a^2-x^2=f(x)$

Therefore by prop V

$$\text{Area}=2\,\frac{3}{a}\int_0^a (a^2-x^2)\,dx=\frac{6}{a}\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$=\frac{6}{a}\left[a^3-\frac{a^3}{3}\right]=\frac{6}{a}\cdot\frac{2}{3}a^3=4a^2 \text{ sq units}$$

Ex 3 Show that the area cut off a parabola by any double ordinate is two thirds of the corresponding rectangle contained by that double ordinate and its distance from the vertex

We have to prove that

$$\text{Area } PAQP=\frac{2}{3}\text{ area of rectangle } PLMQ$$

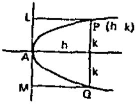


Fig 4

The equation of the parabola is

$$y^2=4ax$$

Let the ordinate be drawn through

$$P(h, k)$$

Then $k^2=4ah$ or $k=2\sqrt{(ah)}$

Area of rectangle $=2k\cdot h=2\cdot 2\sqrt{(ah)}\,h=4\sqrt{(ah)}\,h$ (1)

Area $PAQP=2$ area above x axis

$$=2\int_0^h y\,dx=2\int_0^h 2\sqrt{a}\,\sqrt{x}\,dx$$

$$=4\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^h=\frac{2}{3}\cdot 4\sqrt{a}\,h^{3/2}$$

$=\frac{2}{3}$ [Area of rectangle] by (1)

Ex 4 Find the area included between the parabolas

(Roorkee 80)

$y^2=4ax$ and $x^2=4by$

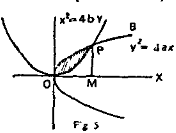


Fig 5

The two curves meet at P say (h, k)

Solving the two we get

from $x^2=4by$, $y=\dfrac{x^2}{4b}$

and putting in $y^2=4ax$

y_2 is the ordinate of the lower curve i e straight line $y=mx$

$$y_2=mx$$

Putting for y_1 and y_2 in (1) we get

$$A=\int_0^h 2\sqrt{a}\sqrt{x}\,dx-\int_0^h mx\,dx$$

$$=2\sqrt{a}\frac{2}{3}\left[x^{3/2}\right]_0^h-m\left[\frac{x^2}{2}\right]_0^h$$

$$=\frac{4\sqrt{a}}{3}h^{3/2}-\frac{m}{2}h^2$$

Now put $h=\dfrac{4a}{m^2}$

$$A=\frac{4}{3}\sqrt{a}\left(\frac{4a}{m^2}\right)^{3/2}-\frac{m}{2}\left(\frac{4a}{m^2}\right)^2$$

$$=\frac{4}{3}\sqrt{a}\,\frac{8a^{3/2}}{m^3}-\frac{m}{2}\,\frac{16a^2}{m^4}$$

$$=\frac{32}{3}\,\frac{a^2}{m^3}-8\,\frac{a^2}{m^3}=\frac{8}{3}\,\frac{a^2}{m^3}$$

Note If the curves were $y^2=9x$ and $y=x$, then

$4a=9$ or $a=\frac{9}{4}$ and $m=1$

$$A=\frac{8}{3}\,\frac{a^2}{m^3}=\frac{8}{3}\left(\frac{9}{4}\right)^2=\frac{8}{3}\,\frac{81}{16}=\frac{27}{2}\text{ sq units.}$$

Ex 6. AOB is the positive quadrant of the ellipse $x^2/a^2+y^2/b^2=1$ in which $OA=a$, $OB=b$ Show that the area between the arc AB and chord AB of the ellipse is

$$\tfrac{1}{4}ab\,(\pi-2)$$

$$\text{Area}=\int_0^a (y_1-y_2)\,dx \qquad (1)$$

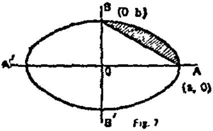


Fig. 7

Clearly ellipse is upper curve and line is lower curve

Therefore y_1 is the ordinate of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\therefore\ \frac{y_1^2}{b^2}=1-\frac{x^2}{a^2}=\frac{(a^2-x^2)}{a^2}\text{ or } y_1=\frac{b}{a}\sqrt{(a^2-x^2)}$$

y_2 is the ordinate of the lower curve i e line AB whose equation in intercepts form is

$$\frac{x}{a}+\frac{y}{b}=1$$

$$I=\int_a^{2a} f(a+t)\,dt=\int_a^{2a} f(a+x)\,dx=\int_a^{2a} f(x)\,dx=\int_0^a f(x)\,dx$$

as shown above

Similarly each integral can be shown to be equal to $\int_0^a f(x)\,dx$

$$\int_0^{na} f(x)\,dx=n\int_0^a f(x)\,dx$$

Deduction $\int_a^{na} f(x)\,dx=\int_a^0 f(x)\,dx+\int_0^{na} f(x)\,dx$

$$=-\int_0^a f(x)\,dx+n\int_0^a f(x)\,dx$$

$$=(n-1)\int_0^a f(x)\,dx$$

(b) If $f(x)=f(x+ma)$ for all integral values of m then Prove that $\int_0^{na} f(x)\,dx=n\int_0^a f(x)\,dx$

Proceed as above

Ex 10 Evaluate $\int_0^\pi \frac{x\,dx}{1+\cos\alpha\sin x}$ $(0<\alpha<\pi)$ (I I T 86)

Solution Let $I=\int_0^\pi \frac{x\,dx}{1+\cos\alpha\sin x}$

$$I=\int_0^\pi \frac{(\pi-x)\,dx}{1+\cos\alpha\sin(\pi-x)}=\int_0^\pi \frac{(\pi-x)\,dx}{1+\cos\alpha\sin x}$$

Adding, $2I=\int_0^\pi \frac{\pi\,dx}{1+\cos\alpha\sin x}=2\pi\int_0^{\pi/2} \frac{dx}{1+\cos\alpha\sin x}$

or $I=\pi\int_0^{\pi/2} \frac{dx}{1+2\cos\alpha\sin\frac{1}{2}x\cos\frac{1}{2}x}$

$$=\pi\int_0^{\pi/2} \frac{\sec^2\frac{1}{2}x\,dx}{1+\tan^2\frac{1}{2}x+2\cos\alpha\tan\frac{1}{2}x}$$

Now put $\tan\frac{1}{2}x=t$ Then $\frac{1}{2}\sec^2\frac{1}{2}x\,dx=dt$

$$I=\pi\int_0^1 \frac{2\,dt}{1+t^2+2t\cos\alpha}=2\pi\int_0^1 \frac{dt}{(t+\cos\alpha)^2+\sin^2\alpha}$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{t+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{0+\cos\alpha}{\sin\alpha}\right)\right]$$

$$=\frac{2\pi}{\sin\alpha}\left[\tan^{-1}\cot\left(\tfrac{1}{2}\alpha\right)-\tan^{-1}\cot\alpha\right]$$

(*b*) Find the area bounded by the curve $x^2=4y$ and the straight line $x=4y-2$ (I.I.T. 81)

Ans $\frac{9}{8}$

Ex 8 Prove that the area in the first quadrant enclosed by the x axis, the line $x=y\sqrt{3}$ and the circle $x^2+y^2=4$ is $\pi/3$

The two curves meet at

$3y^2+y^2=4$

or $y^2=1$

$y=1, -1$

$x=\sqrt{3}, -\sqrt{3}$

Hence the point P in Ist quadrant is $(\sqrt{3}, 1)$

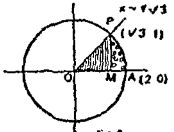


Fig 9

The required common area = Area $APOA$

= Lined area + dotted area

$=\int_0^{\sqrt{3}} y\,dx+\int_{\sqrt{3}}^{2} y\,dx$

Line Circle

$=\int_0^{\sqrt{3}} \frac{x}{\sqrt{3}}\,dx+\int_{\sqrt{3}}^{2} \sqrt{(4-x^2)}\,dx$

$=\frac{1}{\sqrt{3}}\left[\frac{x^2}{2}\right]_0^{\sqrt{3}}+\left[\frac{x}{2}\sqrt{(4-x^2)}+\frac{1}{2}\,4\sin^{-1}\frac{x}{2}\right]_{\sqrt{3}}^{2}$

$=\frac{1}{\sqrt{3}}\,\frac{3}{2}+\left[\left(0-\frac{\sqrt{3}}{2}\,1\right)+2\left(\sin^{-1}1-\sin^{-1}\frac{\sqrt{3}}{2}\right)\right]$

$=\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}+2\,(\pi/2-\pi/3)=2\,\pi/6=\pi/3$ sq units

Ex 9 Find the area common to the circle $x^2+y^2=16a^2$ and the parabola $y^2=6ax$ Hence find the larger of the the areas into *which the circle is divided by the parabola*

Solving the two we get $x^2+6ax-16a^2=0$

$(x+8a)(x-2a)=0$

$x=2a$ $y=\pm 2\sqrt{3}a$

The required common area

$=2\ [APOA]$

$=2$ [Lined area + dotted area]

$=2\int y\,dx+2\int y\,dx$

Parab Circle

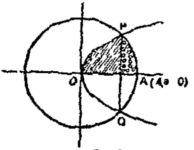


Fig 10

Problem Set (H)

1 $\int_0^1 \log \sin\left(\frac{\pi}{2} x\right) dx = \log \frac{1}{2}$

Put $\frac{\pi}{2} x = t$ and it becomes Ex 1 Page 716

2. $\int_0^{\pi/2} \log \tan x \, dx = \int_0^{\pi/2} \log \cot x \, dx = 0$

$$\log \tan x = \log \frac{\sin x}{\cos x} = \log \sin x - \log \cos x$$

$$I = \int_0^{\pi/2} \log \sin x \, dx - \int_0^{\pi/2} \log \cos x \, dx = 0 \text{ by Prop 4}$$

3 $\int_0^{\pi/4} \log (1+\tan x) \, dx = \pi/8 \log 2$

We have already done this in solved Ex 4 P 718

4 $\int_0^1 \frac{\log x}{\sqrt{(1-x^2)}} dx = \pi/4 \log \frac{1}{2}$

Put $x = \sin \theta$ and it becomes solved Ex 1 P 716

5 $\int_0^\infty \log\left(x+\frac{1}{x}\right) \frac{dx}{1+x^2} = \pi \log 2$

Put $x = \tan \theta$

$$I = \int_0^{\pi/2} \log (\tan \theta + \cot \theta) \, d\theta$$

$$= \int_0^{\pi/2} \log \frac{\sin^2 \theta + \cos^2 \theta}{\sin \theta \cos \theta} \, d\theta = -\int_0^{\pi/2} \log \sin \theta \cos \theta \, d\theta$$

$$= -\left[\int_0^{\pi/2} \log \sin \theta \, d\theta + \int_0^{\pi/2} \log \cos \theta \, d\theta\right]$$

$$= -\left[\frac{\pi}{2} \log \frac{1}{2} + \frac{\pi}{2} \log \frac{1}{2}\right] \text{ by Ex 1 page 716}$$

$$= -\pi \log \frac{1}{2} = \pi \log 2$$

6 $\int_0^\pi \frac{x \, dx}{1 - \cos^2 x} = \frac{\pi^2}{2\sqrt{2}}$,

$$=\int_{-\sqrt{3}}^{\sqrt{3}} (y_1-y_2)\, dx=\int_{-\sqrt{3}}^{\sqrt{3}} \{(2x^2+9)-5x^2\}\, dx$$

$$=\int_{-\sqrt{3}}^{\sqrt{3}} (9-3x^2)\, dx=2\int_{0}^{\sqrt{3}} (9-3x^2)\, dx$$

$f(x)$ is an even function i e $f(-x)=-f(x)$

Hence by prop 5 $\int_{-a}^{a} f(x)\, dx=2\int_{0}^{a} f(x)\, dx$

Common area $=2\left[9x-x^3\right]_0^{\sqrt{3}}$

$=2\,[9\sqrt{3}-3\sqrt{3}]=2\cdot 6\sqrt{3}=12\sqrt{3}$ sq units

Ex 11 Find the area bounded by the x-axis, part of the curve $y=\left(1+\frac{8}{x^2}\right)$ and the ordinates at $x=2$ and $x=4$ If the ordinate at $x=a$ divides the area into two equal parts, find a (I I T 83)

$$A=\int_2^4 y\, dx=\int_2^4\left(1+\frac{8}{x^2}\right)dx=\left[x-\frac{8}{x}\right]_2^4=4$$

$$A_1=\int_2^a y\, dx=\tfrac{1}{2}A=2 \qquad \left[x-\frac{8}{x}\right]_2^a=2$$

or $(a-2)-8\left(\frac{1}{a}-\frac{1}{2}\right)=2$ or $a-\frac{8}{a}=0$

$a^2-8=0 \qquad a=2\sqrt{2}$

Ex 12 (a) find the area included between the parabola $y=\frac{x^2}{4a}$ and the which $y=\frac{8a^3}{x^2+4a^2}$ (Roorkee 83)

(b) Find area bounded by the witch of Agnesi $y^2=\frac{4a^2(2a-x)}{x}$ and its asymptotes

(a) $y(x^2+4a^2)=8a^3$ represents a curve which is symmetrical about y-axis and cuts it at the point $(0, 2a)$ tangent at which is parallel to x axis Also x axis is the asymptote This curve meets the parabola $x^2=4ay$ where

$$\frac{x^2}{4a}=\frac{8a^3}{x^2+4a^2}$$

or $x^4+4a^2x^2-32a^4=0 \quad (x^2-4a^2)(x^2+8a^2)=0$

$x=\pm 2a$ and $y=a$ Thus the two points are

$(2a, a)$ and $(-2a, a)$

Let $f(x)=\log_e \frac{2-x}{2+x}$ Then

$$f(-x)=\log\frac{2+x}{2-x}=\log\left(\frac{2-x}{2+x}\right)^{-1}=-\log\frac{2-x}{2+x}=-f(x)$$

Hence $\int_{-1}^{1}\log_e\left(\frac{2-x}{2+x}\right)dx=0$ by prop 5

(c) Evaluate

$$\int_{-1}^{1}[\sqrt{(1+x+x^2)}-\sqrt{(1-x+x^2)}]\,dx$$ *Ans* 0

11 $\int_{-a}^{a} x\sqrt{(a^2-x^2)}\,dx=0$

Here $f(-x)=-f(x)$ and hence by Prop 5, $I=0$

12 (*a*) Find the value of $\int_{-1}^{3/2}|x\sin\pi x|\,dx$ (*IIT 82*)

We know that $\sin\theta$ is +ive when $0\leqslant\theta<\pi$
and $\sin\theta$ is −ive when $\pi\leqslant\theta\leqslant 3\pi/2$

Hence we write

$$\int_{-1}^{3/2}|x\sin\pi x|\,dx=\int_{-1}^{1}x\sin\pi x\,dx+\int_{1}^{3/2}(-x\sin\pi x)\,dx$$

$$=2\int_0^1 x\sin\pi x\,dx-\int_1^{3/2}x\sin\pi x\,dx$$

(Prop 5)

Integrate by parts

$$I=2\left[-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi^2}\sin\pi x\right]_0^1-\left[-\frac{x}{\pi}\cos\pi x+\frac{1}{\pi^2}\sin\pi x\right]_1^{3/2}$$

$$=2\left(-\frac{\cos\pi}{\pi}\right)-\left(0+\frac{\cos\pi}{\pi}-\frac{1}{\pi^2}\sin\frac{3\pi}{2}\right)$$

$$=-\frac{3}{\pi}(-1)-\frac{1}{\pi^2}(-1)=\frac{3}{\pi}+\frac{1}{\pi^2}$$

(*b*) Find the value of $\int_0^2|(1-x)|\,dx$

We know that $|1-x|=1-x$ if $0\leqslant x<1$
$|1-x|=x-1$ if $1\leqslant x<2$

$$\int_0^2|(1-x)|\,dx=\int_0^1|(1-x)|\,dx+\int_1^2|(1-x)|\,dx$$

$$=\int_0^1(1-x)\,dx+\int_1^2(x-1)\,dx$$

Problem Set (I)

1 Prove that the area bounded by the hyperbola $x^2-y^2=a^2$ between the straight lines $x=a$ and $x=2a$ is

$$2\sqrt{3}a^2-a^2 \log (2+\sqrt{3}) \text{ sq units}$$

$$\int \sqrt{(a^2+x^2)}\, dx=\frac{x}{2} \sqrt{(a^2+x^2)}+\frac{a^2}{2} \log \{x+\sqrt{(a^2+x^2)}\}$$

2 Find the area bounded by the parabola $y=2-x^2$ and the straight line $y+x=0$

Ans $\frac{9}{2}$ sq units Determine upper and lower curves as in Ex 10

3 Prove that the area common to the parabolas $y=2x^2$ and $y=x^2+4$ is $\frac{32}{3}$ sq units

4 Prove that the area bounded by the parabola $y=x^2$ and the line $y=2x$ is 4/3 sq units

5 Prove that the area bounded by the parabolas $y^2=5x+6$ and $x^2=y$ is 81/15 sq units

6 Show that the area included between the parabolas

$y^2=4a(x+a)$ and $y^2=4b(b-x)$ is $\frac{8}{3}\sqrt{(ab)}\,(a+b)$

7 (*a*) Prove that the area between the parabolas $y^2=4x$, $y^2=x$ and $x=1$, $x=4$ is 28/3 sq units

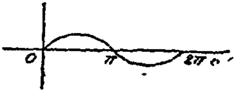
(*b*) Find the area of the figure bounded by the curve

Ans 8/3 Since the curve is symmetrical about both the axis, the required area $=4\int_0^1 (1-x^2)dx$

$$|y|=1-x^2$$

8 Find the area of the region by the method of integration bounded by the curve $y=x\sin x$ and the x axis between $x=0$ and $x=2\pi$ (Roorkee 81)

Area $=2\int_0^{\pi} y\, dx=2\int_0^{\pi} x\sin x\, dx$



14 (a) Evaluate $\int_0^{100\pi} \sqrt{(1-\cos 2x)}\,dx$

Solution We have $\sqrt{(1-\cos 2x)}=\sqrt{2}\,|\sin x|$

Since $|\sin x|$ has a period π we have

$\int_0^{100\pi} \sqrt{(1-\cos 2x)}\,dx=\sqrt{2}\int_0^{100\pi} |\sin x|\,dx$

$=100\sqrt{2}\int_0^{\pi} |\sin x|\,dx$

$=100\sqrt{2}\int_0^{\pi} \sin x\,dx$

$=100\sqrt{2}\left[-\cos x\right]_0^{\pi}$

$=100\sqrt{2}\,[1-(-1)]=200\sqrt{2}$

(b) Evaluate $\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}\right)^2+\left(\frac{x-1}{x+1}\right)^2-2\right]^{1/2} dx$

Sol $I=\int_{-1/2}^{1/2} \left[\left(\frac{x+1}{x-1}-\frac{x-1}{x+1}\right)^2\right]^{1/2} dx$

$=\int_{-1/2}^{1/2} \left|\frac{x+1}{x-1}-\frac{x-1}{x+1}\right| dx$

$\int_{-1/2}^{1/2} \left|\frac{4x}{x^2-1}\right| dx$

$=2\int_0^{1/2} \left|\frac{4x}{x^2-1}\right| dx$ by prop V

$=2\int_0^{1/2} \frac{4x}{1-x^2}\,dx$ by def of modulus

$=-4\left[\log(1-x^2)\right]_0^{1/2}=-4\left[\log\left(1-\frac{1}{4}\right)-0\right]$

$=-4\log 3/4=4\log(4/3)$

15 Since the integrand of the integral $I=\int_{-1}^{1}\frac{1}{1+x^2}\,dx$ is positive, it follows that $I>0$ However, if we make the substitution $x=\frac{1}{u}$, then

$I=-\int_{-1}^{1}\frac{1}{1+u^2}\,du=-I$ whence $2I=0$ or $I=0$

Explain the reason for this paradox

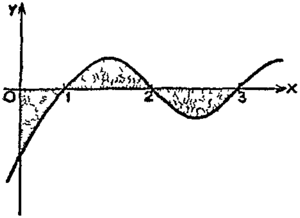
Solution Substitution $x=1/u$ is not valid since the function $1/u$ is discontinuous at $x=0$ which is a point in the interval of integration $[-1\ 1]$

$=\int (x^3-6x^2+11x-6)\,dx$

$=\frac{1}{4}x^4-2x^3+\frac{11}{2}x^2-6x$

$F(0)=0$, $F(1)=\frac{1}{4}-2+\frac{11}{2}-6=-\frac{9}{4}$,

$F(2)=4-16+22-12=-2$, and $F(3)=\frac{81}{4}-54+\frac{99}{2}-18=-\frac{9}{4}$



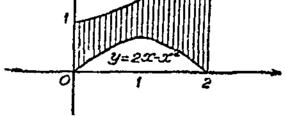
Hence required Area $=|F(1)-F(0)|+|F(2)-F(1)|$
$+|F(3)-F(2)|$

$=\left|-\frac{9}{4}-0\right|+\left|-2-\left(-\frac{9}{4}\right)\right|+\left|-\frac{9}{4}-(-2)\right|$

$=\frac{9}{4}+\frac{1}{4}+\frac{1}{4}=\frac{11}{4}=2\frac{3}{4}$

12 Compute the area of the figure bounded by the straight lines $x=0$, $x=2$ and the curves $y=2^x$, $y=2x-x^2$

Solutions Figure is self explanatory



$2f'(x)f''(x)=2f(x)f'(x)$

Integrating, $f'^2(x)=f^2(x)+A$

When $x=0, f(0)=1, f'(0)=g(0)=0$

$0=1+A$ or $A=-1$

$f'^2(x)=f^2(x)-1$

$$\int\frac{f'(x)}{\sqrt{\{f^2(x)-1\}}}\,dx=\int dx$$

or $\log[f(x)+\sqrt{\{f^2(x)-1\}}]=x+B$

When $x=0$, we have $\log[f(0)+\sqrt{\{f^2(0)-1\}}]$

$=0+B$, $B=0$

Hence $\log[f(x)+\sqrt{\{f^2(x)-1\}}]=x$

or $f(x)+\sqrt{\{f^2(x)-1\}}=e^x$

Hence $f(1)+\sqrt{\{f^2(1)-1\}}=e$

or $f^2(1)-1=\{e-f(1)\}^2=e^2-2e\,f(1)+f^2(1)$

This gives $f(1)=\dfrac{e^2+1}{2e}=\dfrac{1}{2}(e+e^{-1})=\cosh 1$

$$=1+\frac{1}{2!}+\frac{1}{4!}+\frac{1}{6!}=1.543 \text{ Approx}$$

20. Evaluate $\int_0^1 (tx+1-x)^n\,dx$ where n is a positive integer and t is a parameter independent of x. Hence show that

$\int_0^1 x^k(1-x)^{n-k}\,dx=[{}^nC_k\,(n+1)]^{-1}$ (I.I.T. 81)

Solution We have

$$\int_0^1 (tx+1-x)^n\,dx=\left[\frac{1}{t-1}\cdot\frac{(tx+1-x)^{n+1}}{n+1}\right]_0^1$$

$$=\frac{t^{n+1}-1}{(t-1)(n+1)} \qquad (1)$$

For the second part, we write (1) as

$$\int_0^1 (tx+1-x)^n\,dx=\frac{1}{n+1}\left[1+t+t^2+\dots+t^{k-1}+t^k+t^{k+1}+\dots+t^n\right] \qquad (2)$$

Differentiating (2), k times w.r.t. t we get

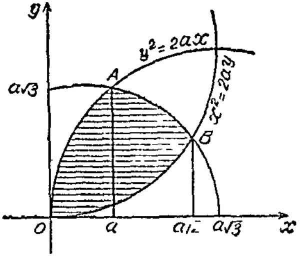
$\int_0^1 n(n-1)\dots(n-k+1)\,x^k(tx+1-x)^{n-k}\,dx$

$=\dfrac{1}{n+1}[k!+$ terms containing t and its higher powers$]$

Putting $t=0$ in the above identity, we get

14 Find the area of the figure which lies in the first quadrant inside the circle $x^2+y^2=3a^2$ and bounded by the parabolas $x^2=2ay$ and $y^2=2ax$ $(a>0)$

Solution Solving the equations $x^2+y^2=3a$, $y^2=2ax$ we find the positive value of the abscissa of their points of intersection A as $x_A=a$. Similarly we find the abscissa of the point B of the intersection of $x^2+y^2=3a^2$, $x^2=2ay$ as $x_B=a\sqrt{2}$



Thus, the required area

$$\int_0^{a\sqrt{2}} (y_2-y_1)\,dx,$$

where $y_1=\dfrac{x^2}{2a}$,

$$y_2=\begin{cases}\sqrt{(2ax)} \text{ for } 0\leqslant x\leqslant a\\ \sqrt{(3a^2-x^2)} \text{ for } a<x\leqslant a\sqrt{2}\end{cases}$$

$$=\int_0^a\left(\sqrt{(2ax)}-\frac{x^2}{2a}\right)dx+\int_a^{a\sqrt{2}}\left[\sqrt{(3a^2-x^2)}-\frac{x^2}{2a}\right]dx$$

$$=\left[\sqrt{(2a)}\ \frac{2}{3}\ x^{3/2}-\frac{x^3}{6a}\right]_0^a$$

$$+\left[\tfrac{1}{2}x\ \sqrt{(3a^2-x^2)}+\tfrac{1}{2}.3a^2\ \sin^{-1}\frac{x}{a\sqrt{3}}-\frac{x^3}{6a}\right]_a^{a\sqrt{2}}$$

Hence $\int_a^x f(t)\,dt$ is an even function if $f(t)$ is an odd function

(*b*) Ans No if $a \neq 0$ yes if $a=0$

23 Given a function $f(x)$ such that

(i) it is integrable over every interval on the real line

(ii) $f(t+x)=f(x)$ for every x and real t

Then show that the integral $\int_a^{a+t} f(x)\,dx$ is independent of a

(IIT 84)

Solution Let $\int f(x)\,dx = F(x)+c$

Then $F'(x)=f(x)$

Now $I=\int_a^{a+t} f(x)\,dx = F(a+t)-F(a)$

$$\frac{dI}{da}=F'(a+t)-F'(a)=f(a+t)-f(a)$$

$=f(a)-f(a)$ by condition (ii)

$=0$

This shows that I is independent of a

Alternative $\int_a^{a+t} f(x)\,dx=\int_a^0 f(x)\,dx+\int_0^t f(x)\,dx$

$+\int_t^{a+t} f(x)\,dx$

In the last integral, put $x=t+u$ so that $dx=du$

Then $\int_t^{a+t} f(x)\,dx=\int_0^a f(t+u)\,du=\int_0^a f(u)\,du$

$=\int_0^a f(x)\,dx \qquad f(t+u)=f(u)$

Hence $\int_a^{a+t} f(x)\,dx=-\int_0^a f(x)\,dx+\int_0^t f(x)\,dx$

$+\int_0^a f(x)\,dx=\int_0^t f(x)\,dx$ which is independent of a

24 It is known that $f(x)$ is an odd function in the interval $[-\frac{1}{2}T, \frac{1}{2}T]$ and has a period equal T Prove that $\int_a^x f(t)\,dt$ is also a periodic function with the same period

Solution We have to prove

$$\int_a^x f(t)\,dt=\int_a^{x+T} f(t)\,dt$$

or $\int_a^x f(t)\,dt-\int_a^{x+T} f(t)\,dt=0$

$$=\log\sqrt{2}-\log 1+\log 1-\log\frac{1}{\sqrt{2}}$$

$$=2\log\sqrt{2}=\log 2 \quad [\ \log 1=0,\ \log\frac{1}{\sqrt{2}}=-\log\sqrt{2}]$$

6 Find the area of one of the curvilinear triangles bounded by the x axis and the curves $y=\sin x$ and $y=\cos x$

Ans $2-\sqrt{2}$

7 Compute the area of the curvilinear triangle bounded by the y axis and the curves $y=\tan x$ and $y=\frac{2}{3}\cos x$

Ans $\frac{1}{3}+\log\left(\frac{\sqrt{3}}{2}\right)$

8 Find the area bounded by the curves $x^2+y^2=25$, $4y=|4-x^2|$ and $x=0$ above the x axis (I I T 87)

$|4-x^2|=4-x^2$ when $4-x^2>0$ i e $x^2<4$

$4y=4-x^2$ or $x^2=-4(y-1)$ for $-2\leqslant x\leqslant 2$

Above represents a parabola with vertex at (0, 1) symmetrical about y axis

$|4-x^2|=-(4-x^2)$ when $4-x^2<0$ i e $x^2>4$

$4y=x^2-4$ or $x^2=4(y+1)$ $x\geqslant 2$, or $x\leqslant -2$

Above represents a parabola with vertex at $(0, -1)$ symmetrical about y axis

Thus we have three curves

I circle $x^2+y^2=25$

II Parabola $x^2=-4(y-1)$ for $-2\leqslant x\leqslant 2$

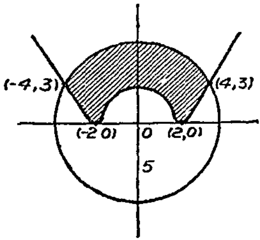
III Parabola $x^2=4(y+1)$ for $x\geqslant 2$ or $x\leqslant -2$

I and II intersect at $-4y+4+y^2=25$

or $(y-2)^2=5^2$ $\quad y-2=\pm 5 \quad y=7, y=-3$ $y=7$ is rejected $y=-3$ is below x axis

I and III intersect at $4y+4+y^2=25$ or $(y+2)^2=5^2$

$y+2=\pm 5 \qquad y=3, \ -7$



3 Area between two curves $y=f(x)$ and $y=\phi(x)$ and the two ordinates drawn at the points $x=a$ and $x=b$, then

$$A=\int_a^b y_1\,dx-\int_a^b y_2\,dx$$

Upper Lower

$$=\int_a^b (f(x)-\phi(x))\,dx$$

where y_1 is the ordinate of $y=f(x)$ which is upper curve and y_2 is the ordinate of $y=\phi(x)$ which is lower curve

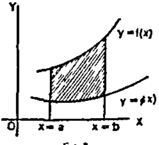


Fig. 2

It is shown by shaded area in the figure above

Solved Examples

1 Find the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

The required area is four times the area in the first quadrant shown shaded in the figure This is the area between the curve, x axis and two ordinates drawn at the points $x=0$ and $x=a$

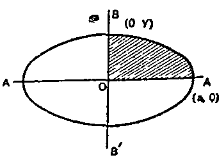


Fig 3

$$A=4\int_0^a y\,dx$$

From the equation of the ellipse we shall find the value of y in terms of x

$$\frac{y^2}{b^2}=1-\frac{x^2}{a^2}=\frac{a^2-x^2}{a^2} \qquad y=\frac{b}{a}\sqrt{(a^2-x^2)}$$

$$A=4\,\frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\,dx$$

$$=4\,\frac{b}{a}\left[\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$=4\,\frac{b}{a}\left[0+\frac{a^2}{2}(\sin^{-1}1-\sin^{-1}0)\right]$$

$$=4\,\frac{b}{a}\cdot\frac{a^2}{2}\left(\frac{\pi}{2}\right)=\pi ab \text{ sq units}$$

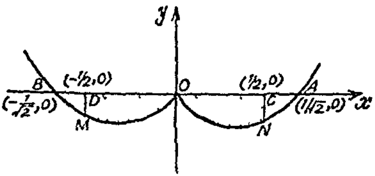
Deduction Area of Circle

Putting $b=a$ the ellipse becomes circle $x^2+y^2=a^2$ and hence its area will be πa^2 sq units

Solution The shape of the curve is as shown

$$y=2x^4-x^2$$

$$\frac{dy}{dx}=8x^3-2x=0 \text{ for max or min}$$



This gives $x=0,\ x=\pm\frac{1}{2}$

$$\frac{d^2y}{dx^2}=24x^2-2>0 \text{ at } x=\pm\frac{1}{2}$$

Hence y is min when $x=\pm\frac{1}{2}$

So the required area is as shown shaded in the figure

$$\text{Area}=\int_{-1/2}^{1/2} y\,dx=\int_{-1/2}^{1/2}(2x^4-x^2)\,dx=2\int_0^{1/2}(2x^4-x^2)\,dx$$

$$=2\left[2\frac{x^5}{5}-\frac{x^3}{3}\right]_0^{1/2}=2\left[\frac{2}{5}\,\frac{1}{32}-\frac{1}{3}\,\frac{1}{8}\right]$$

$$=\frac{1}{40}-\frac{1}{12}=-\frac{7}{120}$$

So area $=\frac{7}{120}$

21 *Find the area of the region bounded by the curve C $y=\tan x$ tangent drawn to C at $x=\pi/4$ and the x axis* **(IIT 88)**

Solution The shape of the curve is as shown in the adjoining figure The required area is as shown shaded in the figure and is bounded by the part OP of the curve $y=\tan x$, the tangent PT at P and the part OT of x axis We now first find the equation of tangent at P We have

$$\frac{dy}{dx}=\cot y=1 \quad \text{at} \quad x=\pi/4$$

Also $y=\tan\frac{\pi}{4}=1$ at P

we get, $\left(\frac{x^2}{4b}\right)^2 = 4ax$

or $x^4 - 64b^2 ax = 0$ or $x(x^3 - 64b^2 a) = 0$

$x = 0,\ x = 4a^{1/3}b^{2/3} = h$

Required area = Shaded area

$$= \int_0^h (y_1 - y_2)\, dx \qquad (1)$$

where y_1 is the ordinate of the upper parabola $y^2 = 4ax$

$$y_1 = 2\sqrt{a}\sqrt{x}$$

y_2 is the ordinate of the lower parabola $x^2 = 4ay$

$$y_2 = \frac{x^2}{4b}$$

Putting for y_1 and y_2 in (1) the required area is

$$A = \int_0^h \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right) dx$$

$$= \left[2\sqrt{a}\ \frac{2}{3}\ x^{3/2} - \frac{1}{4b}\ \frac{x^3}{3}\right]_0^h$$

$$= \left[4\ \frac{\sqrt{a}}{3}\ h^{3/2} - \frac{1}{12b}\ h^3\right] \quad \text{Now put } h = 4a^{1/3}b^{2/3}$$

$$= \frac{4}{3}\sqrt{a}\ (4a^{1/3}\ b^{2/3})^{3/2} - \frac{1}{12b}\ (4a^{1/3}b^{2/3})^3$$

$$= \frac{4}{3}\sqrt{a}\ (8a^{1/2}b) - \frac{1}{12b}\ 64ab^2$$

$$= \frac{32}{3}\ ab - \frac{16}{3}\ ab = \frac{16}{3}\ ab$$

Note In case the two parabolas be $y^2 = 4ax$ and $x^2 = 4ay$ then putting $b = a$ the required area is $\frac{16}{3}a^2$

Ex 5 Find the area between the parabola $y^2 = 4ax$ and the line $y = mx$

Fig 6

The line meets the parabola where

$(mx)^2 = 4ax$ or $x(m^2x - 4a) = 0$

$$x = 0,\ x = \frac{4a}{m^2} = h \text{ say}$$

The required area $= \int_0^h (y_1 - y_2)\, dx$ (1)

where y_1 is the ordinate of the upper curve i.e. parabola $y^2 = 4ax$

$$y_1 = 2\sqrt{a}\sqrt{x}$$

$$=2\int_0^{\pi/2}\sin x\,dx, \qquad |\sin x|=\sin x$$

on the interval $\left[0,\frac{\pi}{2}\right]$

$$=2\left(-\cos x\right)_0^{\pi/2}=2$$

8 Ans (*A*)

[Hint If $f(\theta)=\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)$, then

$$f(-\theta)=\log\left(\frac{2+\sin\theta}{2-\sin\theta}\right)=\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)^{-1}$$

$$=-\log\left(\frac{2-\sin\theta}{2+\sin\theta}\right)=-f(\theta)$$

Hence by prop 5 of § 7, the value of given integral is zero]

9 Ans $\frac{1}{\pi^2}(\pi\sqrt{2}+4\sqrt{2}-8)$

[Hint If $f(x)=|x\cos\frac{1}{2}\pi x|$, then

$f(-x)=|-x\cos\frac{1}{2}\pi(-x)|=|-x\cos\frac{1}{2}\pi x|$

$=|x\cos\frac{1}{2}\pi x|=f(x)$

Hence by prop 5 of § 7 we have

$$I=2\int_0^{1/2}|x\cos\tfrac{1}{2}\pi x|\,dx$$

$$=2\int_0^{1/2}x\cos\tfrac{1}{2}\pi x\,dx$$

$$=2\left[\frac{x\sin\frac{1}{2}\pi x}{\frac{1}{2}\pi}+\frac{\cos\frac{1}{2}\pi x}{\frac{1}{4}\pi^2}\right]_0^{1/2}$$

$$=2\left[\frac{2}{\pi}\cdot\frac{1}{2\sqrt{2}}+\frac{4}{\pi^2}\cdot\frac{1}{\sqrt{2}}-\frac{4}{\pi^2}\right]$$

$$=\frac{1}{\pi^2}(\pi\sqrt{2}+4\sqrt{2}-8)$$

10 Ans $\frac{\pi}{2}$

11 Ans $\frac{1}{2\sqrt{(x^2-1)}}\log\frac{t+x-\sqrt{(x^2-1)}}{t+x+\sqrt{(x^2-1)}}+c$

12 $I=\int\frac{x^{1/2}}{\sqrt{(1+x^3)}}\,dx,$ Put $x^{3/2}=t$ so that $\frac{3}{2}x^{1/2}\,dx=dt$

$$=\frac{2}{3}\int\frac{dt}{\sqrt{(1+t^2)}}=\frac{2}{3}\log t+\sqrt{(1+t^2)}$$

$$=\tfrac{2}{3}\log\{x^{3/2}+\sqrt{(1+x^3)}\}+c$$

$$\frac{y_2}{b}=1-\frac{x}{a},\ i\ e\ \ y_2=\frac{b}{a}(a-x)$$

Putting for y_1 and y_2 in (1) we get

$$A=\frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\,dx-\frac{b}{a}\int_0^a (a-x)\,dx$$

$$=\frac{b}{a}\left[\left\{\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}_0^a-\left\{ax-\frac{x^2}{2}\right\}_0^a\right]$$

$$=\frac{b}{a}\left[\left(0+\frac{a^2}{2}\sin^{-1}1\right)-\left\{a^2-\frac{a^2}{2}\right\}\right]$$

$$=\frac{b}{a}\left[\frac{a^2}{2}\frac{\pi}{2}-\frac{a^2}{2}\right]=\frac{b}{a}\frac{a^2}{4}(\pi-2)$$

$$=\tfrac{1}{4}ab\,(\pi-2)$$

Ex 7 Find the area cut off the parabola $4y=3x^2$ by the straight line $2y=3x+12$

Eliminating y we get

$$2\,(3x+12)=3x^2$$

or $\quad 3x^2-6x-24=0,\ 3x^2-12x+6x-24=0$

$$3x\,(x-4)+6\,(x-4)=0 \text{ or } (x-4)\,(3x+6)=0$$

$$\therefore\ x=-2,\ x=4$$

$$y=3\ \ y=12,$$

i e the points of intersection are $P\,(-2, 3)$ and $(4, 12)$

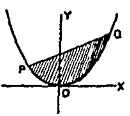


Fig 8

The required area $=\int_{x\ for\ P}^{x\ for\ Q}(y_1-y_2)\,dx$

y_1 is the ordinate for upper curve *i e* straight line $2y=3x+12$

$$y_1=\frac{3x+12}{2}$$

y_2 is the ordinate for the lower curve *i e* parabola $4y=3x^2$

$$y_2=\frac{3x^2}{4}$$

$$\therefore\ A=\int_{-2}^{4}\left(\frac{3x+12}{2}-\frac{3x^2}{4}\right)dx$$

$$=\left[\frac{1}{2}\left(3\,\frac{x^2}{2}+12x\right)-\frac{3x^3}{12}\right]_{-2}^{4}=\left[\frac{3}{4}x^2+6x-\frac{1}{4}x^3\right]_{-2}^{4}$$

$$=\tfrac{3}{4}\,(16-4)+6\,(4+2)-\tfrac{1}{4}\,(64+8)$$

$$=\tfrac{3}{4}\ 12+6\ 6-\tfrac{1}{4}\ 72$$

$$=9+36-18=27 \text{ sq units}$$

IV
VECTORS

$$=2\int_0^{2a} \sqrt{(6a)}\,\sqrt{x}\,dx$$

$$+2\int_{2a}^{4a} \sqrt{[(4a)^2-x^2]}\,dx$$

$$=2\sqrt{(6a)}\,\frac{2}{3}\left[x^{3/2}\right]_0^{2a}+2\left[\frac{x}{2}\sqrt{[(4a)^2-x^2]}+\frac{1}{2}(4a)^2\sin^{-1}\frac{x}{4a}\right]_{2a}^{4a}$$

$$=2\sqrt{(6a)}\,\tfrac{2}{3}\,2a\sqrt{(2a)}+2\,[(0-a\,2\sqrt{3}a)+8a^2(\sin^{-1}1-\sin^{-1}\tfrac{1}{2})]$$

$$=2\,2\sqrt{3}\,\tfrac{4}{3}a^2+2\,[-2\sqrt{3}a^2+8a^2(\pi/2-\pi/6)]$$

$$=\frac{16}{3}\sqrt{3}a^2-4\sqrt{3}a^2+16a^2\pi/3$$

$$=\frac{4\sqrt{3}a^2}{3}+\frac{16\pi a^2}{3}=\frac{4a^2}{3}(4\pi+\sqrt{3})$$

Larger area = Area of circle − Common area

$$=\pi(4a)^2-\frac{4a^2}{3}(4\pi+\sqrt{3})$$

$$=16\pi a^2-\frac{4a^2}{3}(4\pi+\sqrt{3})$$

$$=\frac{4a^2}{3}[12\pi-4\pi-\sqrt{3}]$$

$$=\frac{4a^2}{3}(8\pi-\sqrt{3}$$

Ex 10 Find the area inside the parabola $5x^2-y=0$ but outside the parabola $2x^2-y+9=0$

Eliminating y we get

$$5x^2-(2x^2+9)=0$$

or $\quad 3x^2=9, \qquad x=-\sqrt{3}, \sqrt{3}$

Required area $=\int_{-\sqrt{3}}^{\sqrt{3}} (y_1-y_2)\,dx$ (1)

In order to decide about the values of y_1 and y_2 we have to decide about the upper and lower curves

Now any point between $x=-\sqrt{3}$ and $x=\sqrt{3}$ is $x=0$

For $x=0$, $y=0$ for $5x^2-y=0$

$y=9$ for $2x^2-y+9=0$

The ordinate of $2x^2-y+9=0$ i e 9 is greater than ordinate of $5x^2-y=0$ *i e*

Hence $2x^2-y+9=0$ is upper curve and *is lower*

$$y_1=2x^2+9$$

Therefore from (1)

1
Addition of Vectors

§ 1 Definitions

(i) Scalars and Vectors Those quantities which have only magnitude and are not related to any direction in space are called scalars whereas those which have both magnitude and direction are called vectors quantities The speed of a train is scalar quantity as it is not associated with any direction, it simply gives the rate of motion but its velocity or acceleration is a vector quantity as it gives both the magnitude and direction

(ii) Representation and notation of vectors

Symbolically a vector is often denoted by two letters with an arrow over them i e $\overrightarrow{AB}$ A is called the origin and B the terminus Its magnitude is given by the length AB and direction is from A to B as indicated by the arrow We write vector quantities also in single latter notation like **a**, **b** **c** and the corresponding letters a, b, c denote their magnitudes Thus if $\overrightarrow{AB}=\mathbf{a}$ then $|\overrightarrow{AB}|=a$, where $|\overrightarrow{AB}|$ means the magnitude of vector **a**

Since it is difficult to write bold type letters **a**, **b**, **c** we may use $\bar{a}, \bar{b}, \bar{c}$ i e place a bar over ordinary letters a, b, c or else we may take α, β, γ to represent the vectors and a, b, c to represent their mignitudes Another alternative is to use A, B, C letters for vectors and small letters a, b, c to represent their magnitudes

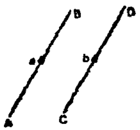
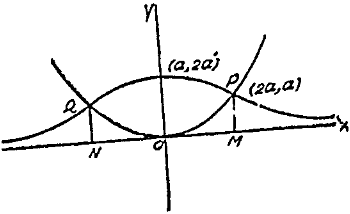


Fig. 1

(iii) Like and unlike vectors Vectors are said to be like when they have the same sense of direction and unlike when they have opposite directions



Required common area

$$2\left[\int_0^{2a} y_1\,dx-\int_0^{2a} y_2\,dx\right]=2\left[\int_0^{2a}\frac{8a^3}{x^2+4a^2}\,dx-\int_0^{2a}\frac{x^2}{4a}\,dx\right]$$

curve parabola

$$=2\left[8a^3\,\frac{1}{2a}\left\{\tan^{-1}\frac{x}{2a}\right\}_0^{2a}-\frac{1}{4a}\left\{\frac{x^3}{3}\right\}_0^{2a}\right]$$

$$=2\left[4a^2\,\frac{\pi}{4}-\frac{1}{12a}\,8a^3\right]$$

$$=2\left[\pi a^2-\frac{2}{3}a^2\right]=a^2\left[2\pi-\frac{4}{3}\right]$$

(b) The given equation is $x\,(y^2+4a^2)=8a^3$

Its shape as discussed in part (a) is as shown in the adjoining figure

$$A=2\int_0^{\infty} x\,dy=2\int_0^{\infty}\frac{8a^3}{y^2+4a^2}\,dy$$

$$=16a^3\,\frac{1}{2a}\left[\tan^{-1}\frac{y}{2a}\right]_0^{\infty}=8a^2\left[\frac{\pi}{2}-0\right]=4\pi a^2$$

(xiii) Addition of Vectors

Let a and b any two vectors. Choose any point O as origin and draw the vectors a and b so that the terminus of a coincides with the origin of b i.e. $\vec{OA}=$ a and $\vec{AB}=$ b

Then the vector given by $\vec{OB}$ is defined as the sum of the vectors a and b

The above law is called triangle law of addition. Since $\vec{OC}=\vec{AB}=b$, we can also say

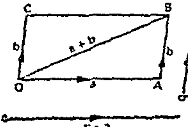


Fig. 2

$\vec{OA}+\vec{OC}=\vec{OB}$, which is known as parallelogram law of addition

Note It should be noted that the magnitude of a+b is not equal to the sum of the magnitudes of a and b

(xiv) Properties

(i) a+b=b+a (Commutative law)

(ii) (a+b)+c=a+(b+c) (Asssociative law)

(xv) Sum of any number of Vectors

If we are to find the sum of any number of vectors a, b, c, d, e, say then form a broken line whose segments in length and direction represent these vectors. Then the vector joining the origin of the first vector to the terminal point of the last vector will represent the vector sum

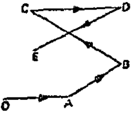


Fig. 3

$$a+b+c+d+e=\vec{OA}+\vec{AB}+\vec{BC}+\vec{CD}+\vec{DE}=\vec{OD}$$

In particular if the terminal point of last vector coincides with the origin of the first vector, then the sum will be the zero vector

Thus $\vec{OA}+\vec{AC}+\vec{BC}+\vec{CD}$

$+\vec{DO}=\vec{OO}=0$

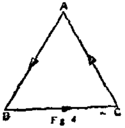


Fig. 4

Similarly,

$$\vec{AB}+\vec{BC}+\vec{CA}=\vec{AA}=0$$

$$=2\left[x(-\cos x)+\int \cos x \, 1 \, dx\right]$$

$$=2\left[-x\cos x+\sin x\right]_0^{\pi}=2\pi$$

9 Find the area of the surface generated by rotating the circle $x=b\cos\theta$, $y=a+b\sin\theta$, $0\leqslant\theta\leqslant 2\pi$ about x axis. (Roorkee 81)

$A=$ Area of the surface $=2\pi\int_0^{2\pi} y\frac{ds}{d\theta}$

Now $\frac{dx}{d\theta}=-b\sin\theta$, $\frac{dy}{d\theta}=b\cos\theta$ so that $\frac{ds}{d\theta}$

$$=\sqrt{(b^2\sin^2\theta+b^2\cos^2\theta)}=b.$$

Hence $A=2\pi\int_0^{2\pi}(a+b\sin\theta)\; b\,d\theta$

$$=2\pi b\left[a\theta-b\cos\theta\right]_0^{2\pi}$$

$$=2\pi b\,[(2a\pi-0)-b(\cos 2\pi-\cos 0)]$$

$=4\pi^2 ab$ Since $[\cos 2\pi=\cos 0=1]$

10 For any real t, $x=\frac{1}{2}(e^t+e^{-t})$, $y=\frac{1}{2}(e^t-e^{-t})$ is a point on the hyperbola $x^2-y^2=1$. Show that the area bounded by the hyperbola and the lines joining its centre to the points corresponding to t_1 and $-t_1$ is t_1 (I I T 82)

[Hint Let $x_1=\cosh t_1$, $y_1=\sinh t_1$, then $(x_1\; y_1)$ represents point P on hyperbola $x^2-y^2=1$ whose centre is the origin O and let PM be perpendicular on x axis Then required area

$$=2\left[\Delta\, OPM-\int_0^{t_1} y\frac{dx}{dt}\,dt\right]$$

$$=2\left[\tfrac{1}{2}x_1 y_1-\int_0^{t_1}\sinh t\sinh t\,dt\right]$$

$$=2\left[\tfrac{1}{2}\cosh t_1\sinh t_1-\tfrac{1}{2}\int_0^{t_1}(\cosh 2t-1)\,dt\right]$$

$$=\sinh t_1\cosh t_1-\left[\frac{\sinh 2t_1}{2}-t_1\right]$$

$$=\sinh t_1\cosh t_1-\sinh t_1\cosh t_1+t_1=t_1$$

11 Find the area bounded by the curve $y=(x-1)(x-2)(x-3)$ lying between the ordinates $x=0$ and $x=3$ (Roorkee 86)

Solution

In the question, it must also be mentioned, that the area is bounded by the curve, lying between the ordinates $x=0$, $x=3$ and x-axis So we have to find the shaded area Now let

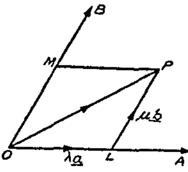
$$F(x)=\int(x-1)(x-2)(x-3)\,dx$$

which shows that a and b are collinear But this is contrary to the hypothesis Hence $\lambda=0$ Similary we can show that $\mu=0$

§ 3 **Resolution of Vectors**

(i) Coplanar Vectors

Theorem I *Let* a b *be two* non-zero non collinear *vectors and* r *any vector coplanar with* a *and* b *Then* r *can be represented uniquely as combination of* a *and* b



Proof Let $\vec{OA}=$ a, $\vec{OB}=$b,

and $\vec{OP}=$r

The lines OA, OB and OP are coplanar

Through the point P draw lines parallel to OA and OB to meet OB and OA in M and L respectively Then we have

$$\mathbf{r}=\vec{OP}=\vec{OL}+\vec{LP}=\vec{OL}+\vec{OM}=\lambda\mathbf{a}+\mu\mathbf{b}$$

[$\vec{OL}$, $\vec{OM}$ are collinear with a and b, we can find scalars λ and μ such that $\vec{OL}=\lambda\mathbf{a}$ and $\vec{OM}=\mu\mathbf{b}$]

Uniqueness Suppose, if possible, there is another representation of r as $\mathbf{r}=\lambda'\mathbf{a}+\mu\mathbf{b}$

Then $\lambda\mathbf{a}+\mu\mathbf{b}=\lambda\mathbf{a}+\mu'\mathbf{b}$

or $(\lambda-\lambda')\mathbf{a}+(\mu-\mu')\mathbf{b}=0$

Since a and b are non collinear vectors, we have $\lambda-\lambda'=0$ and $\mu-\mu'=0$ *i e* $\lambda=\lambda'$ and $\mu=\mu'$ Thus the resolution of r as a linear combination of a and b is unique

(ii) Non Coplanar Vectors

Theorem II If a, b c are non-coplanar vectors and λ, μ, ν are scalars such that

$$\lambda\mathbf{a}+\mu\mathbf{b}+\nu\mathbf{c}=0,$$

then $\lambda=\mu=\nu=0$

Proof Suppose $\lambda\neq0$ Then dividing by λ, the given relation can be written as

$$\mathbf{a}=-\frac{\mu}{\lambda}\mathbf{b}-\frac{\nu}{\lambda}\mathbf{c},$$

which shows that a is coplanar with b and c But this is contrary to the hypothesis Hence we must have $\lambda=0$ Similarly we can prove that $\mu=0$ and $\nu=0$

The required area

$= \int_0^2 (y_1 - y_2)\, dx$

where $y_1 = 2^x$ and $y_2 = 2x - x^2$

$= \int_0^2 (2^x - 2x + x^2)\, dx$

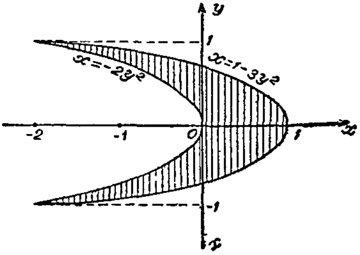
$= \left[\frac{2^x}{\log 2} - x^2 + \frac{1}{3} x^3\right]_0^2 = \left(\frac{4}{\log 2} - 4 + \frac{8}{3}\right) - \frac{1}{\log 2}$

$= \frac{3}{\log 2} - \frac{4}{3}$

13 Compute the area of the figure bounded by the parabolas $x = -2y^2$, $x = 1 - 3y^2$

Solution Solving the equations $x = -2y^2$, $x = 1 - 3y^2$ we find that ordinates of the points of intersection of the two curves as $y_1 = -1$, $y_2 = 1$

Since $1 - 3y^2 \geqslant -2y^2$ for $-1 \leqslant y \leqslant 1$, the required



area $= \int_{-1}^{1} (x_1 - x_2)\, dy$, where $x_1 = 1 - 3y^2$, $x_2 = -2y^2$

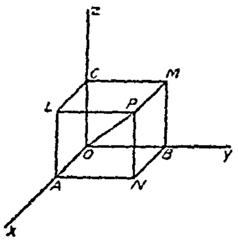
$= \int_{-1}^{1} (1 - 3y^2 + 2y^2)\, dy = \int_{-1}^{1} (1 - y^2)\, dy$

$= 2 \int_0^1 (1 - y^2)\, dy = 2\left[y - \frac{1}{3} y^3\right]_0^1$

$= 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$,

§ 4 The unit vectors i, j, k (Orthonormal system of unit vectors)

Let OX, OY and OZ be three mutually perpendicular straight lines in the right-handed orientation This statement means that



when one rotates from OX to OY, then OZ lies in the direction in which a right handed screw will advance These three mutually perpendicular lines can uniquely determine the position of a point Hence these lines can be taken as the coordinate axes with O as origin The planes XOY, YOZ, and ZOX are called co-ordinate planes

Let **i**, **j**, **k** denote unit vectors along OX OY, OZ respectively

Let $\overrightarrow{OP}$ represent a vector **r** With OP as diagonal construct a rectangular parallelopiped whose three coterminous edges OA, OB OC lie along OX, OY and OZ respectively Let $OA=x$, $OB=y$ and $OC=z$ Then $\overrightarrow{OA}=x\mathbf{i}$, $\overrightarrow{OB}=y\mathbf{j}$ and $\overrightarrow{OC}=z\mathbf{k}$ Now we have

$$\mathbf{r}=\overrightarrow{OP}=\overrightarrow{ON}+\overrightarrow{NP}=\overrightarrow{OA}+\overrightarrow{AN}+\overrightarrow{NP}$$

$$=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

Thus $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$

Here x, y, z are called the coordinates of the point P referred to the axes OX, OY and OZ Also $x\mathbf{i}$, $y\mathbf{j}$, $z\mathbf{k}$ are called the resolved parts of the vector **r** in the directions of **i** **j** and **k** respectively

Direction cosines If α β, γ are the angles which OP makes with the coordinate axes OX, OY and OZ respectively, then $\cos\alpha$ $\cos\beta$ and $\cos\gamma$ are called the direction cosines (d c 's) of the line OP These d c's are usually denoted by l, m, n

$$= \frac{2\sqrt{2}}{3}a^2 - \frac{a^2}{6} + \left[\frac{a\sqrt{2}}{2}a + \frac{3}{2}a^2 \sin^{-1}\frac{\sqrt{2}}{\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}}a^2\right.$$

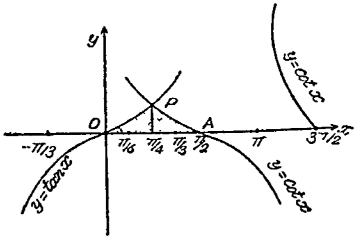
$$\left. - \frac{\sqrt{2}}{2}a^2 - \frac{3}{2}a^2 \sin^{-1}\frac{1}{\sqrt{3}} + \frac{a^2}{6}\right]$$

$$= \frac{\sqrt{2}}{3}a^2 + \frac{3}{2}a^2 \sin^{-1}\left\{\frac{\sqrt{2}}{\sqrt{3}}\sqrt{\left(1-\frac{1}{3}\right)} - \sqrt{\left(1-\frac{2}{3}\right)}\frac{1}{\sqrt{3}}\right\}$$

$$= \left(\frac{\sqrt{2}}{3} + \frac{3}{2}\sin^{-1}\frac{1}{3}\right)a^2$$

15 Find the area of the region bounded by the x axis and the curves defined by $y=\tan x$, $-\frac{1}{3}\pi \leqslant x \leqslant \frac{1}{3}\pi$, $y=\cot x$, $\frac{1}{6}\pi \leqslant x \leqslant \frac{3}{2}\pi$ (IIT 84)

Solution The curves intersect at P, where $\tan x=\cot x$, which is satisfied at $x=\pi/4$ within the given domain of x. The required area POA is given by



$$A = \int_0^{\pi/4} y_1\,dx + \int_{\pi/4}^{\pi/2} y_2\,dx,$$

where $y_1=\tan x$ and $y_2=\cot x$.

$$= \int_0^{\pi/4} \tan x\,dx + \int_{\pi/4}^{\pi/2} \cot x\,dx$$

$$= \Big[\log \sec x\Big]_0^{\pi/4} + \Big[\log \sin x\Big]_{\pi/4}^{\pi/2}$$

(iii) **Method to prove four points to be coplanar**—

(a) **First Method** *To prove that the four points A, B, C, D* coplanar find the vectors $\vec{AB}$, $\vec{AC}$ and $\vec{AD}$ and then prove n to be coplanar by the method given above

(b) *Second Method To prove that the points A, B C, D* position vectors **a**, **b** **c**, **d** are coplanar, use the following rem

Four points with position vectors **a**, **b**, **c** **d** *are coplanar if and only if there exist scalars x y, z, u not zero such that*

(i) $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+u\mathbf{d}=0$ (ii) $x+y+z+u=0$

For proof of this theorem see problem 31

(iv) **Ratio Formula** If A B are points **a** and **b** *i e* $\vec{OA}=\mathbf{a}$, $=\mathbf{b}$ then the point P which divides AB in the ratio m n is n by

$$\mathbf{r}=\frac{m\mathbf{b}+n\mathbf{a}}{m+n}$$

Mid point of AB is $\frac{\mathbf{a}+\mathbf{b}}{2}$

Centroid of triangle ABC is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

re **a** **b** **c** are the position vectors of the vertices w r t, to an gin O

Problem Set (A)

If **a**, **b**, are the vectors forming consecutive sides of a regular hexagon *ABCD*, *express the vectors* $\vec{CD}$, $\vec{DE}$, $\vec{EF}$, $\vec{FA}$, $\vec{AC}$ $\vec{AD}$, $\vec{AE}$ and $\vec{CE}$ in terms of **a** and **b**

Five forces $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ $\vec{AE}$, $\vec{AF}$ act at the vertex A of a regular hexagon *ABCDEF* Prove that their resultant is a $6\,\vec{AO}$ where O is the centroid of the hexagon

ABCDE is a pentagon, prove that the resultant of forces $\vec{AB}$, $\vec{AE}$, $\vec{BC}$, $\vec{DC}$, $\vec{ED}$ and $\vec{AC}$ is $3\,\vec{AC}$

Prove that the resultant of two forces acting at a point O and *represented by* $\vec{OB}$ and $\vec{OC}$ *is given by* $2\,\vec{OD}$ *where D is the* mid point of BC

$y=-7$ is rejected $y=3$ gives the point above x axis
When $y=3$, $x=\pm 4$ Hence the points of intersection of I and III are (4, 3) and (−4, 3) Thus we have the shape of the curve as under

Required Area

$$=2\left[\int_0^4 y\,dx-\int_0 y\,dx-\int_-^4 y\,dx\right]$$
I II III

$$2\left[\int_0^4\sqrt{(25-x^2)}\,dx-\tfrac{1}{4}\int_0^2(4-x^2)\,dx-\tfrac{1}{4}\int_2^4(x^2-4)\,dx\right]$$

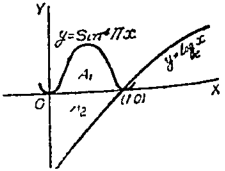
$$2\left[\left\{\frac{x}{2}\sqrt{(25-x^2)}+\frac{25}{2}\sin^{-1}\frac{x}{5}\right\}_0^4-\tfrac{1}{4}\left(4x-\frac{x^3}{3}\right)_0^2-\tfrac{1}{4}\left(\frac{x^3}{3}-4x\right)_2^4\right]$$

$$=25\sin^{-1}4/5+4$$

19 Find the area of the region bounded by the curves $y=\log_e x$, $y=\sin^4\pi x$ and $x=0$ (Roorkee 87)

$$A_1=\int_0^1 y\,dx=\int_0^1\sin^4\pi x\,dx$$

Put $\pi x=t$



$$A_1=\frac{1}{\pi}\int_0^\pi\sin^4 t\,dt$$

$$=\frac{2}{\pi}\int_0^{\pi/2}\sin^4 t\,dt$$

$$=\frac{2}{\pi}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{3}{8}$$

$$A=\int_0^1 y\,dx=\int_0^1\log x\,dx$$

$$=x\log x-\int x\cdot 1/x\,dx=\Big[x\log x-x\Big]_0^1$$

$$(0-1)-(0-0)=-1 \qquad |A|=1$$

$$\text{Required area}=\frac{3}{8}+1=\frac{11}{8}$$

Note $\underset{x\to 0}{\text{Lt}}\ x\log x=0\times\infty=\underset{x\to 0}{\text{Lt}}\ \dfrac{\log x}{1/x}=\dfrac{\infty}{\infty}$

$$=\underset{x\to 0}{\text{Lt}}\ \frac{1/x}{-1/x^2}=\underset{x\to 0}{\text{Lt}}\ -x=0$$

20 Find the area between the curve $y=2x^4-x^2$, the x axis and the ordinates of the two minima of the curve [Roorkee 88]

11 If O is the circumcentre and O' the orthocentre of a triangle ABC then prove that

(i) $\vec{OA}+\vec{OB}+\vec{OC}=\vec{OO'}$

(ii) $\vec{O'A}+\vec{O'B}+\vec{O'C}=2\vec{O'O}$

(iii) $\vec{AO'}+\vec{O'B}+\vec{O'C}=2\vec{AO}=\vec{AP}$
where AP is the diameter of circum circle

12 ABC is a triangle and P any point in BC If $\vec{PQ}$ is the resultant of $\vec{AP}$, $\vec{PB}$ and $\vec{PC}$, show that $ABQC$ is a parallelogram and Q therefore a fixed point

13 If two concurrent forces be represented by $n\,\vec{OP}$ and $m\,\vec{OQ}$ respectively, prove that their resultant is given by $(m+n)\vec{OR}$ where R divides PQ such that $n\,PR=m\,RQ$

14 (a) Forces P, Q act at O and have a resultant R If any transversal cuts their lines of action at A, B and C respectively Prove that

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}$$

(b) A transversal cuts the sides OL, OM and diagonal ON of a parallelogram at A, B and C respectively, prove that

$$\frac{OL}{OA}+\frac{OM}{OB}=\frac{ON}{OC}$$

15 If G is the centroid of the triangle ABC show that $\vec{GA}+\vec{GB}+\vec{GC}=0$ and conversely if $\vec{GA}+\vec{GB}+\vec{GC}=0$, then G is the centroid of triangle ABC

16 Prove that the medians of a triangle are concurrent and find the point of concurrency

17 (a) The lines joining the vertices of a tetrahedron to the centroids of opposite faces are concurrent

(b) The joins of the mid points of the opposite edges of a tetrahedron intersect and bisect each other

18 The four diagonals of parallelopiped and the joins of the mid points of opposite edges are concurrent at a common point of bisection

15 $\int \dfrac{e^{\log(1+1/x^2)}}{x^2+1/x^2}\,dx = \dfrac{1}{\sqrt{2}}\tan^{-1}\left(x-\dfrac{1}{x}\right)$

(*a*) True (*b*) False

16 The area cut off by the parabola $y^2=4ax$ and its latus rectum is

17 The area common to the curves $y^2=x$ and $x^2=y$ is

(*i*) 1 (*ii*) $\frac{2}{3}$ (*iii*) $\frac{1}{3}$, (*iv*) None of these

18 The value of the integral $\int_0^{2a} \dfrac{f(x)}{f(x)+f(2a-x)}\,dx$ is equal to a

(*a*) True (*b*) False (I I T 88)

19 The integral $\int_0^{1.5} [x^2]\,dx$ where [ ] denotes the greatest integer functions equals (I I T 88)

20 The value of $\int_{-2}^{2} (ax^3+bx+c)\,dx$ depends on

(*a*) the value of *b*, (*b*) the value of *c*,
(*c*) the value of *a* (*d*) the value of *a* and *b*

(M N R 88)

21 The value of $\int_0^{\pi/2} \dfrac{\sqrt{\cos x}}{\sqrt{\sin x}+\sqrt{\cos x}}\,dx$ is

(*a*) 0, (*b*) $\pi/2$ (*c*) $\pi/4$, (*d*) None of these

(M N R 88)

Answers and Hints to Problem Set (J)

1 Ans (*b*) 2 Ans (*C*)

3 Ans (*ii*) [Hint $\sqrt{(1+\sin 2x)}=\sin x+\cos x$]

4 Ans (*ii*) [Hint Put $e^x=t$ etc]

5 Ans (*iii*)

[Hint $\cos(2n+1)(\pi-x)=\cos\{(2n+1)\pi-(2n+1)x\}$

$=-\cos(2n+1)x$

and $\cos^2(\pi-x)=\cos^2 x$, so that $f(2a-x)=-f(x)$ and hence by prop 6 of § 7, we have

$\int_0^{\pi} e^{\cos^2 x}\cos^3(2n+1)x\,dx=0$

6 Ans (C) [Hint Use prop 5 of § 7]

7 Ans (*B*)

[Hint $I=\int_{-\pi/2}^{\pi/2} \sqrt{\{\frac{1}{2}\,2\sin^2 x\}}\,dx=\int_{-\pi/2}^{\pi/2} |\sin x|\,dx$

$=2\int_0^{\pi/2} |\sin x|\,dx$, by prop 5 of § 7

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m & m_3 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

(h) The vectors a and b are non collinear Find for what value of x, the vectors $c=(x-2)$ a+b and $d=(2x+1)$ a − b are collinear ?

31 Prove that the necessary and sufficient condition for any four points in three dimensional space to be coplanar is that there exists a linear relation connecting their position vectors such that the algebraic sum of the coefficients (not all zero) in it is zero

32 If a, b c are non coplanar vectors prove that the following vectors are coplanar

(a) 3a 7b−4c 3a 2b+c, a−b+2c

(b) a−6b+7c 7a 8b+9c 3a−20b+5c

(c) a−2b+3c −2a+3b−4c, a−3b+5c

(d) 5a+6b+7c, 7a−8b+9c, a−20b+5c

33 If a b c are non-coplanar vectors prove that the four points as given below are coplanar

(a) 2a−3b−c, a−2b+3c, a+4b−2c, a−6b+6c

(b) 6a+2b−c 2a−b+3c −a+2b−4c −12a b−3c

(c) 6a−4b+10c −5a+3b−10c, 4a−6b−10c, 2b+10c

34 If a b c be any three non-coplanar vectors, then prove that the points

l_1a+m_1b+n_1c l_2a+m_2b+n_2c l_3a+m_3b+n_3c l_4a+m_4b+n_4c are coplanar if

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m_2 & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

35 Find all the values of λ such that $(x, y, z) \neq (0, 0, 0)$ and $(i+j+3k)x+(3i-3j+k)y+(-4i+5j)z=\lambda(xi+yj+zk)$

(I I T 82)

36 A vector A has components A_1 A_2 A_3 in a right handed rectangular cartesian coordinate system Ox Oy, Oz The co or-

13 Ans (*b*)

[Hint $x=\tan\theta$ etc Correct Ans $=1/\sqrt{2}$]

14 Ans (*a*)

15 Ans (*b*) correct Ans is $\frac{1}{\sqrt{2}}\log\frac{x^2-1}{x\sqrt{2}}$

16 Ans $\frac{8}{3}a^3$ 17 Ans (*iii*)

18 Ans (*A*)

Let $I=\int_0^{2a}\frac{f(x)}{f(x)+f(2a-x)}dx$

Then also $I=\int_0^{2a}\frac{f(2a-x)}{f(2a-x)+f(x)}dx$ by prop

Adding $2I=\int_0^{2a}dx=2a$

$I=a$

19 Ans $2-\sqrt{2}$

We have

$$\int_0^{1.5}[x^2]\,dx=\int_0^1[x^2]\,dx+\int_1^{\sqrt{2}}[x^2]\,dx+\int_{\sqrt{2}}^{1.5}[x^2]\,dx$$
$$=\int_0^1 0\,dx+\int_1^{\sqrt{2}}1\,dx+\int_{\sqrt{2}}^{1.5}2\,dx$$
$$=0+\sqrt{2}-1+2(1.5-\sqrt{2})$$
$$=2-\sqrt{2}$$

[Note that $0<x<1\Rightarrow 0<x^2<1\Rightarrow[x^2]=0$

$1<x<\sqrt{2}\Rightarrow 1<x^2<2\Rightarrow[x^2]=1$,

and $\sqrt{2}<x<1.5\Rightarrow 2<x^2<2.25\Rightarrow[x^2]=2$

by the definition of the function]

20 Ans (b)

Note that the integral of x^3 and x is 0 for limits -2 to 2 so that the value of the integral is $4c$ which depends upon (c)

21 Ans (c)

or $\vec{CE}=\vec{AE}-\vec{AC}=2\mathbf{b}-\mathbf{a}-(\mathbf{a}+\mathbf{b})=\mathbf{b}-2\mathbf{a}$ by (1) and (7)

2 Refer fig Ex 1 If R be the resultant then

$$R=\vec{AB}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{AF}=\vec{ED}+\vec{AC}+\vec{AD}+\vec{AE}+\vec{CD}$$

$$[\because \vec{AB}=\vec{ED} \text{ and } \vec{AF}=\vec{CD}]$$

$$=(\vec{AC}+\vec{CD})+(\vec{AE}+\vec{ED})+\vec{AD}$$

$$=\vec{AD}+\vec{AD}+\vec{AD}=3\vec{AD}=6\vec{AO}$$

3 $R=\vec{AB}+\vec{AE}+\vec{BC}+\vec{DC}+\vec{ED}+\vec{AC}$

$$=(\vec{AB}+\vec{BC})+(\vec{AE}+\vec{ED}+\vec{DC})+\vec{AC}$$

$$=\vec{AC}+\vec{AC}+\vec{AC}=3\vec{AC}$$

4 $R=\vec{OB}+\vec{OC}$

$$=(\vec{OD}+\vec{DB})+(\vec{OD}+\vec{DC})$$

$$=2\vec{OD}+(\vec{DB}+\vec{DC})$$

$$=2\vec{OD}+0=2\vec{OD}$$

[D being mid point of BC, we have $\vec{DB}=-\vec{DC}$]

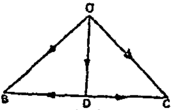


Fig 7

5 Ist Method

$\vec{AB}+\vec{AC}=2\vec{AD}$ by Q 4

$\vec{BC}+\vec{BA}=2\vec{BE}$,

$\vec{CA}+\vec{CB}=2\vec{CF}$

Adding, we get

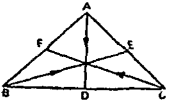


Fig 8

$$(\vec{AB}+\vec{BA})+(\vec{AC}+\vec{CA})+(\vec{BC}+\vec{CB})$$

$$=2(\vec{AD}+\vec{BE}+\vec{CF})$$

or $0+0+0=2(\vec{AD}+\vec{BE}+\vec{CF})$ or $\vec{AD}+\vec{BE}+\vec{CF}=0$

2nd Method

Let the position vectors of A, B, C w r t an origin O be $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ respectively so that those of mid points D, E and F are respectively

$$\frac{\mathbf{b}+\mathbf{c}}{2},\ \frac{\mathbf{c}+\mathbf{a}}{2},\ \frac{\mathbf{a}+\mathbf{b}}{2}$$

$$\frac{5a}{R\sqrt{2}}, \frac{4a}{R\sqrt{2}}, \frac{3a}{R\sqrt{2}}$$

or $$\frac{1}{\sqrt{2}}, \frac{4}{5\sqrt{2}}, \frac{3}{5\sqrt{2}} \qquad R=5a$$

Again $\vec{OR}+\vec{OS}+\vec{OT}=\mathbf{j}+\mathbf{k}+\mathbf{i}+\mathbf{k}+\mathbf{i}+\mathbf{j}=2(\mathbf{i}+\mathbf{j}+\mathbf{k})$

Also $\vec{OP}=\vec{OT}+\vec{TP}=(\mathbf{i}+\mathbf{j}+\mathbf{k})$

$[\ \vec{OT}=\mathbf{i}+\mathbf{j}$ and $\vec{TP}=\vec{OC}=\mathbf{k}]$

$$\vec{OR}+\vec{OS}+\vec{OT}=2\vec{OP}$$

(b) $\vec{AB}=\vec{OB}-\vec{OA}=(5\mathbf{i}+3\mathbf{j}-2\mathbf{k})-(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})$
$=3\mathbf{i}+7\mathbf{j}-\mathbf{k}$

$\vec{BC}=\vec{OC}-\vec{OB}=(-2\mathbf{i}+2\mathbf{j}+3\mathbf{k})-(5\mathbf{i}+3\mathbf{j}-2\mathbf{k})$
$=-7\mathbf{i}-\mathbf{j}+5\mathbf{k}$

and $\vec{CA}=\vec{OA}-\vec{OC}=(2\mathbf{i}-4\mathbf{j}+3\mathbf{k})-(-2\mathbf{i}+2\mathbf{j}+3\mathbf{k})$
$=4\mathbf{i}-6\mathbf{j}$

$$\vec{S}=\vec{P}+\vec{Q}+\vec{R}=15\left[\frac{\vec{AB}}{|\vec{AB}|}+\frac{\vec{BC}}{|\vec{BC}|}+\frac{\vec{CA}}{|\vec{CA}|}\right]$$

$$=15\left[\frac{3\mathbf{i}+7\mathbf{j}-5\mathbf{k}}{\sqrt{83}}+\frac{-7\mathbf{i}-\mathbf{j}+5\mathbf{k}}{5\sqrt{3}}+\frac{4\mathbf{i}-6\mathbf{j}}{2\sqrt{13}}\right]$$

7 In terms of unit vectors $A\ (1, 2, -1)$ is $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ and similarly we can write down the other points with respect to O as origin

$$\vec{AB}=\vec{OB}-\vec{OA}=(-4\mathbf{i}+2\mathbf{j}-2\mathbf{k})-(\mathbf{i}+2\mathbf{j}-\mathbf{k})$$

or $\vec{AB}=-5\mathbf{i}+0\mathbf{j}-\mathbf{k} \qquad |\vec{AB}|=\sqrt{(25+1)}=\sqrt{26}$

Unit vector along AB

$$=\frac{1}{\sqrt{26}}(-5\mathbf{i}-\mathbf{k})$$

A force of magnitude 2 lbs along AB is

$$2\ \frac{1}{\sqrt{26}}(-5\mathbf{i}-\mathbf{k})=\frac{1}{\sqrt{26}}(-10\mathbf{i}-2\mathbf{k}) \qquad (1)$$

Similarly forces of magnitudes 3 and 2 lbs along AC and AD are respectively

$$3\ \frac{1}{\sqrt{26}}(3\mathbf{i}-\mathbf{j}-4\mathbf{k})=\frac{1}{\sqrt{26}}(9\mathbf{i}-3\mathbf{j}-12\mathbf{k}) \qquad (2)$$

(iv) **Collinear or parallel vectors** Two or more vectors are said to be collinear when they act along the same line or along parallel lines

(v) **Coplanar Vectors** Three or more vectors are said to be coplanar when they are parallel to the same plane or lie in the same plane whatever their magnitudes be

Note that two vectors are always coplanar

(vi) **Equal Vectors** *Two vectors are said to be equal when they have the same length (magnitude and are parallel having the same sense of direction)*

(vii) **Zero Vector** If the origin and terminal points of a vector coincide then it is said to be a zero vector It is denoted by **0**, its magnitude is zero and its direction indeterminate

(viii) **Unit Vector** A vector whose magnitude is of unit length is called a unit vector If **a** is a vector whose magnitude is a, then unit vectors in the direction of **a** is denoted by $\hat{\mathbf{a}}$ and is obtained by dividing the vector **a** by its magnitude a Thus $\hat{\mathbf{a}}=\dfrac{\mathbf{a}}{a}$.

Any two unit vectors $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ should not be taken to be equal because although both of them have their magnitude unity but their directions may be different

(ix) **Position Vector** If O be a fixed origin and P any point then the vector $\overrightarrow{OP}$ is called the position vector of the point P w r t the origin O

(x) **Scalar multiple of a vector** If **a** be a given vector, then k **a** is a vector whose magnitude is $|k|$ times the magnitude of vector **a** and whose direction is the same as that of **a** or opposite of **a** according as k is positive or negative

Negative of a Vector If **a** be a given vector then $-$**a** is a vector whose magnitude is same as that of **a** but direction opposite to that of **a**

(xi) **Co initial Vectors** The vectors having the same initial point are called co-initial vectors

(xii) **Localized and free Vectors** A vectors drawn parallel to a given vector through a specified point in space is called a localized vector But if the origin of vectors is not specified, the vectors are called free vectors

$$=2\,(\overrightarrow{AF}+\overrightarrow{CF})$$
$$=-2\,(\overrightarrow{FA}+\overrightarrow{FC})$$
$$=-2\,[2\,\overrightarrow{FE}],$$

where E is the mid point of AC

$$=-4\,\overrightarrow{FE}=4\,\overrightarrow{EF}$$

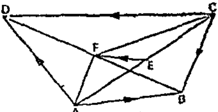


Fig 11

10 We know that the figure formed by the lines joining the mid points of the sides of quadrilateral is ∥^{m} Hence $MPNQ$ is a parallelogram, whose diagonals are MN and PQ intersecting at E which is mid point of both MN and PQ

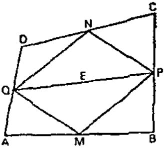


Fig 12

$$\overrightarrow{OA}+\overrightarrow{OB}=2\,(\overrightarrow{OM}),\qquad M \text{ is mid point of } AB$$

$$\overrightarrow{OC}+\overrightarrow{OD}=2\,(\overrightarrow{ON}),\qquad N \text{ is mid point of } CD$$

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}=2\,(\overrightarrow{OM}+\overrightarrow{ON})=2\,[2\,\overrightarrow{OE}]=4\overrightarrow{OE}$$

where E is mid point of MN as it is the intersection of diagonals of a ∥^{m}

11 O is the circumcentre which is the intersection of right bisectors of sides of the triangle and O' is orthocentre which is the point of intersection of altitudes drawn from the vertices Also from geometry we know that

F 13

$$2OD=AO' \qquad 2\overrightarrow{OD}=\overrightarrow{AO'} \qquad (1)$$

(i) $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OO'}$

Now $\overrightarrow{OB}+\overrightarrow{OC}=2\overrightarrow{OD}=\overrightarrow{AO'}$

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OA}+\overrightarrow{AO'}=\overrightarrow{OO'} \qquad \text{by (1)}$$

(xvi) Expression of a vector in terms of position vectors of its end points

The position vectors of the points A and B w r t O as origin are $\mathbf{a}$ and $\mathbf{b}$ respectively i e $\overrightarrow{OA}=\mathbf{a}$, $\overrightarrow{OB}=\mathbf{b}$ then we know that

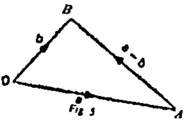


Fig 5

$$\overrightarrow{OA}+\overrightarrow{AB}=\overrightarrow{OB}$$

$$\overrightarrow{AB}=\overrightarrow{OB}-\overrightarrow{OA}=\mathbf{b}-\mathbf{a}$$

Thus $\overrightarrow{AB}$=(P V of B)−(P V of A)

Similarly,

$\overrightarrow{BA}=\mathbf{a}-\mathbf{b}$=(P V of A)−(P V of B) (V Imp)

(xvii) Linear Combination of Vectors

If a vector $\mathbf{r}$ is expressible as

$$\mathbf{r}=x\mathbf{a}+y\mathbf{b}+z\mathbf{c}+$$

when x, y, z are scalars then $\mathbf{r}$ is said to be linear combination of vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$

(xviii) Relation between two Collinear Vectors

Any vector $\mathbf{r}$ collinear with a given vector $\mathbf{a}$ can be expressed as $x\mathbf{a}$ where x is a scalar i e $\mathbf{r}=x\mathbf{a}$

§ 2 Linearly independent and dependent systems of vectors

(i) Definition *A system of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, , $\mathbf{a}_n$ is said to be linearly dependent if there exists a system of scalars λ_1, λ_2, , λ_n not all zero such that*

$$\lambda_1\mathbf{a}_1+\lambda_2\mathbf{a}_2+\quad\lambda_n\mathbf{a}_n=0$$

A system of vectors which is not linearly dependent is called linearly independent Thus a set of vectors $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_n$ is said to be linearly independent if every relation of the type $\lambda_1\mathbf{a}_1+\lambda_2\mathbf{a}_2+$ $+\lambda_n\mathbf{a}_n=0$ implies that

$$\lambda_1=0,\ \lambda_2=0\quad,\lambda_n=0$$

Theorem **If $\mathbf{a}$, $\mathbf{b}$ are two non zero non-collinear vectors and λ, μ are two scalars such that**

$$\lambda\mathbf{a}+\mu\mathbf{b}=0, \text{ then } \lambda=0,\ \mu=0$$

Proof Suppose $\lambda\neq 0$ Then given relation can b- as

$$\mathbf{a}=-\frac{\mu}{\lambda}\mathbf{b},$$

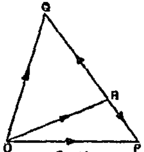


Fig 15

and $\vec{OQ}=\vec{OR}+\vec{RQ}$

$m\vec{OQ}=m\vec{OR}+m\vec{RQ}$ (2)

Adding (1) and (2), we get

$n\vec{OP}+m\vec{OQ}=(m+n)\vec{OR}+(n\vec{RP}+m\vec{RQ})$

$=(m+n)\ \vec{OR}$

We are given that $n\ \vec{PR}=m\vec{RQ}$

$n\ (-\vec{RP})=m\vec{RQ}$ or $n\vec{RP}+m\vec{RQ}=0$

i e The point R divides PQ in the ratio $m\ \ n$

Cor In case the forces be $1\ \vec{OP}$ and $1\ \vec{OQ}$ then their resultant will be $(1+1)\ \vec{OR}$ *i e* $2\vec{OR}$ where R divides PQ in the ratio $1\ \ 1$ *i e* R is the middle point of PQ

14 (a) Let the forces P and Q be represented by $\vec{OL}$ and $\vec{OM}$ so that the diagonal $\vec{ON}$ represents the resultant R

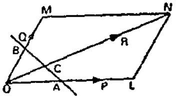


Fig 16

$$\mathbf{P}+\mathbf{Q}=\mathbf{R} \quad (1)$$

Again let $\mathbf{P}=\vec{OL}=m\ \vec{OA}$ $\mathbf{Q}=\vec{OM}=n\vec{OB}$ and $\mathbf{R}=\vec{ON}=t\ \ \vec{OC}$

$$m=\frac{P}{OA},\ n=\frac{Q}{OB},\ t=\frac{R}{OC} \quad (2)$$

Hence from (1), we get

$m\ \vec{OA}+n\ \vec{OB}=t\ \vec{OC}$ or $m\ \vec{OA}+n\vec{OB}-t\vec{OC}=0$ (3)

But $A\ \ B\ \ C$ are collinear and we know that if there exists a relation of the form $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$ between the P V s of three collinear points then $x+y+z=0$

Hence from (3) we must have

$m+n-t=0$ $\quad m+n=t$

Now putting the values of m, n and t from (2) in $m+n=t$ we get

$$\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}$$

(b) From the fig part (*a*), we have $\vec{ON}=\vec{OL}+\vec{LN}$ (1)

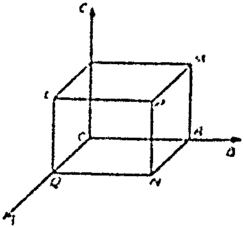
Let $\vec{OL}=\lambda\vec{OA}$, $\vec{OM}=\mu\vec{OB}$ and $\vec{ON}=\nu\vec{OC}$, then

$$\lambda=\frac{OL}{OA},\ \mu=\frac{OM}{OB} \text{ and } \nu=\frac{ON}{OC}$$

From (1) $\nu\ \vec{OC}=\lambda\ \vec{OA}+\mu\ \vec{OB}$

Theorem III. If a, b, c are three non-coplanar vectors, then any vector r can be expressed uniquely as a linear combination of vectors a, b and c.

Proof. Let $\vec{OA}=$ a, $\vec{OB}=$ b and $\vec{OC}=$ c.



The lines OA, OB and OC are non coplanar.

Let $\vec{OP}=$ r.

With OP as diagonal construct a parallelo-piped whose three coterminus edges OQ, OR and OS are along OA, OB and OC respectively.

Now r $=\vec{OP}=\vec{ON}+\vec{NP}$

$=\vec{OQ}+\vec{QN}+\vec{NP}=\vec{OQ}+\vec{OR}+\vec{OS}$

$=\lambda$a$+\mu$b$+\nu$c

[$\because$ $\vec{OQ}$, $\vec{OR}$, $\vec{OS}$ are collinear respectively with a, b and c, we can find scalars λ, μ and ν such that

$\vec{OQ}=\lambda$a, $\vec{OR}=\mu$b and $\vec{OS}=\nu$c]

Uniqueness. Let there be another representation of r as

r $=\lambda'$a$+\mu'$b$+\nu'$c. Then

λa$+\mu$b$+\nu$c$=\lambda'$a$+\mu'$b$+\nu'$c

or $(\lambda-\lambda')$a$+(\mu-\mu')$b$+(\nu-\nu')$c$=0$

Since a, b, c are non-coplanar, we must have

$\lambda-\lambda'=0$, $\mu-\mu'=0$ and $(\nu-\nu')=0$, that is, $\lambda=\lambda'$, $\mu=\mu'$ and $\nu=\nu'$. Hence the resolution is unique.

2 : 1 Therefore the three medians are concurrent at the point $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$ which is also the centroid of the triangle ABC.

17. G_1 the centroid of Δ BCD is $\frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}$ and A is $\mathbf{a}$.

The position vector of a point G which divides AG_1 in the ratio 3 : 1 is

$$\frac{3\ \frac{\mathbf{b}+\mathbf{c}+\mathbf{d}}{3}+1\ \mathbf{a}}{3+1}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$$

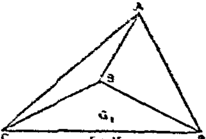


Fig 18

The symmetry of the result shows that this point will also lie on BG_2, CG_3 and DG_4. Hence these four lines concur at the point $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ which is called the centroid of the tetrahedron.

(ii) The mid point of DA is $\frac{\mathbf{a}+\mathbf{d}}{2}$ and that of BC is $\frac{\mathbf{b}+\mathbf{c}}{2}$

and the mid point of these mid points is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}+\mathbf{d}}{4}$ and symmetry of the result proves the theorem.

18 Taking O as origin let the position vectors of A, B and C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively, so that those of D, E and F are $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, $\mathbf{c}+\mathbf{a}$ respectively and that of G is $\mathbf{a}+\mathbf{b}+\mathbf{c}$

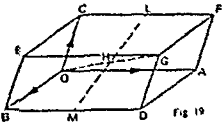


Fig 19

If M_1 be the mid point of diagonal OG then M_1 is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$

If M_2 be the mid point of diagonal AE, then M_2 is

$$\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2} \text{ which is same as } M_1$$

Similarly mid point of other diagonals DC and BF is also the point whose position vector is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{2}$.

Again mid point M of BD is $\frac{1}{2}(\mathbf{b}+\mathbf{a}+\mathbf{b})=\frac{\mathbf{a}+2\mathbf{b}}{2}$

If $OP=r$, then clearly $x=r\cos\alpha=lr$, $y=r\cos\beta=mr$ and $z=r\cos\gamma=nr$. Substituting these values in the relation $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, we get

$$\mathbf{r}=lr\mathbf{i}+mr\mathbf{j}+nr\mathbf{k}$$

$$\hat{\mathbf{r}}=\frac{\mathbf{r}}{r}=l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$$

Hence the direction cosines of vector $\mathbf{r}$ *are the coefficients of* $\mathbf{i}, \mathbf{j}, \mathbf{k}$ *in the rectangular resolution of the unit vector* $\hat{\mathbf{r}}$.

Also $OP^2=ON^2+NP^2=OA^2+AN^2+NP^2=OA^2+OB^2+OC^2$

or $r^2=x^2+y^2+z^2$ i.e. $|\mathbf{r}|^2=x^2+y^2+z^2$

Hence *magnitude of the* vector $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ *is* $\sqrt{(x^2+y^2+z^2)}$

Now dividing $r^2=x^2+y^2+z^2$ by r^2, we get

$$1=\frac{x^2}{r^2}+\frac{y^2}{r^2}+\frac{z^2}{r^2}=l^2+m^2+n^2$$

Thus the sum of the squares of direction cosines is equal to unity.

Note. We often write the vector $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ as $\mathbf{r}=(x, y, z)$. For example, if $\mathbf{a}=(1, 2, -7)$, then $\mathbf{a}=\mathbf{i}+2\mathbf{j}-7\mathbf{k}$. In fact, any vector in three dimensional space may be defined as an ordered triad of real numbers.

§ 6. Collinearity and Coplanarity

(i) collinearity of three points

(a) First method. To prove that three points A, B, C are collinear find the vectors $\overrightarrow{AB}$ and $\overrightarrow{BC}$ and show that one of them is a scalar multiple of the other.

(b) Second method. To prove the collinearity of three points A, B, C with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$, respectively, we may use the following theorem.

Three points with position vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ *are collinear if and only if there exist scalars* x, y, z *not all zero such that*

(*i*) $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$ (*ii*) $x+y+z=0$

For proof of this theorem see problem 28.

(ii) Method to prove three vectors to be coplanar.

Three vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be coplanar if one of them can be expressed as a linear combination of the remaining two vectors. Thus $\mathbf{a}, \mathbf{b}, \mathbf{c}$ will be coplanar if we can find two scalars λ and μ such that

$$\mathbf{a}=\lambda\mathbf{b}+\mu\mathbf{c}$$

$\frac{1\cdot c+2\cdot 0}{1+2}=\frac{c}{3}$ and the point which divides DM in the ratio

$2:1$ is $\frac{2\cdot\frac{b}{2}+1\cdot d}{2+1}=\frac{b+d}{3}=\frac{c}{3}$ by (1) and this is same as E

Similarly the point F which divides AC in the ratio $2:1$ is $\frac{2c+1\cdot 0}{3}=\frac{2c}{3}$ and the point which divides NB in the ratio

$1:2$ is $\frac{1\cdot b+2\cdot\frac{c+d}{2}}{3}=\frac{b+(c+d)}{3}=\frac{2c}{3}$ by (1) which is same as the point F

Thus E and F are the points of trisection of dioganal AC and also the points of trisection of DM and BN respectively

22 Do yourself

23 Let the position vectors B, C, D w r t A as origin be **b**, **c** and **d** and hence those of L, M, N, P are $\frac{b}{2}, \frac{b+c}{2}, \frac{c+d}{2}, \frac{d}{2}$ respectively

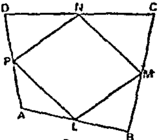


Fig. 21

$\vec{LM}$ = P V of M − P V of L

$=\frac{b+c}{2}-\frac{b}{2}=\frac{c}{2}$

$\vec{PN}$ = P V of N − P V of P = $\frac{c+d}{2}-\frac{d}{2}=\frac{c}{2}$

Above shows that LM is parallel and equal to PN

Similarly we can show MN is parallel and equal to LP

Hence $PLMN$ is a parallelogram

24 Let **a**, **b**, **c** be the position vectors of the vertices A, B and C respectively so that the P Vs of mid points D, E and F are $\frac{b+c}{2}, \frac{c+a}{2}, \frac{a+b}{2}$

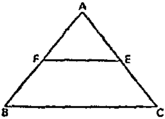


Fig 22

$\vec{BC}$ = P V of C − P V of B

$= c-b$

5 Prove that sum of the three vectors determined by the medians of a triangle directed from the vertices is zero

6 (a) Three vectors of magnitudes a, $2a$, $3a$ meet in a point and their directions are along the diagonals of three adjacent faces of a cube. Determine their resultant R and its direction cosines. Prove also that the sum of the three vectors determined by the diagonals of three adjacent faces of a cube passing through the same corner the vectors being directed from the corner is twice the vector determined by the diagonal of the cube. (I I T 80)

(*b*) Three force vectors $\vec{P}$, $\vec{Q}$, $\vec{R}$ of 15 kg each act along AB, BC and CA respectively. The position vectors of A, B and C are given by

$\vec{OA}=2i-4j+3k$, $\vec{OB}=5i+3j-2k$, $\vec{OC}=-2i+2j+3k$. Find the resultant force vector $\vec{S}$ of the vectors $\vec{P}$, $\vec{Q}$ and $\vec{R}$. (Roorkee 1983)

7 The vertices of a quadrilateral are A (1, 2, −1), B (−4, 2, −2), C(4, 1, −5) and D(2, −1, 3). At the point A forces of magnitudes 2, 3, 2 lbs act along AB, AC and AD respectively. Find their resultant

8 Prove that the system of concurrent forces acting at a point and represented by $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ is equivalent to the system of forces represented by $\vec{OD}$, $\vec{OE}$, $\vec{OF}$ acting at the same point where D, E, F are the middle points of the sides BC, CA and AB respectively of triangle ABC

Also prove that $\vec{AD}+\frac{2}{3}\vec{BE}+\frac{1}{3}\vec{CF}=\frac{1}{2}\vec{AC}$

and $\vec{OE}+\vec{OF}+\vec{DO}=\vec{OA}$

9 Two forces act at the vertex A of a quadrilateral $ABCD$ represented by $\vec{AB}$, $\vec{AD}$ and two at C represented by $\vec{CB}$ and $\vec{CD}$. Prove that their resultant is represented by 4 $\vec{EF}$ where E and F are the middle points of AC and BD respectively

10 $ABCD$ is a quadrilateral and E the point of intersection of the lines joining the middle points of opposite sides. Show that the resultant of $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ and $\vec{OD}$ is equal to 4 $\vec{OE}$ where O is any point

∴ P V of F, the mid point of diagonal AC is $\frac{\mathbf{a}+\mathbf{d}+t\mathbf{b}}{2}$

E is mid point of DB

P V of F is $\frac{\mathbf{d}+\mathbf{b}}{2}$

$\vec{FE}$ = PV of E − P V of $F = \frac{\mathbf{d}+\mathbf{b}}{2} - \frac{\mathbf{d}+t\mathbf{b}}{2} = \frac{(1-t)}{2}\mathbf{b}$

or $\vec{FE} = \frac{1-t}{2}\vec{AB}$

∴ $\vec{FE}$ is scalar multiple of $\vec{AB}$ and hence FE is parallel to AB

Also $\frac{FE}{AB} = \frac{1-t}{2} = \frac{1}{2}\left(1 - \frac{DC}{AB}\right) = \frac{1}{2}\frac{(AB-DC)}{AB}$

$FE = \frac{1}{2}(AB-DC)$

or FE = half the difference of parallel sides

27 We know from geometry that the internal bisector of an angle divides the opposite side in the ratio of the sides containing the angle. If a b c be the position vectors of the vertices A, B, C respectively and the opposite side be of lengths x y, z then if the internal bisector of angle A meets BC in D then by the given condition

$D = \frac{z\mathbf{c}+y\mathbf{b}}{y+z}$ and A is a Therefore the position vector of a point I on AD which divides it in the ratio $y+z$: x is

$$\frac{x\mathbf{a}+(y+z)(z\mathbf{c}+y\mathbf{b})/(y+z)}{x+y+z} = \frac{x\mathbf{a}+y\mathbf{b}+z\mathbf{c}}{x+y+z}$$

The symmetry of the result shows that this point also lies on the internal bisectors of angles B and C Hence the three bisectors are concurrent

28 Let us suppose that the points A, B C are collinear and their position vectors are a b and c respectively Let C divide the join of a and b in the ratio y : x

$\mathbf{c} = \frac{x\mathbf{a}+y\mathbf{b}}{x+y}$ or $x\mathbf{a}+y\mathbf{b}-(x+y)\,\mathbf{c}=0$

or $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$ where $z=-(x+y)$

Also $x+y+z=x+y-(x+y)=0$ Hence proved

Conversely Let $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$ where $x+y+z=0$, 19 Prove that the diagonals of a parallelogram bisect each other and conversely if the diagonals of a quadrilateral bisect each other, it is a parallelogram

20 $ABCD$ is a parallelogram and O the point of intersection of diagonals Show that for any origin the sum of the posi ion vectors of the vertices is equal to four times that of O

21 If M, N are the mid points of the sides AB CD of a parallelogram $ABCD$, prove that DM and BN cut the diagonal AC in its points of trisection which are also the points of trisection of DM and BC respectively

22 $ABCD$ is a parallelogram, P and Q are the mid points of the sides AB and BC respectively Prove that DP and AC meet in a common point of trisection and similarly DQ and AC

23 Prove that the figure formed by joining the mid points of the sides of a quadrilateral taken in order is a parallelogram

24 Prove that in any triangle the line joining the mid points of any two sides is parallel to the third side and half of its length

25 Prove that the straight line joining the mid points of two non parallel sides of a trapezium is parallel to the parallel sides and half of their sum

26 Prove that the straight line joining the mid points of the diagonals of a trapezium is parallel to the parallel sides and half of their difference

27 Prove that internal bisectors of the angles of a triangle are concurrent

28 Prove that three points with position vectors **a**, **b** **c** are collinear if and only if there exist scalars x, y, z not all zero such that

(i) $x\mathbf{a}+y\mathbf{b}+z\mathbf{c}=0$ (ii) $x+y+z=0$

29 Prove that the following points are collinear

(a) $\mathbf{a}-2\mathbf{b}=3\mathbf{c}$, $2\mathbf{a}+3\mathbf{b}$ $4\mathbf{c}$, $-7\mathbf{b}+10\mathbf{c}$

(b) $\mathbf{a}$, $\mathbf{b}$ $3\mathbf{a}-2\mathbf{b}$

(c) $-2\mathbf{a}+3\mathbf{b}+5\mathbf{c}$ $\mathbf{a}+2\mathbf{b}+3\mathbf{c}$, $7\mathbf{a}-\mathbf{c}$

(d) $\mathbf{a}+\mathbf{b}+\mathbf{c}$ $4\mathbf{a}+3\mathbf{b}$ $10\mathbf{a}+7\mathbf{b}-2\mathbf{c}$

(e) $A(1, 2\ 3)$, $B(3\ 4, 7)$, $C(-3, -2\ -5)$

and find the ratio in which B divides AC

(f) If $\vec{AO}+\vec{OB}=\vec{BO}-\vec{OC}$, prove that A, B C are collinear

30 (a) If **a** **b** are two non collinear vectors show that points $l_1\mathbf{a}+m_1\mathbf{b}$ $l_2\mathbf{a}+m_2\mathbf{b}$, $l_3\mathbf{a}+m_3\mathbf{b}$ are collinear if

$$x(l_1\mathbf{a}+m_1\mathbf{b})+y(l_2\mathbf{a}+m_2\mathbf{b})+z(l_3\mathbf{a}+m_3\mathbf{b})=0$$

or $(xl_1+yl_2+zl_3)\mathbf{a}+(xm_1+ym_2+zm_3)\mathbf{b}=0$

Since a and b are two non collinear vectors it follows that

$$xl_1+yl_2+zl_3=0 \quad (1)$$

$$xm_1+ym_2+zm_3=0 \quad (2)$$

Because otherwise one is expressible as a scalar multiple of the other which would mean that a and b are collinear

Also $x+y+z=0$ (3)

Eliminating x, y, z from (1), (2) and (3) we get

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ 1 & 1 & 1 \end{vmatrix}=0$$

(*b*) The vector c is non zero since the coefficient in b is different from zero, and so the vectors c and d are collinear if for some number y we have

$\mathbf{d}=y\mathbf{c}$, that is

$(2x+1)\mathbf{a}-\mathbf{b}=y(x-2)\mathbf{a}+y\mathbf{b}$

or $(yx-2y-2x-1)\mathbf{a}+(y+1)\mathbf{b}=0$

Since a b are non-collinear, we must have

$yx-2y-2x-1=0$

and $y+1=0$

Solving these equations we get

$y=-1$ and $x=1/3$

Ans $x=1/3$

Let us suppose that the points A, B, C and D whose position vectors are a b, c and d respectively are coplanar In that case the lines AB and CD will intersect at some point P (it being assumed that AB and CD are not parallel and if they are then we will choose any other pair of non parallel lines formed by given point) If P divides AB in the ratio q p and CD in the ratio n m then the position vector of P written from AB and CD are

$$\frac{p\,\mathbf{a}+q\,\mathbf{b}}{p+q}=\frac{m\,\mathbf{c}+n\,\mathbf{d}}{m+n}$$

or $\frac{p}{p+q}\mathbf{a}+\frac{q}{p+q}\mathbf{b}-\frac{m}{m+n}\mathbf{c}-\frac{n}{m+n}\mathbf{d}=0$

or $L\mathbf{a}+M\mathbf{b}+N\mathbf{c}+P\mathbf{d}=0$

dinate system is rotated about the z axis through an angle $\pi/2$. Find the components of A in new coordinate system in terms of A_1, A_2, A_3 (I.I.T 83)

37 Let $\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \ldots \mathbf{r}_n$ be the position vectors of points $P_1, P_2, P_3, \ldots P_n$ relative to an origin O. Show that if the vector equation $a_1\mathbf{r}_1+a_2\mathbf{r}_2+\ldots+a_n\mathbf{r}_n=0$ holds, then a similar equation will also hold good with respect to any other origin P if $a_1+a_2+\ldots+a_n=0$ (Roorkee 87)

38 A particle, in equilibrium is subjected to four forces

$$\mathbf{F}_1=-10\,\mathbf{k},\ \mathbf{F}_2=U\left[\frac{4}{13}\,\mathbf{i}-\frac{12}{13}\,\mathbf{j}+\frac{3}{13}\,\mathbf{k}\right],$$

$$\mathbf{F}_3=V\left[-\frac{4}{13}\,\mathbf{i}-\frac{12}{13}\,\mathbf{j}+\frac{3}{13}\,\mathbf{k}\right],\ \mathbf{F}_4=W[\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}]$$

Solve for U, V and W as functions of θ (Roorkee 88)

39 Unit vectors $\mathbf{i}$ and $\mathbf{j}$ are parallel to adjacent edges of a large square table. The directions of $\mathbf{i}$ and $\mathbf{j}$ are referred to as east and north. An ant walking on the table makes the following movements successively

(i) 4 cm 30° east of north, (ii) 12 cm south west, (iii) 6 cm east and (iv) 9 cm west north. Find the magnitude and direction of ant's resultant displacement (Roorkee 88)

Solutions to Problem Set (A)

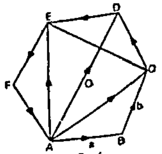


Fig 6

1 $\overrightarrow{AB}=\mathbf{a}$, $\overrightarrow{BC}=\mathbf{b}$

$\overrightarrow{AC}=\overrightarrow{AB}+\overrightarrow{BC}=\mathbf{a}+\mathbf{b}$ (1)

$\overrightarrow{AD}=2\overrightarrow{BC}=2\mathbf{b}$ (2)

(AD is parallel to BC and twice its length)

$\overrightarrow{CD}=\overrightarrow{AD}-\overrightarrow{AC}=2\mathbf{b}-(\mathbf{a}+\mathbf{b})$
$=\mathbf{b}-\mathbf{a}$ (3)

$\overrightarrow{FA}=-\overrightarrow{CD}=\mathbf{a}-\mathbf{b}$ (4)

$\overrightarrow{DE}=-\overrightarrow{AB}=-\mathbf{a}$ (5)

$\overrightarrow{EF}=-\overrightarrow{BC}=-\mathbf{b}$ (6)

$\overrightarrow{AE}=\overrightarrow{AD}+\overrightarrow{DE}=2\mathbf{b}-\mathbf{a}$ by (2) and (5) (7)

$\overrightarrow{CE}=\overrightarrow{CD}+\overrightarrow{DE}=\mathbf{b}-\mathbf{a}-\mathbf{a}=\mathbf{b}-2\mathbf{a}$ by (3) and (5)

(d) Proceed as above

33 (a) Let the given points be A, B, C, and D If they are coplanar then the three coterminous vectors $\vec{AB}$, $\vec{AC}$, $\vec{AD}$ should be coplanar

$$\vec{AB}=\vec{OB}-\vec{OA}=-\mathbf{a}-5\mathbf{b}+4\mathbf{c},$$
$$\vec{AC}=\vec{OC}-\vec{OA}=\mathbf{a}+\mathbf{b}-\mathbf{c}$$
and $$\vec{AD}=\vec{OD}-\vec{OA}=-\mathbf{a}-9\mathbf{b}+7\mathbf{c}$$

If they are coplanar, then as in Q 32 one of them is expressible as a linear combination of the other two

Let $$\vec{AD}=x\,\vec{AB}+y\,\vec{AC}$$
$$-\mathbf{a}-9\mathbf{b}+7\mathbf{c}=x\,(-\mathbf{a}-5\mathbf{b}+4\mathbf{c})+y\,(\mathbf{a}+\mathbf{b}-\mathbf{c})$$

Since **a**, **b**, **c**, are non coplanar, equating the coefficients, we get

$$-x+y=-1,\ -5x+y=-9,\ 4x-y=7$$

Solving first two we get $x=2$, $y=1$ and these values satisfy the third also Hence $\vec{AB}$, $\vec{AC}$, and $\vec{AD}$ are coplanar which in turn means that the four points A, B, C, D are coplanar

2nd Method

We know (Q 31) that the four points are coplanar if we can choose four scalars l, m, n, p such that

$$l\,(2\mathbf{a}+3\mathbf{b}-\mathbf{c})+m\,(\mathbf{a}-2\mathbf{b}+3\mathbf{c})+n\,(3\mathbf{a}+4\mathbf{b}-2\mathbf{c})$$
$$+p\,(\mathbf{a}-6\mathbf{b}+6\mathbf{c})=0 \quad (1)$$

where $$l+m+n+p=0 \quad (2)$$

The relation (1) can be put in the form

$$2\mathbf{a}+3\mathbf{b}-\mathbf{c}=(1-s-t)\,(\mathbf{a}-2\mathbf{b}+3\mathbf{c})+s\,(3\mathbf{a}+4\mathbf{b}-2\mathbf{c})$$
$$+t\,(\mathbf{a}-6\mathbf{b}+6\mathbf{c}) \quad (3)$$

which satisfies the condition (2)

For taking L H S to R H S the sum of the coefficients is

$$(1-s-t)+s+t-1, \text{ i e } 0$$

Since **a**, **b**, **c** ar non-coplanar, equating the coefficients of **a**, **b**, **c** in (3), we get

$$1-s-t+3s+t=2 \text{ or } 2s=1, \qquad s=\tfrac{1}{2}$$
$$(1-s-t)\,(-2)+4s-6t=3 \text{ or } 6s-4t=5$$
$$-4t=5-6=5-3=2, \qquad t=-\tfrac{1}{2}$$

and $(1-s-t)\,(3)-2s+6t=-1$ or $-5s+3t=-4$

The last one is satisfied for $s=\frac{1}{2}$ and $t=-\frac{1}{2}$

or $(\Sigma x l_1)\, \mathbf{a} + (\Sigma x m_1)\, \mathbf{b} + (\Sigma x n_1)\, \mathbf{c} = 0$

Now a, b, c are three non coplanar vectors and hence if there exists a relation of the form $l\,\mathbf{a} + m\,\mathbf{b} + n\,\mathbf{c} = 0$ between them, then $l=0$, $m=0$ $n=0$ (for otherwise one would be expressible as a linear combination of the other two which would mean that a, b, c are coplanar)

$$x l_1 + y l_2 + z l_3 + w l_4 = 0 \qquad (1)$$

$$x m_1 + y m_2 + z m_3 + w m_4 = 0 \qquad (2)$$

$$x n_1 + y n_2 + z n_3 + w n_4 = 0 \qquad (3)$$

Also $\quad x + y + z + w = 0 \qquad (5)$

Eliminating x, y, z and w, we get

$$\begin{vmatrix} l_1 & l_2 & l_3 & l_4 \\ m_1 & m & m_3 & m_4 \\ n_1 & n_2 & n_3 & n_4 \\ 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

35 $(\mathbf{i} + \mathbf{j} + 3\mathbf{k})\, x + (3\mathbf{i} - 3\mathbf{j} + \mathbf{k})\, y + (-4\mathbf{i} + 5\mathbf{j})\, z = \lambda\, (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$

$$(x + 3y - 4z - \lambda x)\, \mathbf{i} + (x - 3y + 5z - \lambda y)\, \mathbf{j} + (3x + y + 0z - \lambda z)\mathbf{k} = 0$$

Above is a relation of the form

$$l\,\mathbf{i} + m\,\mathbf{j} + n\,\mathbf{k} = 0$$

where i j and k are non-coplanar and hence each of th coefficients l, m, n is zero

$$(1-\lambda)\, x + 3y - 4z = 0$$

$$x - (3+\lambda)\, y + 5z = 0$$

$$3x + y - \lambda z = 0$$

Eliminating x, y, z *we get*

$$\begin{vmatrix} 1-\lambda & 3 & -4 \\ 1 & -3-\lambda & 5 \\ 3 & 1 & -\lambda \end{vmatrix} = 0$$

or $(1-\lambda)\, (3\lambda + \lambda^2 - 5) - 3\, (-\lambda - 15) - 4\, (1 + 9 + 3\lambda) = 0$

$$-\lambda^3 + \lambda^2\, (1-3) + \lambda\, (5+3+3-12) + (-5+45-40) = 0$$

$$-\lambda^3 - 2\lambda^2 - \lambda = 0 \text{ or } \lambda\, (\lambda^2 + 2\lambda + 1) = 0$$

$$\lambda\, (\lambda+1)^2 = 0 \text{ which gives } \lambda = 0, -1$$

and $\quad 2\ \frac{1}{\sqrt{26}}(i-3j+4k)=\frac{1}{\sqrt{26}}(2i-6j+8k)$ (3)

Adding 1, 2 and 3 the resultant R is

$$R=\frac{1}{\sqrt{26}}(i-9j-6k)$$

Its magnitude is

$$\sqrt{\left(\frac{1+81+36}{26}\right)}=\sqrt{\left(\frac{118}{26}\right)}$$

Also d c's of the resultant are

$$\frac{1}{\sqrt{(118)}},\frac{-9}{\sqrt{(118)}},\frac{-6}{\sqrt{(118)}}$$

8 Refer Q 4, $\overrightarrow{OB}+\overrightarrow{OC}=2\overrightarrow{OD}$,

$\overrightarrow{OC}+\overrightarrow{OA}=2\overrightarrow{OE}$ and $\overrightarrow{OA}+\overrightarrow{OB}=2\overrightarrow{OF}$

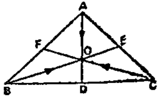


Fig 10

Adding we get

$2(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC})=2(\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF})$

or $\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OD}+\overrightarrow{OE}+\overrightarrow{OF}$

2nd part $\frac{2}{3}BE=BO$, $\frac{1}{3}CF=OF$

$\overrightarrow{AD}+\frac{2}{3}\overrightarrow{BE}+\frac{1}{3}\overrightarrow{CF}$,

$=\overrightarrow{AD}+\overrightarrow{BO}+\overrightarrow{OF}$

$=\overrightarrow{AD}+\overrightarrow{BF}$ But $BF=FA$. $\overrightarrow{BF}=\overrightarrow{FA}$

$=\overrightarrow{FA}+\overrightarrow{AD}=\overrightarrow{FD}=\frac{1}{2}\overrightarrow{AC}$

FD is the line joining the mid points of the sides of a triangle and hence parallel to AC and half its length

Finally, $\overrightarrow{OE}+\overrightarrow{OF}+\overrightarrow{DO}$

$=\frac{1}{2}(\overrightarrow{OA}+\overrightarrow{OC})+\frac{1}{2}(\overline{C}\ 1+\overrightarrow{OB})+\frac{1}{2}(\overrightarrow{BO}+\overrightarrow{OC})$

$=\overrightarrow{OA}+\frac{1}{2}(\overrightarrow{OB}+\overrightarrow{BO})+\frac{1}{2}(\overrightarrow{OC}+\overrightarrow{CO})$

$=\overrightarrow{OA}+0+0=\overrightarrow{OA}$

9 $\overrightarrow{AB}+\overrightarrow{AD}=2\overrightarrow{AF}$ where F is mid point of BD by Q 4

$\overrightarrow{CB}+\overrightarrow{CD}=2\overrightarrow{CF}$,

$\overrightarrow{AB}+\overrightarrow{AD}+\overrightarrow{CB}+\overrightarrow{CD}$

$\vec{OR}=\vec{OQ}+\vec{QR}=\vec{OQ}+6i=(8-6\sqrt{2})\,i+(2\sqrt{3}-6\sqrt{2})\,j$

and finally we have

$\vec{OS}=\vec{OR}+\vec{RS}=\vec{OR}+(-9\cos 45°\,i+9\sin 45°\,j)$

$=\left(8-6\sqrt{2}-\frac{9}{\sqrt{2}}\right)i+\left(2\sqrt{3}-6\sqrt{2}+\frac{9}{\sqrt{2}}\right)j$

$=\frac{1}{\sqrt{2}}(8\sqrt{2}-21)\,i+\frac{1}{\sqrt{2}}(2\sqrt{6}-3)\,j$

$OS=\frac{1}{\sqrt{2}}\sqrt{\left[(8\sqrt{2}-21)^2+(2\sqrt{6}-3)^2\right]}$

$=\frac{1}{\sqrt{2}}\sqrt{\left[602-336\sqrt{2}-12\sqrt{6}\right]}$

$=\sqrt{(301-168\sqrt{2}-6\sqrt{6})}$

which is the magnitude of the resultant displacement and if OS makes an angle θ with OA, then $\tan\theta=\frac{2\sqrt{6}-3}{8\sqrt{2}-21}$, which gives the direction

§ 7 Scalar Products or dot products

(i) **Definition** *The scalar product of two vectors* **a** *and* **b** *of module a and b respectively is equal to ab cos θ where θ is the angle between the directions of* **a** *and* **b**

This product is expressed as **a**·**b** *i e* by putting a dot between the vectors **a** and **b**   **a**·**b**$=ab\cos\theta$

(ii) **Geometrical Interpretation of dot product**

OL is the projection of vector **b** in the direction of [See the figure] vector **a** and $OL=b\cos\theta$

a·**b**$=a\,(b\cos\theta)=OA\cdot OL$

$=b\,(a\cos\theta)=OB\cdot OM$

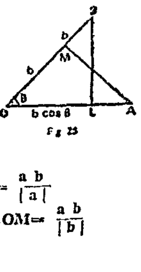


Fig 25

i e **a**·**b** *is equal to the product of the length of one of them with the resolved part of the other (i e projection) in the direction of the other Thus*

Projection of b in the direction of OA=OL= $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

and Projection of a in the direction of OB=OM= $\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$

(iii) **Properties**

(a) **Commutative**   **a**·**b**$=$**b**·**a**$=ab\cos\theta$

(b) If **a**·**b**$=0$ then $ab\cos\theta=0$ which implies either $a=0$ or $b=0$ *or* $\cos\theta=0$ *(i e* $\theta=\pi/2$) Hence *if the scalar product of two vectors is zero, then at least one of the vectors is a zero vector or they are perpendicular.*

(ii) $\vec{OA}+\vec{OB}+\vec{OC}=2\vec{OO}$

$\vec{OB}+\vec{OC}=2\vec{OD}=2(\vec{OO}+\vec{OD})=2\vec{OO}+2\vec{OD}$

$=2\vec{OO}+\vec{AO}$ by (1)

$=2\vec{OO}-\vec{OA}$

$\vec{OA}+\vec{OB}+\vec{OC}=2\vec{OO}$

(iii) We have to prove

$\vec{AO}+\vec{OB}+\vec{OC}=2\vec{AO}=\vec{AP}$

L H S $=2\vec{AO}-\vec{AO}+\vec{OB}+\vec{OC}$ But $-\vec{AO}=+\vec{O'A}$

$=2\vec{AO}+(\vec{OA}+\vec{OB}+\vec{OC})=2\vec{AO}+2\vec{OO}$ by part (2)

$=2\vec{AO}$

$=2$ (The vector represented by the radius through A of the circumcircle)

$=\vec{AP}$ (where AP is diameter through A of the circum circle)

12 We are given that

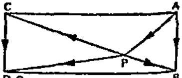


Fig 14

$(\vec{AP}+\vec{PB})+\vec{PC}=\vec{PQ}$

or $\vec{AB}+\vec{PC}=\vec{PQ}$

$\vec{AB}=\vec{PB}-\vec{PC}=\vec{CQ}$

AB is parallel and equal to CQ

Again writing the given relation as $(\vec{AP}+\vec{PC})+\vec{PB}=\vec{PQ}$

we get $\vec{AC}=\vec{PQ}-\vec{PB}=\vec{BQ}$

AC is parallel and equal to BQ (2)

Hence from (1) and (2) the figure $ABQC$ is a parallelogram

Again $\vec{AQ}=\vec{AB}+\vec{BQ}=\vec{AB}+\vec{AC}$, $\vec{BQ}=\vec{AC}$ by (2)

Above relation shows that with the change in the position of P in BC the position vector of Q does not change and hence Q is a fixed point

13 $\vec{OP}=\vec{OR}+\vec{RP}$

$n\vec{OP}=n\vec{OR}+n\vec{RP}$ (1)

Again if the point of application of a force **F** moves first O to B and then from B to C then the work done is

$$\mathbf{F}\cdot\overrightarrow{OB}+\mathbf{F}\cdot\overrightarrow{BC}=\mathbf{F}\cdot(\overrightarrow{OB}+\overrightarrow{BC})=\mathbf{F}\cdot\overrightarrow{OC}$$

Above shows that sum of the works done by a force in two consecutive displacements OB and BC is equal to the work done by the force in single displacement OC

§ 8 **Vector product or cross product** (Roorkee 77)

(i) **Definition** *The vector product of two non-null and non-parallel vector* **a** *and* **b** *is a vector whose module is* $ab \sin \theta$, θ *being the angle between the directions of* **a** *and* **b** *and whose direction is that of a unit vector* $\hat{\mathbf{n}}$ *perpendicular to both* **a** *and* **b** *such that* **a**, **b** $\hat{\mathbf{n}}$ *are in the right handed orientation By the right handed orientation we mean that if we turn the vector* **a** *into the vector* **b** *then,* **n** *will point in the direction in which a right handed screw would move if turned in the same manner*

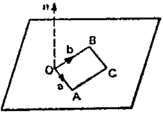


Fig 6

Thus $\mathbf{a}\times\mathbf{b}=ab\sin\theta\ \hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector perpendicular to the plane of **a** and **b** such that **a**, **b**, $\hat{\mathbf{n}}$ form a right handed orientation

When **a** or **b** or both are null vectors or **a** is parallel to **b** then $\hat{\mathbf{n}}$ is not defined

In this case, we agree to write

$$\mathbf{a}\wedge\mathbf{b}=0$$

(ii) **Geometrical Interpretation**

The modulus $ab \sin \theta$ of $\mathbf{a}\times\mathbf{b}$ is the area of the parallelogram whose adjacent sides are represented by **a** and **b** or it is twice the area of the triangle OAB

(iii) **Properties**

(a) **Vector Product is not commutative**

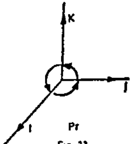
i e $\mathbf{a}\times\mathbf{b}\neq\mathbf{b}\times\mathbf{a}$ but $\mathbf{a}\times\mathbf{b}=-(\mathbf{b}\times\mathbf{a})$

(b) $(m\mathbf{a})\times\mathbf{b}=m(\mathbf{a}\times\mathbf{b})=\mathbf{a}\times(m\mathbf{b})$

(c) If the vectors are collinear, then

$\mathbf{a}\times\mathbf{b}=0$, $\theta=0$ or π

As a consequence of above $\mathbf{a}\times\mathbf{a}=0$



Pr
Fig 27

(d) **Unit Vectors i, j, k**

Since A, B, C are collinear, we have

$v=\lambda+\mu\, i\, c$, $\frac{ON}{OC}=\frac{OL}{OA}+\frac{OM}{OB}$

15 Let the position vectors of the vertices be **a**, **b** and **c** respectively so that the position vector of G, the centroid, is $\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

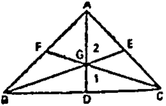


Fig 17

$\overrightarrow{GA}$ = P V of A − P V of G

$$=\mathbf{a}-\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}=\frac{2\mathbf{a}-\mathbf{b}-\mathbf{c}}{3}$$

Similarly $\overrightarrow{GB}=\frac{2\mathbf{b}-\mathbf{c}-\mathbf{a}}{3}$, $\overrightarrow{GC}=\frac{2\mathbf{c}-\mathbf{a}-\mathbf{b}}{3}$

$$\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\tfrac{1}{3}\,(2\Sigma\mathbf{a}-2\Sigma\mathbf{a})=0$$

Alt $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\tfrac{2}{3}[\overrightarrow{DA}+\overrightarrow{EB}+\overrightarrow{FC}]$

But $\overrightarrow{DA}=\overrightarrow{DB}+\overrightarrow{BA}=\tfrac{1}{2}\,\overrightarrow{CB}+\overrightarrow{BA}$ Similarly $\overrightarrow{EB}=\tfrac{1}{2}\,\overrightarrow{CA}+\overrightarrow{AB}$

and $\overrightarrow{FC}=\tfrac{1}{2}\,\overrightarrow{BA}+\overrightarrow{AC}$

$$\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=\tfrac{2}{3}[\tfrac{1}{2}\overrightarrow{BA}+\tfrac{1}{2}\,\overrightarrow{AC}+\tfrac{1}{2}\,\overrightarrow{CB}]=-\tfrac{1}{3}\,(\overrightarrow{AB}+\overrightarrow{BC}+\overrightarrow{CA})=0$$

Conversely Let $\overrightarrow{GA}+\overrightarrow{GB}+\overrightarrow{GC}=0$

Then $(\overrightarrow{OA}-\overrightarrow{OG})+(\overrightarrow{OB}-\overrightarrow{OG})+(\overrightarrow{OC}-\overrightarrow{OG})=0$

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=3\overrightarrow{OG}$$

or $\overrightarrow{OG}=\frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{3}$

Hence G is the centriod of the points A, B and C

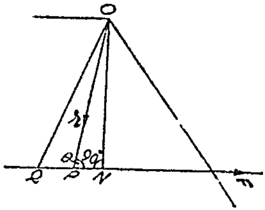
16 Refer fig Q 15

The mid points D, E and F are $\frac{\mathbf{b}+\mathbf{c}}{2}$, $\frac{\mathbf{c}+\mathbf{a}}{2}$, $\frac{\mathbf{a}+\mathbf{b}}{2}$

We know that centroid divides a median AD in the ratio 2 1

Hence the point G is $\frac{2\,\frac{\mathbf{b}+\mathbf{c}}{2}+1\,\mathbf{a}}{2+1}=\frac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{3}$

The symmetry of the result shows that the point G lies on the median BE and CF as well and divides them in the ratio



Thus $|\mathbf{M}| = F\,ON$

$= F\,OP \sin\theta$

If $\mathbf{r}$ be the position vector of any point P on the line of action of force $\mathbf{F}$, then

$|\mathbf{r}\times\mathbf{F}| = F\,OP\sin\theta$

$|\mathbf{M}| = |\mathbf{r}\times\mathbf{F}|$

The direction of vector moment is to be determined by the right handed orientation as explained earlier Thus in the figure the direction of the vector moment is along the normal drawn through O on the plane of the paper and pointing towards the reader

$\mathbf{M}=\mathbf{r}\times\mathbf{F}$ and its magnitude is $F\,ON$ or $Fr\sin\theta$

Moment is independent of the position P on the line of action of F

We have shown that moment of force $\mathbf{F}$ about a point O is $\mathbf{r}\times\mathbf{F}$ where $\mathbf{r}$ is the position vector of any point P on the line of action of $\mathbf{F}$ relative to O

If in place of P we choose any other point Q on the line of action of $\mathbf{F}$ whose position vector relative to O be $\mathbf{q}$ then moment is

$$\mathbf{M} = \mathbf{q}\times\mathbf{F}$$

But $$\mathbf{q} = \overrightarrow{OQ} = \overrightarrow{OP} + \overrightarrow{PQ} = \mathbf{r} + \overrightarrow{PQ}$$

$$\mathbf{M} = \mathbf{q}\times\mathbf{F} = (\mathbf{r} + \overrightarrow{PQ})\times\mathbf{F} = \mathbf{r}\times\mathbf{F} = \mathbf{M}$$

since $\overrightarrow{PQ}\times\mathbf{F} = 0$ as both $\overrightarrow{PQ}$ and $\mathbf{F}$ are in the same line

Working Rule In order to get the vector moment of a force $\mathbf{F}$ about any point O find the vector product $\mathbf{r}\times\mathbf{F}$ where $\mathbf{r}$ is the position vector of any point P on the line of action of the force relative to origin O

and mid point L of CF is $\frac{1}{2}(c+c+a)=\frac{2c+a}{2}$

Mid point of LM is $\frac{a+2b+2c+a}{4}=\frac{a+b+c}{2}$

In a similar manner we can show that the mid points of the join of other opposite edges are also the same

19 Let A be taken as origin and the position vectors of B, C and D be taken as **b**, **c** and **d**

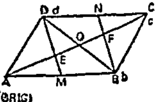


Fig. 19

$\vec{BC}=c-b$ and $\vec{AD}=d$

But BC is parallel and equal to AD

$\vec{BC}=\vec{AD}$

or $c-b=d$ or $c=b+d$ or $\frac{c}{2}=\frac{b+d}{2}$

i.e. mid point of diagonal AC is same as mid point of diagonal BD. Hence the diagonals bisect each other

Converse We are given that diagonals bisect each other

i.e. $\frac{c}{2}=\frac{b+d}{2}$ or $c=b+d$ (1)

$\vec{AB}=b$, $\vec{DC}=c-d=b$ by (1)

AB is parallel and equal to DC

$\vec{AD}=d$ and $\vec{BC}=c-b=d$ by (1)

AD is parallel and equal to BC

Hence the figure is a parallelogram

20 Let O be any origin. Then

$\vec{OA}+\vec{OC}=2\vec{OO}$ [O is the mid pt of AC]

and $\vec{OB}+\vec{OD}=2\vec{OO}$ [O is also the mid pt of BD]

Hence $\vec{OA}+\vec{OB}+\vec{OC}+\vec{OD}=4\vec{OO}$

as required

21 Refer figure Q 19

The figure being a parallelogram we have as in Q 19

$c=b+d$ (1)

The P.V. of a point E which divides AC in the ratio 1 : 2 is

$=b\,(c\times a)=(c\times a)\,b$ (2)

$=c\,(a\times b)=(a\times b)\,c$ (3)

From (1) and (3) we conclude that $a\,(b\times c)=(a\times b)\,c$, and similarly from (2) and (1) $b\,(c\times a)=(b\times c)\,a$ etc

From above we observe that in scalar triple product $[a\ b\ c]=a\,(b\times c)=(a\times b)\,c$, the position of dot and cross can be interchanged at pleasure provided we maintain the cyclic order of a, b, c (Roorkee 1981)

With every change of cyclic order there will be a change of sign

i e $[a\ b\ c]=-[a\ c\ b]$ or $=-[b\ a\ c]$

This follows from the fact that if in a determinant we interchange any two adjacent rows or columns the determinant retains its value but changes its sign

(d) Scalar triple product is zero when two vectors are equal

i e $[a\ a\ c]=0$ or $[a\ b\ b]=0$ etc

This follows from the fact that a determinant is zero when two rows or columns are identical

(e) *Condition for Coplanarity*

In case the three vectors are coplanar then the volume of the parallelopiped is zero and hence $[a\ b\ c]=0$ (M N R 1982)

or $\Delta=0$

(f) Distributive law

$a\times(b+c)=a\times b+a\times c$

§ 11 *Vector Triple Product*

$a\times(b\times c)=(a\,c)\,b-(a\,b)\,c$

It is expressible in terms of b and c as a c and a b both are scalars

Similarly $(b\times c)\times a=(a\,b)\,c-(a\,c)\,b$

$a\times(b\times c)=\ (b\times c)\times a$

Even though the cyclic order of the vectors is maintained but the sign is changed In dot product $a\,(b\times c)=(b\times c)\,a$

Similarly $b\times(c\times a)=(b\,a)\,c-(b\,c)\,a$

Thus $a\times(b\times c)\neq b\times(c\times a)$,

since L H S is linear combination of b and c whereas R H S is linear combination of c and a

§ 12 *Scalar product of four vectors*

$$(a\times b)\,(c\times d)=\begin{vmatrix} a\,c & b\,c \\ a\,d & b\,d \end{vmatrix}$$

$$\overrightarrow{FE} = \text{P V of } E - \text{P V of } F = \frac{c+a}{2} - \frac{a+b}{2} = \tfrac{1}{2}(c-b)$$

$$\overrightarrow{FE} = \tfrac{1}{2}\overrightarrow{BC}$$

Above shows that FE is parallel to BC and half its length

25 Take A as origin and the position vector of B and D as b and d respectively

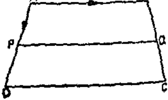


Fig. 23

Now DC is parallel to AB

$$\overrightarrow{DC} = t\overrightarrow{AB} = t\text{b}$$

$$\frac{DC}{AB} = t \qquad (1)$$

$$\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC} = \text{d} + t\text{b}$$

A is 0, B (b), D (d) C (d$+t$b)

P, the mid point of AD, is $\frac{\text{d}}{2}$ and Q, the mid point of BC, is $\frac{\text{b}+\text{d}+t\text{b}}{2}$

$$\overrightarrow{PQ} = \text{P V of } Q - \text{P V of } P = \frac{\text{b}+\text{d}+t\text{b}}{2} - \frac{\text{d}}{2} = \tfrac{1}{2}(1+t)\,\text{b}$$

or $\overrightarrow{PQ} = \frac{1}{2}(1+t)\,\overrightarrow{AB}$

$\overrightarrow{PQ}$ is scalar multiple of $\overrightarrow{AB}$ and hence PQ is parallel to AB

Also $\frac{PQ}{AB} = \frac{1}{2}(1+t) = \frac{1}{2}\left(1 + \frac{DC}{AB}\right) = \frac{1}{2}\frac{(AB+DC)}{AB}$

$PQ = \frac{1}{2}(AB+DC)$

i e PQ = half the sum of parallel sides,

26 Proceeding as in Q 25 the position vector of C is d$+t$b where

$$t = \frac{DC}{AB} \qquad (1)$$

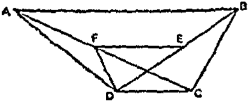
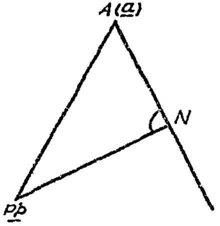


Fig 24

§ 14 To find the perpendicular distance of a given point from a given line



Let the given line pass through a given point A (a) and be parallel to the unit vector **b** and let P (**p**) be the given point From P draw PN perpendicular to the given line Then

$\vec{PA}=\mathbf{a}-\mathbf{p}$ or $\vec{AP}=\mathbf{p}-\mathbf{a}$

Now $\vec{AP}\times\mathbf{b}=|\vec{AP}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\angle PAN=\theta$

or $|\vec{AP}\times\mathbf{b}|=AP\sin\theta,\qquad |\mathbf{b}|=1$

$|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|=PN$ since $PN=AP\sin\theta$

Thus required perpendicular distance $PN=|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|$

Alter $AN=$ Projection of AP on the given line which is parallel to the unit vector **b**

$=\vec{AP}\cdot\mathbf{b}=(\mathbf{p}-\mathbf{a})\cdot\mathbf{b}$

Then $PN^2=AP^2-AN^2=(\vec{AP})^2-(\vec{AN})^2$

$=(\mathbf{p}-\mathbf{a})^2-[(\mathbf{p}-\mathbf{a})\cdot\mathbf{b}]^2$

which is the square of the length of perpendicular PN

Problem Set (B)

1 Find the cosine of the angle between the directions of the vectors $\mathbf{a}=4\mathbf{i}+3\mathbf{j}+\mathbf{k},\quad \mathbf{b}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$
Also find a unit vector perpendicular to both **a** and **b**
What is the sine of the angle between **a** and **b**

2 Find a unit vector perpendicular to both the vectors
(a) $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$

$x\mathbf{a}+y\mathbf{b}=-z\mathbf{c}=(x+y)\,\mathbf{c}$, $\because$ $x+y=-z$

or $\mathbf{c}=\dfrac{x\,\mathbf{a}+y\,\mathbf{b}}{x+y}$

Above relation shows that **b** divides the join of **a** and **b** in the ratio $y : x$. Hence the three points A, B, C are collinear

29 (*a*) Let the given points be A, B and C

$\vec{AB}=$ P V of B − P V of A

$=(2\mathbf{a}+3\mathbf{b}-4\mathbf{c})-(\mathbf{a}-2\mathbf{b}+3\mathbf{c})=\mathbf{a}-5\mathbf{b}-7\mathbf{c}$

$\vec{AC}=$ P V of C − P V of A

$=(-7\mathbf{b}+10\mathbf{c})-(\mathbf{a}-2\mathbf{b}+3\mathbf{c})=-\mathbf{a}-5\mathbf{b}+7\mathbf{c}=-\vec{AB}$

Since $\vec{AC}=-\vec{AB}$, it follows that the points A, B and C are collinear

2nd Method

Let $\mathbf{l}=\mathbf{a}-2\mathbf{b}+3\mathbf{c}$, $\mathbf{m}=2\mathbf{a}+3\mathbf{b}-4\mathbf{c}$, $\mathbf{n}=-7\mathbf{b}+10\mathbf{c}$

Now if we are able to choose three scalars x, y, z such that $x\mathbf{l}+y\mathbf{m}+z\mathbf{n}=0$ and also $x+y+z=0$

then the three points will be collinear. Choosing (by trial) $x=2$, $y=-1$, $z=-1$ we get $x+y+z=2-1-1=0$ and also $x\mathbf{l}+y\mathbf{m}+z\mathbf{n}=2\,(\mathbf{a}-2\mathbf{b}+3\mathbf{c})-1\,(2\mathbf{a}+3\mathbf{b}-\mathbf{c})4-1\,(-7\mathbf{b}+10\mathbf{c})$
$=0$

Hence the three points are conllinear

(b), (c), (d) Do yourself

(e) In terms of unit vectors

$\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$ etc

$\vec{AB}=$ P V of B − P V of $A=2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$

$\vec{BC}=$ P V of C − P V of $B=-6\mathbf{i}-6\mathbf{j}-12\mathbf{k}$

$=-3\,(2\mathbf{i}+2\mathbf{j}+4\mathbf{k})=-3\vec{AB}$

$3\vec{BA}=\vec{BC}$ and hence the three points are collinear

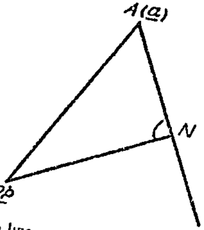
From the relation $3\vec{BA}=\vec{BC}$ we conclude that

$\dfrac{BC}{BA}=\dfrac{3}{1}$ (See Q 28)

30 (*a*) We know that three points whose P Vs are α, β, γ are collinear if there exists a relation of the form $x\alpha+y\beta+z\gamma=0$ where $x+y+z=0$

Now $x\alpha+y\beta+z\gamma=0$ gives

§ 14 To find the perpendicular distance of a given point from a given line



Let the given line pass through a given point A (a) and be parallel to the unit vector b and let P (p) be the given point. From P draw PN perpendicular to the given line. Then

$\vec{PA}=\mathbf{a}-\mathbf{p}$ or $\vec{AP}=\mathbf{p}-\mathbf{a}$

Now $\vec{AP}\times\mathbf{b}=|\vec{AP}|\,|\mathbf{b}|\sin\theta\,\hat{\mathbf{n}}$ where $\angle PAN=\theta$

or $|\vec{AP}\times\mathbf{b}|=AP\sin\theta$ $\quad |\mathbf{b}|=1$

$|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|=PN$ since $PN=AP\sin\theta$

Thus required perpendicular distance $PN=|(\mathbf{p}-\mathbf{a})\times\mathbf{b}|$

Aliter $AN=$ Projection of AP on the given line which is parallel to the unit vector b

$=\vec{AP}\cdot\mathbf{b}=(\mathbf{p}-\mathbf{a})\cdot\mathbf{b}$

Then $PN^2=AP^2-AN^2=(\vec{AP})^2-(\vec{AN})^2$

$=(\mathbf{p}-\mathbf{a})^2-[(\mathbf{p}-\mathbf{a})\cdot\mathbf{b}]^2$

which is the square of the length of perpendicular PN

Problem Set (B)

Find the cosine of the angle between the directions of the vectors $\mathbf{a}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$

Also find a unit vector perpendicular to both a and b

What is the sine of the angle between a and b

Find a unit vector perpendicular to both the vectors

(a) $\mathbf{a}=2\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=3\mathbf{i}+4\mathbf{j}-\mathbf{k}$

where $L+M+N+P=\frac{p}{p+q}+\frac{q}{p+q}-\frac{m}{m+n}-\frac{n}{m+n}=1-1=0$

Hence the condition is necessary

Converse Let $l\mathbf{a}+m\mathbf{b}+n\mathbf{c}+p\mathbf{d}=0$ where

$$l+m+n+p=0 \quad (1)$$

and we will show that the points A, B, C, D are coplanar

Now of the three scalars $l+m, l+n, l+p$ one at least is not zero for if all of them are zero then

$$l+m=0,\ l+n=0,\ l+p=0$$

$$m=n=p=-l$$

Hence $l+m+n+p=0 \Rightarrow l-3l=0 \Rightarrow l=0$

Hence $m=n=p=-l=0$

Thus $l=0, m=0, n=0, p=0$ which is against the hypothesis

Let us sppose that $l+m$ is not zero and therefore

$$l+m=-(n+p)\neq 0 \text{ by (1)} \quad (2)$$

Also from the given relation, we have

$$l\,\mathbf{a}+m\,\mathbf{b}=-(n\,\mathbf{c}+p\,\mathbf{d})$$

or $$\frac{l\,\mathbf{a}+m\,\mathbf{b}}{l+m}=\frac{(n\,\mathbf{c}+p\,\mathbf{d})}{n+p} \text{ by (2)} \quad (3)$$

L H S represents a point which divides AB in the ratio m l and R H S represents a point which divides CD in the ratio p n These points being the same, it follows that a point on AB is the same as a point on CD showing that the lines AB and CD intersect Hence the four points A, B, C and D are coplanar

32 (a) If the given vectors are coplanar then we should be able to express one of them as a linear combination of the other two

Let us assume that

$3\mathbf{a}-7\mathbf{b}-4\mathbf{c}=x\,(3\mathbf{a}-2\mathbf{b}+\mathbf{c})+y\,(\mathbf{a}+\mathbf{b}+2\mathbf{c})$

where x and y are scalars Since a b c are non coplanar equating the coefficients of a b and c we get ,

$$3x+y=3 \quad -2x+y=-7,\ x+2y=-4$$

Solving the first two, we find that $x=2$ and $y=-3$ These values of x and y satisfy the third equation as well Hence the given vectors are coplanar

(b) Here $5\mathbf{a}+6\mathbf{b}+7\mathbf{c}=\frac{1}{2}(7\mathbf{a}-8\mathbf{b}+9\mathbf{c})+\frac{1}{2}(3\mathbf{a}+20\mathbf{b}+5\mathbf{c})$

(c) $\mathbf{a}-2\mathbf{b}+3\mathbf{c}=\frac{1}{3}(\mathbf{a}-3\mathbf{b}+5\mathbf{c})-\frac{1}{3}(-2\mathbf{a}+3\mathbf{b}-4\mathbf{c})$

angles with the vectors $\mathbf{a}=\frac{1}{3}(\mathbf{i}-2\mathbf{j}+2\mathbf{k})$, $\mathbf{b}=\frac{1}{5}(-4\mathbf{i}-3\mathbf{k})$ and $\mathbf{c}=\mathbf{j}$ (Roorkee 87)

9 (a) The position vector of the points A, B, C, D are $\mathbf{i}+\mathbf{j}+\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}$, $3\mathbf{i}+5\mathbf{j}-2\mathbf{k}$, $\mathbf{k}-\mathbf{j}$ respectively. Show that AB and CD are parallel.

(b) The position vectors of A, B, C, D with respect to the origin are respectively $2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$, $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $-5\mathbf{i}+4\mathbf{j}-2\mathbf{k}$, $\mathbf{i}+10\mathbf{j}+10\mathbf{k}$. Show that AB and CD are parallel. (M N R 82)

10 P, Q, R, S are the points $\mathbf{i}-\mathbf{k}$, $-\mathbf{i}+\mathbf{j}$, $2\mathbf{i}-3\mathbf{k}$ and $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$ respectively. Show that the projection of PQ on RS is equal to that of RS on PQ each being $-4/3$. Also find the cosine of their inclination.

11 (a) Find the projection of $\mathbf{b}=2\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ in the direction of vector $\mathbf{a}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$. What is the vector determined by the projection ?

(b) If $\mathbf{a}=4\mathbf{i}+6\mathbf{j}$ and $\mathbf{b}=3\mathbf{j}+4\mathbf{k}$ find the vector form of the component of $\mathbf{a}$ along $\mathbf{b}$. (M N R 87)

(c) Let $\beta=4\mathbf{i}+3\mathbf{j}$ and γ be two vectors perpendicular to each other in the $x-y$ plane. All the vectors in the same plane having projections 1 and 2 along β and γ respectively are given by (I I T 87)

12 Prove by vector method that in a right angled triangle ABC, $AB^2+AC^2=BC^2$, angle A being a right angle. Also prove that mid point of the hypotenuse is equidistant from the vertices.

13 If P be the middle point of the side BC of a triangle ABC prove that $AB^2+AC^2=2(AP^2+BP^2)$

14 Prove that the medians to the base of an isosceles triangle is perpendicular to the base.

15 Show that in a parallelogram the sum of the squares on the diagonals is twice the sum of the squares on two adjacent sides or sum of the squares on all the sides. The difference of that squares on the diagonals is four times the rectangle contained by either of these sides and the projection of the other upon it. The difference of the squares on two adjacent sides is equal to the rectangle contained by either diagonal and the projection of the other upon it.

16 Prove that the parallelogram whose diagonals are equal is a rectangle.

Thus the four points can be put in the form

$l\alpha+m\beta+n\gamma+p\delta=0$ where $l+m+n+p=0$

Therefore the four points are coplanar

(b) A, $6\mathbf{a}+3\mathbf{b}-\mathbf{c}$, B, $2\mathbf{a}-\mathbf{b}+3\mathbf{c}$, C, $-\mathbf{a}+2\mathbf{b}-4\mathbf{c}$, D, $-12\mathbf{a}-\mathbf{b}-3\mathbf{c}$

$$\vec{AB}=\vec{OB}-\vec{OA}=-4\mathbf{a}-3\mathbf{b}+4\mathbf{c}$$

$$\vec{BC}=\vec{OC}-\vec{OB}=-3\mathbf{a}+3\mathbf{b}-7\mathbf{c}$$

$$\vec{CD}=\vec{OD}-\vec{OC}=-11\mathbf{a}-3\mathbf{b}+\mathbf{c}$$

If the given four points are coplanar then the three vectors written above are coplanar and as such one of them is expressible as a linear combination of the other two

Let $-4\mathbf{a}-3\mathbf{b}+4\mathbf{c}=x(-3\mathbf{a}+3\mathbf{b}-7\mathbf{c})+y(-11\mathbf{a}-3\mathbf{b}+\mathbf{c})$

Comparing the coefficients of a, b, c we get

$$-3x-11y=-4,\ 3x-3y=-3,\ -7x+y=4$$

Solving the first two we get $y=\frac{1}{2}$, $x=-\frac{1}{2}$ and these values satisfy the third also Hence the given vectors are coplanar or the given four points are coplanar

2nd Method

Let us write

$$6\mathbf{a}+3\mathbf{b}-\mathbf{c}=(1-s-t)(2\mathbf{a}-\mathbf{b}+3\mathbf{c})+s(-\mathbf{a}+2\mathbf{b}-4\mathbf{c})+t(-12\mathbf{a}-\mathbf{b}-3\mathbf{c})$$

It is of the form $l\alpha+m\beta+n\gamma+p\delta=0$ when L H S is brought to R H S and

$$l+m+n+p=(1-s-t)+s+t-1=0$$

Equating the coefficients of a b and c on both sides we get

$$6=(1-s-t)(2)+s(-1)+t(-12) \text{ or } -3s-14t=4$$

$$2=(1-s-t)(-1)+s(2)+t(-1) \text{ or } 3s=3$$

$s=1$ and $t=-\frac{1}{2}$

$$-1=(1-s-t)(3)+s(-4)+t(-3) \text{ or } -7s-6t=-4$$

The last one is also satisfied by $s=1$ and $t=-\frac{1}{2}$

Hence the four points are coplanar

(c) Proceed as in part (a) or (b)

34 We knwon that the four points α β, γ, δ are coplanar if there exists a relation of the form

$$x\alpha+y\beta+z\gamma+w\delta=0 \qquad \text{such that}$$

$$x+y+z+w=0$$

$$x(l_1\mathbf{a}+m_1\mathbf{b}+n_1\mathbf{c})+y(l_2\mathbf{a}+m_2\mathbf{b}+n_2\mathbf{c})+z(l_3\mathbf{a}+m_3\mathbf{b}+n_3\mathbf{c})+u(l_4\mathbf{a}+m_4\mathbf{b}+n_4\mathbf{c})=0$$

34 If a, b, c are coplanar vectors, prove that

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{c} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}=0$$

35 Prove that $(\mathbf{a}\times\mathbf{b})^2=\mathbf{a}^2\mathbf{b}^2-(\mathbf{a}\cdot\mathbf{b})^2=a^2b^2-(\mathbf{a}\cdot\mathbf{b})^2$

or $(\mathbf{a}\times\mathbf{b})^2=\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}$ (Roorkee 85, 81, 79, 75)

36 Prove that $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=2\mathbf{a}\times\mathbf{b}$ and interpret it

37 Prove that $\mathbf{a}\times(\mathbf{b}+\mathbf{c})+\mathbf{b}\times(\mathbf{c}+\mathbf{a})+\mathbf{c}\times(\mathbf{a}+\mathbf{b})=0$ (Roorkee 81)

38 If $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{d}$ and $\mathbf{a}\times\mathbf{c}=\mathbf{b}\times\mathbf{d}$, show that $\mathbf{a}-\mathbf{d}$ is parallel to $\mathbf{b}-\mathbf{c}$

39 (*a*) If $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$, then prove that b differs from c by a vector which is parallel to a (Roorkee 80)

(*b*) If $\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}\neq 0$, then show that

$\mathbf{a}-\mathbf{c}=k\mathbf{b}$, where k is a scalar (Roorkee 1985)

40 Interpret the equations—

(*i*) $\mathbf{a}\cdot\mathbf{b}=\mathbf{a}\cdot\mathbf{c}$ (*ii*) $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$

and prove that if both the equations hold simultaneously, then $\mathbf{b}=\mathbf{c}$ if $\mathbf{a}\neq 0$

41 (*a*) Find the vector area of a triangle OAB where $\vec{OA}=\mathbf{a}$, $\vec{OB}=\mathbf{b}$ and they are inclined at an angle θ and hence find the vector area of a triangle whose vertices are the points a, b and c (M N R 86)

(*b*) Find the perpendicular distance of the vertex A from the base of a $\triangle ABC$ with a, b, c as position vectors of A, B, C respectively

(*c*) If A, B, C, D are any four points in space, prove that

$|\vec{AB}\times\vec{CD}+\vec{BC}\times\vec{AD}+\vec{CA}\times\vec{BD}|=4$ (area of $\triangle ABC$) (I I T 87)

42 If a, b, c are vectors from the origin to the points A, B, C show that $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$ is perpendicular to the plane ABC

36 When rotated through $\pi/2$, the new x axis is along old y axis and new y axis is along the old negative x axis, z axis remains same as before

Hence the components of A in the new system are

$$A_2, -A_1, A_3$$

37 $\mathbf{r}_1$ is the position vector of P_1 w r t origin O and let the position vector of P be α so that $OP_1=\mathbf{r}_1$, $OP=\alpha$

Now if we choose P as origin then position vector of P_1 will be

$\overrightarrow{PP_1}=\overrightarrow{OP_1}-\overrightarrow{OP}=\mathbf{r}_1-\alpha$

The given relation when written w r t P as origin becomes

$\Sigma a_1(\mathbf{r}_1-\alpha)=0$ or $\Sigma a_1\mathbf{r}_1-\alpha\Sigma a_1=0$

It will reduce to

$\Sigma a_1\mathbf{r}_1=0$ if $\Sigma a_1=0$ i e $a_1+a_2+a_3+ \ldots a_n=0$

38 The resultant of the forces is given by

$$\mathbf{F}_1+\mathbf{F}_2+\mathbf{F}_3+\mathbf{F}_4=\left(\frac{4}{13}U-\frac{4}{13}V+W\cos\theta\right)\mathbf{i}$$
$$+\left(-\frac{12}{13}U-\frac{12}{13}V+W\sin\theta\right)\mathbf{j}$$
$$+\left(\frac{3}{13}U+\frac{3}{13}V-10\right)\mathbf{k}$$

Since the forces are in equilibrium, we have

$$\frac{4}{13}U-\frac{4}{13}V+W\cos\theta=0,$$

$$-\frac{12}{13}U-\frac{12}{13}V+W\sin\theta=0,$$

and $$\frac{3}{13}U+\frac{3}{13}V-10=0$$

Solving these equations for U, V and W, we shall get

$U=\frac{65}{3}(1-3\cot\theta)$, $V=\frac{65}{3}(1+3\cot\theta)$, $W=40\operatorname{cosec}\theta$

39 Let $\mathbf{i}$ and $\mathbf{j}$ be parallel to two adjacent edges OA (east) and OB (north) of a large square table and let P, Q, R, S represent the position of the out after the four successive displacements

Then according to the question, we have

$\overrightarrow{OP}=4\sin 30°\,\mathbf{i}+4\cos 30\,\mathbf{j}=2\,\mathbf{i}+2\sqrt{3}\,\mathbf{j}$

$\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PQ}=\overrightarrow{OP}+(-12\cos 45°\,\mathbf{i}-12\sin 45°\,\mathbf{j})$

$=(2-6\sqrt{2})\,\mathbf{i}+(2\sqrt{3}-6\sqrt{2})\,\mathbf{j}$

Another form of (vi)

If $a+b+c=0$, show that $a\times b=b\times c=c\times a$ (Roorkee 81)

50 Prove by vector methods the following

(*i*) $\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$

(*ii*) $\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$

51 (*a*) Find the volume of the parallelopiped whose edges are represented by the vectors

(*i*) $a=2i-3j+4k,\ b=i+2j-k,\ c=3i-j+2k$

(*ii*) $a=2i-3j,\ b=i+j-k,\ c=3i-k$ (I I T 83)

(*iii*) $a=2i+j-k,\ b=i+2j+3k,\ c=-3i-j+k$

Also find the area of the face made by a and b

(Roorkee 78)

(*iv*) The volume of the parallelopiped whose edges are represented by $-12i+\lambda k,\ 3j-k,\ 2i+j-15k$ is 546 Find the value of λ (M N R 87)

(*b*) For non zero vectors a, b, c $|(a\times b)\cdot c|=|a||b||c|$ holds if and only if

(*i*) $a\cdot b=0,\ b\cdot c=0$ (*ii*) $b\cdot c=0,\ c\cdot a=0$

(*iii*) $c\cdot a=0,\ a\cdot b=0$ (*iv*) $a\cdot b=b\cdot c=c\cdot a=0$ (I I T 82)

52 Find the constant λ so that the vectors

(*i*) $a=2i-j+k,\ b=i+2j-3k,\ c=3i+\lambda j+5k$ (Roorkee 86)

(*ii*) $a=i-3j+k,\ b=2i+\lambda j+k,\ c=3i+j-2k$

are coplanar

53 If a, b, c be three vectors, prove that

(*i*) $[a+b,\ b+c,\ c+a]=2\,[a\ b\ c]$ and hence prove that the vectors $a+b,\ b+c,\ c+a$ are coplanar if and only if a, b, c are coplanar

(*ii*) $[a-b\ \ b-c\ \ c-a]=0$,

(*iii*) $r=(r\cdot i)\,i+(r\cdot j)\,j+(r\cdot k)\,k$,

(*iv*) $i\times(a\times i)+j\times(a\times j)+k\times(a\times k)=2a$

54 prove that $[a\times b\ \ b\times c\ \ c\times a]=[a\ b\ c]^2$

$$=\begin{vmatrix} a\cdot a & a\cdot b & a\cdot c \\ b\cdot a & b\cdot b & b\cdot c \\ c\cdot a & c\cdot b & c\cdot c \end{vmatrix}$$

Hence show that the vectors $a\times b,\ b\times c,\ c\times a$ are non coplanar if and only if a, b, c are non-coplanar

(c) If a and b are perpendicular vectors, then a b=0

(d) (ma) (nb)=(na) (mb)=mn (a b)

(e) a a=a a cos 0=a^2 This is written as a^2 a^2 i e square of a vector is equal to square of its module

(f) Distributive Law a (b+c)=a b+a c

(g) Orthonormal Triads i j k unit vectors

$$i^2=j^2=k^2=1$$

Also i j=0 j k=0 k i=0

(h) In terms of unit vectors let

$$a=a_1i+a_2j+a_3k, \quad b=b_1i+b_2j+b_3k$$

$$(a\ b)=(a_1i+a_2j+a_3k)(b_1i+b_2j+b_3k)$$

$$=a_1b_1+a_2b_2+a_3b_3=ab\cos\theta, \text{ by relations in (6)}$$

$$\cos\theta=\frac{a_1b_1+a_2b_2+a_3b_3}{ab}=\frac{\Sigma a_1b_1}{\sqrt{(\Sigma a_1^2)}\sqrt{(\Sigma b_1^2)}}$$

(i) (a+b+c+) (p+q+r+)

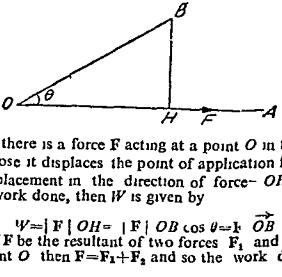
=(a p+a q+a r+

+(b p+b q+b r+)

+(c p+c q+c r+)+

(iv) **Work (Physical interpretation of dot product)**

A force is said to do work when its point of application moves



Suppose there is a force F acting at a point O in the direction OA and suppose it displaces the point of application from O to B. Then the displacement in the direction of force $OH=OB\cos\theta$. If W be the work done, then W is given by

$$W=|F|\ OH=|F|\ OB\cos\theta=F\ \vec{OB}$$

Again if F be the resultant of two forces F_1 and F_2 acting at the same point O then $F=F_1+F_2$ and so the work done by F is given by

$$F\ \vec{OB}=(F_1+F_2)\ \vec{OB}=F_1\ \vec{OB}+F_2\ \vec{OB}$$

Above shows that the work done by the resultant is equal to the sum of the works done by the forces separately

(c) Find the angle between $\vec{OA}$ and $\vec{OC}$

(d) Find the area of triangle OAB (Roorkee 82)

65 If c be a given non zero scalar, and **A** and **B** be given non zero vectors such that $\mathbf{A}\perp\mathbf{B}$, find the vector **X** which satisfies the equations $\mathbf{A}\cdot\mathbf{X}=c$ and $\mathbf{A}\times\mathbf{X}=\mathbf{B}$ (I I T 83)

66 *If the vectors* **a**, **b**, **c** *are coplanar and* **a**, **b** *are non collinear, then show that*

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}\mathbf{c}=\begin{vmatrix} \mathbf{c}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{c}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}\mathbf{a}+\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{b} \end{vmatrix}\mathbf{b}$$

67 *Given the vectors* $\vec{AB}=\mathbf{b}$ and $\vec{AC}=\mathbf{c}$ *coincident with two sides* of a triangle ABC. Find resolution (w r t the basis **b**, **c**) of the vector drawn from the vertex B of the $\triangle ABC$ and coinciding with the altitude BD

68 If $\mathbf{a}\cdot\mathbf{b}\neq 0$, find the vector **r** which satisfies the equations

$$(\mathbf{r}-\mathbf{c})\times\mathbf{b}=\mathbf{0},\ \mathbf{r}\cdot\mathbf{a}=0$$

69 The position vectors of the points A, B, C and D are $3\mathbf{i}-2\mathbf{j}-\mathbf{k}$, $2\mathbf{i}+3\mathbf{j}-4\mathbf{k}$, $-\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $4\mathbf{i}+5\mathbf{j}+\lambda\mathbf{k}$ respectively. If the points A, B, C and D lie on a plane, find the value of λ (I I T 86)

70 *Show that the shortest distance between a diagonal of a rectangular parallelopiped the lengths of whose three coterminous edges are* a, b, c, and the edges not meeting it are

$$\frac{bc}{\sqrt{(b^2+c^2)}},\ \frac{ca}{\sqrt{(c^2+a^2)}},\ \frac{ab}{\sqrt{(a^2+b^2)}}$$

71 Three vectors $\mathbf{a}=(12, 4, 3)$, $\mathbf{b}=(8, -12, -9)$, $\mathbf{c}=(33, -4, -24)$ define a parallelopiped. Evaluate the lengths of its edges, area of its faces and its volume (Roorkee 88)

Solutions to Problem Set (B)

1 $\mathbf{a}\cdot\mathbf{b}=ab\cos\theta=a_1a_2+b_1b_2+c_1c_2$

where $a=\sqrt{(\Sigma a_i^2)}$, $b=\sqrt{(\Sigma b_i^2)}$

$$\sqrt{(16+9+1)}\ \sqrt{(4+1+4)}\cos\theta=4\cdot 2+3(-1)+1\cdot 2$$

$$\sqrt{(26)}\ 3\cos\theta=7 \text{ or } \cos\theta=\frac{7}{3\sqrt{(26)}}$$

$$i \times i = j \times j = k \times k = 0$$

$$i \times j = k = -j \times i, \; j \times k = i = -k \times j, \; k \times i = j = -i \times k$$

(e) $a \times b$ in terms of unit vectors

Let $\quad a = a_1 i + a_2 j + a_3 k \qquad a = \sqrt{(\Sigma a_1^2)}$

$\quad b = b_1 i + b_2 j + b_3 k \qquad b = \sqrt{(\Sigma b_1^2)}$

then $\quad a \times b = (a_1 i + a_2 j + a_3 k) \times (b_1 i + b_2 j + b_3 k)$

or $\quad ab \sin \theta \, \hat{n} = a_1 b_2 i \times j + a_1 b_3 \, i \times k + a_2 b_1 \, j \times i + a_2 b_3 \, j \times k + a_3 b_1 \, k \times i + a_3 b_2 \, k \times j$

$$= (a_2 b_3 - a_3 b_2) \, i + (a_3 b_1 - a_1 b_3) \, j + (a_1 b_2 - a_2 b_1) \, k \qquad (1)$$

We have used the relations given in (d) above

Above can be expressed in determinant form as

$$a \times b = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

(f) Sine of the angle between a and b,

Squaring both sides of (1) and noting that $\hat{n}^2 = 1$

$$a^2 b^2 \sin^2 \theta = \Sigma (a_2 b_3 - a_3 b_2)^2$$

$$\sin^2 \theta = \frac{\Sigma (a_2 b_3 - a_3 b_2)^2}{\Sigma a_1^2 \, \Sigma b_1^2}$$

(g) Condition for vectors to be parallel

In this case $a \times b = 0$ as $\theta = 0$ or π and hence from (e) the last two rows of the determinant must have their corresponding elements proportional

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3}$$

(h) Condition for three points A, B C to be collinear

Determine $\vec{AB}$ and $\vec{BC}$ and show that $\vec{AB} \times \vec{BC}$ is zero

or $\qquad \vec{AB} = k \, \vec{BC}$

where k is a scalar

§ 9 Vector moment of a force about a point

(Physical interpretation of cross product)

The vector moment or torque M of a force F about any given point O is in magnitude equal to F times the perpendicular distance of O from the line of action of force F

3 Ans $\frac{1}{\sqrt{3}}[\mathbf{i}-\mathbf{j}-\mathbf{k}]$, $\sin\theta=\frac{2}{\sqrt{7}}$

4 Calculate $\mathbf{a}\times\mathbf{b}$ as in Ex 1 and it comes out to be

$$42\mathbf{i}+14\mathbf{j}-21\mathbf{k} \quad \text{or} \quad \mathbf{a}\times\mathbf{b}=7\,(6\mathbf{i}+2\mathbf{j}-3\mathbf{k})=7\mathbf{c}$$

Now $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=7\mathbf{c}\times\mathbf{c}=0$ *as cross product of two like vectors* is zero

Again $\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & -6 & 2\\ 6 & 2 & -3\end{vmatrix}=14\mathbf{i}+21\mathbf{j}+42\mathbf{k}=7\,(2\mathbf{i}+3\mathbf{j}+6\mathbf{k})=7\mathbf{a}$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times7\mathbf{a}=7\,(\mathbf{a}\times\mathbf{a})=0$$

Hence $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=0$

5 (*a*) $\mathbf{a}=4\mathbf{i}+3\mathbf{j}+\mathbf{k}$, $\mathbf{b}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$

and $\mathbf{a}\times\mathbf{b}=7\mathbf{i}-6\mathbf{j}-10\mathbf{k}$ by Ex 1

We have to show that $(\mathbf{a}\times\mathbf{b})$ *is perpendicular to both* $\mathbf{a}$ *and* $\mathbf{b}$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=4\cdot7+3\,(-6)+1\,(-10)=28-18-10=0$$

$$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=2\cdot7-1\,(-6)+2\,(-10)=14+6-20=0$$

Hence $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ as its dot product with both is zero

(*b*) If $\mathbf{a}\times\mathbf{b}=0$, then it does not *necessarily* imply that at least one of $\mathbf{a}$ or $\mathbf{b}$ is a null vector for the cross product of two non zero parallel vectors may be zero For example, if $\mathbf{a}=\mathbf{i}+\mathbf{j}+2\mathbf{k}$ and $\mathbf{b}=2\mathbf{i}+2\mathbf{j}+4\mathbf{k}$, then $\mathbf{a}\times\mathbf{b}=0$ as can be easily verified

6 (*a*) $\mathbf{a}\times\mathbf{b}=-\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

Hence the required unit vector is

$$\frac{-\mathbf{i}+2\mathbf{j}+2\mathbf{k}}{\sqrt{(1+4+4)}}=\tfrac{1}{3}\,(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$$

Therefore a vector of magnitude 9 is $9\cdot\tfrac{1}{3}\,(-\mathbf{i}+2\mathbf{j}+2\mathbf{k})$

$$=-3\mathbf{i}+6\mathbf{j}+6\mathbf{k}$$

(*b*) Here P V of point A (1, 2, 5) is $\mathbf{i}+2\mathbf{j}+5\mathbf{k}$ *etc*

$\overrightarrow{AB}=$ P V of B − P V of $A=4\mathbf{i}+5\mathbf{j}+4\mathbf{k}$,

$\overrightarrow{AC}=$ P V of C − P V of $A=2\mathbf{i}+0\mathbf{j}-6\mathbf{k}$

$\overrightarrow{AB}\times\overrightarrow{AC}$ is a vector perpendicular to the plane determined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ *i e* plane ABC

Cor *The algebraic sum of the moments of a system of forces about any point is equal to the moment of their resultant about the same point*

If F b the resultant of a number of forces F_1, F_2, ... acting through a point, then $F = F_1 + F_2 + \ldots$

$$M = r \times F = r \times (F_1 + F_2 + \ldots)$$
$$= r \times F_1 + r \times F_2 + \ldots$$

Hence proved

§ 10 **Scalar triple product**

$$(a \times b) \cdot c = c \cdot (a \times b)$$

is called the scalar triple product of three vectors a, b and c and is written as [a b c]

Properties of Scalar triple product

(a) If a, b, c be expressed in terms of unit vectors as

$$a = a_1 i + a_2 j + a_3 k,\ b = b_1 i + b_2 j + b_3 k,\ c = c_1 i + c_2 j + c_3 k$$

then it is easy to see that

$$[a\ b\ c] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \Delta, \text{ say}$$

(b) **Geometrical interpretation**

The scalar triple product [a b c] represents the volume of the parallelopiped whose coterminous edges a b c form a right handed system of vectors

(c) We know that in a determinant if a line be crossed over two lines the determinant retains its value both in magnitude and sense since it is multiplied by $(-1)^2$ so that it does not change

$$[a\ b\ c] = \begin{vmatrix} c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = [c\ a\ b]$$

$$= \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ a_1 & a_2 & a_3 \end{vmatrix} = [b\ c\ a]$$

$$[a\ b\ c] = [b\ c\ a] = [c\ a\ b]$$

or $$a \cdot (b \times c) = (b \times c) \cdot a \quad (1)$$

(b) Let the required vector be $\alpha = d_1 i + d_2 j + d_3 k$ where $d_1^2 + d_2^2 + d_3^2 = 51$ (given)

Now $\alpha \cdot \beta = |\alpha||\beta| \cos\theta$ or $\cos\theta = \frac{(\alpha \cdot \beta)}{|\alpha|(\beta)}$

Each of the given vectors a, b, c is a unit vector

$$\cos\theta = \frac{d \cdot a}{|d||a|} = \frac{d \cdot b}{|d||b|} = \frac{d \cdot c}{|d||c|}$$

or $d \cdot a = d \cdot b = d \cdot c$ $|d| = \sqrt{51}$ cancels out

$\frac{1}{3}(d_1 - 2d_2 + 2d_3) = \frac{1}{5}(-4d_1 + 0d_2 - 3d_3) = d_2$

$d_1 - 5d_2 + 2d_3 = 0$ from Ist and 3rd

$4d_1 + 5d_2 + 3d_3 = 0$ from 2nd and 3rd

$$\frac{d_1}{-15-10} = \frac{d_2}{8-3} = \frac{d_3}{5+20} \text{ or } \frac{d_1}{5} = \frac{d_2}{-1} = \frac{d_3}{-5} = \lambda \text{ say}$$

Putting for d_1, d_2 and d_3 in (1) we get

$(25+1+25)\lambda^2 = 51$ $\lambda = \pm 1$

Hence the required vectors are $\pm(5i - j - 5k)$

9 (a) $\overrightarrow{AB}$ = P V of B P V of $A = i + 2j - k$,

$\overrightarrow{CD}$ = P V of D − P V of $C = -3(i + 2j - k)$

Clearly $\overrightarrow{CD} = -3\overrightarrow{AB}$ and hence they are parallel

(b) Proceed as above

10 $\overrightarrow{PQ} = -2i + 2j + k$, $PQ = 3$

$\overrightarrow{RS} = i - 2j + 2k$ $RS = 3$,

Now $a \cdot b = ab\cos\theta = |a|$ Projection of b in the direction of a
$= |b|$ Projection of a in the direction of b

Now $\overrightarrow{PQ} \cdot \overrightarrow{RS} = -2 \cdot 1 + 2(-2) + 1 \cdot 2 = -4$

Hence $-4 = 3$ Projection of $\overrightarrow{RS}$ in the direction of $\overrightarrow{PQ}$
$= 3$ Projection of $\overrightarrow{PQ}$ in the direction of $\overrightarrow{RS}$

Hence either projection $= -\frac{4}{3}$

Again $\overrightarrow{PQ} \cdot \overrightarrow{RS} = PQ\, RS \cos\theta = 3 \cdot 3\cos\theta$

or $-\frac{4}{9} = \cos\theta$, $\theta = \cos^{-1}(-\frac{4}{9})$

11 (a) Ans $\frac{2}{\sqrt{(14)}}, \frac{2}{\sqrt{(14)}} \left(\frac{i + 2j + 3k}{\sqrt{(14)}}\right) = \frac{1}{7}(i + 2j + 3k)$

(b) Projection of a along b is $\frac{a \cdot b}{|b|}$

$$= \frac{4 \cdot 0 + 6 \cdot 3 + 0 \cdot 4}{5} = \frac{18}{5}$$

Hence the vector determined by this projection is

Let $\mathbf{c}\times\mathbf{d}=\mathbf{n}$, L H S $=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{n}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})$

[The position of dot and cross can be changed if cyclic order is maintained]

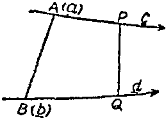
$$=\mathbf{a}\cdot\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{a}\cdot\{(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}\}$$

$$=(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\cdot\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})$$

$$=\begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c} \\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}=\begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix}$$

§ 13 To find the shortest distance between two non intersecting lines passing through the points whose position vectors are a and b and are parallel to the vectors c and d respectively

Let PQ be the shortest distance between the two lines so that PQ is perpendicular to both $\mathbf{c}$ and $\mathbf{d}$ But $\mathbf{c}\times\mathbf{d}$ is a vector perpendicular to both $\mathbf{c}$ and $\mathbf{d}$ Hence PQ is parallel to the vector $\mathbf{c}\times\mathbf{d}$



Let $\mathbf{c}\times\mathbf{d}=\mathbf{n}$ and let $|\mathbf{n}|=n$

If A, B are the points $\mathbf{a}$ and $\mathbf{b}$ respectively, the length p of this common perpendicular PQ is equal to the length of of the projection AB on the vector $\mathbf{n}$ Hence

$(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}=$(modulus of $\mathbf{n}$) (Projection of $(\mathbf{a}-\mathbf{b})$ on $\mathbf{n}$)

$$=np$$

$$p=\frac{1}{n}[(\mathbf{a}-\mathbf{b})\cdot\mathbf{n}]=\frac{1}{n}[(\mathbf{a}-\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})]$$

$$=\frac{[\mathbf{a}-\mathbf{b}\;\;\mathbf{c}\;\;\mathbf{d}]}{|\mathbf{c}\times\mathbf{d}|}$$

Working Rule *The shortest distance between two non intersecting lines parallel respectively to* $\mathbf{c}$ *and* $\mathbf{d}$ *will be projection of the line joining any two points (one on each line) on the vector* $\mathbf{c}\times\mathbf{d}$ *which is perpendicular to both the lines*

$= 2\,(\overrightarrow{AP}^2 + \overrightarrow{PB}^2) = 2\,(AP^2 + PB^2)$

14 Refer fig Q 12

The triangle being isosceles, we have $AB = AC$

Also $AP = \dfrac{\mathbf{b}+\mathbf{c}}{2}$ where P is mid point of BC

Also $\overrightarrow{BC} = \mathbf{c} - \mathbf{b}$

$\overrightarrow{AP}\,\overrightarrow{BC} = \dfrac{\mathbf{b}+\mathbf{c}}{2}\,(\mathbf{c}-\mathbf{b}) = \frac{1}{2}\,(\mathbf{c}^2 - \mathbf{b}^2)$

$= \frac{1}{2}\,(AC^2 - AB^2) = 0$ by (1)

$\overrightarrow{AP}$ is perpendicular to $\overrightarrow{BC}$ or $AP \perp BC$

i e median is at right angles to the base

15 Taking A as origin let the position vectors of B and C be $\mathbf{b}$ and $\mathbf{c}$ so that, that of D is $\mathbf{b}+\mathbf{c}$ by parallelogram law

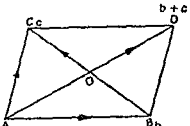


Fig 30

Let O be the intersection of the diagonals also BC = P V of C - P V of B $= \mathbf{c} - \mathbf{b}$

Now $AD^2 + BC^2 = \overrightarrow{AD}^2 + \overrightarrow{BC}^2 = (\mathbf{b}+\mathbf{c})^2 + (\mathbf{c}-\mathbf{b})^2$

$= 2\,(\mathbf{b}^2 + \mathbf{c}^2) = 2\,(AB^2 + AC^2)$

$= \{(AB^2 + AC^2) + (AB^2 + AC^2)\}$

$= AB^2 + AC^2 + CD^2 + BD^2$

= Sum of the squares on the four sides

Again

$\overrightarrow{AD}^2 - \overrightarrow{BC}^2 = (\mathbf{b}+\mathbf{c})^2 - (\mathbf{c}-\mathbf{b})^2 = 4\mathbf{b}\,\mathbf{c}$

$= 4\,\overrightarrow{AB}\,\overrightarrow{AC} = 4AB$ (Projection of AC on AB)

$= 4$ (rectangle contained by AB and projection of AC on AB)

Similarly

$\overrightarrow{AC}^2 - \overrightarrow{AB}^2 = \mathbf{c}^2 - \mathbf{b}^2 = (\mathbf{c}+\mathbf{b})\;(\mathbf{c}-\mathbf{b}) = \overrightarrow{AD}\;\overrightarrow{BC}$

$=$ diag AD (Projection of diag BC on AD)

Let $\mathbf{c}\times\mathbf{d}=\mathbf{n}$ L.H.S. $=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{n}=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{n})$

[The position of dot and cross can be changed if cyclic order is maintained]

$$=\mathbf{a}\cdot\{\mathbf{b}\times(\mathbf{c}\times\mathbf{d})\}=\mathbf{a}\cdot\{(\mathbf{b}\cdot\mathbf{d})\,\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{d}\}$$

$$=(\mathbf{b}\cdot\mathbf{d})\,(\mathbf{a}\cdot\mathbf{c})-(\mathbf{b}\cdot\mathbf{c})\,(\mathbf{a}\cdot\mathbf{d})$$

$$=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}=\begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d}\\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}$$

§ 13 To find the shortest distance between two non intersecting lines passing through the points whose position vectors are a and b and are parallel to the vectors c and d respectively.

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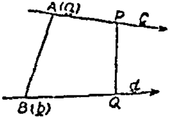

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$$=np$$

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$$=\frac{[\mathbf{a}-\mathbf{b}\ \ \mathbf{c}\ \ \mathbf{d}]}{|\mathbf{c}\times\mathbf{d}|}$$

Working Rule *The shortest distance between two non intersecting lines parallel respectively to* **c** *and* **d** *will be projection of the line joining any two points (one on each line) on the vector* **c**×**d** *which is perpendicular to both the lines.*

$$PM=\frac{\vec{PA}\cdot\vec{n}}{|\vec{n}|}=\frac{(2i-4j+2k)\cdot(4i+7j-4k)}{\sqrt{(16+49+16)}}$$

$$=\frac{8-28-8}{9}=-\frac{28}{9}=\frac{28}{9}\text{ units}$$

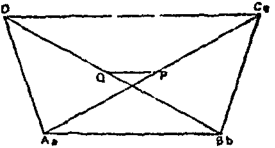


Fig 33

20 Choose D as origin and let the position vectors of A, B and C be a, b and c respectively

$\vec{AB}$= b − a, $\vec{AC}$= c − b, $\vec{CD}$= − c, $\vec{CA}$= a

$\vec{AC}$= c − a, $\vec{BD}$= − b

Also points P and Q are mid points of AC and BD

P is $\frac{\mathbf{a}+\mathbf{c}}{2}$ and Q is $\frac{\mathbf{b}}{2}$

$\vec{PQ}=\frac{1}{2}$ (b − a − c)

Again square of a vector is equal to square of its module

L H S = $(\mathbf{b}-\mathbf{a})^2+(\mathbf{c}-\mathbf{b})^2+\mathbf{c}^2+\mathbf{a}^2$

$=2\,(\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-\mathbf{b}\,\mathbf{a}-\mathbf{c}\,\mathbf{b})$

R H S = $(\mathbf{c}-\mathbf{a})^2+(-\mathbf{b})^2+4\cdot\frac{1}{4}(\mathbf{b}-\mathbf{a}-\mathbf{c})^2$

$=2\,[\mathbf{a}^2+\mathbf{b}^2+\mathbf{c}^2-\mathbf{b}\,\mathbf{a}-\mathbf{b}\,\mathbf{c}]$

Hence L H S = R H S, c b = b c

21 Let the position vectors of A, B, C be a, b, c relative to *O as origin, then those of D, E and F are respectively* a + b, b + c and c + a

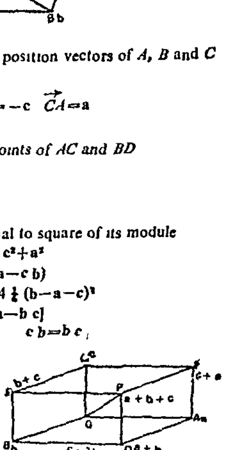


Fig 34

The four diagonals are

OP, *CD*, *AE* and *BF*

$\vec{OP}=\vec{OD}+\vec{DP}=$ a + b + c $\quad \vec{CD}=\vec{OD}-\vec{OC}=$ a + b − c

$\vec{AE}=\vec{OE}-\vec{OA}=$ b + c − a $\quad \vec{BF}=\vec{OF}-\vec{OB}=$ c + a − b

Also determine the sine of the angle between a and b

(*b*) $a=3i+2j-k$, $b=12i+5j-5k$ (Roorkee 1978)

3 Two vectors **a** and **b** are expressed in terms of unit vectors as follows

$a=3i+j+2k, \quad b=2i-2j+4k$

What is the unit vector perpendicular to each of the vectors. Also determine the sine of the angle between the given vectors

4 If $a=2i+3j+6k$, $b=3i-6j+2k$ and $c=6i+2j-3k$, show that $a\times b=7c$

Also prove that $a\times(b\times c)=(a\times b)\times c=0$ in this case

5 (*a*) By taking vectors a and b from either Ex 3 or Ex 1 prove that $a\times b$ is a vector which is perpendicular to both a and b

(*b*) a and b are two vectors satisfying the condition $a\times b=0$. Does it imply that one of the vectors a or b must be a null vector. Give an example in support of your answer

(Roorkee 1984)

6 (*a*) Prove that a vector of magnitude 9 perpendicular to both the vectors

$a=4i-j+3k$ and $b=-2i+j-2k$ is $-3i+6j+6k$

(*b*) Given $A=(1, 2, 5)$, $B=(5, 7, 9)$ and $C=(3, 2, -1)$. Find a unit vector normal to the plane of the triangle *ABC*

(*c*) The unit vector perpendicular to the plane determined by $P(1, -1, 2)$, $Q(2, 0, -1)$, $R(0, 2, 1)$ is

(I I T 83)

(*d*) Find the value of the constant *S* such that the scalar product of the vector $i+j+k$ with the unit vector parallel to the sum of the vectors $2i+4j-5k$ and $Si+2j+3k$ is equal to one (Roorkee 1985)

7 Show that the vectors

$a=\frac{1}{7}(2i+3j+6k)$, $b=\frac{1}{7}(3i-6j+2k)$, $c=\frac{1}{7}(6i+2j-3k)$

form an orthonormal triad. Also find two vectors of magnitude 2 each being normal to the plane containing the vector a and b

8 (*a*) Prove that the points

$2j-j+k$, $i-3j-5k$, $3i-4j-4k$

are the vertices of a right angled triangle. Find also the other two angles

(*b*) Find a vector of magnitude $\sqrt{(51)}$ which makes equal

Hence any pair of opposite edges are perpendicular

24 Let the point of intersection O of two altitudes BQ and CR be taken as origin and the position vectors of the vertices A, B, C be **a**, **b**, **c** respectively Let AO produced meet BC at P We will show that AP is perpendicular to

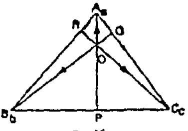


Fig 36

BC, showing thereby that the three altitudes are concurrent,

$$\vec{OB}=\mathbf{b}, \qquad \vec{BQ}=\mu\,\mathbf{b}$$

as it is collinear with $\vec{OB}$

Similarly since $\vec{OC}=\mathbf{c}$, $\qquad \vec{CR}=\nu\,\mathbf{c}$

Now $\vec{AC}=\mathbf{c}-\mathbf{a}$ and $\vec{AB}=\mathbf{b}-\mathbf{a}$

Since $BQ \perp AC$ we have $\mu\,\mathbf{b}\cdot(\mathbf{c}-\mathbf{a})=0$ and so $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}$

Again since $CR \perp$ to AB, $\nu\,\mathbf{c}\cdot(\mathbf{b}-\mathbf{a})=0$, $\qquad \mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$

$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}$ or $\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})=0$

or $\lambda\mathbf{a}\cdot(\mathbf{c}-\mathbf{b})=0$

$\vec{AP}\cdot\vec{BC}=0$ or AP is $\perp$ to BC

25 Let the right bisectors of sides BC and CA meet at O and taking O as origin let the position vectors of A, B and C be taken as **a**, **b**, **c** respectively Hence the mid points D, E, F are

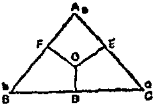


Fig 37

$$\frac{\mathbf{b}+\mathbf{c}}{2}, \frac{\mathbf{c}+\mathbf{a}}{2}, \frac{\mathbf{a}+\mathbf{b}}{2}$$

$OD \perp BC$, $\qquad \dfrac{\mathbf{b}+\mathbf{c}}{2}\cdot(\mathbf{c}-\mathbf{b})=0$ i e $\mathbf{b}^2=\mathbf{c}^2$

Again since $OE \perp CA$, $\qquad \dfrac{\mathbf{c}+\mathbf{a}}{2}\cdot(\mathbf{a}-\mathbf{c})=0$, or $\mathbf{a}^2=\mathbf{c}^2$

$$\mathbf{a}^2=\mathbf{b}^2=\mathbf{c}^2 \qquad (1)$$

Now we have to prove that OF is also $\perp$ to AB which will be true if $\dfrac{\mathbf{a}+\mathbf{b}}{2}\cdot(\mathbf{b}-\mathbf{a})=0$ i e $\mathbf{b}^2=\mathbf{a}^2$ which is true by (1)

Hence proved

17 Prove that the diagonals of a rhombus are at right angles

18, Prove that the angle in a semi circle is right angle

19 Find the equation of the plane passing through the point A (3, -2 1) and perpendicular to the vector $4\mathbf{i}+7\mathbf{j}-4\mathbf{k}$ If PM be the perpendicular from the point P (1, 2, -1) to this plane, find its length

20 In a quadrilateral $ABCD$, prove that

$$AB^2+BC^2+CD^2+DA^2=AC^2+BD^2+4PQ^2$$

where P and Q are the middle points of the diagonals AC and BD

21 Prove that in a parallelopiped the sum of the squares on the diagonals is equal to four times the sum of the squares on three coterminous edges

22 Prove that in a tetrahedron if two pairs of opposite edges are perpendicular the third pair is also perpendicular Also show that sum of the squares on two opposite edges is the same for each pair

23 Prove that any two opposite edges of a regular tetrahedron are perpendicular

24 Prove that in any triangle the perpendiculars from the vertices upon the opposite sides are concurrent

25 Prove that the right bisectors of the sides of a triangle are concurrent

26 If two medians of a triangle are equal, the triangle is isosceles

27 If $|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|$, then show that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular (Roorkee 86)

28 Prove that $\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)^2=\left(\frac{\mathbf{a}-\mathbf{b}}{ab}\right)^2$

29 If $\mathbf{a}$ and $\mathbf{b}$ are unit vectors and θ is the angle between them show that $\sin(\theta/2)=\frac{1}{2}|\mathbf{a}-\mathbf{b}|$

30 If $\mathbf{a}$ $\mathbf{b}$ $\mathbf{c}$ are non coplanar vectors and $\mathbf{n}\cdot\mathbf{a}=\mathbf{n}\cdot\mathbf{b}=\mathbf{n}\cdot\mathbf{c}=0$ Show that $\mathbf{n}$ is a null vector

31 If a straight line is equally inclined to three coplanar straight lines prove that it is perpendicular to their plane

32 If $\mathbf{a}$ $\mathbf{b}$, $\mathbf{c}$ are mutually perpendicular vectors of equal magnitude, show that $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is equally inclined to $\mathbf{a}$ $\mathbf{b}$ and $\mathbf{c}$

33 A line makes angles α, β γ δ with the diagonals of a cube prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta=4/3$ Also prove that the angle between two diagonals of a cube is $\cos^{-1}(1/3)$

31 Let the vectors representing the three lines be **a**, **b**, **c** and **n** be a unit vector along the given straight line which is equally inclined to above three lines

$\mathbf{n}\cdot\mathbf{a}=1\ a\cos\theta$ or $\mathbf{n}\cdot\frac{\mathbf{a}}{a}=\cos\theta$ or $\mathbf{n}\cdot\hat{\mathbf{a}}=\cos\theta$

Hence by the given condition, we have

$\mathbf{n}\cdot\hat{\mathbf{a}}=\mathbf{n}\cdot\hat{\mathbf{b}}=\mathbf{n}\cdot\hat{\mathbf{c}}=\cos\theta$

Since the three unit vectors are all different (as their directions are different) and coplanar the above relation will hold good only when $\cos\theta=0$ *i e* $\theta=\pi/2$ Hence **n** is perpendicular to **a**, **b** and **c** and as such it is perpendicular to their plane

32 We have

$$\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a}=0 \qquad (1)$$

as **a**, **b**, **c** are mutually perpendicular

Again their magnitudes are same *i e* $a=b=c$

Also $(\mathbf{a}+\mathbf{b}+\mathbf{c})^2=\Sigma\mathbf{a}^2+2\Sigma\,\mathbf{a}\cdot\mathbf{b}=a^2+b^2+c^2=3a^2$ (2)

$|\mathbf{a}+\mathbf{b}+\mathbf{c}|=\sqrt{3}a$

Now $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{a}=a^2+\mathbf{b}\cdot\mathbf{a}+\mathbf{c}\cdot\mathbf{a}=a^2$ by (1),

or $a\sqrt{3}\,a\cos\theta_1=a^2$, $\cos\theta_1=1/\sqrt{3}$ by (2)

Similarly $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{b}$ and $(\mathbf{a}+\mathbf{b}+\mathbf{c})\cdot\mathbf{c}$ will give

$\cos\theta_2=1/\sqrt{3}$ and $\cos\theta_3=1/\sqrt{3}$

Hence $\theta_1=\theta_2=\theta_3$ In other words $\mathbf{a}+\mathbf{b}+\mathbf{c}$ is equally inclined to **a**, **b** and **c**

33 Let a be the edge of the cube so that

$\vec{OA}=a\mathbf{i}$, $\vec{OB}=a\mathbf{j}$, $\vec{OC}=a\mathbf{k}$

$\vec{OD}=a(\mathbf{i}+\mathbf{j})$, $\vec{OE}=a(\mathbf{j}+\mathbf{k})$

$\vec{OF}=a(\mathbf{k}+\mathbf{i})$

The four diagonals are

$\vec{OP}=a(\mathbf{i}+\mathbf{j}+\mathbf{k})$, $\vec{CD}=a(\mathbf{i}+\mathbf{j}-\mathbf{k})$

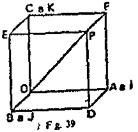


Fig. 39

$\vec{AE}=a(\mathbf{j}+\mathbf{k}-\mathbf{i})$, $\vec{CF}=a(\mathbf{k}+\mathbf{j}-\mathbf{i})$,

Let L (x, y, z) be a point on the line drawn through O parallel to the given line which makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals

$$\vec{OL}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \text{ and } |\vec{OL}|=\sqrt{(x^2+y^2+z^2)}$$

43 Find the area of the parallelogram whose adjacent sides are

$$a = i + 2j + 3k, \quad b = 3i - 2j + k$$ (Roorkee 79)

44 Show that the area of a triangle whose two adjacent sides are determined by the vectors $a = 3i + 4j$ $b = -5i + 7j$ is $20\frac{1}{2}$ square units

45 (a) Find the area of the parallelogram having diagonals $3i + j - 2k$ and $i - 3j + 4k$

(b) If $a = 2i - 3j + k$ $b = -i + k$, $c = 2j - k$, find the area of the parallelogram having diagonals $a + b$ and $b + c$

46 Find two unit vectror parallel to the diagonals of a parallelogram whose sides are

$$2i + 4j - 5k \text{ and } i + 2j + 3k$$ (Roorkee 76)

Find the vector product of the two vectors given above Can this product be of use in finding the area of the ||^{m} If so find this area (Roorkee 76)

47 Given $a = i + 2j + 3k$ $b = -i + 2j + k$ and $c = 3i + j$, find a unit vector in the direction of the resultant of these vectors *Also find a vectors* **r** *which is normal to both* **a** and **b** What is the inclination of **r** and **c** (Roorkee 80)

48 (a) AC and BD are two diagonals of a quadrilateral, prove that its area is $\frac{1}{2}\ \vec{AC} \times \vec{BD}$

(b) Determine the lengths of the diagonals of a parallelogram constructed on the vectors $a = 2m + n$ and $b = m - 2n$, where **m** and **n** are unit vectors forming an angle of 60

49 Prove by vectors that in any triangle ABC

(i) $a^2 = b^2 + c^2 - 2bc \cos A$

(ii) $a = b \cos C + c \cos B$,

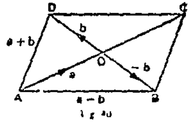
(iii) $a \cos B - b \cos A = \dfrac{a^2 - b^2}{c}$

(iv) $2(bc \cos A + ca \cos B + ab \cos C) = a^2 + b^2 + c^2$

(v) $(a + b + c)(\cos A + \cos B + \cos C)$
$= a(1 + \cos A) + b(1 + \cos B) + c(1 + \cos C)$,

(vi) $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$

diagonals



and $\vec{AO}=\mathbf{a}$, $\vec{OD}=\mathbf{b}$ so that

$\vec{OB}=-\mathbf{b}$

$\vec{AD}=\vec{AO}+\vec{OD}=\mathbf{a}+\mathbf{b}$,

$\vec{AB}=\vec{AO}+\vec{OB}=\mathbf{a}-\mathbf{b}$

Now $(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})=\mathbf{a}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}-\mathbf{b}\times\mathbf{b}$

$=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{b}=2(\mathbf{a}\times\mathbf{b})$, $\quad \mathbf{b}\times\mathbf{a}=-(\mathbf{a}\times\mathbf{b})$

We know that $\mathbf{a}\times\mathbf{b}$ represents the vector area of ‖ᵐ whose adjacent sides are a and b

$(\mathbf{a}-\mathbf{b})\times(\mathbf{a}+\mathbf{b})$ represents the vector area of ‖ᵐ *ABCD* whose abjacent sides are *AB* and *AD* and it is equal to $2(\mathbf{a}\times\mathbf{b})$ *i e* twice the vector area of ‖ᵐ whose adjacent sides are semi diagonals of the first ‖ᵐ or $=4(\frac{1}{2}\mathbf{a}\times\mathbf{b})$=four times the vector are of the triangle whose adjacent sides are semi-diagonals of the ‖ᵐ

37 L H S $=(\mathbf{a}\times\mathbf{b})+(\mathbf{a}\times\mathbf{c})+(\mathbf{b}\times\mathbf{c})+(\mathbf{b}\times\mathbf{a})+(\mathbf{c}\times\mathbf{a})+(\mathbf{c}\times\mathbf{b})$

$=(\mathbf{a}\times\mathbf{b})-(\mathbf{c}\times\mathbf{a})+(\mathbf{b}\times\mathbf{c})-(\mathbf{a}\times\mathbf{b})+(\mathbf{c}\times\mathbf{a})-(\mathbf{b}\times\mathbf{c})$

$=0$

38 We have to prove that $\mathbf{a}-\mathbf{d}$ is parallel to $\mathbf{b}-\mathbf{c}$ and hence their cross product should be shown to be zero

Now $(\mathbf{a}-\mathbf{d})\times(\mathbf{b}-\mathbf{c})=(\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c})-(\mathbf{d}\times\mathbf{b}-\mathbf{d}\times\mathbf{c})$

$=(\mathbf{a}\quad \mathbf{b}+\mathbf{d}\times\mathbf{c})+(\mathbf{a}\times\mathbf{c}+\mathbf{d}\times\mathbf{b})$

$=(\mathbf{a}\times\mathbf{b}-\mathbf{c}\times\mathbf{d})-(\mathbf{a}\times\mathbf{c}-\mathbf{b}\times\mathbf{d})$

$=0-0=0$, by given relation.

Hence $\mathbf{a}-\mathbf{d}$ as parallel to $\mathbf{a}-\mathbf{c}$

39 (*a*) $\mathbf{a}\times\mathbf{b}=\mathbf{a}\times\mathbf{c}$ *i e* $\mathbf{a}\times(\mathbf{b}-\mathbf{c})=0$,

a is parallel to $\mathbf{b}-\mathbf{c}$

$\mathbf{b}-\mathbf{c}=\lambda\mathbf{a}$, that is b differs from c by a vectors which is parallel to a

(*b*) Proceed is in part (*a*)

40 We have $\mathbf{a}\ \mathbf{b}=\mathbf{a}\ \mathbf{c}\Rightarrow \mathbf{a}\ (\mathbf{b}-\mathbf{c})=0$

Either $\mathbf{a}=0$, or $\mathbf{b}-\mathbf{c}=0$ or a is $\perp$ to $\mathbf{b}-\mathbf{c}$

Again $\mathbf{a}\times\mathbf{b}-\mathbf{a}\times\mathbf{c}\Rightarrow\mathbf{a}\times(\mathbf{b}-\mathbf{c})=0$

Either $\mathbf{a}=0$ or $\mathbf{b}-\mathbf{c}=0$ or is a parallel to $\mathbf{b}-\mathbf{c}$

i e $\mathbf{b}-\mathbf{c}=k\mathbf{a}$

Interpretation

But $\mathbf{a}\neq0$ and hence if both the equations hold simultaneously then $\mathbf{b}-\mathbf{c}=0$ *i e* $\mathbf{b}=\mathbf{c}$

41 We know that

$\mathbf{a}\times\mathbf{b}=ab\sin\theta\,\hat{\mathbf{n}}$

55 Prove that if **l**, **m**, **n** be three non-coplanar vectors then

$$[\mathbf{l}\ \mathbf{m}\ \mathbf{n}]\ (\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{l}\cdot\mathbf{a} & \mathbf{l}\cdot\mathbf{b} & \mathbf{l} \\ \mathbf{m}\cdot\mathbf{a} & \mathbf{m}\cdot\mathbf{b} & \mathbf{m} \\ \mathbf{n}\cdot\mathbf{a} & \mathbf{n}\cdot\mathbf{b} & \mathbf{n} \end{vmatrix}$$

56 Express (*a*) **a**, **b**, **c** in terms of **b**×**c**, **c**×**a**, **a**×**b**, and
(*b*) **b**×**c**, **c**×**a**, **a**×**b** in the terms of **a**, **b**, **c**

57 Prove that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})=0$

58 If **a**, **b**, **c** be three unit vectors such that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\tfrac{1}{2}\mathbf{b},$$

find the angles which **a** makes with **b** and **c**, **b** and **c** being non parallel

59 Show that $(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ if and only if **a** and **c** are collinear or $(\mathbf{a}\times\mathbf{c})\times\mathbf{b}=0$

60 Let A_r ($r=1$ 2, 3, 4) be the area of four faces of a tetrahedron Let, $\mathbf{n}_r$ be the outward drawn normals to the respective Regarding integral as the limit of a sum, prove the following faces with magnitudes equal to corresponding areas
Prove that $\mathbf{n}_1+\mathbf{n}_2+\mathbf{n}_3+\mathbf{n}_4=0$.

61 Prove that the following four points are coplanar
(*a*) (4, 5, 1), (0, −1, −1), (3, 9, 4) and (−4, 4, 4)
(*b*) $4\mathbf{i}+8\mathbf{j}+12\mathbf{k}$, $2\mathbf{i}+4\mathbf{j}+6\mathbf{k}$, $3\mathbf{i}+5\mathbf{j}+4\mathbf{k}$, $5\mathbf{i}+8\mathbf{j}+4\mathbf{k}$
(*c*) $\mathbf{i}+2\mathbf{j}+3\mathbf{k}$, $3\mathbf{i}-\mathbf{j}+2\mathbf{k}$, $6\mathbf{i}-4\mathbf{j}+2\mathbf{k}$ and $-2\mathbf{i}+3\mathbf{j}+\mathbf{k}$

62 If **a**, **b**, **c** and **d** be four vectors, then prove that

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})+(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{d})+(\mathbf{c}\times\mathbf{a})\cdot(\mathbf{b}\times\mathbf{d})=0$$

(*b*) If the four points **a**, **b**, **c**, **d** are coplanar, prove that

$$[\mathbf{abc}]=[\mathbf{bcd}]+[\mathbf{cad}]+[\mathbf{abd}]$$

63 If $A_1 A_2 \ldots A_n$ are the vertices of regular plane polygon with n sides and O is its centre, show that

$$\sum_{i=1}^{n-1}(\vec{OA_i}\times\vec{OA_{i+1}})=(1-n)\,(\vec{OA_n}\times\vec{OA_1})\qquad \text{(I I T 82)}$$

64 (*a*) Find out the volume of a prism on triangular base the three sides of the prism meeting on a vertex are given below

$\vec{OA}=3\mathbf{i}+4\mathbf{j}+12\mathbf{k}$, $\vec{OB}=12\mathbf{i}+3\mathbf{j}+4\mathbf{k}$ $\vec{OC}=4\mathbf{i}+12\mathbf{j}+3\mathbf{k}$

(*b*) Find the unit vector perpendicular to $\vec{OA}$ and $\vec{OB}$

$|a \times b| = 8\sqrt{(1+1+1)} = 8\sqrt{3}$ sq units

44. Area of $\Delta = \frac{1}{2} |a \times b| = 20\frac{1}{2}$ sq units

45. We know that area of a ||^{m} is four times the area of Δ whose adjacent sides are semi diagonals of the ||^{m} by Q 36

Refer fig Q 36 ||^{m} $ABCD = 4\Delta OCD$

$= 4$ [$\frac{1}{2}$ semi diagonal $OC \times$ semi diagonal OD]

$= 2\,[\frac{1}{2}(3i+j-2k) \times \frac{1}{2}(i-3j+4k)]$

$$= \frac{1}{2}\begin{vmatrix} i & j & k \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix} = \frac{1}{2}(\ \ 2i - 14j - 10k)$$

$= -i - 7j - 5k$

Area of the parallelogram
$= |-i-7j-5k| = \sqrt{(1+49+25)} = \sqrt{75} = 5\sqrt{3}$

(*b*) Proceed as above First find $a+b$ and $b+c$

Area $= \frac{1}{2}\sqrt{21}$

46. If a and b be the adjacent sides of a ||^{m} then the diagonals are

$\vec{AC} = a+b$, $\vec{BD} = b-a$

Unit vectors are $\dfrac{\vec{BC}}{|\vec{AC}|}$ and $\dfrac{\vec{BD}}{|\vec{BD}|}$

Area of parallelogram,

$= |a \times b|$

Ans $\frac{1}{7}(3i+6j-2k)$ $\frac{1}{\sqrt{69}}(-i-2j+8k)$, $11\sqrt{(5)}$ sq units

47. Resultant $= a+b+c = 3i+5j+4k$
Unit vector in the direction of resultant

$$= \frac{3i+5j+4k}{\sqrt{(9+25+16)}} = \frac{1}{5\sqrt{2}}(3i+5j+4k),$$

Also $a \times b = -4\,(i+j+k)$, $|a \times b| = 4\sqrt{3}$

$(a \times b) \cdot c = -4\,(i+j+k) \cdot (3i+j) = -4\,(3+1) = -16$

$4\sqrt{3}\,\sqrt{10}\cos\theta = -16$ or $\cos\theta = \dfrac{-4}{\sqrt{30}}$

Unit vector perpendicular to both a and b

Let $\mathbf{r}=l\,\mathbf{i}+m\,\mathbf{j}+n\,\mathbf{k}$ be unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ so that $\mathbf{r}\cdot\mathbf{a}=0$, $\mathbf{r}\cdot\mathbf{b}=0$ where $l^2+m^2+n^2=1$

$$4l+3m+n=0 \text{ and } 2l-m+2n=0$$

Solving by cross multiplication,

$$\frac{l}{7}=\frac{m}{-6}=\frac{n}{-10}=\sqrt{\left(\frac{l^2+m^2+n^2}{49+36+100}\right)}=\frac{1}{\sqrt{(185)}}$$

Putting the values of l, m n from above we get the unit vector perpendicular to both a and b as

$$\mathbf{r}=\frac{1}{\sqrt{(185)}}[7\mathbf{i}-6\mathbf{j}-10\mathbf{k}]$$

Alternative method

We know that $\mathbf{a}\times\mathbf{b}$ represents a vector which is perpendicular to both a and b,

$$\text{Now } \mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 4 & 3 & 1 \\ 2 & -1 & 2 \end{vmatrix}=7\mathbf{i}-6\mathbf{j}-10\mathbf{k}$$

Hence a unit vector will be obtained by dividing $7\mathbf{i}-6\mathbf{j}-10\mathbf{k}$ by its module $\sqrt{(49+36+100)}=\sqrt{(185)}$

The required unit vector is $\frac{1}{\sqrt{(185)}}[7\mathbf{i}-6\mathbf{j}-10\mathbf{k}]$ as before

Again $\mathbf{a}\times\mathbf{b}=ab\sin\theta\,\hat{\mathbf{n}}$

$$7\mathbf{i}-6\mathbf{j}-10\mathbf{k}=\sqrt{(26)}\sqrt{(9)}\sin\theta\,\hat{\mathbf{n}}$$

Square both sides and we know that square of a vector is equal to square of its module i e $\mathbf{a}^2=a^2$ and $\hat{\mathbf{n}}^2=1$

$$42+36+100=26\times 9\sin^2\theta\ 1$$

$$\sin\theta=\sqrt{\left(\frac{185}{26\times 9}\right)}=\frac{1}{3}\sqrt{\left(\frac{185}{26}\right)}$$

2 (a) Ans $\frac{1}{\sqrt{(155)}}[-3\mathbf{i}+5\mathbf{j}+11\mathbf{k}]$, $\sin\theta=\sqrt{\left(\frac{155}{156}\right)}$

(b) Ans $\frac{1}{\sqrt{(115)}}\left[-5\mathbf{i}+3\mathbf{j}-9\mathbf{k}\right]$

(*ii*) Multiplying both sides of (1) scalarly by $\overrightarrow{BC}$, we get

$$\overrightarrow{BC}\,\overrightarrow{BC}=-(\overrightarrow{CA}+\overrightarrow{AB})\,\overrightarrow{BC}$$

$$a^2=-ba\cos(\pi-C)-ca\cos(\pi-B)$$

or $a^2=ab\cos C+ca\cos B$

$$a=b\cos C+c\cos B$$

(*iii*) As in (1) $\quad \mathbf{a}+\mathbf{b}+\mathbf{c}=0$ (1)

Also

$$\left.\begin{aligned}\mathbf{c}\,\mathbf{a}&=ca\cos(\pi-B)=-ac\cos B\\ \mathbf{a}\,\mathbf{b}&=ab\cos(\pi-C)=-ab\cos C\\ \mathbf{b}\,\mathbf{c}&=bc\cos(\pi-A)=-bc\cos A\end{aligned}\right\}\quad (A)$$

$\mathbf{c}\,\mathbf{a}-\mathbf{b}\,\mathbf{c}=c\,(b\cos A-a\cos B)$ [Put $\mathbf{c}=-(\mathbf{a}+\mathbf{b})$]

$$-(\mathbf{a}+\mathbf{b})\,(\mathbf{a}-\mathbf{b})=c\,(b\cos A-a\cos B)$$

or $\mathbf{a}^2-\mathbf{b}^2=c\,(a\cos B-b\cos A)$

$$\frac{a^2-b^2}{c}=a\cos B-b\cos A$$

(*iv*) Since $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ Square

$(\mathbf{a}+\mathbf{b}+\mathbf{c})^2=0$ or $\Sigma\mathbf{a}^2+2\Sigma\,(\mathbf{b}\,\mathbf{c})=0$

or $a^2+b^2+c^2=-2\,(-bc\cos A-ca\cos B-ab\cos C)$

$$=2\,(bc\cos A+ca\cos B+ab\cos C)$$

by (*A*) of part (*iii*)

(*v*) On simplification it reduces to

$$(b\cos C+c\cos B)+(c\cos A+a\cos C)+(a\cos B+b\cos A)$$

$=a+b+c$ (by part ii)

(*vi*) $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

$\therefore\ \mathbf{a}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=0$ or $\mathbf{a}\times\mathbf{a}+\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}=0$

or $\mathbf{a}\times\mathbf{b}=-\mathbf{a}\times\mathbf{c}$ or $\mathbf{a}\times\mathbf{b}=\mathbf{c}\times\mathbf{a}$ (1)

Again $\mathbf{b}\times(\mathbf{a}+\mathbf{b}+\mathbf{c})=0$ $\quad \mathbf{b}\times\mathbf{a}+\mathbf{b}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}=0$

$\mathbf{b}\times\mathbf{c}=-\mathbf{b}\times\mathbf{a}$ or $\mathbf{b}\times\mathbf{c}=\mathbf{a}\times\mathbf{b}$ (2)

Hence from (1) and (2), we get

$$\mathbf{a}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}=\mathbf{c}\times\mathbf{a}$$

$$ab\sin(\pi-C)=bc\sin(\pi-A)=ca\sin(\pi-B)$$

or $ab\sin C=bc\sin A=ca\sin B$

Dividing throughout by *abc*, we get

$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}$$

$$\vec{AB}\times\vec{AC}=\begin{vmatrix} i & j & k \\ 4 & 5 & 4 \\ 2 & 0 & -6 \end{vmatrix}=-30i+32j-10k$$

Unit vector normal to the plane ABC is

$$\frac{-30i+32j-10k}{\sqrt{(30^2+32^2+10^2)}}=\frac{-30i+32j-10k}{2\sqrt{(506)}}=\frac{-15i+16j-5k}{\sqrt{(506)}}$$

(c) Ans $\frac{1}{\sqrt{6}}(2i+j+k)$

(d) Ans $S=1$

7 Clearly each vector is of unit modulus and also a b=0, b c=0, c a=0 Hence they form an orthonormal triad Also the two vectors of magnitude 3 normal to the plane of a and b are $3\,\frac{a\times b}{|a\times b|}$ and $3\,\frac{b\times a}{|b\times a|}$

8 Let the given points be A, B and C

$\vec{AB}=\vec{OB}-\vec{OA}=-i-2j-6k$

$|\vec{AB}|=\sqrt{(1+4+36)}=\sqrt{(41)}$,

$\vec{BC}=2i-j+k, |\vec{BC}|=\sqrt{6}$,

$\vec{CA}=-i+3j+5k$,

$|\vec{CA}|=\sqrt{(35)}$

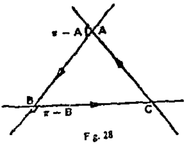


Fig. 28

Clearly $35+6=41$, i e $BC^2+CA^2=AB^2$ and hence the triangle is right angled at C

Alternately, $\vec{BC}\,\vec{CA}=2(-1)-1\,(3)+1\,5=0$,

$\vec{BC}$ is $\perp$ to $\vec{CA}$

Again $\vec{AB}\,\vec{BC}=AB\,BC\cos(\pi-B)$

or $-2+2-6=-\sqrt{(41)}\sqrt{6}\cos B$

$\cos B=\sqrt{\left(\frac{6}{41}\right)}$ or $B=\cos^{-1}\sqrt{\left(\frac{6}{41}\right)}$

B is the angle between $\vec{BA}$ and $\vec{BC}$ and hence $\pi-B$ is the angle between $\vec{AB}$ and $\vec{BC}$

Similarly $A=\cos^{-1}\sqrt{\left(\frac{35}{41}\right)}$

$=\sin\alpha\cos\beta\,\hat{n}-\cos\alpha\sin\beta\,\hat{n}$

$=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\,\hat{n}$

$i\times j=\hat{n}$ and $j\times i=-\hat{n}$

Equating the two values of $\overrightarrow{OB}\times\overrightarrow{OA}$ we get

$\sin(\alpha-\beta)\,\hat{n}=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\,\hat{n}$

$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$

The other formula for $\sin(\alpha+\beta)$ can be similarly proved if we take the cross product $\overrightarrow{OB'}\times\overrightarrow{OA}$

51 (a) (i) $\text{Volume}=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$

$$\text{Volume}=\begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$=2(4-1)+3(2+3)+4(-1-6)$

$=6+15-28=-7=7$, numerically

(ii) Ans 4 (iii) Ans 5, $\frac{5}{2}\sqrt{(6)}$

Note $[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$

Your could first find $\mathbf{b}\times\mathbf{c}$ and then take the dot product of $\mathbf{a}$ and $\mathbf{b}\times\mathbf{c}$

$$546=\begin{vmatrix} -12 & 0 & \lambda \\ 0 & 3 & -1 \\ 2 & 1 & -15 \end{vmatrix}=12\times 44-6\lambda$$

$\lambda=\ldots\times 44\quad 91=-3$

(b) We have

$|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}|=|\mathbf{a}|\ |\mathbf{b}|\ |\mathbf{c}|$

$\Leftrightarrow |\ |\mathbf{a}|\ |\mathbf{b}|\sin\theta\hat{\mathbf{n}}\cdot\mathbf{c}\,|=|\mathbf{a}|\ |\mathbf{b}|\ |\mathbf{c}|$

$\Leftrightarrow |\ |\mathbf{a}|\,|\,|\mathbf{b}|\ |\mathbf{n}|\sin\theta\cos\alpha\,|=|\mathbf{a}|\ |\mathbf{b}|\,|\mathbf{c}|$

$\Leftrightarrow |\sin\theta|\,|\cos\alpha|=1$

$\frac{18}{5}\hat{b}=\frac{18}{5}\left(\frac{3j+4k}{5}\right)=\frac{18}{25}(3j+4k)$

(c) $\beta=4i+3j \qquad \gamma=3\lambda i-4\lambda j$ for all values of λ

We have chosen γ such that $\beta\cdot\gamma=0$ as β and γ are perpendicular

Let the required vector be $\alpha=pi+qj$

Projection of α in the direction of $\beta=\frac{\alpha\cdot\beta}{|\beta|}$

$1=\frac{4p+3q}{5} \qquad 4p+3q=5$

$2=\frac{3\lambda p-4\lambda q}{5\lambda} \qquad 3p-4q=10$

Solving we get $p=2 \quad q=-1 \qquad \alpha=2i-j$

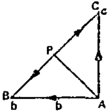


Fig 29

12 Let the position vectors of points B and C w r t A as origin be b and c so that $\vec{AB}=\mathbf{b}$, $\vec{AC}=\mathbf{c}$

Since they are at right angles therefore $\mathbf{b}\cdot\mathbf{c}=0=\mathbf{c}\cdot\mathbf{b}$

Now $\vec{BC}=\vec{AC}-\vec{AB}=\mathbf{c}-\mathbf{b}$

Again square of a vector is square of its module

$$BC^2=\vec{BC}^2=(\mathbf{c}-\mathbf{b})^2=\mathbf{c}^2+\mathbf{b}^2-2\mathbf{c}\cdot\mathbf{b}$$

$$\vec{AC}^2+\vec{AB}^2-0=AC^2+AB^2$$

If P be the mid point of BC so that $PC=PB$

Also $\vec{PB}=-\vec{PC}$ (1)

Now $\mathbf{b}=\vec{AB}=\vec{AP}+\vec{PB}$, $\mathbf{c}=\vec{AC}=\vec{AP}+\vec{PC}=\vec{AP}-\vec{PB}$ by (1)

$$\mathbf{b}\cdot\mathbf{c}=(\vec{AP}+\vec{PB})\cdot(\vec{AP}-\vec{PB})=\vec{AP}^2-\vec{PB}^2$$

or $\quad 0=AP^2-PB^2 \qquad AP=PB=PC$

Hence the mid point P of BC is equidistant from the three vertices

13 $AB^2+AC^2=\vec{AB}^2+\vec{AC}^2=(\vec{AP}+\vec{PB})^2+(\vec{AP}+\vec{PC})^2$

$$=(\vec{AP}+\vec{PB})^2+(\vec{AP}-\vec{PB})^2$$

$$=2(\vec{AP}^2+\vec{PB}^2)=2(AP^2+PB^2)$$

14 Refer fig Q 12

The triangle being isosceles, we have $AB=AC$ (1)

Also $AP=\frac{\mathbf{b}+\mathbf{c}}{2}$ where P is mid point of BC

Also $\vec{BC}=\mathbf{c}-\mathbf{b}$

$$\vec{AP}\cdot\vec{BC}=\frac{\mathbf{b}+\mathbf{c}}{2}\cdot(\mathbf{c}-\mathbf{b})=\tfrac{1}{2}(\mathbf{c}^2-\mathbf{b}^2)$$

$$=\tfrac{1}{2}(AC^2-AB^2)=0 \qquad \text{by (1)}$$

L H S $=(\mathbf{a}+\mathbf{a}+\mathbf{a})-[(\mathbf{a}.\mathbf{i})\,\mathbf{i}+(\mathbf{a}.\mathbf{j})\,\mathbf{j}+(\mathbf{a}.\mathbf{k})\,\mathbf{k}]$

$=3\mathbf{a}-\mathbf{a}$ by (iii)

$=2\mathbf{a}$

54 $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$, etc

$\mathbf{a}.\mathbf{a}=\Sigma a_1^2$, $\mathbf{a}.\mathbf{b}=\Sigma a_1 b_1$

$$[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}=\Delta$$

$$\mathbf{a}\times\mathbf{b}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=(a_2 b_3-a_3 b_2)\,\mathbf{i}+(a_3 b_1-a_1 b_3)\,\mathbf{j}+(a_1 b_2-a_2 b_1)\,\mathbf{k}$$

Similarly we can write down for $\mathbf{b}\times\mathbf{c}$ and $\mathbf{c}\times\mathbf{a}$

$[\mathbf{a}\times\mathbf{b},\ \mathbf{b}\times\mathbf{c}\ \ \mathbf{c}\times\mathbf{a}]$

$$=\begin{vmatrix} a_2b_3-a_3b_2 & b_1a_3-b_3a_1 & a_1b_2-a_2b_1 \\ b_2c_3-b_3c_2 & c_1b_3-c_3b_1 & b_1c_2-b_2c_1 \\ c_2a_3-c_3a_2 & a_1c_3-a_3c_1 & c_1a_2-c_2a_1 \end{vmatrix}=\Delta'$$

or $$\Delta'=\begin{vmatrix} C_1 & C_2 & C_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{vmatrix}=\begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}=\Delta^2=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]^2$$

The capital letters above denote the cofactors of the corresponding small letters in Δ Also from determinants we know that $\Delta'=\Delta^2$

Again $$\Delta^2=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$=\begin{vmatrix} \Sigma a_1^2 & \Sigma a_1 b_1 & \Sigma a_1 c_1 \\ \Sigma b_1 a_1 & \Sigma b_1^2 & \Sigma b_1 c_1 \\ \Sigma c_1 a_1 & \Sigma c_1 b_1 & \Sigma c_1^2 \end{vmatrix}$$

$$=\begin{vmatrix} \mathbf{a}.\mathbf{a} & \mathbf{a}.\mathbf{b} & \mathbf{a}.\mathbf{c} \\ \mathbf{b}.\mathbf{a} & \mathbf{b}.\mathbf{b} & \mathbf{b}.\mathbf{c} \\ \mathbf{c}.\mathbf{a} & \mathbf{c}.\mathbf{b} & \mathbf{c}.\mathbf{c} \end{vmatrix}$$

16 Here $AD = BC$ since diagonals are equal

$$\vec{AD}^2 = \vec{BC}^2 \quad \text{or} \quad (b+c)^2-(c-b)^2=0 \quad \text{or} \quad 4b\cdot c=0$$

i e $\vec{AB}\cdot\vec{AC}=0$ $\quad AB \perp AC$

and hence the parallelogram is a rectangle

17 If the figure of Q 15 is a rhombus, then $AB=AC$

$$\vec{AD}\cdot\vec{BC}=(b+c)\cdot(c-b)=c^2-b^2$$

$$=\vec{AC}^2-\vec{AB}^2=AC^2-AB^2=0$$

$\vec{AD}\perp\vec{BC}$ or $AD\perp BC$ i e diagonals are perpendicular in a rhombus

18 Take the centre O as origin and AB is the diameter so that

$$OA=OB$$

If the point A be a then B is

a and $|a|=r=r$ dius

Let P be any point r on the circumference so that $|r|=OP=r$

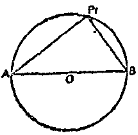


Fig 31

Then $\vec{AP}$ = P V of P − P V of A = r − a

and $\vec{BP}$ = P V of P − P V of B = r + a

$$\vec{AP}\cdot\vec{BP}=(r-a)\cdot(r+a)=r^2-a^2=r^2-r^2=0$$

Hence AP is $\perp$ to BP, i e angle in a semi-circle is a right angle

19 Let $A=a=3i-2j+k$ be the point through which the plane passes Let us choose $L=r=(x\,i+y\,j+z\,k)$ any point $(x\ y\ z)$ on this plane

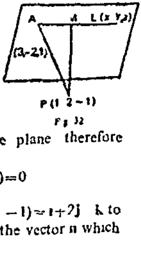


Fig 32

Therefore $\vec{AL}=r-a$

or $\vec{AL}=(x-3)\,i+(y+2)\,j+(z-1)\,k$

Since $4i+7j-4k=n$, say is normal to the plane therefore

$$(r-a)\cdot n=0$$

or $$(x-3)\,4+(y+2)\,7+(z-1)\,(-4)=0$$

or $$4x+7y-4z+6=0$$

Again PM is perpendicular from $P(1, 2, -1)=i+2j-k$ to the plane so that PM is projection of AP along the vector n which is normal to the plane

Multiply both sides scalarly by $\mathbf{b}\times\mathbf{c}$ we get

$$(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})=l\,\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})+m\,\mathbf{b}\cdot(\mathbf{b}\times\mathbf{c})+n\,\mathbf{c}\cdot(\mathbf{b}\times\mathbf{c})$$

$$=l\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+m\,[\mathbf{b}\ \mathbf{b}\ \mathbf{c}]+n\,[\mathbf{c}\ \mathbf{b}\ \mathbf{c}]=l\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$$

$$l=\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{b}\times\mathbf{c})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Similarly multiplying both sides scalarly by $(\mathbf{c}\times\mathbf{a})$ and $(\mathbf{a}\times\mathbf{b})$ we get

$$m=\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{c}\times\mathbf{a})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\quad n=\frac{(\mathbf{b}\times\mathbf{c})\cdot(\mathbf{a}\times\mathbf{b})}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Substituting the values of l, m, n in (2) we get the required expression for $\mathbf{b}\times\mathbf{c}$ in terms of $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

In a similar manner we can express $(\mathbf{c}\times\mathbf{a})$ and $(\mathbf{a}\times\mathbf{b})$.

57 $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

L.H.S. $=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}+(\mathbf{b}\cdot\mathbf{a})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}+(\mathbf{c}\cdot\mathbf{b})\mathbf{a}-(\mathbf{c}\cdot\mathbf{a})\mathbf{b}=0$

$\because\ \mathbf{a}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{a},\ \mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ and $\mathbf{b}\cdot\mathbf{c}=\mathbf{c}\cdot\mathbf{b}$

58 $(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=\frac{1}{2}\mathbf{b}$

or $(\mathbf{a}\cdot\mathbf{c}-\frac{1}{2})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}=0$ or $\lambda\mathbf{b}-\mu\mathbf{c}=0$

Above is a relation between $\mathbf{b}$ and $\mathbf{c}$ which are non parallel

$\lambda=0,\ \mu=0 \qquad \mathbf{a}\cdot\mathbf{c}=\frac{1}{2}$ or $1\cdot 1\cos\theta=\frac{1}{2}$

$\theta=60^\circ$

$\mathbf{a}\cdot\mathbf{b}=0 \qquad \mathbf{a}$ and $\mathbf{b}$ are at right angles

$\mathbf{a}$ makes with $\mathbf{b}$ an angle 90° and with $\mathbf{c}$ an angle of 60°

59 We are given that

$(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$

$(\mathbf{a}\cdot\mathbf{b})\mathbf{c}-(\mathbf{b}\cdot\mathbf{c})\mathbf{a}=0$ or $(\mathbf{a}\times\mathbf{c})\times\mathbf{b}=0$

From above we conclude that either $\mathbf{b}=0$ or $\mathbf{a}\times\mathbf{c}=0$

But $\mathbf{b}\neq 0$ $\quad$ $\mathbf{a}\times\mathbf{c}=0$ *and hence* $\mathbf{c}=t\,\mathbf{a}$ *i.e.* $\mathbf{a}$ and $\mathbf{c}$ are collinear

Converse. If $\mathbf{c}=t\mathbf{a}$, then

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=(\mathbf{a}\times\mathbf{b})\times t\mathbf{a}=t\,[(\mathbf{a}\cdot\mathbf{a})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{a}]$

$\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\mathbf{a}\times(\mathbf{b}\times t\mathbf{a})=t\,[(\mathbf{a}\cdot\mathbf{a})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{a}]$

$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}=\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$

60 **Hint** With one vertex as origin, let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be the position vectors of the other vertices. Then

$\mathbf{n}_1=\frac{1}{2}(\mathbf{a}\times\mathbf{b})\quad \mathbf{n}_2=\frac{1}{2}(\mathbf{b}\times\mathbf{c})\quad \mathbf{n}_3=\frac{1}{2}(\mathbf{c}\times\mathbf{a}),$

and $\mathbf{n}_4=\frac{1}{2}(\mathbf{b}-\mathbf{c})\times(\mathbf{a}-\mathbf{c})$ etc.

61 (*a*) Let the given points be *A, B, C* and *D*

$OP^2+CD^2+AE^2+BF^2=\vec{OP}^2+\vec{CD}^2+\vec{AE}^2+\vec{BF}^2$

$=(a+b+c)^2+(a+b-c)^2+(b+c-a)^2+(c+a-b)^2$

$=4(a^2+b^2+c^2)=4(\vec{OA}^2+\vec{OB}^2+\vec{OC}^2)$

$=4(OA^2+OB^2+OC^2)$, $\quad a^2=\vec{OA}^2=OA^2$ etc

22. Taking D as origin, let the position vectors of A, B, C be a, b and c respectively

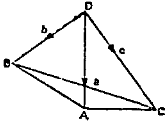


Fig. 35

$AC \perp$ to DB $\quad (c-a)\cdot b=0$

or $\quad b\cdot c=b\cdot a \quad (1)$

Again $AB \perp$ to DC,

$(b-a)\cdot c=0$

or $\quad b\cdot c=a\cdot c \quad (2)$

Hence from (1) and (2), we get $\quad a\cdot b=b\cdot c=c\cdot a \quad (3)$

$a\cdot b=c\cdot a$, it follows that

$a\cdot(b-c)=0$,

which shows that DA is $\perp$ CB

Hence if two pairs of opposite edges are perpendicular then the third pair is also perpendicular

Again $\quad AB^2+CD^2=\vec{AB}^2+\vec{CD}^2=(b-a)^2+c^2$

$=a^2+b^2+c^2-2a\cdot b$

$=a^2+b^2+c^2-2b\cdot c=a^2+b^2+c^2-2c\cdot a$ by (3)

$=BC^2+DA^2=CA^2+DB^2$

Hence sum of the squares on the two opposite edges is the same for each pair

23 Since the tetrahedron is regular therefore

$DA=DB=DC=AB=BC=CA$

$a^2=b^2=c^2=(b-a)^2=(c-b)^2=(a-c)^2 \quad (1)$

Let us prove that the opposite edges DA and BC are perpendicular

or $\quad \vec{DA}\cdot\vec{BC}=0 \quad$ or $a\cdot(c-b)=0 \quad (2)$

Now from (1) we have $(b-a)^2=(a-c)^2$

or $\quad b^2+a^2-2b\cdot a=a^2+c^2-2a\cdot c$

or $\quad 2a^2-2b\cdot a=2a^2-2a\cdot c$ by (1)

or $\quad a\cdot(b-c)=0$ which is (2)

$$\Sigma \overrightarrow{OA_i} \times \overrightarrow{OA_{i+1}} = \Sigma\, |\overrightarrow{OA_i}|\, |\overrightarrow{OA_{i+1}}| \sin\frac{2\pi}{n}\,\mathbf{n}$$

$$= \Sigma\, r^2 \sin\frac{2\pi}{n}\,\hat{\mathbf{n}} = (n-1)\, r^2 \sin\frac{2\pi}{n}\,\hat{\mathbf{n}}$$

and $(1-n)\,\overrightarrow{OA_n} \times \overrightarrow{OA_1} = (1-n)\, r^2 \sin\frac{2\pi}{n}(-\hat{\mathbf{n}})$

$= (n-1)\, r^2 \sin\frac{2\pi}{n}\,\hat{\mathbf{n}}$ where $\hat{\mathbf{n}}$ is a unit vector $\perp$ to plane of the polygon

Hence L H S = R H S

64 (*a*) Volume $=\frac{1}{2}\begin{vmatrix} 3 & 4 & 12 \\ 12 & 3 & 4 \\ 4 & 12 & 3 \end{vmatrix} = \frac{1}{2}[3(9-48)-4(36-16)+12(144-12)]$

$= \dfrac{1387}{2} = 693\ 5$

(b) $\overrightarrow{OA} \times \overrightarrow{OB} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 12 \\ 12 & 3 & 4 \end{vmatrix} = -20\mathbf{i} + 132\mathbf{j} - 39\mathbf{k}$

Unit vector $\perp$ to both $\overrightarrow{OA}$ and $\overrightarrow{OB}$

$= \dfrac{-20\mathbf{i}+132\mathbf{j}-39\mathbf{k}}{\sqrt{[(-20)^2+(132)^2+(-39)^2]}} = \dfrac{-20\mathbf{i}+132\mathbf{j}-39\mathbf{k}}{\sqrt{(19345)}}$

(c) $\overrightarrow{OA} \cdot \overrightarrow{OC} = (3\mathbf{i}+4\mathbf{j}+12\mathbf{k}) \cdot (4\mathbf{i}+12\mathbf{j}+3\mathbf{k})$

$|\overrightarrow{OA}|\,|\overrightarrow{OC}| \cos\theta = 3\cdot 4+4\cdot 12+12\cdot 3 = 96$

or $\cos\theta = \dfrac{96}{\sqrt{(3^2+4^2+12^2)}\sqrt{(4^2+12^2+3^2)}} = \dfrac{96}{169}$

$\theta = \cos^{-1}\dfrac{96}{169}$

(*d*) Area of $\triangle\, OAB = \frac{1}{2}\,|(\overrightarrow{OA} \times \overrightarrow{OB})| = \sqrt{(19345)}$ from (*b*)

65 We are given $\mathbf{A}\cdot\mathbf{X} = c$ (1) $\mathbf{A}\times\mathbf{X} = \mathbf{B}$ (2)

Taking cross product with A on both sides of (2),

$\mathbf{A}\times(\mathbf{A}\times\mathbf{X}) = \mathbf{A}\times\mathbf{B}$ *i e* $(\mathbf{A}\cdot\mathbf{X})\,\mathbf{A} - (\mathbf{A}\cdot\mathbf{A})\,\mathbf{X} = \mathbf{A}\times\mathbf{B}$

Now using (1), we get $c\mathbf{A} - |\mathbf{A}|^2\mathbf{X} = \mathbf{A}\times\mathbf{B}$

or $\mathbf{X}\,|\mathbf{A}|^2 = c\mathbf{A} - \mathbf{A}\times\mathbf{B}$ or $\mathbf{X} = \dfrac{c\mathbf{A}-\mathbf{A}\times\mathbf{B}}{|\mathbf{A}|^2}$

66 Since a b are non collinear we can express c as a linear combinations of a and b as

$$\mathbf{c} = x\mathbf{a} + y\mathbf{b}$$

26 Taking A as origin, let P Vs of B and C be **b** and **c** respectively and hence those of the mid points F and E are $\mathbf{b}/2$ and $\mathbf{c}/2$

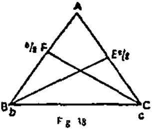


Fig 18

$\vec{BE}=\mathbf{c}/2-\mathbf{b}$ $\vec{CF}=\mathbf{b}/2-\mathbf{c}$

$BE=CF$ (given), $BE^2=CF^2$

or $\vec{BE}^2=\vec{CF}^2$

or $\left(\frac{\mathbf{c}}{2}-\mathbf{b}\right)^2=\left(\frac{\mathbf{b}}{2}-\mathbf{c}\right)^2$

or $\frac{\mathbf{c}^2}{4}+\mathbf{b}^2-\mathbf{c}\cdot\mathbf{b}=\frac{\mathbf{b}^2}{4}+\mathbf{c}^2-\mathbf{b}\cdot\mathbf{c}$

or $\frac{3}{4}(\mathbf{b}^2-\mathbf{c}^2)=0$ or $\mathbf{b}^2=\mathbf{c}^2$ or $\vec{AB}^2=\vec{AC}^2$

or $AB^2=AC^2$ $AB=AC$ Hence the triangle is isosceles

27 We know that square of a vector is square of its module

$|\mathbf{a}+\mathbf{b}|=|\mathbf{a}-\mathbf{b}|^2$ or $(\mathbf{a}+\mathbf{b})^2=(\mathbf{a}-\mathbf{b})^2$

or $4\,\mathbf{a}\cdot\mathbf{b}=0$ which implies that **a** and **b** are perpendicular

28 We know that $\frac{\mathbf{a}}{a}=\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^2=1$

Hence we have to prove that

$$\left(\frac{\hat{\mathbf{a}}}{a}-\frac{\hat{\mathbf{b}}}{b}\right)^2=\left(\frac{\hat{\mathbf{a}}}{b}-\frac{\hat{\mathbf{b}}}{a}\right)^2$$

or $(b\hat{\mathbf{a}}-a\hat{\mathbf{b}})^2=(a\hat{\mathbf{a}}-b\hat{\mathbf{b}})^2$

or $b^2\,1+a^2\,1-2\,ab\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}=a^2\,1+b^2\,1-2ab\hat{\mathbf{a}}\cdot\hat{\mathbf{b}}$

Clearly L H S = R H S

29 *Here* **a** *and* **b** *are unit vectors inclined at an angle* θ so that

$\mathbf{a}^2=\mathbf{b}^2=1$ and $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=\cos\theta$ (1)

Now $|\mathbf{a}-\mathbf{b}|^2=(\mathbf{a}-\mathbf{b})^2$

$=\mathbf{a}^2+\mathbf{b}^2-2\mathbf{a}\cdot\mathbf{b}=1+1-2\cdot1\cdot1\cos\theta$ by (1)

$=2(1-\cos\theta)=2\cdot2\sin^2\theta/2=4\sin^2\theta/2$

30 Let us suppose that **n** is not a null vector $\mathbf{n}\cdot\mathbf{a}=0$ and $\mathbf{n}\cdot\mathbf{b}=0$ so that **n** is perpendicular to both **a** and **b** and hence perpendicular to the plane of **a** and **b** Further $\mathbf{n}\cdot\mathbf{c}=0$ so that **n** is also perpendicular to **c** also and hence **c** will lie in the plane of **a** and **b** This would mean that **n**, **b**, **c** are coplanar and this is contrary to the hypothesis Hence the vector **n** must be a null vector

or $(\mathbf{a}\ \mathbf{b})\,\mathbf{r}-(\mathbf{a}\ \mathbf{r})\,\mathbf{b}-(\mathbf{a}\ \mathbf{b})\,\mathbf{c}+(\mathbf{a}\ \mathbf{c})\,\mathbf{b}=0$

or $(\mathbf{a}\ \mathbf{b})\,\mathbf{r}-0-(\mathbf{a}\ \mathbf{b})\,\mathbf{c}+(\mathbf{a}\ \mathbf{c})\,\mathbf{b}=0$ $[\ \mathbf{a}\ \mathbf{r}=0]$

Hence $\mathbf{r}=\dfrac{(\mathbf{a}\ \mathbf{b})\,\mathbf{c}-(\mathbf{a}\ \mathbf{c})\,\mathbf{b}}{\mathbf{a}\ \mathbf{b}}$

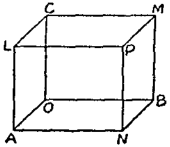
69 $\vec{AB}=-\mathbf{i}+5\mathbf{j}-3\mathbf{k}$, $\vec{AC}=-4\mathbf{i}+3\mathbf{j}+3\mathbf{k}$ and $\vec{AD}=\mathbf{i}+7\mathbf{j}+(\lambda+1)\,\mathbf{k}$

Now A, B, C, D will be coplanar if

$$\begin{vmatrix} -1 & 5 & -3 \\ -4 & 3 & 3 \\ 1 & 7 & \lambda+1 \end{vmatrix}=0$$

This on expansion will give $\lambda=-\dfrac{146}{17}$

70 Let a, b, c be the lengths of the sides OA, OB and OC respectively of the rectangular parallelopiped ($OANB$, $MCLP$)



Take O as the origin of vectors and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ denote unit vectors along OA, OB and OC respectively Then $\vec{OA}=a\mathbf{i}$, $\vec{OB}=b\,\mathbf{j}$, $\vec{OC}=c\,\mathbf{k}$ Also $\vec{OP}=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ The edges which do not meet the diagonals OP are AL, AN, BN and their parallels BM CM and CL

Suppose we are to find the distance between the diagonal OP and the edge BN

Now OP is the line passing through O whose position vector is $\mathbf{0}$ and parallel to the vector $a\mathbf{i}+b\mathbf{j}+c\mathbf{k}$ And BN is the line through B whose position vector is $b\mathbf{j}$ and parallel to $\mathbf{i}$ Hence by § 13, the shortest distance between OP and

$\therefore \quad \overrightarrow{OP}\,\overrightarrow{OL} = |\overrightarrow{OP}|\ |\overrightarrow{OL}| \cos\alpha$

$a(\mathbf{i}+\mathbf{j}+\mathbf{k})\ (x\mathbf{i}+y\mathbf{j}+z\mathbf{k}) = (x^2+y^2+z^2)\sqrt{(a^2+a^2+a^2)} \cos\alpha$

$$\frac{x+y+z}{\sqrt{3}\ (x^2+y^2+z^2)} = \cos\alpha \qquad (1)$$

Similarly taking the dot product of $\overrightarrow{OL}$ with the vectors represented by the other diagonals, we get

$$\frac{x+y-z}{\sqrt{3}\sqrt{(\Sigma x^2)}} = \cos\beta, \frac{y+z-x}{\sqrt{3}\sqrt{(\Sigma x^2)}} = \cos\gamma, \frac{z+x-y}{\sqrt{3}\sqrt{(\Sigma x^2)}} = \cos\delta$$

$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$$

$$= \frac{(x+y+z)^2+(x+y-z)^2+(y+z-x)^2+(z+x-y)^2}{3\ (x^2+y^2+z^2)}$$

$$= \frac{4\ (x^2+y^2+z^2)}{3\ (x^2+y^2+z^2)} = \frac{4}{3}$$

Again if angle between the diagonals OP and CD be θ, then

$\overrightarrow{OP}\,\overrightarrow{CD} = |\overrightarrow{OP}|\ |\overrightarrow{CD}| \cos\theta$

$a(\mathbf{i}+\mathbf{j}+\mathbf{k}).a\ (\mathbf{i}+\mathbf{j}-\mathbf{k}) = a\sqrt{3}\ a\sqrt{3} \cos\theta$

$a^2 = 3a^2 \cos\theta$ i e $\cos\theta = \frac{1}{3}$

Similarly angles between any other two diagonals is $\cos^{-1}(1/3)$

34 *We know that between any three coplanar vectors* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *there exists a relation of the form*

$$x\,\mathbf{a}+y\,\mathbf{b}+z\,\mathbf{c}=0 \qquad (1)$$

where x, y, z are scalars

Multiplying both sides of (1) scalarly by $\mathbf{a}$ and $\mathbf{b}$ respectively

$$x\,\mathbf{a}\,\mathbf{a}+y\,\mathbf{a}\,\mathbf{b}+z\,\mathbf{a}\,\mathbf{c}=0 \qquad (2)$$

$$x\,\mathbf{b}\,\mathbf{a}+y\,\mathbf{b}\,\mathbf{b}+z\,\mathbf{b}\,\mathbf{c}=0 \qquad (3)$$

Eliminating x, y, z from (1), (2) and (3), we get

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\,\mathbf{a} & \mathbf{a}\,\mathbf{b} & \mathbf{a}\,\mathbf{c} \\ \mathbf{b}\,\mathbf{a} & \mathbf{b}\,\mathbf{b} & \mathbf{b}\,\mathbf{c} \end{vmatrix}$$

35 We know that

$$\mathbf{a}\times\mathbf{b} = ab \sin\theta\hat{\mathbf{n}}$$

Squaring both sides, we get

$$(\mathbf{a}\times\mathbf{b})^2 = a^2b^2 \sin^2\theta\hat{\mathbf{n}}^2 = a^2b^2\ (1-\cos^2\theta), \qquad \hat{\mathbf{n}}^2=1$$

$$= a^2b^2 - a^2b^2\cos^2\theta = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a}\ \mathbf{b})^2$$

($a^2 = \mathbf{a}^2$, $\mathbf{a}\ \mathbf{b} = ab\cos\theta$)

36 Let O be the intersection of

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 12 & 4 & 3 \\ 8 & -12 & -9 \end{vmatrix} = 0\mathbf{i}+132\mathbf{j}-176\,\mathbf{k}$$

Hence area of the face determined by OA and OB
$=\sqrt{(32^2+176^2)}=44\sqrt{(3^2+4^2)}=220$

Similarly areas of other two faces can be found This is left for the reader

$$\text{Volume}=\begin{vmatrix} 12 & 4 & 3 \\ 8 & -12 & -9 \\ 33 & -4 & -24 \end{vmatrix}=12\begin{vmatrix} 12 & 1 & 1 \\ 8 & -3 & -3 \\ 33 & -1 & -8 \end{vmatrix}$$

$$=12\begin{vmatrix} 12 & 1 & 1 \\ 44 & 0 & 0 \\ 33 & -1 & -8 \end{vmatrix}$$

$$= -12\times 44\,(-8+1)=12\times 44\times 7=3696$$

§ 13 Reciprocal system of vectors

The three vectors $\mathbf{a}'$, $\mathbf{b}$, $\mathbf{c}'$ defined by the equations

$$\mathbf{a} =\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]},\ \mathbf{b}'=\frac{\mathbf{c}\times\mathbf{a}}{[\mathbf{abc}]},\ \mathbf{c} =\frac{\mathbf{a}\times\mathbf{b}}{[\mathbf{abc}]}$$

are called reciprocal system to the vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ which are non coplanar, *i e* $[\mathbf{abc}]\neq 0$

Property 1 *If* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *and* $\mathbf{a}'$ $\mathbf{b}'$, $\mathbf{c}'$ *be reciprocal system of vectors, then* $\mathbf{a}\ \mathbf{a}'=\mathbf{b}\ \mathbf{b}'=\mathbf{c}\ \mathbf{c}'=1$

We have $\mathbf{a}\ \mathbf{a} =\mathbf{a}\ \dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\dfrac{[\mathbf{abc}]}{[\mathbf{abc}]}=1$

Similarly $\mathbf{b}\ \mathbf{b}'=\mathbf{c}\ \mathbf{c} =1$

$\mathbf{a}\ \mathbf{a} + \mathbf{b}\ \mathbf{b} +\mathbf{c}\ \mathbf{c} =3$

Note Because of the property $\mathbf{a}\ \mathbf{a}'=\mathbf{b}\ \mathbf{b}'=\mathbf{c}\ \mathbf{c}'=1$ the two systems of vectors are called reciprocal systems

Property 2 *The product of any vector of one system with a vector of the other system which does not correspond to it is zero i e* $\mathbf{a}\ \mathbf{b} =\mathbf{a}\ \mathbf{c}'=\mathbf{b}\ \mathbf{c}'$
$=\mathbf{c}\ \mathbf{a}'=\mathbf{c}\ \mathbf{b} =0$

Area of

$\triangle OAB = \frac{1}{2} OA \; OB \sin\theta$

$= \frac{1}{2} ab \sin\theta$

$\mathbf{a}\times\mathbf{b} = 2\triangle\, \hat{\mathbf{n}}$

Hence vector area of triangle OAB

is $\frac{1}{2}(\mathbf{a}\times\mathbf{b})$

$= \frac{1}{2}$ (area of ||$^{\text{m}}$ whose adjacent sides are given by $\mathbf{a}$ and $\mathbf{b}$

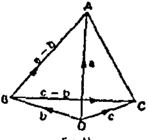


Fig 41

Now referred to O as origin let the position vectors of A, B, C be $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively

Then $\overrightarrow{BC} = \mathbf{c}-\mathbf{b}$, $\overrightarrow{BA} = \mathbf{a}-\mathbf{b}$

Vector area of $\triangle ABC$ is $\frac{1}{2}\overrightarrow{BC}\times\overrightarrow{BA}$

$= \frac{1}{2}(\mathbf{c}-\mathbf{b})\times(\mathbf{a}-\mathbf{b}) = \frac{1}{2}(\mathbf{c}\times\mathbf{a}-\mathbf{b}\times\mathbf{a}-\mathbf{c}\times\mathbf{b}+\mathbf{b}\times\mathbf{b})$

$= \frac{1}{2}(\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a})$

($\mathbf{b}\times\mathbf{b}=0$ and $-\mathbf{c}\times\mathbf{b}=\mathbf{b}\times\mathbf{c}$)

Condition of Collinearity

If the three points be collinear, then $\triangle = 0$

$\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}=0$

(b) By part (a) 2 Area of $\triangle AB = |\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|$ and $BC = |\mathbf{b}-\mathbf{c}|$

Hence length of perpendicular, from A on BC

$= 2\triangle ABC/BC = \dfrac{|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}|}{|\mathbf{b}-\mathbf{c}|}$

(c) Without loss of generality we may choose the point D as origin and the position vectors of A, B, C be $\mathbf{a}, \mathbf{b}, \mathbf{c}$ respectively

$|\overrightarrow{AB}\times\overrightarrow{CD}+\overrightarrow{BC}\times\overrightarrow{AD}+\overrightarrow{CA}\times\overrightarrow{BD}|$

$= |(\mathbf{b}-\mathbf{a})\times(-\mathbf{c})+(\mathbf{c}-\mathbf{b})\times(-\mathbf{a})+(\mathbf{a}-\mathbf{c})\times(-\mathbf{b})|$

$= 2|\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}| = 2(2\triangle ABC) = 4\triangle$ by part (a)

42 As seen in Q 41, we have

$\overrightarrow{BC}\times\overrightarrow{BA} = \mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$

and $\overrightarrow{BC}\times\overrightarrow{BA}$ represents a vector perpendicular to both $\overrightarrow{BC}$ and BA and hence it is normal to the plane defined by BC and BA or $\mathbf{a}\times\mathbf{b}+\mathbf{b}\times\mathbf{c}+\mathbf{c}\times\mathbf{a}$ is perpendicular to the plane ABC

43 Area of ||$^{\text{m}}$ $= |(\mathbf{a}\times\mathbf{b})|$

$$\text{But } \mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & -2 & 1 \end{vmatrix} = 8(\mathbf{i}+\mathbf{j}+\mathbf{k})$$

$$r = r\,a'a + r\,b'b + r\,c'c$$

Also in the two systems of vectors [abc], [a b c'], each is reciprocal of the other and as such any vector r can also be written as $r = r\,aa' + r\,bb' + r\,cc'$

Again $i\,i = j\,j = k\,k = 1$ and $[i\,j\,k] = 1$

the system of vectors i, j k is its own reciprocal

Hence in terms of unit vectors, i, j, k we have

$$r = r\,i\,i + r\,j\,j + r\,k\,k$$

Examples

1 *If* a b, c *are non coplanar, show that*

$$r = \frac{(r\,a)\,b \times c}{[abc]} + \frac{(r\,b)\,c \times a}{[abc]} + \frac{(r\,c)\,a \times b}{[abc]}$$

We have proved that

$$r = r\,a'a + r\,b'b + r\,c'c$$

$$r = r\,aa' + r\,bb' + r\,cc'$$

Now put $a' = \frac{b \times c}{[abc]}$ etc and we get the result

2 *If* $a' = \frac{b \times c}{[abc]}$, $b' = \frac{c \times a}{[abc]}$, $c' = \frac{a \times b}{[abc]}$ *then prove that*

$$a = \frac{b' \times c'}{[a'\,b'\,c']},\ b = \frac{c' \times a'}{[a'\,b'\,c']},\ c = \frac{a' \times b'}{[a'\,b'\,c']}$$

$$b' \times c' = \frac{(c \times a) \times (a \times b)}{[abc]^2}$$

$$= \frac{(c \times a)\,b\,a - (c \times a)\,a\,b}{[abc]^2} \quad \text{Put } (c \times a)\,a = [caa] = 0$$

$$= \frac{[cba]a}{[abc]^2} = \frac{[abc]a}{[abc]^2} = \frac{1}{[abc]}\,a \qquad (1)$$

$$[a'\,b'\,c'] = a'\,(b' \times c')$$

$$= \frac{b \times c}{[abc]} \cdot \frac{1}{[abc]}\,a = \frac{[bca]}{[abc]^2}$$

$$= \frac{[abc]}{[abc]^2} = \frac{1}{[abc]} \qquad (2)$$

and $\frac{b' \times c'}{[a'\,b'\,c']} = \frac{1}{[abc]}\,a[abc] = a$ by (1) and (2)

Hence proved Similarly we can prove other results

48 (a) Vector area of the quadrilateral $ABCD$

= Vector area of $\triangle ABC$ + vector area of $\triangle ACD$

$=\frac{1}{2}\vec{AB}\times\vec{AC}+\frac{1}{2}\vec{AC}\times\vec{AD}$

$=\frac{1}{2}\vec{AC}\times\vec{AD}-\frac{1}{2}\vec{AC}\times\vec{AB}$

$=\frac{1}{2}\vec{AC}\times(\vec{AD}-\vec{AB})$

$=\frac{1}{2}\vec{AC}\times\vec{BD}$

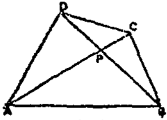


Fig. 42

(b) $ABCD$ is the parallelogram with $\mathbf{a}=\vec{AB}=2\mathbf{m}+\mathbf{n}$

and $\mathbf{b}=\vec{AD}=\mathbf{m}-2\mathbf{n}$

Then $\vec{AC}=\vec{AB}+\vec{BC}=\vec{AB}+\vec{AD}=3\mathbf{m}-\mathbf{n}$

and $\vec{BD}=\vec{AD}-\vec{AB}=-\mathbf{m}-3\mathbf{n}$

Hence $AC=\sqrt{\{(\vec{AC}^2)\}}=\sqrt{(3\mathbf{m}-\mathbf{n})^2}$

$=\sqrt{(9\mathbf{m}^2+\mathbf{n}^2-6\mathbf{m}\,\mathbf{n})}$

$=\sqrt{(9+1-6\;1\;1\cos 60^\circ)}$

$[\quad |\mathbf{m}|=1,\ |\mathbf{n}|=1]$

$=\sqrt{7}$

$BD=\sqrt{\{(BE)^2\}}=\sqrt{\{(-\mathbf{m}-3\mathbf{n})^2\}}$

$=\sqrt{(\mathbf{m}^2+9\mathbf{n}^2+6\mathbf{m}\,\mathbf{n})}$

$=\sqrt{(1+9+6\;1\;1\cos 60^\circ)}$

$=\sqrt{13}$

49 (i) Let the vectors represented by sides BC, CA and AB be denoted by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ respectively and their modules be denoted by a, b and c

Also we know that

$\vec{BC}+\vec{CA}+\vec{AB}=\vec{BB}=0$

or $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$

$\vec{BC}=-(\vec{CA}+\vec{AB})$ (1)

Squaring both sides we get

$\vec{BC}^2=\vec{CA}^2+\vec{AB}^2+2\vec{CA}\,\vec{AB}$

or $a^2=b^2+c^2+2bc\cos(\pi-A)$

or $a^2=b^2+c^2-2bc\cos A$

Angle between AB and AC is A and hence angle between AB and CA is $\pi-A$

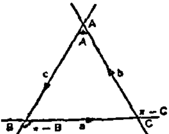


Fig 43

2 The force represented by $5i + j$ is acting through the point $9i - j + 2k$. Find the moment about the point $3i - 2j + k$.

3 A force $F = 3i + 2j - 4k$ is acting at the point $(1, -1, 2)$. Find the moment of F about the point $(2, -1, 3)$.

4 Find the moment about the point $i + 2j + 3k$ of a force represented by $i + j + k$ acting through the point $2i + 3j + k$.

5 A force $P = 4i - 3k$ passes through the point A whose position vector is $2i - 2j + 5k$. Find the moment of P about the point B whose position vector is $i - 3j + k$.

6 (a) *A force $F = 3j - 6k$ passes through the point $A, 4i - 2j - 9k$* find the moment of P about the point B whose position vector is $6i - 7k$.

(b) Find the moment of the force $5i + 10j + 16k$ acting at the point $2i - 7j + 10k$ about the point $-5i + 6j - 10k$.

(c) Find the vector moment of three forces $i + 2j - 3k$, $2i + 3j + 4k$ and $-i - j + k$ acting on a particle at a point *P (0, 1, 2) adout the point A (1, −2, 0)* (M N R 84)

7 (a) *A* particle acted on by constant forces $4i + j - 3k$ and $3i + j - k$ is displaced from the point $i + 2j + 3k$ to the point $5i + 4j + k$. Find the total work done by the forces.

(b) A force $F = 2i + j - k$ acts at a point A whose position vector is $2i - j$. Find the moment of F about the origin. If the point of application of F moves from the point A *to the point B (position vector $2i + j$), find the work done* by F (Roorkee 78)

8 Find the work done in moving an object along a straight line from $(3, 2, -1)$ to $(2, -1, 4)$ in a force field given by *$F = 5i - 3j + 2k$*

9 Find the work done in moving an object along a vector $\vec{PQ} = 3i + 2j - 5k$ if the force applied is $F = 2i - j - k$

10 *Forces acting on a particle have magnitudes 5, 3, 1 lbs wt* and act in the directions of the vectors $6i + 2j + 3k$, $3i - 2j + 6k$ and $2i - 3j - 6k$ respectively. These remain constant while the paritcle is displaced from the point A $(2i - j - 3k)$ to B $(5i - j + k)$. Find the work done.

11 A particle is displaced from the point whose position vector is $5i - 5j - 7k$ to the point whose position vector is $6i + 2j - 2k$ under the action of constant forces $10i - j + 11k$, $4i + 5j + 6k$ and $-2i + j - 9k$. Show that the total work done by the forces is 87 units.

50 (i) Let there be two unit vectors **i** and **j** along OX and OY, two perpendicular lines in the plane of the paper. If OP and OQ be any two lines in the same plane making angles α and β with OX respectively then $\angle POQ=\alpha-\beta$

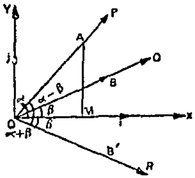


Fig 44

Again let $\vec{OA}$ and $\vec{OB}$ represent unit vectors along OP and OQ respectively

$$\vec{OA}\cdot\vec{OB}=1\cdot1\cos(\alpha-\beta)=\cos(\alpha-\beta) \qquad (1)$$

Again $\vec{OA}=\vec{OM}+\vec{MA}=OA\cos\alpha\,\mathbf{i}+OA\sin\alpha\,\mathbf{j}$

$$\vec{OA}=\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j}, \qquad OA=1 \qquad (2)$$

Similarly $\vec{OB}=\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j}$ (3)

$$\vec{OA}\cdot\vec{OB}=(\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j})\cdot(\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j})$$

$$=\cos\alpha\cos\beta+\sin\alpha\sin\beta \qquad \mathbf{i}\cdot\mathbf{i}=\mathbf{j}\cdot\mathbf{j}=1,\ \mathbf{i}\cdot\mathbf{j}=0$$

or $\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$ by (1)

In case we are to find the value of $\cos(\alpha+\beta)$ then we shall draw a line OR making an angle β in the clock wise sense so that $\angle AOB=\alpha+\beta$, and $\vec{OB}$ represents a unit vector along OR

$$\vec{OB}=\cos(-\beta)\,\mathbf{i}+\sin(-\beta)\,\mathbf{j}=\cos\beta\,\mathbf{i}-\sin\beta\,\mathbf{j}$$

$$\vec{OA}\cdot\vec{OB}=1\cdot1\cos(\alpha+\beta)=\cos(\alpha+\beta)$$

Also $\vec{OA}\cdot\vec{OB}=(\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j})\cdot(\cos\beta\,\mathbf{i}-\sin\beta\,\mathbf{j})$

$$=\cos\alpha\cos\beta-\sin\alpha\sin\beta \quad \text{as above}$$

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

(ii) $\vec{OB}\times\vec{OA}=1\cdot1\sin(\alpha-\beta)\,\hat{\mathbf{n}}$ Anti clockwise where **n** is a unit vector perpendicular to XY plane

Again $\vec{OB}\times\vec{OA}=(\cos\beta\,\mathbf{i}+\sin\beta\,\mathbf{j})\times(\cos\alpha\,\mathbf{i}+\sin\alpha\,\mathbf{j})$

$$=\cos\beta\sin\alpha\,\mathbf{i}\times\mathbf{j}+\sin\beta\cos\alpha\,\mathbf{j}\times\mathbf{i}$$

$$\mathbf{i}\times\mathbf{i}=\mathbf{j}\times\mathbf{j}=0$$

$\mathbf{r}=\vec{OP}=$ P V of $P-$ P V of $O=(9\mathbf{i}-\mathbf{j}+2\mathbf{k})-(3\mathbf{i}+2\mathbf{j}+\mathbf{k})$

$=6\mathbf{i}-3\mathbf{j}+\mathbf{k}$

$\mathbf{M}=\mathbf{r}\times\mathbf{F}=(6\mathbf{i}-3\mathbf{j}+\mathbf{k})\times(5\mathbf{i}+\mathbf{k})$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & -3 & 1 \\ 5 & 0 & 1 \end{vmatrix}=-3\mathbf{i}-\mathbf{j}+15\mathbf{k}$$

3 Here O is $(2, -1, 3)$ or $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$.

P is $(1, -1, 2)$ or $\mathbf{i}-\mathbf{j}+2\mathbf{k}$

Ans $2\mathbf{i}-7\mathbf{j}-2\mathbf{k}$,

4 Ans $3\mathbf{i}+\mathbf{j}-4\mathbf{k}$

5 Ans $\mathbf{M}=\vec{BA}\times\mathbf{P}=-3\mathbf{i}+19\mathbf{j}-4\mathbf{k}$

6 (a) Ans $6(2\mathbf{i}-6\mathbf{j}+\mathbf{k})$

(b) $\mathbf{r}=\vec{OP}=$ P V of $P-$ P V of O

$=(2\mathbf{i}-7\mathbf{j}+10\mathbf{k})-(-5\mathbf{i}+6\mathbf{j}-10\mathbf{k})$

$=7\mathbf{i}-13\mathbf{j}+20\mathbf{k}$.

$\mathbf{M}=\mathbf{r}\times\mathbf{F}=(7\mathbf{i}-13\mathbf{j}+20\mathbf{k})\times(5\mathbf{i}+10\mathbf{j}+16\mathbf{k})$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 7 & -13 & 20 \\ 5 & 10 & 16 \end{vmatrix}=480\mathbf{i}-12\mathbf{j}+135\mathbf{k}$$

(c) Ans Use cor § 9 page 798

7 (a) We know that the work done by several forces is equal to the work done by their resultant

$\mathbf{R}=\mathbf{F}_1+\mathbf{F}_2=(4\mathbf{i}+\mathbf{j}-3\mathbf{k})+(3\mathbf{i}+\mathbf{j}-\mathbf{k})=7\mathbf{i}+2\mathbf{j}-4\mathbf{k}$

Again if the point is displaced from P to Q then

$\vec{PQ}=$ P V of $Q-$ P V of $P=(5\mathbf{i}+4\mathbf{j}+\mathbf{k})-(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$

$=4\mathbf{i}+2\mathbf{j}-2\mathbf{k}$

Hence the work done $=\mathbf{R}\cdot\vec{PO}$

$=(7\mathbf{i}+2\mathbf{j}-4\mathbf{k})\cdot(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})$

$=28+4+8=40$ units

8 Here $\vec{PQ}=(2\mathbf{i}-\mathbf{j}+4\mathbf{k})-(3\mathbf{i}+2\mathbf{j}-\mathbf{k})=-\mathbf{i}-3\mathbf{j}+5\mathbf{k}$

Work done $=\mathbf{F}\cdot\vec{PQ}=(4\mathbf{i}-3\mathbf{j}+2\mathbf{k})\cdot(-\mathbf{i}-3\mathbf{j}+5\mathbf{k})$

$=-4+9+10=15$ units

$\Leftrightarrow \theta=\frac{\pi}{2}$ and $\alpha=0$

$\Leftrightarrow$ a $\perp$ b and c $\parallel \hat{n}$

$\Leftrightarrow$ a $\perp$ b and c $\perp$ to both a and b

$\Leftrightarrow$ a, b, c are mutually perpendicular

$\Leftrightarrow$ a b=b c=c a=0

Hence (iv) is correct

52 Three vectors, a b, c are said to be coplanar if

$$[a\ b\ c]=0 \quad \text{or} \quad \begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & \lambda & 5 \end{vmatrix}=0$$

or $\quad 2(10+3\lambda)+1(5+9)+1(\lambda-6)=0$

or $\quad 7\lambda+28=0 \qquad \lambda=-4$

(b) Proceed as above $\lambda=0$

53 (i) L H S $=(a+b)\,[(b+c)\times(c+a)]$

$=(a+b)\,[b\times c+b\times a+c\times c+c\times a]$

$=(a+b)\,[b\times c+b\times a+c\times a] \qquad c\times c=0$

$=a\,(b\times c)+a\,(b\times a)+a.(c\times a)$

$\quad +b\,(b\times c)+b\,(b\times a)+b\,(c\times a)$

$=[a\ b\ c]+[a\ b\ a]+[a\ c\ a]$

$\quad +[b\ b\ c]+[b\ b\ a]+[b\ c\ a]$

$=[a\ b\ c]+[a\ b\ c]=2\,[a\ b\ c]$

Because scalar triple product is zero when two vectors are equal and also [b c a]=[a b c] as the cyclic order is maintained

In case a b, c are coplanar then [a b c]=0 and hence

$[a+b\ \ b+c\ \ c+a]=0$

a+b, b+c, c+a are also coplanar

(ii) Proceed as above

(iii) $r=xi+yj+zk$

Multiply both sides scalarly by i

$r\ i=xi\ i+0+0=x\ 1=x$

Similarly $r\ j=y$, $r\ k=z$

$r=(r\ i)\,i+(r\ j)\,j+(r\ k)\,k$

(iv) $i\times(a\times i)=(i\ i)a-(i\ a)\,i$

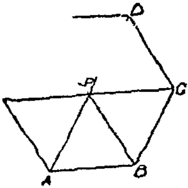
$=a-(a\ i)\,i$

Hence work done $= 1 \times 2 + (-3)(4) + 5(-1)$
$= -15$

15 Ans work done $= 9$, vector Moment $= -6i - 6j$

Let P be any point in the plane of the polygon Then the vector moment (torque) of the force represented by $\vec{AB}$ about P is $\vec{PA} \times \vec{AB} =$ twice the vector area of $\triangle PAB$

Similarly moment of force represented by BC about P = twice the vector area of $\triangle BPC$



Sum of moments

= twice [sum of vector areas of $\triangle PAB, PBC, PCD$]

= twice the vector area of polygon

55 Let $\mathbf{l}=l_1\,\mathbf{i}+l_2\,\mathbf{j}+l_3\,\mathbf{k}$ etc

$$[\mathbf{l}\ \mathbf{m},\ \mathbf{n}]=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix},\ (\mathbf{a}\times\mathbf{b})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$[\mathbf{l},\ \mathbf{m},\ \mathbf{n}]\ (\mathbf{a}\times\mathbf{b})=\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

$$=\begin{vmatrix} l_1\,\mathbf{i}+l_2\,\mathbf{j}+l_3\,\mathbf{k} & \Sigma l_1 a_1 & \Sigma l_1 b_1 \\ m_1\,\mathbf{i}+m_2\,\mathbf{j}+m_3\,\mathbf{k} & \Sigma m_1 a_1 & \Sigma m_1 b_1 \\ n_1\,\mathbf{i}+n_2\,\mathbf{j}+n_3\,\mathbf{k} & \Sigma n_1 a_1 & \Sigma n_1 b_1 \end{vmatrix}$$

$$=\begin{vmatrix} \mathbf{l} & \mathbf{l}\,\mathbf{a} & \mathbf{l}\,\mathbf{b} \\ \mathbf{m} & \mathbf{m}\,\mathbf{a} & \mathbf{m}\,\mathbf{b} \\ \mathbf{n} & \mathbf{n}\,\mathbf{a} & \mathbf{n}\,\mathbf{b} \end{vmatrix}=\begin{vmatrix} \mathbf{l}\,\mathbf{a} & \mathbf{l}\,\mathbf{b} & \mathbf{l} \\ \mathbf{m}\,\mathbf{a} & \mathbf{m}\,\mathbf{b} & \mathbf{m} \\ \mathbf{n}\,\mathbf{a} & \mathbf{n}\,\mathbf{b} & \mathbf{n} \end{vmatrix}$$

56 Let $\mathbf{a}=l\,(\mathbf{b}\times\mathbf{c})+m\,(\mathbf{c}\times\mathbf{a})+n\,(\mathbf{a}\times\mathbf{b})$ (1)

We have to find the values of l, m and n

Multiply both sides of (1) scalarly by $\mathbf{a}$

$$\mathbf{a}\,\mathbf{a}=l\,\mathbf{a}\ (\mathbf{b}\times\mathbf{c})+m\,\mathbf{a}\ (\mathbf{c}\times\mathbf{a})+n\,\mathbf{a}\,(\mathbf{a}\times\mathbf{b})$$

or $\mathbf{a}\,\mathbf{a}=l\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]+m\,[\mathbf{a}\ \mathbf{c}\ \mathbf{a}]+n\,[\mathbf{a}\ \mathbf{a}\ \mathbf{b}]=l\,[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

Scalar triple product is zero when two vectors are equal

$$l=\frac{\mathbf{a}\ \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

Similarly multiplying both sides of (1) scalarly by $\mathbf{b}$ and $\mathbf{c}$, we get

$$m=\frac{\mathbf{a}\ \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]},\ n=\frac{\mathbf{a}\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}$$

$$\mathbf{a}=\frac{\mathbf{a}\ \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{b}\times\mathbf{c})+\frac{\mathbf{a}\ \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{c}\times\mathbf{a})+\frac{\mathbf{a}\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{a}\times\mathbf{b})$$

Similarly we can write the values $\mathbf{b}$ and $\mathbf{c}$ as

$$\mathbf{b}=\frac{\mathbf{b}\ \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{b}\times\mathbf{c})+\frac{\mathbf{b}\ \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{c}\times\mathbf{a})+\frac{\mathbf{b}\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{a}\times\mathbf{b})$$

and $$\mathbf{c}=\frac{\mathbf{c}\ \mathbf{a}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{b}\times\mathbf{c})+\frac{\mathbf{c}\ \mathbf{b}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{c}\times\mathbf{a})+\frac{\mathbf{c}\ \mathbf{c}}{[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]}\ (\mathbf{a}\times\mathbf{b})$$

(*ii*) Let $\mathbf{b}\times\mathbf{c}=l\,\mathbf{a}+m\,\mathbf{b}+n\,\mathbf{c}$ (2)

7 θ is the angle between two vectors **a** and **b** then **a b** $\geqslant 0$ only if
(a) $0 \leqslant \theta \leqslant \pi$, (b) $\pi/2 \leqslant \theta \leqslant \pi$, (c) $0 \leqslant \theta \leqslant \pi/2$
(d) $0 < \theta < \pi/2$

8 If **a** be a non zero vector then which of the following is correct ?
(a) **a a**$=0$, (b) **a a** > 0, (c) **a a** $\geqslant 0$, (d) **a a** $\leqslant 0$

9 **a** and **b** are two non zero vectors, then (**a** + **b**) (**a**−**b**) is equal to
(a) $a+b$, (b) $(a-b)^2$, (c) $(a+b)^2$, (d) a^2-b^2,
(e) $a-b$

10 **a b**$=0$ implies only
(a) **a**$=0$, (b) **b**$=0$, (c) $\theta=90°$,
(d) either **a**$=0$ or **b**$=0$ or $\theta=90°$,
(e) either **a**$=0$ or **b**$=0$

11 If **a**, **b**, **c** be three non zero vectors then the equation
a b$=$**a c** implies
(a) **b**$=$**c**,
(b) **a** is orthogonal to both **b** and **c**,
(c) **a** is orthogonal to **b** **c**,
(d) Either **a** is orthogonal to both **b** and **c** or **a** is orthogonal to **b**−**c**
(e) Either **b**$=$**c** or **a** is orthogonal to **b**−**c**

12 If **a** and **b** include an angle of 120° and their magnitudes are 2 and $\sqrt{3}$, then **a b** is equal
(a) 3, (b) $-\sqrt{3}$, (c) $\sqrt{3}$, (d) -3, (e) $-\sqrt{3}/2$

13 If {**i**, **j**, **k**} be a set of orthonormal unit vectors, then fill up the blanks
(a) **i i**+**j j**+**k k**=
(b) **i j**+**j k**+**k i**=
(c) **i i**−**j j**=**k k**=
(d) **i j**=**j k**=**k i**=

14 If θ be the angle between the vectors 4 (**i**−**k**) and **i**+**j**+**k** then θ is
(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) $\cos^{-1}(1/\sqrt{3})$

15 If θ be the angle between the vectors **i**+**j** and **j**−**k**, then θ is
(a) 0 (b) $\pi/4$, (c) $\pi/2$ (d) $\pi/3$ (e) $\pi/6$

16 If **a** and **b** are two unit vectors, then **a**×**b** is a unit vector if

17 If {**i** **j** **k**} be an orthonormal set of unit vectors, then fill in the blanks
(a) **i**×**j**= (b) **j**×**i**= (c) **i**×(**j**×**k**)=
(d) **i**×(**j**×**k**)+**j**×(**k**×**i**)+**k**×(**i**×**j**)
(e) **i** (**j**×**k**)+**j** (**k**×**i**)+**k** (**i**×**j**)

18 [**a b c**] is the scalar triple product of three vectors **a**, **b** and **c**, then [**a b c**] is equal to

A is $(4, 5, 1) = 4\mathbf{i}+5\mathbf{j}+\mathbf{k}$

Similarly express B, C, D in terms of unit vectors

$$\vec{AB} = \vec{OB} - \vec{OA} = -4\mathbf{i}-6\mathbf{j}-2\mathbf{k}$$

$$\vec{BC} = \vec{OC} - \vec{OB} = 3\mathbf{i}+10\mathbf{j}+5\mathbf{k}$$

$$\vec{CD} = \vec{OD} - \vec{OC} = -7\mathbf{i}-5\mathbf{j}+0\mathbf{k}$$

If the four points are coplanar then the vectors $\vec{AB}$, $\vec{BC}$ and $\vec{CD}$ are coplanar

$$[\vec{AB}\ \vec{BC}\ \vec{CD}] = 0$$

or $$\begin{vmatrix} -4 & -6 & 2 \\ 3 & 10 & 5 \\ -7 & -5 & 0 \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 2 & 3 & 1 \\ 3 & 10 & 5 \\ 7 & 5 & 0 \end{vmatrix} = 0$$

or $$2(0-25)-3(0-35)+1(15-70)=0$$
$$=-50+105-55=0$$

which is true

Hence the four points are coplanar

(b) Prove that $[\vec{BA}\ \vec{CA}\ \vec{DA}]=0$

or $$\begin{vmatrix} 2 & 4 & 6 \\ 1 & 3 & 8 \\ -1 & 0 & 7 \end{vmatrix} = 0$$

or $$2(21-0)-4(7+8)+6(0+3)=0$$

or $$42-60+18=0$$

62 (a) Refer § 12 page 799 the left hand side

$$\begin{vmatrix} \mathbf{a}\cdot\mathbf{c} & \mathbf{a}\cdot\mathbf{d} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{c}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{d} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{a}\cdot\mathbf{d} \end{vmatrix} + \begin{vmatrix} \mathbf{b}\cdot\mathbf{a} & \mathbf{b}\cdot\mathbf{d} \\ \mathbf{c}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{d} \end{vmatrix}$$

keeping in view that $\mathbf{a}\cdot\mathbf{b}=\mathbf{b}\cdot\mathbf{a}$ etc above will be of the form

$$(x-y)+(y-z)+(z-x)=0$$

(b) If we denote the points by A, B, C, D then

$$\vec{AB}=\mathbf{b}-\mathbf{a},\ \vec{AC}=\mathbf{c}-\mathbf{a},\ \vec{AD}=\mathbf{d}-\mathbf{a}$$

Now A, B, C, D will be coplanar if $(\mathbf{b}-\mathbf{a})\times(\mathbf{c}-\mathbf{a})\cdot(\mathbf{d}-\mathbf{a})=0$

This gives on simplifications $[\mathbf{a}\,\mathbf{b}\,\mathbf{c}]=[\mathbf{b}\,\mathbf{c}\,\mathbf{d}]+[\mathbf{a}\,\mathbf{b}\,\mathbf{d}]+[\mathbf{c}\,\mathbf{a}\,\mathbf{d}]$

63 If r is the radius of circum circle then

$$|\vec{OA_i}| = r \text{ for } i=1, 2, \ldots n$$

32 If a and b are position vectors of A and B respectively the position vector of a point C on AB produced such that $\vec{AC} = 3\vec{AB}$ is
(a) 3a − b (b) 3b − a (c) 3a − 2b (d) 3b − 2a
(M N R 80)

33 The projection of the vector i − 2j + k on the vector 4i − 4j + 7k is
(a) $\frac{5\sqrt{6}}{10}$ (b) 2½ (c) $\frac{9}{19}$ (d) $\frac{\sqrt{6}}{19}$ (M N R 80)

34 If A = 2i + 2j + 3k, B = −i + 2j + k and C = 3i + j, then A + tB is perpendicular to C if t is equal to
(a) 5 (b) 4 (c) 6 (d) 2 (M N R 79)

35 (*a*) If A = 2i + 2j − k, B = 6i − 3j + 2k then A × B will be given by
(i) 2i − 2j − k (ii) 6i − 3j + 2k
(iii) i − 10j − 18k (iv) i + j + k (M N R 78)
(*b*) The number of vectors of unit length perpendicular to vectors a = (1, 1, 0) and b = (0, 1, 1) is
(a) one (b) two (c) three (d) Infinite
(e) None (I I T 87)

36 The point with position vectors 60i + 3j, 40i − 8j, ai − 52j are collinear if
(i) $a = -40$ (ii) $a = 40$ (iii) $a = 20$
(iv) None of these (I I T 83)

37 The area of a triangle whose vertices are A (1, −1, 2), B (2, 1, −1), C (3, −1, 2) is (I I T 83)

38 The points with position vectors a + b, a − b and a + λb are collinear for all real values of λ
(a) True (b) False (I I T 84)

39 If $\begin{vmatrix} a & a^2 & 1+a^3 \\ b & b^2 & 1+b^3 \\ c & c^2 & 1+c^3 \end{vmatrix} = 0$ and the vectors

$\vec{A} = (1, a, a^2)$, $\vec{B} = (1, b, b^2)$, $\vec{C} = (1, c, c^2)$ are non coplanar then the product abc = (I I T 85)
(*b*) If the vectors ai + j + k, i + bj + k and i + j + ck ($a \neq b$, $c \neq 1$) are coplanar then the value of

$\frac{1}{1-a} + \frac{1}{1-b} + \frac{1}{1-c}$ is (I I T 87)

40 If $\vec{A}$, $\vec{B}$, $\vec{C}$ are three non coplanar vectors, then

$\frac{\vec{A} \cdot \vec{B} \times \vec{C}}{\vec{C} \times \vec{A} \cdot \vec{B}} + \frac{\vec{B} \cdot \vec{A} \times \vec{C}}{\vec{C} \cdot \vec{A} \times \vec{B}} =$ (I I T 85) or $$x\mathbf{a}+y\mathbf{b}-\mathbf{c}=0 \tag{1}$$

Taking dot product with a and b respectively, we get

$$x\mathbf{a}\cdot\mathbf{a}+y\mathbf{b}\cdot\mathbf{a}-\mathbf{c}\cdot\mathbf{a}=0 \tag{2}$$

and $$x\mathbf{a}\cdot\mathbf{b}+y\mathbf{b}\cdot\mathbf{b}-\mathbf{c}\cdot\mathbf{b}=0 \tag{3}$$

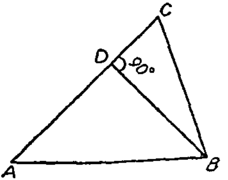
Eliminating x, y from (1), (2), and (3), we get

$$\begin{vmatrix} \mathbf{a} & \mathbf{b} & -\mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & -\mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & -\mathbf{b}\cdot\mathbf{c} \end{vmatrix}=0 \text{ or } \begin{vmatrix} \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}=0$$

$$\text{or } \mathbf{a}\begin{vmatrix} \mathbf{a}\cdot\mathbf{b} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{b}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}-\mathbf{b}\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}+\mathbf{c}\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}=0$$

$$\text{or } \begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}\mathbf{c}=\begin{vmatrix} \mathbf{c}\cdot\mathbf{a} & \mathbf{a}\cdot\mathbf{b} \\ \mathbf{b}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{b} \end{vmatrix}\mathbf{a}+\begin{vmatrix} \mathbf{a}\cdot\mathbf{a} & \mathbf{c}\cdot\mathbf{a} \\ \mathbf{a}\cdot\mathbf{b} & \mathbf{b}\cdot\mathbf{c} \end{vmatrix}\mathbf{b}$$

67 We are given $\overrightarrow{AB}=\mathbf{b}$, $\overrightarrow{AC}=\mathbf{c}$ and $BD \perp AC$. To find the resolution of $\overrightarrow{BD}$ w r t b and c



$$\overrightarrow{BD}=\overrightarrow{BA}+\overrightarrow{AD}=-\mathbf{b}+\overrightarrow{AD} \tag{1}$$

But $|\overrightarrow{AD}|$ = Projection of b on c $=\dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|}$

Also the unit vector in the direction of AC is $\dfrac{\mathbf{c}}{|\mathbf{c}|}$. Hence

$$\overrightarrow{AD}=\left(\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|}\right)\left(\frac{\mathbf{c}}{|\mathbf{c}|}\right)=\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}$$

Hence from (1) we get

$$BD=\frac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{c}|^2}\mathbf{c}-\mathbf{b}$$

68 From $(\mathbf{r}-\mathbf{c})\times\mathbf{b}=0$ we have

$$\mathbf{r}\times\mathbf{b}-\mathbf{c}\times\mathbf{b}=0$$

Taking cross product with a we get

$$\mathbf{a}\times(\mathbf{r}\times\mathbf{b})-\mathbf{a}\times(\mathbf{c}\times\mathbf{b})=\mathbf{a}\times 0=0$$

$(\vec{a}+\vec{b})\cdot\vec{p}+(\vec{b}+\vec{c})\cdot\vec{q}+(\vec{c}+\vec{a})\cdot\vec{r}$
is equal to
(A) 0 (B) 1 (C) 2 (D) 3 (IIT 88)

47 The components of a vector $\vec{a}$ along and perpendicular to a non zero vector $\vec{b}$ are and respectively

Solutions

1 (c)

2 Parallel does not imply the same sense of direction Hence they are equal if the sense of direction is same and not equal if the sense of direction is opposite Hence (c) is the correct answer

3 a is the modulus of vector a Modulus of ma is

$$|m|a=1 \text{ if } a=\frac{1}{|m|}$$

Hence (c) is the correct answer

4 a+b is a unit vector if $|(a+b)|^2=1$ or $(a+b)^2=1$
{ square of a vector is square of its module }
$a^2+b^2+2|a||b|\cos\theta=1$ or $1+1+2\cdot1\cdot1\cos\theta=1$
$\cos\theta=-\frac{1}{2}=-\cos\pi/3=\cos(\pi-\pi/3)=\cos 2\pi/3,$
$\theta=2\pi/3$
Hence (d) is the correct answer

5 (b) is the correct answer

6 $a=\frac{2x+3(a+2b)}{2+3}$, $5a-3a-6b=2x$,
$x=a-3b$ Hence (c) is the correct answer

7 $a\cdot b=ab\cos\theta\geq0$ if $\cos\theta\geq0$, a and b are +ive
Hence (c) is the correct answer i e $0\leqslant\theta\leqslant\pi/2$

8 Square of vector is square of its module, $a\cdot a=a^2>0$
Hence (b) is the correct answer

9 Clearly (d) is the correct answer

10 (d) is the correct answer

11 We have $a\cdot b=a\cdot c \Rightarrow a\cdot(b-c)=0$
It follows that a is orthogonal to b−c or b−c=0 i e b=c or a=0 But a is non zero vector Hence the correct answer is (e) which includes all the above cases

12 $\cos 120°=-\frac{1}{2}$
$a\cdot b=2\sqrt{3}\,(-\frac{1}{2})=-\sqrt{3}$ Hence (b) is correct answer

13 (a) 3, (b) 0, (c) 1, (d) 0

14 $4(i-k)\cdot(i+j+k)=4(i^2-k^2)+4(i\cdot j-k\cdot j)$
$=4(1-1)+4(0-0)=0,\quad \theta=\pi/2$
Hence (c) is the correct answer

15 $(i+j)\cdot(j+k)=\sqrt{2}\,\sqrt{2}\cos\theta$
or $i\cdot j+i\cdot k+j\cdot j+j\cdot k=2\cos\theta$

BN is given by

$$p=\frac{[0-b\mathbf{j}\ \ \mathbf{i}\ \ a\mathbf{i}+b\mathbf{j}+c\mathbf{k}]}{|\mathbf{i}\times(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})|}$$

Now $[0-b\mathbf{j},\ \mathbf{i},\ a\mathbf{i}+b\mathbf{j}+c\mathbf{k}]$

$$=\begin{vmatrix} 0 & -b & 0 \\ 1 & 0 & 0 \\ a & b & c \end{vmatrix}=bc$$

and $\mathbf{i}\times(a\mathbf{i}+b\mathbf{j}+c\mathbf{k})=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ a & b & c \end{vmatrix}$

$$=b\mathbf{k}-c\mathbf{j}$$

so that $|\mathbf{i}\times(a\mathbf{i}+b\mathbf{j}+c\mathbf{k}|=\sqrt{(b^2+c^2)}$

$$p=\frac{bc}{\sqrt{(b^2+c^2)}}$$

Similarly, it can be shown that the shortest distance between OP and AN is $\dfrac{ca}{\sqrt{(c^2+a^2)}}$ and that between OP and AL is $\dfrac{ab}{\sqrt{(a^2+b^2)}}$

71 Let the three coterminus edges OA, OB, OC be represented by the vectors **a**, **b** and **c** respectively Then

$\vec{OA}=12\mathbf{i}+4\mathbf{j}+3\mathbf{k}$ $\vec{OB}=8\mathbf{i}-12\mathbf{j}-9\mathbf{k}$

and $\vec{OC}=33\mathbf{i}-4\mathbf{j}-24\mathbf{k}$

$OA=\sqrt{(12^2+4^2+3^2)}=13,$

$OB=\sqrt{[8^2+(-12)^2+(-9)^2]}=17$

and $OC=\sqrt{[33^2+(-4^2)+(-24)^2]}=41$

Hence lengths of the edges of the parallelopiped are 13, 17 and 41

Now vector area of the face determined by OB is given by

Then (1) gives $abc=-1$

(b) The given vectors are coplanar if

$$\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} \quad \text{or} \quad abc-(a+b+c)+2=0 \qquad (1)$$

$$\frac{1}{1-a}+\frac{1}{1-b}+\frac{1}{1-c}=\frac{\Sigma(1-b)(1-c)}{1-\Sigma a+\Sigma ab-abc}$$

$$=\frac{3-2(a+b+c)+(bc+ca+ab)}{1-(a+b+c)+(ab+bc+ca)\quad a+b+c)+2} \text{ by } 1$$

$$=\frac{3-2\Sigma a+\Sigma ab}{3-2\Sigma a+\Sigma ab}=1$$

40 Ans 0

Since $\vec{A}, \vec{B}, \vec{C}$ are non coplanar, we have

$$[\vec{A}\ \vec{B}\ \vec{C}]\neq 0$$

Also $\vec{A}\ \vec{B}\times\vec{C}=[\vec{A}\ \vec{B}\ \vec{C}]$, $[\vec{B}\ \ \vec{A}\times\vec{C}]=[\vec{B}\ \vec{A}\ \vec{C}]$

$$=-[\vec{A}\ \vec{B}\ \vec{C}]$$

$$\vec{C}\times\vec{A}\ \ \vec{B}=[\vec{C}\ \vec{A}\ \vec{B}]=[\vec{A}\ \vec{B}\ \vec{C}]$$

and $C\ \ A\times\vec{B}=[\vec{C}\ \vec{A}\ \vec{B}]=[\vec{A}\ \vec{B}\ \vec{C}]$

Hence

$$\frac{\vec{A}\ \ \vec{B}\times\vec{C}}{\vec{C}\times\vec{A}\ \vec{B}}+\frac{\vec{B}\ \ \vec{A}\times\vec{C}}{\vec{C}\ \ \vec{A}\times\vec{B}}=\frac{[\vec{A}\ \vec{B}\ \vec{C}]}{[\vec{A}\ \vec{B}\ \vec{C}]}+\frac{-[\vec{A}\ \vec{B}\ \vec{C}]}{[\vec{A}\ \vec{B}\ \vec{C}]}=0$$

41 Ans $B\left(\frac{5}{3},\frac{2}{3},\frac{2}{3}\right)$

Let $\vec{B}=(x,y,z)=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ Also $\vec{A}=\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\vec{C}=\mathbf{j}-\mathbf{k}$

Then $\vec{A}\times\vec{B}=\vec{C}$ on equating the coeff of $\mathbf{i}, \mathbf{j}\ \mathbf{k}$ gives

$z-y=0$, $x-z=1$ and $y-x=-1$

These three equations are equivalent to two equations

$z-y=0$ (1) $x-z=1$ (2)

Also $\vec{A}\ \vec{B}=3$ gives $x+y+z=3$ (3)

Solving (1), (2) and (3) we get

$$x=\frac{5}{3},\ y=z=\frac{2}{3}$$

Hence $\vec{B}=\left(\frac{5}{3}\ \frac{2}{3},\frac{2}{3}\right)$

42 Ans (c) See Objective Questions on determinants

$$a \cdot b' = a \cdot \frac{c \times a}{[abc]} = \frac{[aca]}{[abc]} = 0$$

as the numerator is the scalar triple product of three vectors two of which are equal and hence it is zero

Similarly $a \cdot c' = b \cdot a' = b \cdot c' = 0$, etc

Cor *Thus we conclude from the two properties that if* a', b', c' *be reciprocal system to* a, b, c *then* a, b, c *is a reciprocal system to* a', b', c'

Property 3 *The scalar triple product* [abc] *of any three non-coplanar vectors is reciprocal to the corresponding scalar triple product formed out of the reciprocal system of vectors* a', b', c'

i e $$[a'\ b'\ c'] = \frac{1}{[abc]}$$

$$[a\ b\ c'] = a' \cdot (b' \times c') \qquad (1)$$

Now substitute the values of a', b', and c' in terms of a, b and c

$$[a'b'c'] = \frac{(b \times c) \cdot [(c \times a) \times (a \times b)]}{[abc]^3} \text{ from (1)} \qquad (2)$$

Now $(c \times a) \times (a \times b) = (c \times a) \times m$ say $= (m \cdot c)a - (m \cdot a)c$

$$= (a \times b) \cdot c\, a - (a \times b) \cdot a\, c = [abc]\, a$$

$$(a \times b) \cdot a = [a\ b\ a] = 0,$$

$$[a'b'c'] = \frac{(b \times c) \cdot [abc]\, a}{[abc]^3} = \frac{[abc]\ (b \times c) \cdot a}{[abc]^3} \text{ from (2)}$$

$$= \frac{[abc]\ [bca]}{[abc]^3} = \frac{[abc]\ [abc]}{[abc]^3} = \frac{1}{[abc]}$$

$$[abc]\ [a'\ b'\ c'] = 1$$

Cor *From above, we conclude that*

$$[a \times b,\ b \times c,\ c \times a] = [abc]^2,$$

Cor We have done before that any vector r can be expressed in terms of three non coplanar vectors a, b, c

as $$r = \frac{[rbc]\ a + [rca]\ b + [rab]\ c}{[abc]}$$

But $$\frac{[rbc]a}{[abc]} = \frac{r \cdot (b \times c)}{[abc]} a = (r \cdot a')\, a$$

where a', b', c' form a reciprocal system of vectors to a, b, c

PART-2

ALGEBRA

3 *If* a, b, c *and* a′, b′, c′ *are reciprocal system of vectors prove that*

(i) $\mathbf{a}\times\mathbf{a}'+\mathbf{b}\times\mathbf{b}'+\mathbf{c}\times\mathbf{c}'=0$

(ii) $\mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'=\dfrac{\mathbf{a}+\mathbf{b}+\mathbf{c}}{[\mathbf{abc}]}$

(iii) $\mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}'+\mathbf{c}\cdot\mathbf{c}'=3$

(1) $\mathbf{a}'=\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}$ because of reciprocal system

$$\mathbf{a}\times\mathbf{a}'+\mathbf{b}\times\mathbf{b}'+\mathbf{c}\times\mathbf{c}'$$

$$=\frac{1}{[\mathbf{abc}]}[\mathbf{a}\times(\mathbf{b}\times\mathbf{c})+\mathbf{b}\times(\mathbf{c}\times\mathbf{a})+\mathbf{c}\times(\mathbf{a}\times\mathbf{b})]=0$$

see Ex 57 P 808

(ii) $\mathbf{b}'\times\mathbf{c}'=\dfrac{1}{[\mathbf{abc}]}\mathbf{a}$ as proved in (1) of Ex 2 above

$$\mathbf{a}'\times\mathbf{b}'+\mathbf{b}'\times\mathbf{c}'+\mathbf{c}'\times\mathbf{a}'=\frac{1}{[\mathbf{abc}]}[\mathbf{a}+\mathbf{b}+\mathbf{c}]$$

(iii) $\mathbf{a}\cdot\mathbf{a}'=\mathbf{a}\cdot\dfrac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\dfrac{[\mathbf{abc}]}{[\mathbf{abc}]}=1$

$$\mathbf{a}\cdot\mathbf{a}'+\mathbf{b}\cdot\mathbf{b}'+\mathbf{c}\cdot\mathbf{c}'=3$$

4 (a) *Find a set of vectors reciprocal to the set*

(a) $2\mathbf{i}+3\mathbf{j}-\mathbf{k},\ \mathbf{i}-\mathbf{j}-2\mathbf{k},\ -\mathbf{i}+2\mathbf{j}+2\mathbf{k}$

(b) $-\mathbf{i}+\mathbf{j}+\mathbf{k},\ \mathbf{i}-\mathbf{j}+\mathbf{k},\ \mathbf{i}+\mathbf{j}-\mathbf{k}$

Let the given vectors be a, b, c so that

$$[\mathbf{abc}]=\begin{vmatrix}2 & 3 & -1\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix}=3,$$

$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & -2\\ -1 & 2 & 2\end{vmatrix}=2\mathbf{i}+\mathbf{k}$$

$$\mathbf{a}'=\frac{\mathbf{b}\times\mathbf{c}}{[\mathbf{abc}]}=\frac{2\mathbf{i}+\mathbf{k}}{3}$$

Similarly $\mathbf{b}'=-\dfrac{8\mathbf{i}+3\mathbf{j}-7\mathbf{k}}{3}$ and $\mathbf{c}'=\dfrac{-7\mathbf{i}+3\mathbf{j}-5\mathbf{k}}{3}$

Problem Set (C)

1 Find the moment about the point $\mathbf{i}+2\mathbf{j}-\mathbf{k}$ of a force represented by $3\mathbf{i}+\mathbf{k}$ acting through the point $2\mathbf{i}-\mathbf{j}+3\mathbf{k}$

1

Surds

§ 1 Some Definitions

Any root of a number which cannot be exactly found is called a **surd**

For example, $\sqrt{3}$, $\sqrt{7}$, $\sqrt[3]{4}$, $\sqrt[4]{5}$ are all surds

A surd which consists wholly of an irrational number is called a **pure surd**, whereas a surd consisting of the product of a rational and an irrational number, is called a **mixed surd** Thus $\sqrt[3]{(11)}$, $\sqrt{(27)}$, $\sqrt[3]{7}$ are pure surds while $3\sqrt{3}$, $7\sqrt[3]{(11)}$ are mixed surds

Note that a mixed surd can be converted into a pure surd Thus the mixed surd $3\sqrt{3}$ can be written as $\sqrt{(27)}$ which is a pure surd A pure surd can sometimes be expressed as a mixed surd For example, the pure surd $\sqrt{8}$ can be written as a mixed surd $2\sqrt{2}$

Two or more surds are said to be similar or like when they can be so reduced so as to have the same irrational factor Thus $\sqrt{(27)}$ and $\sqrt{(48)}$ are similar surds, for they are respectively equivalent to $3\sqrt{3}$ and $4\sqrt{3}$

The order of a surd is indicated by the number denoting the root Thus $\sqrt{7}$, $\sqrt[3]{9}$, $(11)^{2/5}$, $\sqrt[n]{3}$ are surds of second, third, fifth and n^{th} order respectively A surd of second order is often called a quadratic surd, a surd of third order is called a cubic surd

Surds consisting of one term only are called simple surds or monomial surds An expression consisting of two or more simple surds connected by the sign + or − is called a compound surd Thus $3\sqrt{2}$ is a simple surd whereas the surd $\sqrt{5}+\sqrt{3}$ is a compound surd

Rationalisation. If two surds be such that their product is rational, then each of them is said to be rationalised when multiplied by the other and either of them is said to be a rationalising factor of the other Thus $2\sqrt{3}$ and $\sqrt{7}+\sqrt{5}$ are rationalised

12 Constant forces $(2i-5j+6k)$, $-i+2j-k$ and $2i+7j$ act on a particle. Determine the total work done by the forces in a displacement of the particle from the point $4i-3j-2k$ to the point $6i+j-3k$.

13 Constant forces $P=2i-5j+6k$ and $Q=-i+2j-k$ act on a particle. Determine the work done when the particle is displaced from a point A with position vector $4i-3j-2k$ to point B with position vector $6i+j-3k$. (Roorkee 84)

14 Constant forces $P=(2i-5j+6k)$ and $Q=-i+2j-k$ act on a particle at the point A $(4, -3, -2)$. Find the moment of the resultant force about the origin O $(0, 0, 0)$. Also find the work done when the particle is displaced from A to B $(6, 1, -3)$.

15 Constant forces $P_1=i-j+k$, $P_2=-i+2j-k$ and $P_3=j-k$ act on a particle at a point A. Determine the work done when the particle is displaced from the point A to B where $A=4i-3j-2k$ and $B=6i+j-3k$ are the position vectors of A and B. Also find the moment of the force P_1 about the point $(1, 0, 1)$. (Roorkee 80)

16 Show that twice the vector area of a closed plane polygon is equal to the sum of the torques about any point in the plane of the polygon of forces represented by the sides of the polygon taken in order.

Solutions to Problem Set (C)

1 $F=3i+k$

Point O about which moment is to be taken is

$$i+2j-k$$

$$r=\overrightarrow{OP}=\overrightarrow{P.V.}\text{ of }P-P.V.\text{ of }O=(2i-j+3k)-(i+2j-k)$$

or $$r=i-3j+4k$$

Hence the moment is

$$r\times F=(i-3j+4k)\times(3i+k)$$

$$\begin{vmatrix} i & j & k \\ 1 & -3 & 4 \\ 3 & 0 & 1 \end{vmatrix}=-3i+11j+9k$$

2 $F=5i+k$, $O=3i+2j+k$, $P=9i-j+2k$.

For example, under these conditions

$$\sqrt{a^2}=a \text{ if } a>0$$
$$\sqrt{a^2}=a- \text{ if } a<0$$

The following two transformations are useful to remember

$$\sqrt{(a+\sqrt{b})}=\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}+\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$$

and

$$\sqrt{(a-\sqrt{b})}=\sqrt{\left(\frac{a+\sqrt{(a^2-b)}}{2}\right)}-\sqrt{\left(\frac{a-\sqrt{(a^2-b)}}{2}\right)}$$

Problem Set

1 Take factors outside the radical sign

(i) $\sqrt[3]{\{54\,(1-\sqrt{5})^4\}}$, (ii) $\sqrt[3]{(5-\sqrt{5})^7}$

(iii) $\sqrt{\left(8\,\frac{8}{63}\right)}$ (iv) $\sqrt{\left(11\,\frac{11}{120}\right)}$

2 Which of the given numbers is greatest ?

$$6\sqrt[3]{5},\ 8\sqrt[3]{2},\ 2\sqrt[3]{130},\ \sqrt[3]{900}$$

3 Evaluate the expression

(i) $4x^3+2x^2-8x+7$ for $x=\frac{1}{2}(\sqrt{3}+1)$

(ii) $3x^2+4xy-3y^2$ for $x=\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}},\ y=\dfrac{\sqrt{5}-\sqrt{2}}{\sqrt{5}+\sqrt{2}}$,

(iii) x^2+xy+y^2 for $x=\dfrac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}},\ y=\dfrac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}}$

(Roorkee 1981)

4 Find the value of

$$\frac{\sqrt{(a+x)}+\sqrt{(a-x)}}{\sqrt{(a+x)}+\sqrt{(a-x)}} \quad \text{when} \quad x=\frac{2ab}{b^2+1}$$

(Roorkee 1979)

5 If $x=\dfrac{\sqrt{(a+2b)}+\sqrt{(a-2b)}}{\sqrt{(a+2b)}-\sqrt{(a-2b)}}$, show that

$$bx^2-ax+b=0$$

(Roorkee 1980)

Express as an equivalent fraction with rational denominator

6 $\dfrac{8\sqrt{3}-3\sqrt{5}}{9\sqrt{3}-4\sqrt{5}}$ 7 $\dfrac{11}{2+\sqrt{3}+\sqrt{5}}$

8 $\dfrac{\sqrt{(10)}+\sqrt{5}-\sqrt{3}}{\sqrt{3}+\sqrt{(10)}-\sqrt{5}}$ 9 $\dfrac{(\sqrt{3}+\sqrt{5})\,(\sqrt{5}+\sqrt{2})}{\sqrt{2}+\sqrt{3}+\sqrt{5}}$

10 $\dfrac{1}{2-\sqrt{2}+\sqrt{3}-\sqrt{6}}$

11 $\dfrac{1}{\sqrt{(10)}+\sqrt{(14)}+\sqrt{(15)}+\sqrt{(21)}}$

12 $\dfrac{15}{\sqrt{(10)}+\sqrt{(20)}+\sqrt{(40)}-\sqrt{5}-\sqrt{(80)}}$

9 Ans 9 units

10 Unit vectors in the direction of the forces are

$$\frac{6i+2j+3k}{\sqrt{(36+4+9)}}, \frac{3i-2j+6k}{7}, \frac{2i-3j-6k}{7}$$

Hence if F_1 F_2 F_3 be the forces of magnitudes 5, 3 1, then they are $\frac{5}{7}(6i+2j+3k)$, $\frac{3}{7}(3i-2j+6k)$ and $\frac{1}{7}(2i-3j-6k)$

Again if R be their resultant, then $R=F_1+F_2+F_3$

$$R=\frac{1}{7}[(30+9+2)i+(10-6-3)j+(15+18-6)k]$$
$$=\frac{1}{7}(41i+j+27k)$$

Again $\vec{AB}$=P V of B−P V of A

$$=(5i-j+k)-(2i-j-3k)=3i+0j+4k$$

The work done by various forces is equal to work done by their resultant

$$\text{Work done}=R\ \vec{AB}$$
$$=\frac{1}{7}[41i+j+27k]\ [3i+0j+4k]$$
$$=\frac{1}{7}[123+0+108]=\frac{231}{7}=33 \text{ units,}$$

11 87 units of work

12 17 units of work

13 If R is the resultant of two forces P and Q then

$R=P+Q=i-3j+5k$ Also the displacement of the particle

$$=\vec{AB}=(6i+j-3k)-(4i-3j-2k)=2i+4j-k$$

Hence required work done$=R\ \vec{AB}=(i-3j+5k)\ (2i+4j-k)$
$$=2-12-5=-15$$

14 Resultant force $R=P+Q=i-3j+5k$ and vector

$\vec{OA}=4i-3j-2k$ Hence moment of R about $O=\vec{OA}\wedge R$

$$=\begin{vmatrix} i & j & k \\ 4 & -3 & -2 \\ 1 & -3 & 5 \end{vmatrix}=-21i-22j-9k$$

Magnitude of moment$=\sqrt{\{(-21)^2+(-22)^2+(-9)^2\}}$
$$=\sqrt{(1006)}$$

Also $\vec{AB}=2i+4j-k$

$$\frac{3\sqrt{(26-15\sqrt{3})}}{5\sqrt{2}-\sqrt{(38+5\sqrt{3})}}$$

is a rational number (I.I.T 1978)

34 (i) Simplify the following to a rational number

$$\frac{\{4+\sqrt{(15)}\}^{3/2}+\{4-\sqrt{(15)}\}^{3/2}}{\{6+\sqrt{(35)}\}^{3/2}-\{6-\sqrt{(35)}\}^{3/2}}$$ (I.I.T 1977)

(ii) Simplify

(a) $\sqrt{(9-6a+a^2)}+\sqrt{(9+6a+a^2)}$ if $a \leqslant -3$,

(b) $\dfrac{1}{\sqrt{[x+2\sqrt{(x-1)}]}}+\dfrac{1}{\sqrt{[x-2\sqrt{(x-1)}]}}$ if $1 \leqslant x < 2$.

35 Show that $\dfrac{\sqrt{7}}{\sqrt{(16+6\sqrt{7})}-\sqrt{(16-6\sqrt{7})}}$

is a rational number (I I T 73)

36 (a) Show that

$$\frac{1}{\sqrt{\{11-2\sqrt{(30)}\}}}-\frac{3}{\sqrt{\{7-2\sqrt{(10)}\}}}-\frac{4}{\sqrt{\{8+4\sqrt{(3)}\}}}=0$$

(I I T 75)

(b) Express $\dfrac{4+3\sqrt{3}}{\sqrt{(7+4\sqrt{3})}}$ in the form $A+\sqrt{B}$, where A and B are integers (I I T 71)

(c) Evaluate $(97+56\sqrt{3})^{1/4}$

37 Given $\sqrt{5}=2.23607$, find the value of

$$\frac{10\sqrt{2}}{\sqrt{(18)}-\{(3+\sqrt{5})\}}-\frac{\sqrt{(10)}+\sqrt{(18)}}{\sqrt{8}+\sqrt{(3-\sqrt{5})}},$$

38 (a) If $ax=\dfrac{2pq}{1+q^2}$, find the value of

$$\frac{\sqrt{\left(\frac{p}{a}+x\right)}+\sqrt{\left(\frac{p}{a}-x\right)}}{\sqrt{\left(\frac{p}{a}+x\right)}-\sqrt{\left(\frac{p}{a}-x\right)}}.$$

(b) If $x=a\left(\dfrac{m^2+n^2}{2mn}\right)^{1/2}$, $a, m, n > 0, m > a$,

find the value of

$$\left[\frac{(x^2+a^2)^{1/2}+(x^2-a^2)^{1/2}}{(x^2+a^2)^{1/2}-(x^2-a^2)^{1/2}}\right]$$

39 If $x=\dfrac{\sqrt{7}-\sqrt{5}}{\sqrt{7}+\sqrt{5}}, y=\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}-\sqrt{5}}$

find the value of x^3+y^3

2

Objective Questions

Choose the correct answer

1 Two vectors are said to be equal if

(a) their magnitudes are same,

(b) direction is same

(c) originate from the same point,

(d) they meet at the same point,

(e) they have same magnitude and same sense of direction

2 Two vectors **a** and **b** are parallel and have equal magnitudes Then

(a) They are equal

(b) They are not equal

(c) They may or may not be equal,

(d) They have the same sense of direction

(e) They do not have the same direction,

3 If **a** is a non zero vector of modulus a and m is a non zero scalar, then m **a** is a unit vector if

(a) $m=\pm 1$ (b) $a=|m|$ (c) $a=\frac{1}{|m|}$

4 **a** and **b** are two unit vectors and θ is the angle between them Then **a** **b** is a unit vector if

(a) $\theta=\pi/3$ (b) $\theta=\pi/4$ (c) $\theta=\pi/2$, (d) $\theta=2\pi/3$

5 The position vectors A and B are **a** and **b** respectively, then the position vector of a point P which divides AB in the ratio 1 2 is

(a) $\frac{\mathbf{a}+\mathbf{b}}{3}$ (b) $\frac{\mathbf{b}+2\mathbf{a}}{3}$, (c) $\frac{\mathbf{a}-2\mathbf{b}}{3}$, (d) $\frac{\mathbf{b}-2\mathbf{a}}{3}$

6 Point A is **a**+2**b**, P is **a** and P divides AB in the ratio 2 3 The position vector of B is

(a) 2**a** **b** (b) **b**−2**a** (c) **a**−3**b** (d) **b**

58 Rationalize the denominator of

$$\frac{1}{\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a}+\sqrt{b'}+\sqrt{c}}$$

if $\frac{a}{a}=\frac{b}{b}=\frac{c}{c}$

59 To prove the equality

$$\sqrt[3]{(5\sqrt{2}+7)}-\sqrt[3]{(5\sqrt{2}-7)}=2$$

a student reasoned as follows Cubing both sides of the equality we get

$$14-3\sqrt[3]{(5\sqrt{2}+7)}\sqrt[3]{(5\sqrt{2}-7)}\,\{\sqrt[3]{(5\sqrt{2}+7)}-\sqrt[3]{(5\sqrt{2}-7)}\}=8$$

But, since $\sqrt[3]{(5\sqrt{2}+7)}-\sqrt[3]{(5\sqrt{2}-7)}=2$, it follows that

$$\sqrt[3]{(5\sqrt{2}+7)}\,\sqrt[3]{(5\sqrt{2}-7)}=1$$

or $\sqrt[3]{\{(5\sqrt{2})^2-7^2\}}=1$ or $\sqrt[3]{1}=1$ or $1=1$

and that is what we had to prove

Did the student prove the equality ?

60 Let $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}$ Prove that

$$\sqrt{(Aa)}+\sqrt{(Bb)}+\sqrt{(Cc)}+\sqrt{(Dd)}=\sqrt{[(a+b+c+d)\,(A+B+C-D)]}$$

Solutions

1 Ans (i) $3\,(1-\sqrt{5})\,\sqrt[3]{2}$,

(ii) $(5-\sqrt{5})\,\sqrt[5]{(5-\sqrt{5})^2}$,

(iii) $\sqrt{\left(8\frac{8}{63}\right)}=\sqrt{\left(\frac{512}{63}\right)}=\sqrt{\left(\frac{256\times 2}{9\times 7}\right)}=\frac{16}{3}\sqrt{\left(\frac{2}{7}\right)}$,

(iv) $\sqrt{\left(11\frac{11}{120}\right)}=\sqrt{\left(\frac{1331}{120}\right)}=\sqrt{\left(\frac{11\times 11\times 11}{4\times 30}\right)}=\frac{11}{2}\sqrt{\left(\frac{11}{30}\right)}$

2 We have

$6\times\sqrt[3]{5}=\sqrt[3]{(6^3\times 5)}=\sqrt[3]{(1080)}$,

$8\times\sqrt{2}=\sqrt{(8^3\times 2)}=\sqrt[3]{(1024)}$,

$2\times\sqrt{(130)}=\sqrt{(2^3\times 130)}=\sqrt[3]{(1040)}$, and $\sqrt[3]{(900)}$

Hence out of the four given numbers, $6\times\sqrt[3]{5}$ is the greatest

3 (i) Since $x=\frac{1}{2}(\sqrt{3}+1)$, we have

$(2x-1)^2=3$ or $4x^2-4x-2=0$

or $2x^2-2x-1=0$ (1)

Then $4x^3+2x^2-8x+7$

$=2x\,(2x^2-2x-1)+3\,(2x^2-2x-1)+10$

$=2x\times 0+3\times 0+10$ by (1) $=10$

(ii) $x=\frac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}=\frac{(\sqrt{5}+\sqrt{2})^2}{(\sqrt{5}-\sqrt{2})\,(\sqrt{5}+\sqrt{2})}=\frac{7+2\sqrt{(10)}}{5-2}$

(a) [b a c], (b) [c b a] (c) [b c a] (d) [a c b]

19 If θ is the angle between vectors a and b, then | a×b |=| a.b | when θ is equal to
(a) 0 (b) 180°, (c) 135° (d) 45°

20 a×(b×c) is equal to
(a) (a b) c (a c) b (b) (a b) a+(a b) c,
(c) (b c) a (b c) b, (d) (a c) b−(a b) c,
(e) (c a) a−(b a) c

21 u=a×(b×c)+b×(c×a)+c×(a×b), then
(a) u is a unit vector (b) u=a+b+c, (c) u=0, (d) u≠0

22 If a=4i+2j−5k, b=−12i−6j+15k, then the vectors a, b are
(i) orthogonal (ii) parallel, (iii) non coplanar,
(iv) None of these

23 If the position vectors of three points are
a−2b+3c, 2a+3b−4c, −7b+10c,
then the three points are
(a) collinear, (b) coplanar, (c) non collinear
(d) neither

24 (a) *If a+b+c=0 | a |=3, | b |=5 | c |=7,* then the angle between a and b is
(a) π/6 (b) 2π/3, (c) 5π/3, (d) π/4
(*b*) *If a b c are unit vectors such that a+b+c=0, then the value of a b+b c+c a is*
(a) 1, (b) 3 (c) $-\frac{3}{2}$ (d) none of these

25 If a b c are any three coplanar unit vectors then
(i) a (b×c)=1, (ii) a (b×c)=3
(iii) (a×b) c=0 (iv) (c×a) b=1

26 If a b=a c and a×b=a×c, then
(i) a is perpendicular to b−c, (ii) a is parallel to b c,
(iii) either a=0 or b=c, (iv) None of these

27 If | a |=| b | then (a+b) (a−b) is
(i) +ive, (ii) −ive, (iii) zero, (iv) None of these

28 The vector 2i+j−k is perpendicular to i−4j−λk if λ is equal to
(a) 0, (b) −1 (c) 2 (d) −3 (M N R 83)

29 If X A=0 X B=0 X C=0 for some non zero vector X then [A B C]=0
(a) True (b) False (I I T 83)

30 The vectors 2i+3j−4k and ai+bj+ck are perpendicular when
(i) a=2 b=3 c=−4 (ii) a=4, b=4 c=5
(iii) a=4 b=4, c=−5 (iv) None of these (M N R 82)

31 The vectors A=3i−k B=i+2j are adjacent sides of a parallelogram Its area is
(a) $\frac{1}{2}\sqrt{17}$ (b) $\frac{1}{2}\sqrt{14}$, (c) $\sqrt{41}$, (d) $\frac{1}{2}\sqrt{7}$ (M N R 81)

$$=\frac{4(x+1)}{x^3+1}=\frac{4(3^{1/3}+1)}{3+1}=3^{1/3}+1$$

20 Let $\sqrt{8}=2^{3/2}=x$ and $4^{1/3}=2^{2/3}=y$
so that $x^6=2^9$ and $y^6=4^3=2^4$

Given expression

$$=\frac{x+y}{x-y}=\frac{(x+y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)}{(x-y)(x^5+x^4y+x^3y^2+x^2y^3+xy^4+y^5)}$$

$$=\frac{x^6+2x^5y+2x^4y^2+2x^3y^3+2x^2y^4+2xy^5+y^6}{x^6-y^6}$$

$$=\frac{2^9+2\cdot 2^{15/2}\cdot 2^{2/3}+2\cdot 2^6\cdot 2^{4/3}+2\cdot 2^{9/2}\cdot 2^2+2\cdot 2^3\cdot 2^{8/3}+2\cdot 2^{3/2}\cdot 2^{10/3}+2^4}{2^9-2^4}$$

$$=\frac{1}{31}[2^5+2^{31/6}+2^{13/3}+2^{7/2}+2^{8/3}+2^{11/6}+1]$$

(cancelling the factor 2^4 from N^r and D^r)

21 We should break $\sqrt{(15)}$ into two factors the sum of whose squares is 15 These factors are clearly $\sqrt{3}$ and $\sqrt{5}$ Hence we may write

$8+2\sqrt{(15)}=5+3+2\sqrt{5}\sqrt{3}=(\sqrt{5}+\sqrt{3})^2$

Hence $\sqrt{\{8+2\sqrt{(15)}\}}=\sqrt{5}+\sqrt{3}$

Otherwise Assume $\sqrt{\{(8+2\sqrt{15})\}}=\sqrt{x}+\sqrt{y}$

Then $8+2\sqrt{(15)}=x+y+2\sqrt{(xy)}$

$x+y=8,\ xy=15$

Solving these equations, we shall get

$x=5,\ y=3,\quad$ or $\quad x=3,\ y=5$

Hence $\sqrt{\{8+2\sqrt{(15)}\}}=\sqrt{5}+\sqrt{3}$

(*b*) $(5+2\sqrt{6})$

22 Ans $(\sqrt{\tfrac{7}{2}}-\sqrt{\tfrac{5}{2}})$

23 Ans $3^{1/4}(\sqrt{\tfrac{1}{2}}+\sqrt{\tfrac{5}{2}})$

24 $\sqrt{(18)}-\sqrt{(16)}=3\sqrt{2}-4=\sqrt{2}\,(3-2\sqrt{2})$
$=\sqrt{2}(\sqrt{2}-1)^2$

Hence $\sqrt{\{\sqrt{(18)}-\sqrt{(16)}\}}=\pm 2^{1/4}(\sqrt{2}-1)$

25 $\sqrt{(32)}-\sqrt{(24)}=4\sqrt{2}-2\sqrt{6}$
$=\sqrt{2}(4-2\sqrt{3})=\sqrt{2}(\sqrt{3}-1)^2$

$\sqrt{\{\sqrt{(32)}-\sqrt{(24)}\}}=\pm 2^{1/4}(\sqrt{3}-1)$

26 (i) $\dfrac{2+\sqrt{3}}{2}=\dfrac{4+2\sqrt{3}}{4}=\dfrac{(\sqrt{3}+1)^2}{4}$

Hence $\sqrt{\left(\dfrac{2+\sqrt{3}}{2}\right)}=\left(\dfrac{\sqrt{3}+1}{2}\right)$

(ii) We have $\sqrt{(68+48\sqrt{2})}=2\sqrt{(17+12\sqrt{2})}$
$=2[(3+2\sqrt{2})^2]^{1/2}=2(3+2\sqrt{2})$

41 $\vec{A}=(1, 1, 1)$, $\vec{C}=(0, 1, -1)$ are given vectors, then a vector $\vec{B}$ satisfying the equations $\vec{A}\times\vec{B}=\vec{C}$ and $\vec{A}\cdot\vec{B}=3$ is (I I T 85)

42 Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, $\mathbf{b}=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}$ and $\mathbf{c}=c_1\mathbf{i}+c_2\mathbf{j}+c_3\mathbf{k}$ be three non zero vectors such that c is a unit vector perpendicular to both the vectors a and b If the angle between a and b is $\pi/6$ then

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}^2 \text{ is equal to}$$

(a) 0, (b) 1 (c) $\frac{1}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)$
(d) $\frac{3}{4}(a_1^2+a_2^2+a_3^2)(b_1^2+b_2^2+b_3^2)(c_1^2+c_2^2+c_3^2)$
(e) None of these (I I T 86)

43 A vector a has components $2p$ and 1 with respect to a rectangular cartesian system This system is rotated through a certain angle about the origin in the counter clockwise sence If with respect to new system a has components $p+1$ and 1, then

(a) $p=0$ (b) $p=1$ or $p=-\frac{1}{3}$
(c) $p=-1$ or $p=\frac{1}{3}$, (d) $p=1$ or $p=-1$
(e) None of these (I I T 86)

44 If $|\vec{\alpha}+\vec{\beta}|=|\vec{\alpha}-\vec{\beta}|$ then

(a) $\vec{\alpha}$ is parallel to $\vec{\beta}$ (b) $\vec{\alpha}$ is perpendicular to $\vec{\beta}$
(c) $|\vec{\alpha}|=|\vec{\beta}|$ (d) none of these, (M N R 88)

45 If $\vec{a}$ and $\vec{b}$ are two vectors such that $\vec{a}\cdot\vec{b}=0$ and $\vec{a}\times\vec{b}=\vec{0}$, then

(a) $\vec{a}$ is parallel to $\vec{b}$ (b) $\vec{a}$ is perpendicular to $\vec{b}$
(c) Either $\vec{a}$ or $\vec{b}$ is a null vector, (d) none of these (M N R 88)

46 If $\vec{a}, \vec{b}, \vec{c}$ are three non-copanar vectors and $\vec{p}, \vec{q}, \vec{r}$ are vectors defined by the relations

$$\vec{p}=\frac{\vec{b}\times\vec{c}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{q}=\frac{\vec{c}\times\vec{a}}{[\vec{a}\ \vec{b}\ \vec{c}]},\quad \vec{r}=\frac{\vec{a}\times\vec{b}}{[\vec{a}\ \vec{b}\ \vec{c}]}$$

Then the value of the expression

38 (*a*) The given expression

$$=\frac{\left[\sqrt{\left(\frac{p}{a}+x\right)}+\sqrt{\left(\frac{p}{a}-x\right)}\right]^2}{\left(\frac{p}{a}+x\right)-\left(\frac{p}{a}-x\right)}=\frac{1}{2x}\left[\frac{2p}{a}+2\sqrt{\left(\frac{p^2}{a^2}-x^2\right)}\right]$$

$$=\frac{p}{ax}+\sqrt{\left(\frac{p^2}{a^2x^2}-1\right)}$$

Now since $ax=\frac{2pq}{1+q^2}$ so that $\frac{p}{ax}=\frac{1+q^2}{2q}$, the expression

$$=\frac{1+q^2}{2q}+\sqrt{\left(\frac{(1+q^2)^2}{4q^2}-1\right)}=\frac{1+q^2}{2q}+\sqrt{\left[\left(\frac{1-q^2}{2q}\right)^2\right]}$$

Now if $-1\leqslant q\leqslant 1$, then

given expression $=\frac{1+q^2}{2q}+\frac{1-q^2}{2q}=\frac{1}{q}$

and if $q<-1$ or $q>1$, then

given expression $=\frac{1+q^2}{2q}+\frac{q^2-1}{2q}=q$

(*b*) Ans n^2/m^2

39 $x=\frac{(\sqrt{7}-\sqrt{5})^2}{7-5}=\frac{12-2\sqrt{(35)}}{2}=6-\sqrt{(35)}$

Similarly $y=6+\sqrt{(35)}$

Then $x+y=12$ and $xy=36-35=1$

Hence $x^3+y^3=(x+y)^3-3xy\ (x+y)=12^3-3\times 1\times 12$

$=1728-36=1692$

40 Ans $\frac{\sqrt{3}}{3}=\frac{1.732}{3}=.577$

41 Assume $\sqrt{\{21-4\sqrt{5}+8\sqrt{3}-4\sqrt{(15)}\}}=\sqrt{x}+\sqrt{y}-\sqrt{z}$

$21-4\sqrt{5}+8\sqrt{3}-4\sqrt{(15)}$

$=x+y+z+2\sqrt{(xy)}-2\sqrt{(yz)}-2\sqrt{(zx)}$

Hence $x+y+z=21$

$2\sqrt{(xy)}=8\sqrt{3},\ 2\sqrt{(yz)}=4\sqrt{5},\ 2\sqrt{(zx)}=4\sqrt{(15)}$

whence by multiplication

$xyz=240$, that is, $\sqrt{(xyz)}=4\sqrt{(15)}$

$\sqrt{x}=\frac{4\sqrt{(15)}}{2\sqrt{5}}=2\sqrt{3},\quad \sqrt{y}=\frac{4\sqrt{(15)}}{2\sqrt{(15)}}=2$

and $\sqrt{z}=\frac{4\sqrt{(15)}}{4\sqrt{3}}=\sqrt{5}$

Since these values of x, y, z satisfy the equation $x+y+z=21$, the required square root is $2\sqrt{3}+2-\sqrt{5}$

42 Assume $\sqrt{\{5-\sqrt{(10)}-\sqrt{(15)}+\sqrt{(6)}\}}=\sqrt{x}+\sqrt{y}-\sqrt{z}$

or $0+0+1+0=2\cos\theta$
or $\frac{1}{2}=\cos\theta \qquad \theta=\pi/3,$
(d) is the correct answer

16 $\mathbf{a}\times\mathbf{b}=1\ 1\ \sin\theta\,\hat{\mathbf{n}}$
It will be a unit vector if $\sin\theta=1$, i e $\theta=\pi/2$, i e if a and b are orthogonal

17 (a) k, (b) $-$k, (c) 0, (d) 0, (e) 3

18 Scalar triple product remains unchanged if cyclic order is maintained Hence (c) is the correct answer

19 $|ab\sin\theta\,\hat{\mathbf{n}}|=|ab\cos\theta|$
$ab\sin\theta=ab\cos\theta$ or $\tan\theta=1, \quad \theta=45°$
Hence (d) is the correct answer

20 Ans (d)

21 Ans (c) Check yourself

22 Ans (ii)

23 Ans (a) If A, B, C are the points, then $\overrightarrow{BC}=-2\overrightarrow{AB}$ and so A B, C are collinear

24 (a) Ans (d) We have $(\mathbf{a}+\mathbf{b})^2=\mathbf{c}^2$
or $\mathbf{a}^2+\mathbf{b}^2+2\mathbf{a}\ \mathbf{b}=\mathbf{c}^2$
or $9+25+2\ 3\ 5\cos\theta=49$
$\cos\theta=\frac{1}{2}$ so that $\theta=\pi/3$
(b) Squaring $\mathbf{a}+\mathbf{b}+\mathbf{c}=0$ and noting that $\mathbf{a}^2=|\mathbf{a}|=1$ etc, we get $\mathbf{a}\ \mathbf{b}+\mathbf{b}\ \mathbf{c}+\mathbf{c}\ \mathbf{a}=-3/2$

25 Ans (iii) 26 Ans (iii) 27 Ans (iii) 28 Ans (c)

29 Ans (a) Since X is non zero, the given conditions will be satisfied if either (i) at least one of the vectors A, B, C is zero or (ii) X is perp to all the vectors A B, C
In case (ii), A B C are coplanar and so [A B C]=0

30 Ans (ii) 31 Ans (c) 32 Ans (d)

33 Ans (b) 34 Ans (a) 35 (a) Ans (iii)
(b) $\mathbf{a}\times\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$, which is perpendicular to both a and b
Hence unit vector$=\pm\dfrac{\mathbf{i}-\mathbf{j}+\mathbf{k}}{\sqrt{3}}$ (b) is correct

36 Ans (a) 37 Ans $\sqrt{(13)}$ 38 Ans (a)

39 Ans $abc=-1$
Refer Q 5 Problem Set C P 436 of determinants we have
$\triangle\,(1+abc)=0$ (1)
Since the vectors $(1\ a\ a^2)$, $(1\ b, b^2)$, $(1, c, c^2)$ are non coplanar we have

$$\triangle=\begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}\neq 0$$

$$= \frac{\{\sqrt{(11)}-\sqrt{5}\}\{\sqrt{(11)}+\sqrt{5}\}\{\sqrt{(19)}+\sqrt{(1}}{\{\sqrt{(19)}-\sqrt{(13)}\}\{\sqrt{(11)}+\sqrt{5}\}\{\sqrt{(19)}+\sqrt{(1}}$$

$$= \frac{11 \quad 5}{19-13} \frac{\sqrt{(19)}+\sqrt{(13)}}{\sqrt{(11)}+\sqrt{(5)}}$$

$$= \frac{\sqrt{(19)}+\sqrt{(13)}}{\sqrt{(11)}+\sqrt{5}} > 1$$

Hence $x > y$, that is, $\sqrt{(11)}-\sqrt{5} > \sqrt{(19)}-\sqrt{(13)}$

48 First note that

$$\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})} = \sqrt{[\sqrt{(a+\sqrt{b})}+\sqrt{(a-\sqrt{b})}]^2}$$

$$= \sqrt{[(a+\sqrt{b}+a-\sqrt{b}+2\sqrt{(a+\sqrt{b})\,(a-\sqrt{b})}]}$$

$$= \sqrt{[2a+2\sqrt{(a^2-b)}]}$$

$$= 2\sqrt{\left[\frac{a+\sqrt{(a^2 \quad b)}}{2}\right]}$$

In our case $a = x \quad b = 4x-4 \quad a^2-b = x^2-4x+4$,

$$\sqrt{(a^2-b)} = \sqrt{(x \quad 2)^2} = \begin{cases} x-2 \text{ if } x > 2 \\ 2-x \text{ if } x < 2 \end{cases}$$

In the first case, we have

$$\sqrt{[x+2\sqrt{(x-1)}]}+\sqrt{[x-2\sqrt{(x-1)}]}$$

$$= 2\sqrt{\left(\frac{x+x-2}{2}\right)} = 2\sqrt{(x \quad 1)}$$

And in the second case

$$\sqrt{[x+2\sqrt{(x-1)}]}+\sqrt{[x-2\sqrt{x-1)}]} = 2\sqrt{\left(\frac{x+2-x}{2}\right)} = 2$$

Finally, it is easy to see that at $x=2$, the expression under consideration is also equal to 2

Hence the given expression is equal to 2 if $x \leqslant 2$, and equal to

$$2\sqrt{(x-1)} \text{ if } x > 2$$

49 Assume $(72-32\sqrt{5})^{1/3} = x-\sqrt{y}$

Then $(72+32\sqrt{5})^{1/3} = x+\sqrt{y}$

By multiplication $(5184-1024\times 5)^{1/3} = x^2-y$

that is, $4 = x^2-y$

Again $72-32\sqrt{5} = x^3-3x^2\sqrt{y}+3xy-y\sqrt{y}$

whence $72 = x^3+3xy$ (1)

From (1) and (2) $72 = x^3+3x\,(x^2-4)$

that is, $x^3-3x-18=0$, (2)

By trial, we find that $x=3$ is a root of the equation Hence

$y = x^2-4 = 9-4 = 5$

The required cube root $= 3-\sqrt{5}$

50 Ans $3-2\sqrt{2}$

51 Ans $\sqrt{3}+\sqrt{2}$

52 $38\sqrt{(14)}-100\sqrt{2} = 2\sqrt{2}\,(-50+19\sqrt{7})$

43 Ans (b) We have $\mathbf{a}=2p\mathbf{i}+\mathbf{j}$ On rotation, let $\mathbf{b}$ be the vector with components $p+1$ and 1 so that $\mathbf{b}=(p+1)\mathbf{i}+\mathbf{j}$ Now under rotation about origin,

$$|\mathbf{a}|=|\mathbf{b}| \Rightarrow |\mathbf{a}|^2=|\mathbf{b}|^2 \Rightarrow 4p^2+1=(p+1)^2+1$$

$$\Rightarrow 4p^2=(p+1)^2 \Rightarrow 2p=\pm(p+1)$$

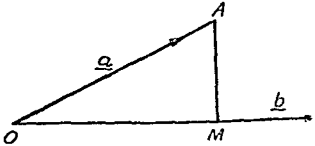
Hence $p=1$ or $-\frac{1}{3}$

44 Ans (b)

45 Ans (c)

46 Ans (d),

47 Let $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent the vectors $\mathbf{a}$ and $\mathbf{b}$ respectively. Draw $\overrightarrow{AM}$ perpendicular to $\overrightarrow{OB}$ Then $\overrightarrow{OM}$



and $\overrightarrow{MA}$ are component vectors of the vector $\mathbf{a}$ along and perpendicular to the vector $\mathbf{b}$ We first find the vector $\overrightarrow{OM}$

Now OM = Projection of vector $\mathbf{a}$ on vector $\mathbf{b}$

$$=\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}$$

Also unit vector in the direction of the vector $\mathbf{b}$ is

$$\frac{\mathbf{b}}{|\mathbf{b}|}$$

Hence $\overrightarrow{OM}=\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\frac{\mathbf{b}}{|\mathbf{b}|}=\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}\right)\mathbf{b}$

Also $\mathbf{a}=\overrightarrow{OA}=\overrightarrow{OM}+\overrightarrow{MA}$

$$\overrightarrow{MA}=\mathbf{a}-\overrightarrow{OM}=\mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

Thus components of vector $\mathbf{a}$ along and perpendicular to the vector $\mathbf{b}$ are

$$\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b} \text{ and } \mathbf{a}-\left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|^2}\right)\mathbf{b}$$

and $$U_n=\frac{1}{\sqrt5}(A^n-B^n)$$

$$U_n+U_{n-1}=\frac{1}{\sqrt5}(A^n-B^n)+\frac{1}{\sqrt5}(A^{n-1}-B^{n-1})$$
$$=\frac{1}{\sqrt5}[(A^n+A^{n-1})-(B^n+B^{n-1})] \quad (3)$$

Now multiplying (1) by A^{n-1}, we get
$A^{n+1}-A^n-A^{n-1}=0$, that is $A^{n+1}=A^n+A^{n-1}$
Similarly from (2) $B^{n+1}=B^n+B^{n-1}$
Substituting these values is (3), we get
$$U_n+U_{n-1}=\frac{1}{\sqrt5}[A^{n+1}-B^{n+1}]=U_{n+1}$$

56 Put $\left[-\frac{q}{2}+\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}\right]^{1/3}=A,$
$$\left[-\frac{q}{2}-\sqrt{\left(\frac{q}{2}+\frac{p^3}{27}\right)}\right]^{1/3}=B$$
Then $x=A+B$
and so $x^3-(A+B)^3=A^3+B^3+3AB(A+B)$ (1)
But $A^3+B^3=-q$ and $AB=-\frac{p}{3}$ (2)
Hence from (2), $x^3=-q-p(A+B)=-q-px$ from (1)
$$x^3+px+q=0$$

57 Suppose, if possible,
$$\sqrt[3]{2}=p+\sqrt q$$
whence cubing $2=p^3+3pq+(3p^2+q)\sqrt q$ (1)
Since q is not a perfect square, the equality (1) can hold if
$$3p^2+q=0$$
which is impossible since it is given that $q>0$ and $p^2\geqslant 0$ always

58 Let $\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}=\frac{1}{\lambda}$
whence
$$a'=a\lambda,\ b'=b\lambda,\ c'=c\lambda$$
Adding these we get $\lambda=\frac{a'+b'+c'}{a+b+c}$

The given expression takes the form
$$=\frac{1}{(1+\sqrt\lambda)(\sqrt a+\sqrt b+\sqrt c)}$$
$$=\frac{(1-\sqrt\lambda)(\sqrt a+\sqrt b-\sqrt c)}{(1-\lambda)[a+b-c+2\sqrt{(ab)}]}$$
$$=\frac{(1-\sqrt\lambda)(\sqrt a+\sqrt b-\sqrt c)[a+b-c-2\sqrt{(ab)}]}{(1-\lambda)(a^2+b^2+c^2-2ab-2bc-2ca)}$$

6 $\sqrt[3]{(\sqrt[3]{2}-1)} = \sqrt[3]{\frac{1}{9}} - \sqrt[3]{\frac{2}{9}} + \sqrt[3]{\frac{4}{9}}$

(a) True (b) False

7 The difference

$\sqrt{|40\sqrt{2}-57|} - \sqrt{(40\sqrt{2}+57)}$ is an integer

(a) True (b) False

Answers to Problem Set (B)

1 (c) 2 (b) 3 (a) 4 $\frac{1}{\sqrt{2}}[\sqrt{(x^2+2)} - \sqrt{(x^2-2)}]$

5 (a) 6 (a)

Put $\sqrt[3]{2}=\alpha$ so that $\alpha^3=2$

Denoting the R H S by x, we get

$x^3 = \frac{1}{9}(1-\alpha+\alpha^2)^3$

$= \frac{1}{9}(1-\alpha+\alpha^2)(1+\alpha^2+\alpha^4+2\alpha-2\alpha^3-2\alpha)$

$= \frac{1}{9}(1-\alpha+\alpha^2)(1+\alpha^2+2\alpha+2\alpha^2-4-2\alpha)$

$= \frac{1}{3}(1-\alpha+\alpha^2)(\alpha^2-1) = \frac{1}{3}(\alpha^2-\alpha+1)(\alpha+1)(\alpha-1)$ $[\because \alpha^3=2 \Rightarrow \alpha^4=2\alpha]$

$= \frac{1}{3}(\alpha^3+1)(\alpha-1) = \frac{1}{3}(2+1)(\alpha-1)$

$= \alpha-1$

R H S $= \sqrt[3]{(\alpha-1)} = \sqrt[3]{(\sqrt[3]{2}-1)}$

7 Ans (a)

Difference $= -\sqrt{\left[\sqrt{|40\sqrt{2}-57|} - \sqrt{(40\sqrt{2}+57)}\right]^2}$

$= -\sqrt{\left[57-40\sqrt{2}+57+40\sqrt{2} - 2\sqrt{(57^2-40^2\times 2)}\right]}$

$= -\sqrt{(114-14)} = -10$

----

when they are respectively multiplied by $\sqrt{3}$ and $\sqrt{7}-\sqrt{5}$, since $2\sqrt{3}\times\sqrt{3}=6$ and $(\sqrt{7}+\sqrt{5})(\sqrt{7}-\sqrt{5})=2$

Binomial quadratic surds are of the form $p\sqrt{q}\pm r\sqrt{s}$ or $p\pm q\sqrt{r}$, Two binomial quadratic surds which differ only in the sign which connects their terms are said to be **conjugate** or **complementary** to each other Thus the conjugate of the surd $2\sqrt{7}+5\sqrt{3}$ is the surd $2\sqrt{7}-5\sqrt{3}$

Important If a, b, c, d, are all rational numbers and b, d are not perfect squares, then

$$a+\sqrt{b}=c+\sqrt{d}$$

if and only if $a=c$ and $b=d$

§ 2 *To find the factor which will rationalize any given binomial surd*

Case I Suppose the given surd is $\sqrt[p]{a}-\sqrt[q]{b}$
Suppose $a^{1/p}=x$ $b^{1/q}=y$ and let n be the L C M of p and q Then x^n and y^n are both rational Now x^n-y^n is divisible by $x-y$ for all values of n, and

$$x^n-y^n=(x-y)\,(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+y^{n-1})$$

Thus the rationalizing factor is

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+\ldots+y^{n-1}$$

and the rational product is x^n-y^n

Case II Let the given surd be $\sqrt[p]{a}+\sqrt[q]{b}$

Let x, y, n have the same meaning as in Case I

(1) If n is even then x^n-y^n is divisible by $x+y$ and

$$x^n-y^n=(x+y)\,(x^{n-1}-x^{n-2}y+x^{n-3}y^2\ldots-y^{n-1})$$

Thus the rationalizing factor is

$$x^{n-1}-x^{n-2}y+x^{n-3}y^2\ldots-y^{n-1}$$

and the rational product is x^n-y^n

(2) If n is odd x^n+y^n is divisible by $x+y$, and

$$x^n+y^n=(x+y)\,(x^{n-1}-x^{n-2}y+\ldots+y^{n-1})$$

Thus the rationalizing factor is

$$x^{n-1}-x^{n-2}y+\ldots-xy^{n-2}+y^{n-1}$$

and the rational product is x^n+y^n

Important Remark The symbol $\sqrt[n]{a}$ is to be understood here (if n is odd) as the only real number whose n^{th} power is equal to a In this case a can be either less or greater than zero If n is even, then the symbol $\sqrt[n]{a}$ is understood as the only positive number the n^{th} power of which is equal to a Here, necessarily $a\geqslant 0$, and a real number Students must note carefully the results (1) and (2) Keeping these results in view the following computation is correct

$\sqrt{(-a)}\ \sqrt{(-b)}=\sqrt{a}\ \sqrt{(-1)}\ \sqrt{b}\ \sqrt{(-1)}=\sqrt{a}\ \sqrt{b}(\sqrt{-1})^2=-ab$

But the computation

$$\sqrt{(-a)}\ \sqrt{(-b)}=\sqrt{\{(-a)\,(-b)\}}=\sqrt{(ab)} \text{ is wrong}$$

Difference of two complex numbers

The difference of two complex numbers $z=(x, y)$ and $z'=(x', y')$ is defined by the equality

$$z-z'=z+(-z')=(x, y)+(-x', -y')$$
$$=(x+(-x'),\ y+(-y'))=(x-x', y-y')$$

Division It is defined by the equality $z/z'=z\,(z')^{-1}$, provided $z' \neq (0, 0)$

we have $\dfrac{z}{z'}=(x, y)\,(x', y')^{-1}$

$$=(x, y)\left(\frac{x'}{x'^2+y'^2}, -\frac{y'}{x'^2+y'^2}\right)$$

$$=\left(\frac{xx'}{x'^2+y'^2}+\frac{yy'}{x'^2+y'^2}, -\frac{xy'}{x'^2+y'^2}+\frac{yx'}{x'^2+y'^2}\right)$$

$$=\left(\frac{xx'+yy'}{x'^2+y'^2}, \frac{yx'-xy'}{x'^2+y'^2}\right)$$

provided $x'^2+y'^2 \neq 0$

§ 2 (A) **Modulus and argument of a complex number**

Let $z=x+iy$ be any complex number

If $x=r\cos\theta$, $y=r\sin\theta$, then $r=+\sqrt{(x^2+y^2)}$ is called the modulus of the complex number z written as $|z|$ and $\theta=\tan^{-1} y/x$ is called the argument or amplitude of z written as arg z

It follows that $|z|=0$ if and only if $x=0$ and $y=0$

Geometrically, $|z|$ is the distance of the point z from the origin.

It can be easily proved that

$$|z|^2=|z^2|,\ \text{Re}\ z \leqslant |z| \text{ and } I_m(z) \leqslant |z|$$

Also argument of a complex number is not unique, since if θ be a value of the argument, so also is $2n\pi+\theta$ where $n=0, \pm 1, \pm 2$

The value of argument which satisfies the inequality

$$-\pi < \theta \leqslant \pi$$

is called the **principal value** of the argument We remark that the argument of 0 is not defined

Simplify

13 $\dfrac{3+\sqrt{6}}{5\sqrt{3}-2\sqrt{(12)}-\sqrt{(32)}+\sqrt{(50)}}$ (Roorkee 1979)

14 $\dfrac{\sqrt{2}\,(\sqrt{3}+1)\,(2-\sqrt{3})}{(\sqrt{2}-1)\,(3\sqrt{3}-5)\,(2+\sqrt{2})}$

15 $\dfrac{\{\sqrt{(45)}-\sqrt{(20)}\}\,\{\sqrt{(12)}+\sqrt{(75)}\}\,\sqrt{3}}{\sqrt{5}+\sqrt{(180)}}$

16 The expression $\dfrac{12}{3+\sqrt{5}+2\sqrt{2}}$ is equal to

(*a*) $1-\sqrt{5}+\sqrt{2}+\sqrt{(10)}$, (*b*) $1+\sqrt{5}+\sqrt{2}-\sqrt{(10)}$

(*c*) $1+\sqrt{5}-\sqrt{2}+\sqrt{(10)}$, (*d*) $1-\sqrt{5}-\sqrt{2}+\sqrt{(10)}$

(I I T 1980)

Prove the following identities

17 $\dfrac{(a+2)^2\,(a-3)+(a^2-4)\,\sqrt{(a^2-9)}}{(a-2)^2\,(a+3)+(a^2-4)\,\sqrt{(a^2-9)}}=\dfrac{(a+2)\,\sqrt{(a-3)}}{(a-2)\,\sqrt{(a+3)}}\,(a\geqslant 3)$

18 $\dfrac{n^3-3n+(n^2-1)\,\sqrt{(n^2-4)}-2}{n^3-3n+(n^2-1)\,\sqrt{(n^2-4)}+2}=\dfrac{(n+1)\,\sqrt{(n-2)}}{(n-1)\,\sqrt{(n+2)}}$

Express with rational denominator,

19 $\dfrac{4}{\sqrt[3]{9}-\sqrt[3]{3}+1}$ 20 $\dfrac{\sqrt{8}+\sqrt[3]{4}}{\sqrt{8}-\sqrt[3]{4}}$

Find the square root of

21 (*a*) $8+2\sqrt{(15)}$ (*b*) $49+20\sqrt{6}$ (M N R 1983)

22 $6-\sqrt{(35)}$ 23 $\sqrt{(27)}+\sqrt{(15)}$

24 $\sqrt{(18)}-\sqrt{(16)}$ 25 $\sqrt{(32)}-\sqrt{(24)}$

26 (i) $\dfrac{2+\sqrt{3}}{2}$ (ii) $1-\sqrt{(68+48\sqrt{2})}$

(*I I T 1972*)

27 $2x-1+2\sqrt{(x^2-x-6)}$ 28 $a+x+\sqrt{(2ax+x^2)}$

29 (i) $(3/2)\,(x-1)+\sqrt{(2x^2-7x-4)}$,

(ii) $1-x+\sqrt{(22x-15-8x^2)}$

30 $1+a^2+\sqrt{(1+a^2+a^4)}$ 31 $x+y+z+2\sqrt{(xz+yz)}$

32 (*a*) Show that the expression

$\sqrt{2}\,[2x+\sqrt{(x^2-y^2)}]\,[\sqrt{\{x-\sqrt{(x^2-y^2)}\}}]$

can be simplified to

$\sqrt{(x+y)^3}-\sqrt{(x-y)^3}$ (I I T 1980)

(*b*) Prove that the expression

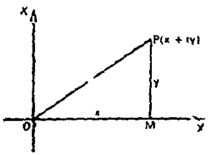
$x\sqrt{\{1+\sqrt[3]{(y^2/x^2)}\}}+y\sqrt{\{1+\sqrt[3]{(x^2/y^2)}\}}$

simplifies to $\sqrt{[(x^{2/3}+y^{2/3})^3]}$

33 Show that the square of

Vector representation of complex numbers

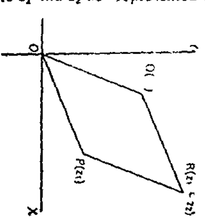
If P is the point (x, y) on the Argand plane corresponding to the complex number $z=x+iy$ referred to OX and OY as co-ordinate axes, the modulus and argument of z are represented by the magnitude and direction of the vector $\overrightarrow{OP}$ respectively and vice-versa



The points on the Argand plane representing the sum, and difference, of two complex numbers

Sum Let the complex numbers z_1 and z_2 be represented by the points P and Q on the Argand plane

Complete the parallelogram $OPRQ$ Then the mid points of PQ and OR are the same But mid point of PQ is



$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

so that the co ordinates of R are (x_1+x_2, y_1+y_2) Thus the point R corresponds to the sum of the complex numbers z_1 and z_2

In vector notation we have

$$z_1+z_2=\overrightarrow{OP}+\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR} \qquad (1)$$

Difference We first represent $-z_2$ by Q' so that QQ' is bisected at O Complete the parallelogram $OPRQ'$ Then the point R represents the complex number z_1-z_2 since the mid point of PQ' and OR are the same

As OQ is equal and parallel to RP, we see that $ORPQ$ is a parallelogram, so that $\overrightarrow{OR}=\overrightarrow{QP}$

Thus we have in vectorial notation

$$z_1-z_2=\overrightarrow{OP}-\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{QO}$$
$$=\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR}=\overrightarrow{QP} \qquad \text{(see fig p 28)} \quad ..(2)$$

40 If $\sqrt{3}=1.732$, find the value of

$$\frac{\sqrt{(26-15\sqrt{3})}}{5\sqrt{2}-\sqrt{(38+5\sqrt{3})}}$$

Find the square root of

41 $21-4\sqrt{5}+8\sqrt{3}-4\sqrt{15}$ (I I T 74)

42 $5-\sqrt{(10)}-\sqrt{(15)}+\sqrt{6}$

43 $6+\sqrt{(12)}-\sqrt{(24)}-\sqrt{8}$

44 $21+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{(24)}-\sqrt{(56)}+2\sqrt{(21)}$

45 Show that

$$\sqrt{[6+2\sqrt{3}+2\sqrt{2}+2\sqrt{6}]}-\frac{1}{\sqrt{(5-2\sqrt{6})}}$$

is a rational number (I I T 76)

46 Prove that

$$\sqrt{\{10+\sqrt{(24)}+\sqrt{(40)}+\sqrt{(60)}\}}=\sqrt{2}+\sqrt{3}+\sqrt{5}$$

47 Without extracting the roots, determine which is greater

$$\sqrt{11}-\sqrt{5} \quad \text{or} \quad \sqrt{19}-\sqrt{13}$$

48 Prove that for $x \geq 1$ the expression

$$\sqrt{[x+2\sqrt{(x-1)}]}+\sqrt{[x-2\sqrt{(x-1)}]}$$

is equal to 2 if $x \leqslant 2$, and to $2\sqrt{(x-1)}$ if $x > 2$

Find the real cube root of

49 $72-32\sqrt{5}$

50 $99-70\sqrt{2}$

51 $9\sqrt{3}+11\sqrt{2}$

52 $38\sqrt{(14)}-100\sqrt{2}$

53 If $\sqrt{3}=1.732$, find the value of

$$(26+15\sqrt{3})^{2/3}-(26+15\sqrt{3})^{-2/3}$$

54 Prove

(i) $\sqrt[3]{(20+14\sqrt{2})}+\sqrt[3]{(20-14\sqrt{2})}=4$

(ii) $\left\{6+\sqrt{\left(\frac{847}{27}\right)}\right\}^{1/3}+\left\{6-\sqrt{\left(\frac{847}{27}\right)}\right\}^{1/3}=3$

55 Let $u_n=\frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n-\left(\frac{1-\sqrt{5}}{2}\right)^n\right]$ $(n=0, 1, 2, 3, \ldots)$

Prove that $u_{n+1}=u_n+u_{n-1}$ $(n \geqslant 1)$

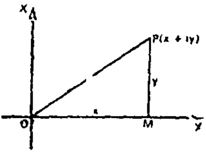
56 If $x=\left[-\frac{q}{2}+\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}\right]^{1/3}+\left[-\frac{q}{2}-\sqrt{\left(\frac{q^2}{4}+\frac{p^3}{27}\right)}\right]^{1/3}$,

prove that $x^3+px+q=0$

57 Prove that $\sqrt[3]{2}$ cannot be expressed in the form $p+\sqrt{q}$ where p and q are rational ($q > 0$ and is not a perfect square)

Vector representation of complex numbers

If P is the point (x, y) on the Argand plane corresponding to the complex number $z=x+iy$ referred to OX and OY as co ordinate axes, the modulus and argument of z are represented by the magnitude and direction of the vector $\overrightarrow{OP}$ respectively and vice-versa



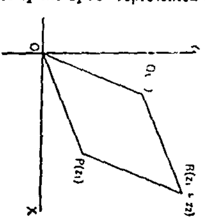
The points on the Argand plane representing the sum, and difference, of two complex numbers

Sum Let the complex numbers z_1 and z_2 be represented by the points P and Q on the Argand plane

Complete the parallelogram $OPRQ$ Then the mid points of PQ and OR are the same But mid point of PQ is

$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

so that the co ordinates of R are (x_1+x_2, y_1+y_2) Thus the point R corresponds to the sum of the complex numbers z_1 and z_2



In vector notation we have

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Difference We first represent $-z_2$ by Q' so that QQ' is bisected at O Complete the parallelogram $OPRQ$ Then the point R represents the complex number z_1-z_2 since the mid point of PQ' and OR are the same

As OQ is equal and parallel to RP, we see that $ORPQ$ is a parallelogram, so that $\overrightarrow{OR}=\overrightarrow{QP}$

Thus we have in vectorial notation

$$z_1-z_2=\overrightarrow{OP}-\overrightarrow{OQ}=\overrightarrow{OP}+\overrightarrow{QO}$$
$$=\overrightarrow{OP}+\overrightarrow{PR}=\overrightarrow{OR}=\overrightarrow{QP}$$

(see fig p 28) ..(2)

11 $$\frac{1}{\sqrt{(10)}+\sqrt{(14)}+\sqrt{(15)}+\sqrt{(21)}}$$
$$=\frac{1}{\sqrt{2}\sqrt{5}+\sqrt{2}\sqrt{7}+\sqrt{3}\sqrt{5}+\sqrt{3}\sqrt{7}}$$
$$=\frac{1}{(\sqrt{3}+\sqrt{2})(\sqrt{7}+\sqrt{5})}=\frac{(\sqrt{3}-\sqrt{2})(\sqrt{7}-\sqrt{5})}{(3-2)(7-5)}$$
$$=\tfrac{1}{2}\,[\sqrt{(21)}+\sqrt{(10)}-\sqrt{(14)}-\sqrt{(15)}]$$

12 $$\frac{15}{\sqrt{(10)}+\sqrt{(20)}+\sqrt{(40)}-\sqrt{5}-\sqrt{(80)}}$$
$$=\frac{15}{\sqrt{(10)}+2\sqrt{5}+2\sqrt{(10)}-\sqrt{5}-4\sqrt{5}}$$
$$=\frac{15}{3\sqrt{(10)}-3\sqrt{5}}=\frac{5\,\{\sqrt{(10)}+\sqrt{5}\}}{10-5}$$
$$=\sqrt{(10)}+\sqrt{5}=\sqrt{5}\,(\sqrt{2}+1)$$

13 The given expression
$$=\frac{3+\sqrt{6}}{5\sqrt{3}-4\sqrt{3}-4\sqrt{2}+5\sqrt{2}}=\frac{3+\sqrt{6}}{\sqrt{3}+\sqrt{2}}$$
$$=\frac{(3+\sqrt{6})(\sqrt{3}-\sqrt{2})}{3-2}=3\sqrt{3}-3\sqrt{2}+\sqrt{(18)}-\sqrt{(12)}$$
$$=3\sqrt{3}-3\sqrt{2}+3\sqrt{2}-2\sqrt{3}=\sqrt{3}$$

14 Ans $2+\sqrt{3}$

15 Ans 3

16 Ans (b)

17 Do yourself

18 Factorizing n^3-3n-2, we get
$$n^3-3n-2=n^2(n+1)-n(n+1)-2(n+1)$$
$$=(n+1)(n^2-n-2)=(n+1)(n+1)(n-2)$$
$$=(n+1)^2(n-2)$$
Similarly $n^3-3n+2=(n-1)^2(n+2)$

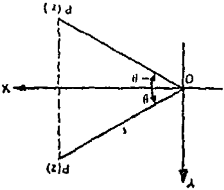
Hence the given expression
$$=\frac{(n+1)^2(n-2)+(n-1)(n+1)\sqrt{[(n-2)(n+2)]}}{(n-1)^2(n+2)+(n-1)(n+1)\sqrt{[(n-2)(n+2)]}}$$
$$=\frac{(n+1)\sqrt{(n-2)}}{(n-1)\sqrt{(n+2)}}\times\frac{(n+1)\sqrt{(n-2)}+(n-1)\sqrt{(n+2)}}{(n-1)\sqrt{(n+2)}+(n+1)\sqrt{(n-2)}}$$
$$=\frac{(n+1)\sqrt{(n-2)}}{(n-1)\sqrt{(n+2)}}$$

19 Let $3^{1/3}=x$ Then $3=x^3$

Given expression
$$=\frac{4}{x^2-x+1}=\frac{4(x+1)}{(x+1)(x^2-x+1)}$$

§ 4 Conjugate to complex numbers

If $z=x+iy$, then the complex number $x-iy$ is called the conjugate of the complex number z and is written as $\bar{z}$. It is easily seen that numbers conjugate to z_1+z_2 and z_1z_2 are $\bar{z}_1+\bar{z}_2$ and $\bar{z}_1\bar{z}_2$ respectively.



Also we have,

$|z|^2=z\bar{z}$, $2R(z)=z+\bar{z}$

and $2iI(z)=z-\bar{z}$

It is clear that $|\bar{z}|=|z|$, $\overline{(\bar{z})}=z$. Geometrically, the conjugate of z is the reflection (or image) of z in the real axis. If (r, θ) are polar co-ordinates of P, then the polar co-ordinates of its reflection P' are $(r, -\theta)$ so that we have

$$\text{amp } z= -\text{amp } \bar{z}$$

§ 5 Properties of Moduli

We now prove some basic results on moduli.

Theorem 1. *The modulus of the sum of two complex numbers can never exceed the sum of their moduli.*

Proof. We shall prove that

$$|z_1+z_2| \leqslant |z_1|+|z_2|$$

We have
$$|z_1+z_2|^2=(z_1+z_2)\overline{(z_1+z_2)}=(z_1+z_2)(\bar{z}_1+\bar{z}_2)$$
$$=z_1\bar{z}_1+z_1\bar{z}_2+z_2\bar{z}_1+z_2\bar{z}_2$$
$$=z_1\bar{z}_1+z_2\bar{z}_2+(z_1\bar{z}_2+\bar{z}_1z_2) \qquad (1)$$

Now $z_1\bar{z}_1=|z_1|^2$, $z_2\bar{z}_2=|z_2|^2$

and $z_1\bar{z}_2+\bar{z}_1z_2=(x_1+iy_1)(x_2-iy_2)+(x_1-iy_1)(x_2+iy_2)$
$$=2(x_1x_2+y_1y_2)=2R(z_1\bar{z}_2) \leqslant 2|z_1\bar{z}_2|$$

since the real part of a complex number can never exceed its modulus.

Then (1) gives $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2R(z_1\bar{z}_2)$
$$\leqslant |z_1|^2+|z_2|^2+2|z_1\bar{z}_2|,$$
$$=|z_1|^2+|z_2|^2+2|z_1||z_2|,$$

since $|z_1\bar{z}_2|=|z_1||\bar{z}_2|=|z_1||z_2|$

$$|z_1+z_2|^2 \leqslant (|z_1|+|z_2|)^2$$

Hence $$|z_1+z_2| \leqslant |z_1|+|z_2|$$

Remark. Since $|z_1+z_2|^2=|z_1|^2+|z_2|^2+2R(z_1\bar{z}_2)$

and $$|z_1-z_2|^2=|z_1|^2+|z_2|^2-2R(z_1\bar{z}_2),$$

and similarly

$$\sqrt{(16-6\sqrt{7})}=3-\sqrt{7},$$

Hence given expression

$$=\frac{\sqrt{7}}{3+\sqrt{7}-(3-\sqrt{7})}=\frac{\sqrt{7}}{2\sqrt{7}}=\tfrac{1}{2},$$

which is rational

36 (*a*) We have

$$\sqrt{[11-2\sqrt{(30)}]}=\sqrt{[6+5-2\sqrt{6}\sqrt{5}]}$$
$$=\sqrt{[(\sqrt{6}-\sqrt{5})^2]}=\sqrt{6}-\sqrt{5}$$

Similarly $\sqrt{[7-2\sqrt{(10)}]}=\sqrt{5}-\sqrt{2}$

and $\sqrt{(8+4\sqrt{3})}=\sqrt{[6+2+2\sqrt{6}\sqrt{2}]}$

$$=\sqrt{[(\sqrt{6}+\sqrt{2})^2]}=\sqrt{6}+\sqrt{2}$$

Hence the expression on the left hand side

$$=\frac{1}{\sqrt{6}-\sqrt{5}}-\frac{3}{\sqrt{5}-\sqrt{2}}-\frac{4}{\sqrt{6}+\sqrt{2}}$$
$$=\frac{\sqrt{6}+\sqrt{5}}{6-5}-\frac{3(\sqrt{5}+\sqrt{2})}{5-2}-\frac{4(\sqrt{6}-\sqrt{2})}{6-2}$$
$$=\sqrt{6}+\sqrt{5}-\sqrt{5}-\sqrt{2}-\sqrt{6}+\sqrt{2}$$
$$=0$$

(*b*) $D^r=2+\sqrt{3},$

• Given expression $=\frac{4+3\sqrt{3}}{2+\sqrt{3}}\cdot\frac{2-\sqrt{3}}{2-\sqrt{3}}=\frac{-1+2\sqrt{3}}{4-3}=-1+\sqrt{12}$

(*c*) Ans $2+\sqrt{3}$

37 We have

$$\sqrt{(3+\sqrt{5})}=\sqrt{\left(\frac{6+2\sqrt{5}}{2}\right)}=\frac{1}{\sqrt{2}}\sqrt{[(\sqrt{5}+1)^2]}=\frac{\sqrt{5}+1}{\sqrt{2}}$$

and similarly

$$\sqrt{(3-\sqrt{5})}=\frac{\sqrt{5}-1}{\sqrt{2}}$$

Hence given expression

$$=\frac{10\sqrt{2}}{\sqrt{(18)}-\frac{\sqrt{5}+1}{\sqrt{2}}}-\frac{\sqrt{(10)}+\sqrt{(18)}}{\sqrt{8}+\frac{\sqrt{5}-1}{\sqrt{2}}}$$
$$=\frac{10\sqrt{2}\sqrt{2}}{\sqrt{(18)}\sqrt{2}-\sqrt{5}-1}-\frac{\sqrt{2}\sqrt{(10)}+\sqrt{2}\sqrt{(18)}}{\sqrt{8}\sqrt{2}+\sqrt{5}-1}$$
$$=\frac{20}{5-\sqrt{5}}-\frac{2\sqrt{\ }+6}{3+\sqrt{5}}$$
$$=\frac{20(5+\sqrt{5})}{25-5}-\frac{(2\sqrt{5}+6)(3-\sqrt{5})}{9-5}$$
$$=5+\sqrt{5}-\tfrac{8}{4}=3+\sqrt{5}=3+2.23607=5.23607$$

Alternative We have

$$z_1 - z_2 = (r_1 \cos \theta_1 - r_2 \cos \theta_2) + i\,(r_1 \sin \theta_1 - r_2 \sin \theta_2)$$

so that $|z_1 - z_2| = \sqrt{[r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_1 - \theta_2)]}$

$$\geqslant \sqrt{(r_1^2 + r_2^2 - 2r_1r_2)} \qquad [\because \cos (\theta_1 - \theta_2) \leqslant 1]$$

$$= |r_1 - r_2| = ||z_1| - |z_2||$$

Geometrical Interpretation

Here $OP = |z_1|$, $OQ = |z_2|$, $QP = |z_1 - z_2|$

(See figure of theorem I)

Since in a triangle difference of any two sides is less than the third side, we have from $\triangle OPQ$, $OP - OQ < QP$

or $\qquad |z_1| - |z_2| < |z_1 - z_2|$

or $\qquad |z_1 - z_2| > |z_1| - |z_2|$

Equality will hold when O, P, Q are in a straight line.

Remark Theorem II can be derived from theorem I. We have

$$|z_1| = |z_1 - z_2 + z_2| \leqslant |z_1 - z_2| + |z_2|$$

or $\qquad |z_1| - |z_2| \leqslant |z_1 - z_2|$

Theorem III The modulus of the product of the complex numbers is the product of their moduli.

Proof We have

$$|z_1z_2|^2 = z_1z_2\bar{z}_1\bar{z}_2 = z_1\bar{z}_1z_2\bar{z}_2 = |z_1|^2|z_2|^2$$

so that $\qquad |z_1z_2| = |z_1||z_2|$

In the same manner it can be proved that

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad \text{provided } (z_2 \neq 0)$$

It is easy to see that

$|z_1z_2 \ldots z_n| = |z_1||z_2| \ldots |z_n|$ and in particular $|z^n| = |z|^n$

§ 6 Properties of Arguments

We now prove the following theorems on arguments of complex numbers.

Theorem I *The argument of the product of any number of complex quantities is equal to the sum of their arguments.*

Proof Let $z_1, z_2, z_3, \ldots, z_n$ be non zero complex numbers. Let $r_1, r_2, \ldots, r_n$ denote their moduli and $\theta_1, \theta_2, \ldots, \theta_n$ their arguments.

We then have

$$z_1z_2 \ldots z_n = r_1 (\cos \theta_1 + i \sin \theta_1)\, r_2\, (\cos \theta_2 + i \sin \theta_2) \ldots r_n\, (\cos \theta_n + i \sin \theta_n)$$

$$= r_1r_2 \ldots r_n\, [\cos (\theta_1 + \theta_2 + \ldots + \theta_n) + i \sin (\theta_1 + \theta_2 + \ldots + \theta_n)]$$

Then $5-\sqrt{(10)}-\sqrt{(15)}+\sqrt{6}$
$=x+y+z+2\sqrt{(xy)}-2\sqrt{(yz)}-2\sqrt{(zx)}$
$x+y+z=5,\ 2\sqrt{(xy)}=\sqrt{6},\ 2\sqrt{(yz)}=\sqrt{(10)}$
and $2\sqrt{(zx)}=\sqrt{(15)}$
$8\,xyz=\sqrt{6}\,\sqrt{(10)}\,\sqrt{(15)}=30$
$$\sqrt{(xyz)}=\frac{\sqrt{(15)}}{2}$$
Hence $x=\dfrac{\sqrt{(15)}}{2}\times\dfrac{2}{\sqrt{(10)}}=\sqrt{\tfrac{3}{2}},\ y=\dfrac{\sqrt{(15)}}{2}\times\dfrac{2}{\sqrt{(15)}}=1$

and $z=\dfrac{\sqrt{(15)}}{2}\times\dfrac{2}{\sqrt{6}}=\sqrt{\tfrac{5}{2}}$

Since these values of x, y, z satisfy $x+y+z=5$, the required square root is $1+\sqrt{\tfrac{3}{2}}-\sqrt{\tfrac{1}{2}}$

43 Ans $1+\sqrt{3}-\sqrt{2}$

44 Assume
$\sqrt{\{(21)+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{(24)}-\sqrt{(56)}+2\sqrt{(21)}\}}$
$=\sqrt{x}+\sqrt{y}-\sqrt{z}-\sqrt{u}$
Then $21+3\sqrt{8}-6\sqrt{3}-6\sqrt{7}-\sqrt{(24)}-\sqrt{(56)}+2\sqrt{(21)}$
$=x+y+z+u+2\sqrt{(xy)}-2\sqrt{(xz)}-2\sqrt{(xu)}$
$-2\sqrt{(yz)}-2\sqrt{(yu)}+2\sqrt{(zu)}$

$x+y+z+u=21$ (1)
$2\sqrt{(xy)}=3\sqrt{8}$ (2)
$2\sqrt{(xz)}=6\sqrt{3}$ (3)
$2\sqrt{(xu)}=6\sqrt{7}$ (4)
$2\sqrt{(yz)}=\sqrt{(24)}$ (5)
$2\sqrt{(yu)}=\sqrt{(56)}$ (6)
$2\sqrt{(zu)}=2\sqrt{(21)}$ (7)

Dividing (2) by (3), we get $\sqrt{\left(\dfrac{y}{z}\right)}=\dfrac{\sqrt{8}}{2\sqrt{3}}$ (8)

Now multiplying (5) and (8), $2y=\sqrt{(24)}\times\dfrac{\sqrt{8}}{2\sqrt{3}}=4$

Hence $\sqrt{y}=\sqrt{2}$ Then from (2), $\sqrt{x}=3$
From (5) $\sqrt{z}=\sqrt{3}$ and from (6), $\sqrt{u}=\sqrt{7}$
Since these values of x, y, z, u satisfy (1), the required square root is $3+\sqrt{2}-\sqrt{3}-\sqrt{7}$

45 Ans 1

46 Do yourself

47 Let $x=\sqrt{(11)}-\sqrt{5}$ and $y=\sqrt{(19)}-\sqrt{(13)}$

Then $\dfrac{x}{y}=\dfrac{\sqrt{(11)}-\sqrt{5}}{\sqrt{(19)}-\sqrt{(13)}}$

If n is any rational number, then the theorem states that

$$(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$$

It is easy to check that

$$(\cos\theta+i\sin\theta)^{-n}=\cos n\theta-i\sin n\theta,$$

$$(\cos\theta-i\sin\theta)^n=\cos n\theta-i\sin n\theta$$

and $$(\cos\theta-i\sin\theta)^{-n}=\cos n\theta+i\sin n\theta$$

Problem Set (A)

1 Under what condition is the sum of two complex numbers x_1+iy_1 and x_2+iy_2 is (*a*) real number (*b*) a purely imaginary number ?

2 (a) A student writes the formula $\sqrt{(ab)}=\sqrt{a}\sqrt{b}$ Then substitutes $a=-1$ and $b=-1$ and finds $1=-1$. Explain where he is wrong ? (Roorkee 82, 1987)

(b) Is the following computation correct ? If not give the correct computation

"$\sqrt{[(-2)\sqrt{(-3)}]}=\sqrt{[(-2)(-3)]}=\sqrt{6}$" (Roorkee 1978)

3 If $x+iy=\sqrt{\left(\dfrac{a+ib}{c+id}\right)}$, prove that $(x^2+y^2)^2=\dfrac{a^2+b^2}{c^2+d^2}$ (I I T 1979)

4 If $x=-5+2\sqrt{(-4)}$, find the value of

$$x^4+9x^3+35x^2-x+4$$

5 Prove the validity of the following formulae by squaring them

(a) $\sqrt{\{(a+bi)\}}+\sqrt{\{(a-bi)\}}=\sqrt{\{2(\sqrt{(a^2+b^2)}+a)\}}$

(b) $\sqrt{\{(a+bi)\}}-\sqrt{\{(a-bi)\}}=i\sqrt{\{2(\sqrt{(a^2+b^2)}-a)\}}$

Use these formulae to transform the following expressions

$\sqrt{(8+6i)}+\sqrt{(8-6i)}$ and $\sqrt{(-35+12i)}-\sqrt{(-35-12i)}$

6 Find the value of $[4+3\sqrt{(-20)}]^{1/2}+[4-3\sqrt{(-20)}]^{1/2}$

7 Prove that $x^4+4=(x+1+i)(x+1-i)(x-1+i)(x-1-i)$

8 Simplify

(i) i^{457} (ii) $(-\sqrt{-1})^{4n+3}$ (n, a +ive integer)

(iii) $\dfrac{1-i}{1+i}$ (iv) $\dfrac{1+2i}{1-(1-i)^2}$

(v) $\dfrac{1+i}{1-i}-\dfrac{1-i}{1+i}$ (vi) $\dfrac{3-i}{2+i}+\dfrac{3+i}{2-i}$

(vii) $\dfrac{(1-i)^3}{1-i^3}$ (viii) $\dfrac{3}{1+i}-\dfrac{2}{2-i}+\dfrac{2}{1-i}$

(ix) $\left(\dfrac{1+i}{1-i}\right)^{4n+1}$ (n, a +ve integer)

(x) $\dfrac{\sqrt{(5+12i)}+\sqrt{(5-12i)}}{\sqrt{(5+12i)}-\sqrt{(5-12i)}}$

$$(38\sqrt{(14)}-100\sqrt{2})^{1/3}=\sqrt{2}\,(-50+19\sqrt{7})^{1/3} \quad (1)$$

Now assume $(-50+19\sqrt{7})^{1/3}=x+\sqrt{y}$

Then $(-50-19\sqrt{7})^{1/3}=x-\sqrt{y}$

By multiplication, $(2500-361\times 7)^{1/3}=x^2-y$

that is, $(-27)^{1/3}=x^2-y$ (2)

or $-3=x^2-y$

Again $-50+19\sqrt{7}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$

whence $-50=x^3+3xy$ (3)

From (2) and (3)

$$-50=x^3+3x\,(x^2+3)$$

or $4x^3+9x+50=0$

By trial $x=-2$ is a root

Then $y=7$

Hence $(-50+19\sqrt{7})^{1/3}=-2+\sqrt{7}$

Finally from (1), we get

$$(38\sqrt{14}-100\sqrt{2})^{1/3}=\sqrt{2}\,(-2+\sqrt{7})=\sqrt{(14)}-2\sqrt{2}$$

53 Ans $8\sqrt{3}=13.856$

54 (i) Do yourself

(ii) Assume $\left[6+\sqrt{\left(\frac{847}{27}\right)}\right]^{1/3}=x+\sqrt{y}$ (1)

Then $\left[6-\sqrt{\left(\frac{847}{27}\right)}\right]^{1/3}=x-\sqrt{y}$ (2)

By multiplication $\left(36-\frac{847}{27}\right)^{1/3}=x^2-y$

or $\left(\frac{125}{27}\right)^{1/3}=x^2-y$, i.e. $x^2-y=\frac{5}{3}$

Also $6+\sqrt{\left(\frac{847}{27}\right)}=x^3+3x^2\sqrt{y}+3xy+y\sqrt{y}$

Hence $x^3+3xy=6$

$x^3+3x\,(x^2-\frac{5}{3})=6$

or $4x^3-5x-6=0$

By trial $x=\frac{3}{2}$ is a root

Now adding (1) and (2), we get

$$\left[6-\sqrt{\left(\frac{847}{27}\right)}\right]^{1/3}+\left[6-\sqrt{\left(\frac{847}{27}\right)}\right]^{1/3}=2x=2\times\frac{3}{2}=3,$$

as required

55 Put $\frac{1}{2}(1+\sqrt{5})=A$, $\frac{1}{2}(1-\sqrt{5})=B$

Then $A+B=1$, $AB=\frac{1}{4}(1-5)=-1$

Again $A^2-A-1=\frac{1}{4}(6+2\sqrt{5})-\frac{1}{2}(1+\sqrt{5})-1=0$ (1)

Similarly $B^2-B-1=0$ (2)

(v) $-\sqrt{3}+i$ (vi) $\frac{1+7i}{(2-i)^2}$ (Roorkee 1981)

(vii) $\frac{(1+i)^{2n+1}}{(1-i)^{2n-1}}, n \in N$

(viii) $1+i\tan\alpha \left(-\pi<\alpha<\pi, \alpha\ne\pm\frac{\pi}{2}\right)$

(ix) $2.5(\cos 300^\circ + i\sin 300^\circ)^2$

16 Find the modulus and argument of the complex number $z_1=z^2-z$ if $z=\cos\phi+i\sin\phi$

17 Find real values of x and y for which the following equations are satisfied

(i) $(1-i)x+(1+i)y=1-3i$ (ii) $\frac{x-1}{3+i}+\frac{y-1}{3-i}=i$

(iii) $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$ (MNR 87, IIT 1980)

(iv) $(x+iy)(2-3i)=4+i$ (Roorkee 1978)

(v) $(1+i)y^2+(6+i)=(2+i)x$

(vi) $\sqrt{(x^2-2x+8)}+(x+4)i=y(2+i)$

(vii) $(x^4+2xi)-(3x^2+yi)=(3-5i)+(1+2yi)$ (Rookree 1984)

18 If $a, b, c, \dots k$ are the roots of the equation $x^n+p_1x^{n-1}+p_2x^{n-2}+\dots+p_{n-1}x+p_n=0$ ($p_1, p_2, \dots p_n$ are real), then prove that $(1+a^2)(1+b^2)\dots(1+k^2)=(1-p_2+p_4-\dots)^2+(p_1-p_3+p_5-\dots)^2$

19 Solve the equation $x^4-4x^2+8x+35=0$ having given that one root is $2+\sqrt{(-3)}$

20 Find real θ such that $\frac{3+2i\sin\theta}{1-2i\sin\theta}$ is (i) real (ii) purely imaginary (IIT 1976)

21 (a) Find the vertices of a regular polygon of n sides if its centre is located at $z=0$ and one of its vertices z_1 is known.

(b) Points z_1 and z_2 are adjacent vertices of a regular polygon of n sides. Find the vertex z_3 adjacent to z_2 ($z_3\ne z_1$)

22 Show that a real value of x will satisfy the equation $\frac{1-ix}{1+ix}=a-ib$ if $a^2+b^2=1$ (a, b real)

23 Prove the following identities

(i) $(x^2+a^2)^4=(x^4-6x^2a^2+a^4)^2+(4x^3a-4xa^3)^2$

$$[\sqrt{(a+b+c)}-\sqrt{(a'+b'+c)}]\times$$
$$=\frac{(\sqrt{a}+\sqrt{b}-\sqrt{c})\,[a+b-c-2\sqrt{(ab)}]\sqrt{(a+b+c)}}{(a+b+c-a-b-c)\times(a^4+b^2+c^2-2ab-2bc-2ca)}$$

after substituting the value of λ

59 The equality is not proved since in the course of "proving" it, the student twice used the equality to be proved

To prove the equality we proceed as follows

Let $\sqrt[3]{(5\sqrt{2}+7)}-\sqrt[3]{(5\sqrt{2}-7)}=a$ (1)

Cubing, $14-3\sqrt{(5\sqrt{2}+7)}\sqrt[3]{(5\sqrt{2}-7)}$
$\{\sqrt[3]{\sqrt{(5\sqrt{2}+7)}}-\sqrt[3]{(5\sqrt{2}-7)}\}=a^3$

or $14-3\sqrt[3]{(5\sqrt{2}+7)}\sqrt[3]{(5\sqrt{2}-7)}\,a=a^3$ by (1)

or $14-a^3=3\sqrt[3]{(50-49)}\,a=3a$

or $a^3+3a-14=0$

Since $a=2$ is the only real root of this equation, we have proved the equality

60 Let $\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}=\lambda$

Then $A=a\lambda$ $B=b\lambda$ $C=\lambda c$

$D=d\lambda$ and

Hence $A+B+C+D=\lambda(a+b+c+d)$ (1)

Now $\sqrt{(Aa)}+\sqrt{(Bb)}+\sqrt{(Cc)}+\sqrt{(Dd)}$
$=\lambda(a+b+c+d)$
$=\sqrt{\left\{\frac{A+B+C+D}{a+b+c+d}\right\}}\,(a+b+c+d)$, by (1)
$=\sqrt{\{(a+b+c+d)\,(A+B+C+D)\}}$

Problem Set (B)

Objective Questions

1 $\sqrt{(3+\sqrt{5})}=$

(a) $\sqrt{5}+1$ (b) $\sqrt{3}+\sqrt{2}$, (c) $\frac{\sqrt{5}+1}{\sqrt{2}}$, (d) $\frac{\sqrt{5}+1}{2}$,

2 $\sqrt{[10+\sqrt{(24)}+\sqrt{(40)}+\sqrt{(60)}]}=$

(a) $\sqrt{2}+\sqrt{3}-\sqrt{5}$, (b) $\sqrt{2}+\sqrt{3}+\sqrt{5}$

(c) $\sqrt{2}-\sqrt{3}+\sqrt{5}$, (d) $2+\sqrt{3}+\sqrt{5}$

3 $\sqrt[3]{(20+14\sqrt{2})}+\sqrt[3]{(20-14\sqrt{2})}=4$

(a) True (b) False

4 $\sqrt{(x^2-\sqrt{(x^4-4)})}=$

5 If $p=\sqrt{7}-\sqrt{5}$ and $q=\sqrt{(13)}-\sqrt{(11)}$ then

(a) $p>q$ (b) $p<q$ (c) $p=q$

(d) none of these

is divisible by x^3+x^2+x+1 where l, m, n, p are positive integers

(*b*) Find the common roots of the equations

$$z^3+2z^2+2z+1=0,$$
$$z^{1985}+z^{100}+1=0$$

33 If $ax+cy+bz=X$, $cx+by+az=Y$, $bx+ay+cz=Z$, show that

(i) $(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$
$$=X^2+Y^2+Z^2-YZ-ZX-XY$$

(ii) $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$
$$=X^3+Y^3+Z^3-3XYZ$$

34 Given $z_1+z_2+z_3=A$, $z_1+z_2\omega+z_3\omega^2=B$, $z_1+z_2\omega^2+z_3\omega=C$

(i) Express z_1, z_2, z_3 in terms of A, B, C

(ii) Prove $|A|^2+|B|^2+|C|^2=3\{|z_1|^2+|z_2|^2+|z_3|^2\}$

35 For what real values of x and y are the numbers
$$-3+ix^2y \text{ and } x^2+y+4i$$
conjugate complex ?

36 (*a*) If $|z|=1$, prove that $\frac{z-1}{z+1}$ $(z\neq-1)$, is a pure imaginary number What will you conclude if $z=1$?

(Roorkee 1982)

(*b*) If the number $\frac{z-1}{z+1}$ is a pure imaginary, then prove that $|z|=1$

37 Locate the points representing the complex numbers z for which

(i) $\arg z=\pi/3$ (ii) $\pi/3<\arg z\leqslant\frac{3\pi}{2}$

(iii) $|z-i|=1$ and $\arg z=\frac{\pi}{2}$ (iv) $|\pi-\arg z|<\frac{\pi}{4}$

38 Find all complex numbers z which satisfy the following equations

(i) $z=\bar{z}$ (ii) $z=-\bar{z}$ (iii) $\bar{z}=2-z$

(iv) $z^2=-\bar{z}$ (v) $z^2=\bar{z}$ (vi) $z^2+|z|=0$

39 Find the complex numbers z which simultaneously satisfy the equations

$$\left|\frac{z-12}{z-8i}\right|=(5/3) \text{ and } \left|\frac{z-4}{z-8}\right|=1$$

40 Locate the points representing to complex numbers z on the Argand diagram which

2

Complex Numbers

§ 1 Definitions

A complex number may be defined as an ordered pair of real numbers and may be denoted by the symbol (x, y) If we write $z=(x, y)$, then x is called the real part and y the imaginary part of the complex number z and may be denoted by $\mathbf{R}(z)$ and $\mathbf{I}(z)$ respectively

It is clear from the definition that two complex numbers $(x\ y)$ and (x', y') are equal if and only if $x=x'$ and $y=y'$ We shall denote the set of all complex numbers by the letter **C**

Sum of two complex numbers The sum of two complex numbers (x, y) and $(x'\ y')$ is defined by the equality

$$(x, y)+(x', y')=(x+x'\ y+y')$$

Product of two complex numbers The product is defined by the equality

$$(x\ y)\ (x', y')=(xx'-yy', xy'-yx')$$

The symbol i It is customary to denote the complex number $(0\ 1)$ by the symbol i With this notation

$$i^2=(0, 1)\ (0, 1)=(0\ 0-1\ 1\ \ 0\ 1-1\ 0)=(-1, 0)$$

so that i may be regarded as the square root of the real number -1

Using the symbol i, we may write the complex number (x, y) as $x+iy$ For, we have

$$(x+iy)=(x, 0)+(0, 1)\ (y, 0)=(x\ \ 0)+(0\ y-1\ 0, 0\ 0+1\ y)$$
$$=(x, 0)+(0, y)=(x+0\ \ 0+y)=(x, y)$$

Remark $i=\sqrt{(-1)}$ and $i^2=-1$, we have

$$(\sqrt{-1})^2=\sqrt{(-1)}\ \sqrt{(-1)}=-1 \tag{1}$$

Again since $(\sqrt{a}\ \sqrt{-1})^2=\{\sqrt{a}\ \sqrt{(-1)}\}\times\{\sqrt{a}\ \sqrt{-1}\}$

$$=(\sqrt{a})^2\ \{\sqrt{(-1)}\}^2=a\ (-1)=-a \tag{2}$$

Hence $\sqrt{(-a)}$ means the product, of $\sqrt{a}$ and $\sqrt{(-1)}$ Thus a pure imaginary number can be expressed as the product of $\sqrt{(-1)}$

(iv) $\left(\frac{1+i}{\sqrt{2}}\right)^8+\left(\frac{1-i}{\sqrt{2}}\right)^8=2$

(v) $\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n=\cos n\phi+i\sin n\phi$

51 (a) If $x_r=\cos\frac{\pi}{2^r}+i\sin\frac{\pi}{2^r}$ prove that $x_1x_2x_3\ldots$ ad inf $=-1$

(b) If α, β are the roots of the equation $x^2-2x+4=0$ prove that $\alpha^n+\beta^n=2^{n+1}\cos\frac{n\pi}{3}$

52 If $2\cos\theta=x+\frac{1}{x}$ and $2\cos\phi=y+\frac{1}{y}$ etc then prove that

(i) $xyz\ldots+\frac{1}{xyz\ldots}=2\cos(\theta+\phi+\ldots)$

(ii) $\frac{x}{y}+\frac{y}{x}=2\cos(\theta-\phi)$ **(M N.R 87)**

(iii) $x^my^n+\frac{1}{x^my^n}=2\cos(m\theta+n\phi)$

(iv) $\frac{x^m}{y^n}+\frac{y^n}{x^m}=2\cos(m\theta-n\phi)$

53 If $(1+x)^n=p_0+p_1x+p_2x^2+\ldots+p_nx^n$, prove that

$$p_0-p_2+p_4-\ldots=2^{n/2}\cos\frac{n\pi}{4}$$

and $p_1-p_3+p_5-\ldots=2^{n/2}\sin\frac{n\pi}{4}$

54 Prove that if $\cos\alpha+i\sin\alpha$ is a solution of the equation

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_n=0,$$

then $p_1\sin\alpha+p_2\sin2\alpha+\ldots+p_n\sin n\alpha=0$ ($p_1, p_2, \ldots p_n$ are real)

55 If $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, Prove that

(i) $\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma)$ **(Roorkee 1985)**

(ii) $\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) $\cos2\alpha+\cos2\beta+\cos2\gamma=\sin2\alpha+\sin2\beta+\sin2\gamma=0$

56 Find all the values of the given root

(i) $(2-2i)^{1/3}$, (ii) $(1-\sqrt{3}i)^{1/4}$, (iii) $(-64a^4)^{1/4}$ (a real)

57 Prove that n, nth roots of unity form a series in G.P.

58 If β is an imaginary root of the equation $z^n-1=0$,

z_1, z_2, z_3 are the vertices of an equilateral triangle inscribed in a unit circle

(iii) Complex numbers z_1, z_2, z_3 are the vertices A, B, C respectively of an isosceles right angled triangle with right angle at C show that

$(z_1-z_2)^2=2(z_1-z_3)(z_3-z_2)$ (I I T 1986)

(iv) The cube roots of unity when represented on the Argand diagram from the vertices of an equilateral triangle

(a) True (b) False

(I I T (1988)

65 Let the complex numbers z_1, z_2, and z_3 be the vertices of an equilateral triangle Let z_0 be the circumcentre of the triangle Prove that $z_1^2+z_2^2+z_3^2=3z_0^2$ (I I T 1981)

66, The complex numbers z_1, z_2, z_3 are the vertices of a triangle Find all the complex numbers z which make the triangle into a parallelogram

67 Show that the triangle whose vertices are z_1, z_2, z_3 and z_1', z_2', z_3' are directly similar if

$$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix}=0$$

68 (a) If $|z_1|=|z_2|=\ldots=|z_n|=1$, prove that

$$|z_1+z_2+\ldots+z_n|=\left|\frac{1}{z_1}+\frac{1}{z_2}+\ldots+\frac{1}{z_n}\right|$$

(b) For any two non zero complex numbers z_1, z_2 prove the inequality

$$(|z_1|+|z_2|)\left|\frac{z_1}{|z_1|}+\frac{z_2}{|z_2|}\right|\leq 2(|z_1|+|z_2|)$$

69 Prove that

$$\left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right|<1 \text{ if } |z_1|<1 \text{ and } |z_2|<1$$

70 Demonstrate that the complex number $x+iy$ whose modulus is unity, $y\neq0$, can be represented as

$$x+iy=\frac{a+i}{a-i}$$

where a is a real number

71 If $a^2+b^2+c^2=1$, $b+ic=(1+a)z$, prove that

Remark It is evident from the definitions of difference and modulus that $|z_1 - z_2|$ is the distance between the points z_1 and z_2. It follows that for fixed complex number z_0 and real number r, the equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r

(B) Polar Form of a Complex Number

If r is the modulus and θ the argument of a complex number z then $z = r(\cos\theta + i\sin\theta)$ which is the polar form or trigonometrical form of z

Since $e^{i\theta} = \cos\theta + i\sin\theta$ we can write $z = re^{i\theta}$ This is known as the exponential form of z

(C) Representation of certain complex numbers of the form $a+ib$ in the form r $(\cos\theta + i\sin\theta)$

$$r = +\sqrt{(a^2+b^2)} \text{ always and } \tan\theta = b/a$$
$$= \tan\alpha \text{ say} \qquad \theta = \alpha$$

(i) a +ive b+ive $\tan\theta = b/a = \tan\alpha$ $\qquad\theta = \alpha$

(ii) a +ive b —ive, then $\theta = -\alpha$

(iii) a —ive b+ive then $\theta = \pi - \alpha$

(iv) a, —ive, b—ive then $\theta = -(\pi - \alpha)$

We shall write $a + ib = (r\ \theta)$

In accordance with the above rule we write below $(r\ \theta)$ form of certain complex numbers

$1 = (1, 0)$ $-1 = (1\ \pi)$, $i = (1, \pi/2)$, $-i = (1\ -\pi/2)$

$1 + i = (\sqrt{2}, \pi/4)$, $1 - i = (\sqrt{2}, -\pi/4)$, $-1 + i = (\sqrt{2}\ 3\pi/4)$,

$(-1 - i) = (\sqrt{2}, -3\pi/4)$

$1 + i\sqrt{3} = (2, \pi/3)$ $(1 - i\sqrt{3}) = (2\ -\pi/3)$

$-1 + i\sqrt{3} = (2\ 2\pi/3,)$ $-1 - i\sqrt{3} = (2, -2\pi/3)$ etc

§ 3 The geometrical representation of complex numbers

We represent the complex number $z = x + iy$ by a point P whose cartesian co ordinates are (x, y) referred to rectangular axes OX and OY usually called real and imaginary axes respectively Clearly the polar co ordinates of P are (r, θ) where r is the modulus and θ the argument of complex number z The plane whose points are represented by complex numbers is called Argand plane or Argand diagram

It is also called Complex plane or Gaussian plane

Note The complex number z is known as the affix of the point (x, y) which represents it

4 We have, $x+5=2\sqrt{(-4)}$

Squaring, $x^2+10x+25=-16$

or $x^2+10x+41=0$ (1)

Now $x^4+9x^3+35x^2-x+4$

$=x^2(x^2+10x+41)-x(x^2+10x+41)+4(x^2+10x+41)-160$

$=x^2\times0-x\times0+4\times0-160$ from (1)

$=-160$

You may divide the expression by $x^2+10x+41$ and the remainder will be -160

5 Square of L H S

$=(a+bi)+(a-bi)+2\sqrt{\{(a+bi)\}}\sqrt{\{(a-bi)\}}$

$=2a+2\sqrt{\{(a^2+b^2)\}}$

$=2[\sqrt{(a^2+b^2)}+a]$

and square of R H S $=2[\sqrt{(a^2+b^2)}+a]$

Thus $[\sqrt{(a+bi)}+\sqrt{(a-bi)}]^2=2[\sqrt{(a^2+b^2)}+a]$

Hence $\sqrt{(a+bi)}+\sqrt{(a-bi)}=[2\{\sqrt{(a^2+b^2)}+a\}]^{1/2}$ (i)

(b) Proceed as in (a) (ii)

Using formula in (a), we have

$\sqrt{(8+6i)}+\sqrt{(8-6i)}=\sqrt{[2\{\sqrt{(8^2+6^2)}+8\}]}$

$=\sqrt{[2(10+8)]}=\sqrt{(36)}=6$

Using formulae in (b), we have

$\sqrt{(-35+12i)}-\sqrt{(-35-12i)}=i\sqrt{[2\{\sqrt{((-35)^2+(12)^2)}-(-35)\}]}$

$=i\sqrt{[2(37+35)]}=i\sqrt{(144)}=12i\ [\because 35^2+12^2=37^2]$

6 Using formula (a) of ex. 5, we have

$[4+3\sqrt{(-20)}]^{1/2}+[4-3\sqrt{(-20)}]^{1/2}$

$=[2\{\sqrt{(4^2+(3\sqrt{(20)})^2}+4\}]^{1/2}=[2\{\sqrt{(196)}+4\}]^{1/2}$

$=[2(14+4)]^{1/2}=(36)^{1/2}=6$

7 $(x+1+i)(x+1-i)(x-1+i)(x-1-i)$

$=[(x+1)^2-i^2][(x-1)^2-i^2]$

$=(x^2+2x+1+1)(x^2-2x+1+1)$

$=[(x^2+2)+2x][(x^2+2)-2x]$

$=(x^2+2)^2-4x^2=x^4+4x^2+4-4x^2$

$=x^4+4$

8 (i) $i^{457}=i^{456}\,i=(i^4)^{114}\,i=i$ $i^4=1$

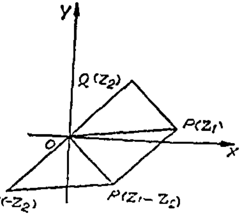
(ii) $(-\sqrt{-1})^{4n+3}=(-1)\ldots$

(iii) $\dfrac{1-i}{1+i}=\dfrac{(1-i)^2}{(1+i)(1-i)}=\dfrac{1+i^2-2i}{1-i^2}=\dfrac{1-1-2i}{1+1}=-i$

(iv) Do yourself. Ans.

(v) Do yourself. Ans. 2

It follows that the complex number $z_1 - z_2$ is represented by vector $\overrightarrow{QP}$, where the points P and Q represent the complex numbers z_1 and z_2 respectively



Important Remark

(i) Obviously $|z_1 - z_2| = QP$ and arg $(z_1 - z_2)$ is the angle through which OX has to rotate in anti clockwise direction as to be parallel to line QP. It is often convenient to use the polar representation about some point z_0 other than the origin. The representation $z - z_0 = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$ means that ρ is the distance between z and z_0 i.e. $\rho = |z - z_0|$, and ϕ is the angle of inclination of vector $z - z_0$ with the real axis. Further if the vector $z - z_0$ is rotated about z_0 in the anti clockwise direction through an angle θ and z is the new position of z then

$$z - z_0 = \rho e^{i(\phi+\theta)} = \rho e^{i\phi} e^{i\theta} = (z - z_0) e^{i\theta} \qquad \text{(Note this)}$$

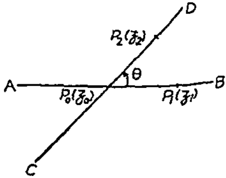
(ii) Multiplication by i

Since $z = r(\cos\theta + i\sin\theta)$ and $i = \cos\frac{1}{2}\pi + i\sin\frac{1}{2}\pi)$ we get

$$iz = r\left[\cos\left(\theta + \frac{\pi}{2}\right) + i\sin\left(\theta + \frac{\pi}{2}\right)\right]$$

Hence multiplication of z with i rotates the vector for z through a right angle in the positive direction

(iii) Let the lines AB and CD intersect at the point P_0 represented by the complex number z_0 and let P_1, P_2 be any two points on AB and CD represented by z_1 and z_2 respectively. Then the angle θ between the lines is given by



$$\theta = \arg(z_2 - z_0) - \arg(z_1 - z_0)$$

$$= \arg\left(\frac{z_2 - z_0}{z_1 - z_0}\right)$$

[Note that here only principal values of the arguments are considered]

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or $x^2+10x+41=0$ (i)
Now $x^4+9x^3+35x^2-x+4$
$=x^2(x^2+10x+41)-x(x^2+10x+41)+4(x^2+10x+41)-160$
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$=-160$
You may divide the expression by $x^2+10x+41$ and the remainder will be -160

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$=(a+bi)+(a-bi)+2\sqrt{(a+bi)}\sqrt{(a-bi)}$
$=2a+2\sqrt{(a^2+b^2)}$
$=2[\sqrt{(a^2+b^2)}+a]$
and square of R H S $=2[\sqrt{(a^2+b^2)}+a]$
Thus $[\sqrt{(a+bi)}+\sqrt{(a-bi)}]^2=2[\sqrt{(a^2+b^2)}+a]$
Hence $\sqrt{(a+bi)}+\sqrt{(a-bi)}=[2\{\sqrt{(a^2+b^2)}+a\}]^{1/2}$
(b) Proceed as in (a)
Using formula in (a), we have
$\sqrt{(8+6i)}+\sqrt{(8-6i)}=\sqrt{[2\{\sqrt{(8^2+6^2)}+8\}]}$
$=\sqrt{[2(10+8)]}=\sqrt{(36)}=6$
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$\sqrt{(-35+12i)}-\sqrt{(-35-12i)}=i\sqrt{[2\{\sqrt{((-35)^2+(12)^2)}\}-(-35)]}$
$=i\sqrt{[2(37+35)]}=i\sqrt{(144)}=12i$ [$\because 35^2+12^2=37^2$]

6 Using formula (a) of ex 5 we have
$[4+3\sqrt{(-20)}]^{1/2}+[4-3\sqrt{(-20)}]^{1/2}$
$=[2\{\sqrt{(4^2+(3\sqrt{(20)})^2)}+4\}]^{1/2}=[2\{\sqrt{(196)}+4\}]^{1/2}$
$=[2(14+4)]^{1/2}=(36)^{1/2}=6$

7 $(x+1+i)(x+1-i)(x-1+i)(x-1-i)$
$=[(x+1)^2-i^2][(x-1)^2-i^2]$
$=(x^2+2x+1+1)(x^2-2x+1+1)$
$=[(x^2+2)+2x][(x^2+2)-2x]$
$=(x^2+2)^2-4x^2=x^4+4x^2+4-4x^2$
$=x^4+4$

8 (i) $i^{457}=i^{456}\,i=(i^4)^{114}\,i=i$, $i^4=1$
(ii) $(-\sqrt{-1})^{4n+3}=(-i)^{4n+3}$ [illegible]
(iii) $\frac{1-i}{1+i}=\frac{(1-i)^2}{(1+i)(1-i)}=\frac{1+i^2-2i}{1-i^2}=\frac{1-1-2i}{1+1}=-i$
(iv) Do yourself. Ans [illegible]
(v) Do yourself. Ans 2

We have by addition

$$|z_1+z_2|^2+|z_1-z_2|^2=2\,(|z_1|^2+|z_2|)$$

Alternative Proof of Theorem I

Writing $z_1=r_1(\cos\theta_1+i\sin\theta_1)$ and $z_2=r_2(\cos\theta_2+i\sin\theta_2)$, we get $z_1+z=(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)$,

so that $|z_1+z_2|=\sqrt{[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]}$

$\leqslant\sqrt{[r_1^2+r_2^2+2r_1r_2]}$ [$\cos(\theta_1-\theta)\leqslant 1$]

$=r_1+r_2=|z_1|+|z_2|$

Hence $|z_1+z_2|<|z_1|+|z|$

Geometrical interpretation

Let P Q be the points of affix of z_1 and z Complete the parallelogram $OPRQ$ Then R is the point of affix (z_1+z_2)

Now $|z_1|=OP$,

$|z_2|=OQ=PR$

and $|z_1+z_2|=OR$

We know that in any triangle sum of any two sides is greater, than the third side

Hence $OP+PR>OR$

or $|z_1|+|z_2|>|z_1+z_2|$

or $|z_1+z|<|z_1|+|z_2|$

Equality will hold when O, P, Q are in a straight line

Hence $|z_1+z_2|\leqslant|z_1|+|z_2|$

As a special case

$|\alpha+\beta|\leqslant|\alpha|+|\beta|$

Remark By the above theorem, we have

$|z_1+z_2+z_3|\leqslant|z_1+z_2|+|z_3|\leqslant|z_1|+|z_2|+|z_3|$

This property can easily be extended, by induction to the form

$$\left|\sum_{k=1}^{n} z_k\right|\leqslant\sum_{k=1}^{n}|z_k|\qquad(n=1,2,3,\ldots)$$

Theorem II *The modulus of the difference of two complex numbers can never be less than the difference of their moduli*

Proof Proceeding as in theorem I we get

$$|z_1-z_2|^2=|z_1|^2-2\,R\,(z_1\bar{z}_2)+|z_2|^2$$

$$\geqslant|z_1|-2\,|z_1\bar{z}_2|+|z_2|^2$$

$$=|z_1|^2-2\,|z_1|\,|z_2|+|z_2|$$

$$=(|z_1|-|z_2|)^2$$

so that $|z_1-z_2|\geqslant||z_1|-|z_2||$

$$=\left(\frac{1+2i}{5}+\frac{3(1-i)}{2}\right)\left(\frac{-10+20i}{20}\right)$$

$$=\left(\frac{2+4i+15-15i}{10}\right)\left(\frac{-1+2i}{2}\right)$$

$$=\frac{(17-11i)(-1+2i)}{20}=\frac{5+45i}{20}=\tfrac{1}{4}+\tfrac{9}{4}i$$

(v) $$\frac{(a+ib)^2}{(a-ib)}-\frac{(a-ib)^2}{(a+ib)}=\frac{(a+ib)^3-(a-ib)^3}{(a-ib)(a+ib)}$$

$$=\frac{2(3a^2ib+i^3b^3)}{a^2+b^2}=\frac{2(3a^2b-b^3)}{a^2+b^2}i=\frac{2b(3a^2-b^2)}{a^2+b^2}i$$

(vi) $$\frac{1}{1-\cos\theta+2i\sin\theta}=\frac{1-\cos\theta-2i\sin\theta}{(1-\cos\theta+2i\sin\theta)(1-\cos\theta-2i\sin\theta)}$$

$$=\frac{(1-\cos\theta-2i\sin\theta)}{(1-\cos\theta)^2+4\sin^2\theta)}$$

$$=\frac{2\sin^2\theta/2-4i\sin\theta/2\cos\theta/2}{4\sin^4\theta/2+16\sin^2\theta/2\cos^2\theta/2}$$

$$=-\frac{2\sin^2\theta/2(1-2i\cot\theta/2)}{2\sin^2\theta/2(2\sin^2\theta/2+8\cos^2\theta/2)}$$

$$=\frac{1-2i\cot\theta/2}{1-\cos\theta+4(1+\cos\theta)}=\frac{1-2i\cot\theta/2}{5+3\cos\theta}$$

$$=\frac{1}{5+3\cos\theta}-\frac{2\cot\theta/2}{5+3\cos\theta}i$$

(vii) Given expression

$$=\frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cos u/\sin u+i)(1+i\sin v/\cos v)}$$

$$=\frac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cos u+i\sin u)(\cos v+i\sin v)}\sin u\cos v$$

$$=\frac{\cos(x+y)+i\sin(x+y)}{\cos(u+v)+i\sin(u+v)}\sin u\cos v$$

$$=\frac{[\cos(x+y)+i\sin(x+y)][\cos(u+v)-i\sin(u+v)]}{\cos^2(u+v)+\sin^2(u+v)}\times\sin u\cos v$$

$$=\sin u\cos v\,[\cos(x+y-u-v)+i\sin(x+y-u-v)]$$

(viii) Using De Moivre's Theorem the given expression

$$=\frac{(\cos\theta+i\sin\theta)^{-16}(\cos\theta+i\sin)^{-15}}{(\cos\theta+i\sin\theta)^{46}(\cos\theta+i\sin\theta)^{-30}}$$

$$=(\cos\theta+i\sin\theta)^{-47}=\cos 47\theta-i\sin 47\theta$$

11 First we note that the length of segment joining the points z_1 and z_2 is $|z_1-z_2|$. Therefore

(i) $|5-(-3)|=|8|=8$

(ii) $|3-(-4i)|=|3+4i|=\sqrt{(3^2+4^2)}=\sqrt{(25)}=5$

$\operatorname{Arg}(z_1, z_2 \ldots z_n) = \theta_1 + \theta_2 + \ldots + \theta_n = \arg z_1 + \arg z_2 + \ldots + \arg z_n$
which proves the theorem. In particular, $\arg z^n = n \arg z$

Theorem II *The argument of the quotient of two complex numbers is equal to the difference of their arguments*

Proof We have
$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$
$$= (r_1/r_2)(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)$$
$$= (r_1/r_2)[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)]$$
$$\operatorname{Arg}(z_1/z_2) = \theta_1 - \theta_2 = \arg z_1 - \arg z_2$$

Important Remark The notion of linear ordering, greater than or less than, does not apply to complex numbers. Thus the statement $z_1 > z_2$ and $z_1 < z_2$ have no meaning unless z_1 and z_2 are both real. Since $|z|$, $R(z)$ and $I_m(z)$ are real numbers, the statements like $|z_1| > |z_2|$, $R(z_1) < R(z_2)$ and $I(z_1) > I(z_2)$ are meaningful. Also since $|z|^2 = R^2(z) + I^2(z)$, it is easy to see that

$$|z| \geqslant |R(z)| \geqslant R(z)$$
and
$$|z| \geqslant |I(z)| \geqslant I(z)$$

§ 7 **Cube roots of unity**

Cube roots of unity are clearly the roots of the equation $x^3 - 1 = 0$, that is, of the equation $(x-1)(x^2+x+1)=0$ whence we get
$$x = 1, \frac{-1 \pm \sqrt{3}i}{2}$$

Thus the cube roots of unity are
$$x_1 = 1 \quad x_2 = \frac{-1+\sqrt{3}i}{2} \text{ and } x_3 = \frac{-1-\sqrt{3}i}{2}$$

Of these x_1 is real and x_2, x_3 are complex conjugates. Further it is *easy to see that* $x_2^2 = x_3$ *and* $x_3^2 = x_2$. *That is why we usually denote the complex roots by* ω and ω^2. It is easy to see that $1+\omega+\omega^2=0$ and $\omega^3=1$ so that $\omega^4=\omega^3\omega=\omega$, $\omega^5=\omega^3\omega^2=\omega^2$, $\omega^6=1$ etc.

§ 8 **De Moivre's Theorem**

1 $(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \cos(\alpha+\beta) + i\sin(\alpha+\beta)$

2 $\cos\beta - i\sin\beta = \cos(-\beta) + i\sin(-\beta)$

3 $(\cos\alpha + i\sin\alpha)(\cos\beta - i\sin\beta) = \cos(\alpha-\beta) + i\sin(\alpha-\beta)$

4 $x = \cos\theta + i\sin\theta$ then $\frac{1}{x} = \cos\theta - i\sin\theta$

$\therefore\ x + \frac{1}{x} = 2\cos\theta \quad x - \frac{1}{x} = 2i\sin\theta$

Hence $\sqrt{\{(a^2-1)+2a\sqrt{(-1)}\}} = \pm(a+i)$

Note The device used in (ix) is very useful In general, the square root of numbers of the form $a+2ib$, can be easily found if we can resolve b into two factors such that the difference of their squares is a For example to find the square root of $-7-24i$, we should resolve $b=12$ into two factors the difference of whose squares is $a=-7$ Clearly these factors are 3 and 4 Hence we may write

$$-7-24i = 3^2+(4i)^2-2\cdot 3\cdot 4i = (3-4i)^2,$$

Hence $\sqrt{(-7-24i)} = \pm(3-4i)$

(x) $x^2+1+i\sqrt{(x^4+x^2+1)}$

$= \frac12[2x^2+2+2i\sqrt{\{(x^2+x+1)(x^2-x+1)\}}]$ [Factorizing x^4+x^2+1]

$= \frac12[\{\sqrt{(x^2+x+1)}\}^2-\{\sqrt{(x^2-x+1)}\}^2+2i\sqrt{\{(x^2+x+1)(x^2-x+1)\}}]$

$= \frac12[\sqrt{(x^2+x+1)}+i\sqrt{(x^2-x+1)}]^2$

Hence $\{\ldots\}^{1/2} = \pm\frac{1}{\sqrt2}\{\sqrt{(x^2+x+1)}+i\sqrt{(x^2-x+1)}\}$

(xi) $4ab-2(a^2-b^2)\sqrt{(-1)} = 4ab-2(a+b)(a-b)i$

$= (a+b)^2-(a-b)^2-2(a+b)(a-b)i$

$= [(a+b)-(a-b)i]^2$

Hence $\sqrt{[4ab-2(a^2-b^2)\sqrt{(-1)}]} = \pm[(a+b)-(a-b)i]$

13 (i) Let $1 = r\cos\theta$, $-1 = r\sin\theta$

Squaring and adding these relations, we get

$$r^2(\cos^2\theta+\sin^2\theta) = 1^2+(-1)^2 = 2,$$

or $r^2 = 2$, $r = \sqrt2$

Then $\cos\theta = \frac{1}{\sqrt2}$, $\sin\theta = -\frac{1}{\sqrt2}$

The value of θ between $-\pi$ and π which satisfies these equations is $-\frac{\pi}{4}$

Thus $|1-i| = r = \sqrt2$ and $\arg(1-i) = -\frac{\pi}{4}$

(ii) Ans $|-1-\sqrt3 i| = 2$, $\arg(-1-\sqrt3 i) = -\frac{2\pi}{3}$

(iii) Let $r\cos\theta = 1+\sqrt2$, $r\sin\theta = 1$

Then $r^2 = (1+\sqrt2)^2+1 = 4+2\sqrt2$, $r = \sqrt{(4+2\sqrt2)}$

Now $\cos\theta = \frac{1+\sqrt2}{\sqrt{(4+2\sqrt2)}} = \frac{1+\sqrt2}{\sqrt{\{2\sqrt2(\sqrt2+1)\}}} = \sqrt{\left(\frac{1+\sqrt2}{2\sqrt2}\right)}$

9 Compute

(i) $(1+i)^{-1}$ (ii) $[(\sqrt{3}+i)(\sqrt{3}-i)]^{-2/3}$

(iii) $[(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)]^{-1}$

(iv) $\dfrac{4(\cos 75^\circ+i\sin 75^\circ)}{0.4(\cos 30^\circ+i\sin 30^\circ)}$

10 Put the following in the form $A+iB$

(i) $\dfrac{(1+i)}{3-i}$ (ii) $\dfrac{5+4i}{4+5i}$

(iii) $\dfrac{(3-2i)(2+3i)}{(1+2i)(2-i)}$ (iv) $\left(\dfrac{1}{1-2i}+\dfrac{3}{1+i}\right)\left(\dfrac{3+4i}{2-4i}\right)$ (Roorkee 1979)

(v) $\dfrac{(a+ib)^3}{(a-ib)}-\dfrac{(a-ib)^3}{(a+ib)}$ (vi) $\dfrac{1}{1-\cos\theta+2i\sin\theta}$ (I I T 1978)

(vii) $\dfrac{(\cos x+i\sin x)(\cos y+i\sin y)}{(\cot u+i)(1+i\tan v)}$ (Roorkee 1980)

(viii) $\dfrac{(\cos 2\theta-i\sin 2\theta)^7(\cos 3\theta+i\sin 3\theta)^{-5}}{(\cos 4\theta+i\sin 4\theta)^{12}(\cos 5\theta+i\sin 5\theta)^{-6}}$

11 Find the lengths of the segments connecting the points represented by the following pairs of numbers

(i) $5, 3i$ (ii) [illegible]

(iii) $-6i, 3i$ (iv) [illegible]

(v) $3-2i, 3+5i$

12 Find the square root of the following numbers

(i) [illegible] (ii) [illegible]

(iii) $5+12i$ (iv) [illegible]

(v) $-8-6i$ (Roorkee 1979) (vi) $4\sqrt{(-5)}-1$

(vii) $1+4\sqrt{(-3)}$ (viii) $-7-24i$

(ix) $a^2-1+2a\sqrt{(-1)}$ (x) $x^2+i\sqrt{(x^4+x^2+1)}$

(xi) $4ab-2(a^2-b^2)\sqrt{(-1)}$

13 Find the modulus and the principal value of the argument of the following numbers

(i) $1-i$ (ii) $-1+\sqrt{3}i$ (iii) [illegible]

14 If $(1+i)(1+2i)(1+3i)\ldots(1+ni)=x+iy$, show that $2\cdot 5\cdot 10\ldots(1+n^2)=x^2+y^2$

15 Put the following numbers in trigonometrical form, that is in the form $r(\cos\theta+i\sin\theta)$ where r is a positive real number and $-\pi<\theta\leqslant\pi$

(i) 3 (ii) -5 (iii) $6i$ (iv) $-2i$

Then $r=\sqrt{\{(-1)^2+1^2\}}=\sqrt{2}$, $\cos\theta=-\frac{1}{\sqrt{2}}$, $\sin\theta=\frac{1}{\sqrt{2}}$.

The principal value of θ satisfying these equations is $\frac{3\pi}{4}$

Hence $\frac{1+7i}{(2-i)^2}=r(\cos\theta+i\sin\theta)=\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)$

(vii) $\frac{(1+i)^{2n+1}}{(1-i)^{2n-1}}=\frac{(1+i)^{2n+1}(1+i)^{2n-1}}{(1^2-i^2)^{2n-1}}=\frac{(1+i)^{4n}}{2^{2n-1}}=\frac{[(1+i)^2]^{2n}}{2^{2n-1}}$

$$=\frac{(2i)^{2n}}{2^{2n-1}}=2\,i^{2n}=2(-1)^n$$

$=2=2(\cos 0°+i\sin 0°)$ if n is even
$=-2=2(\cos\pi+i\sin\pi)$ if n is odd

(viii) Let $r\cos\theta=1$, $r\sin\theta=\tan\alpha$

Then $r=\sqrt{\{1+\tan^2\alpha\}}=|\sec\alpha|=\frac{1}{|\cos\alpha|}$

Then $\cos\theta=|\cos\alpha|$ (1), $\sin\theta=\tan\alpha\,|\cos\alpha|$ (2)

To solve (1) and (2), we consider two cases

(a) $\cos\alpha>0$, that is, α lies in the interval $-\pi/2<\alpha<\pi/2$

In this case $|\cos\alpha|=\cos\alpha$, and the equations (1) and (2) take the form $\cos\theta=\cos\alpha$, $\sin\theta=\sin\alpha$

Clearly one of the values of θ is α

Hence for $-\pi/2<\alpha<\pi/2$, the trigonometric form is

$$1+i\tan\alpha=\frac{1}{\cos\alpha}(\cos\alpha+i\sin\alpha)$$

(b) $\cos\alpha<0$, that is, α lies in the interval $-\pi<\alpha<-\frac{\pi}{2}$

or in the interval $\frac{\pi}{2}<\alpha<\pi$ In this case, $|\cos\alpha|=-\cos\alpha$ and the equation (1) and (2) take the form

$$\cos\theta=-\cos\alpha,\ \sin\theta=-\sin\alpha$$

One value of θ from these is, $\theta=\pi+\alpha$

Hence for $-\pi<\alpha<-\frac{\pi}{2}$ and $\frac{\pi}{2}<\alpha<\pi$ the trigonometric form is

$$1+i\tan\alpha=\frac{1}{-\cos\alpha}[\cos(\pi+\alpha)+i\sin(\pi+\alpha)]$$

Remark. Students may think that the expression can be written as

$$1+i\tan\alpha=1+\frac{i\sin\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}(\cos\alpha+i\sin\alpha)$$

(ii) $(x^2+a^2)^7=(x^7-21x^5a^2+35x^3a^4-7xa^6)^2$
$+(7x^6a-35x^4a^3+21x^2a^5-a^7)^2$

24 If $1, \omega, \omega^2$ are the three cube roots of unity, show that

(i) $(1-\omega+\omega^2)(1+\omega-\omega^2)=4$ (ii) $(1+\omega)^3-(1+\omega^2)^3=0$

(iii) $(1-\omega+\omega^2)^3=(1+\omega-\omega^2)^3=-8$

(iv) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=32$ (I I T 1965)

(v) $(2+5\omega+2\omega^2)^6=(2+2\omega+5\omega^2)^6=729$

(vi) $(1+\omega)(1+\omega^2)(1+\omega^4)(1+\omega^8)=1$

(vii) $(1-\omega)(1-\omega^2)(1-\omega^4)(1-\omega^8)=9$

(viii) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)$ to $2n$ factors$=2^{2n}$

(ix) $x^3+y^3=(x+y)(\omega x+\omega^2 y)(\omega^2 x+\omega y)$

(x) $x^3-y^3=(x-y)(\omega x-\omega^2 y)(\omega^2 x-\omega y)$

(xi) $(x+y)^2+(x\omega+y\omega^2)^2+(x\omega^2+y\omega)^2=6xy$

(xii) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)=a^3+b^3+c^3-3abc$

25 If α and β are the complex cube roots of unity, show that

$$\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=0 \qquad \text{(I I T 1977)}$$

26 If $x=a+b$, $y=a\alpha+b\beta$ and $z=a\beta+b\alpha$ where α and β are complex cube roots of unity, show that

$$xyz=a^3+b^3 \qquad \text{(I I T 1978)}$$

27 If $x=a+b$, $y=a\omega+b\omega^2$, $z=a\omega^2+b\omega$, prove that

$$x^3+y^3+z^3=3(a^3+b^3) \qquad \text{(Roorkee 1977)}$$

28 Prove that

(i) $\left(\frac{\sqrt{3}+i}{2}\right)^6+\left(\frac{i-\sqrt{3}}{2}\right)^6=-2$

(ii) $\left(\frac{-1+\sqrt{3}i}{2}\right)^n+\left(\frac{-1-\sqrt{3}i}{2}\right)^n=-1$

when n is a positive integer but not a multiple of 3

(iii) $z^{14}+1/z^{14}=-1$ where z is a root of the equation

$$z+1/z=1$$

29 Prove that

(i) $1+\omega^n+\omega^{2n}=0$ when n is a positive integer but not a multiple of 3

(ii) $1+\omega^n+\omega^{2n}=3$ when n is a multiple of 3

30 It is given that n is an odd integer greater than 3 but n is not a multiple of 3 Prove that x^3+x^2+x is a factor of

$$(x+1)^n-x^n-1 \qquad \text{(I I T 1980)}$$

31 Prove that

$(x+y)^n-x^n-y^n$ is divisible by $xy(x+y)(x^2+xy+y^2)$

if n is odd but not a multiple of 3

32 (a) Show that the polynomial

$$x^{4l}+x^{4m+1}+x^{4n+2}+x^{4p+3}$$

$$z_1=2\sin\frac{\phi}{2}\left[\cos\left(\frac{\pi+3\phi}{2}\right)+i\sin\left(\frac{\pi+3\phi}{2}\right)\right]$$

Consequently, if ϕ satisfies the condition (1), then

$$\arg z_1=\frac{\pi+3\phi}{2}$$

Case III Let $\sin\frac{\phi}{2}<0$, that is,

$(2n+1)\pi \leqslant \frac{1}{2}\phi < (2n+2)\pi$

or $(4n+2)\pi < \phi < (4n+4)\pi$, (2)

n, any integer

Then $|z_1| = -2\sin\frac{\phi}{2}$ and the trigonometrical form of z_1 is

$$z_1=-2\sin\frac{\phi}{2}\left[\cos\left(\frac{3\pi+3\phi}{2}\right)+i\sin\left(\frac{3\pi+3\phi}{2}\right)\right]$$

Hence, when ϕ satisfies (2), we have

$$\arg z_1=\left(\frac{3\pi+3\phi}{2}\right)$$

17 (i) $(1-i)x+(1+i)y=1-3i$

Equating real and imaginary parts, we get

$x+y=1$ and $-x+y=-3$

Solving these, we get $x=2$, $y=-1$

(ii) Ans $x=-4$, $y=6$

(iii) $\dfrac{(1+i)x-2i}{3+i}+\dfrac{(2-3i)y+i}{3-i}=i$

or $(1+i)(3-i)x-2i(3-i)+(2-3i)(3+i)=y+i(3+i)$

$=i(3+i)(3-i)$

or $(4+2i)x-6i-2+(9-7i)y+3i-1=10i$

or $(4x+9y-3)+i(2x-7y-3)=10i$

Equating real and imaginary parts,

$4x+9y-3=0$ (1)

and $2x-7y-3=10$ or $2x-7y-13=0$ (2)

Solving (1) and (2), we get $x=3$ $y=-1$

(iv) $(x+iy)(2-3i)=4+i$

or $(2x+3y)+i(-3x+2y)=4+i$

Equating real and imaginary parts,

$2x+3y=4$, $-3x+2y=1$

Solving these, we get

$x=(5/13)$, $y=(14/13)$

(v) Ans $x=5$, $y=2$ or $x=5$, $y=-2$

(i) $|z| < 1$, (ii) $|z| > 3$,
(iii) $|z-3| = 1$, (iv) $|z-i| < 1$,
(v) $|i-1-2z| \geq 9$, (vi) $2 \leq |z+i| \leq 3$,
(vii) $|z+i| = |z-2|$, (viii) $|z|-4 = |z-i| - |z+5i| = 0$,
(ix) $|z-1| = |z-3| = |z-i|$, (x) $|z-1|^2 + |z+1|^2 = 4$

41 If $z = x + iy$, $r = \sqrt{(x^2+y^2)} =$ const, what is the location of the points corresponding to
(a) $z+2$ (b) $z-1+i$

42 If $|z| = 3$, then where are the points representing the numbers
(a) $2-z$, (b) $-1+3z$
located?

43 Locate the complex numbers $z = x + iy$ for which
(i) $\log_{1/2}|z-2| \geq \log_{1/2}|z|$
(ii) $\log_{\sqrt{3}} \dfrac{|z|^2 - |z| + 1}{2+|z|} < 2$

44 Find the integral solutions of the equations
(i) $(1-i)^x = 2^x$, (ii) $(1+i)^x = (1-i)^x$

45 Prove that the sum and product of two complex numbers are real if and only if they are conjugate of each other (I I T 1977)

46 (a) Among the complex numbers z which satisfy the condition $|z-25i| \leq 15$, find the numbers having the least positive argument
(b) Find the greatest value of the moduli of complex numbers z satisfying the equation $|z - \frac{4}{z}| = 2$
(c) Find the complex number z, the least in absolute value which satisfies the condition $|z-2+2i| = 1$

47 For every real number $c \geq 0$ find all complex numbers z which satisfy the equation
$$|z|^2 - 2iz + 2c(1+i) = 0$$

48 For every real number $c \geq 1$, find all the complex numbers z that satisfy the equation
$$z + c|z+1| + i = 0$$

49 (a) For every real number $c \geq 0$, find all the complex numbers z which satisfy the equation
$$2|z| - 4cz + 1 + ic = 0$$
(b) For every real number $c > 0$, find all complex numbers z satisfying the equation $z|z| + cz + i = 0$

50 Prove the following
(i) $(\cos 60^\circ + i \sin 60^\circ)^6 = 1$
(ii) $[\sqrt{2}(\cos 56^\circ 15' + i \sin 56^\circ 15')]^8 = 16i$
(iii) $(\sin\theta + i\cos\theta)^n = \cos n(\frac{1}{2}\pi - \theta) + i \sin n(\frac{1}{2}\pi - \theta)$

One root of this equation is given as $2+\sqrt{3}i$. Since the complex roots occur in conjugate pairs the other root must be $2-\sqrt{3}i$.

Hence $(x-2-\sqrt{3}i)(x-2+\sqrt{3}i)$ is a factor of L H S of (1)
But $(x-2-\sqrt{3}i)(x-2+\sqrt{3}i)=(x-2)^2+3=x^2-4x+7$
Then the other quadratic factor of L H S of (1) is of the form $\qquad x^2+px+5$

Hence we have the identity

$$x^4-4x^3+8x+35\equiv(x^2-4x+7)(x^2+px+5)$$

Equating the coefficient of x on both sides of the above identity we get $\qquad 8=7p-20$ or $p=4$

[Note that same value of p will be obtained by equating the coefficient of x^3]

Hence the other two roots of the equation are the roots of the equation $\qquad x^2+4x+5=0$

or $\qquad x=\dfrac{-4\pm\sqrt{(16-20)}}{2}=\dfrac{-4\pm\sqrt{-4}}{2}=-2\pm i$

20 $\quad \dfrac{3+2i\sin\theta}{1-2i\sin\theta}=\dfrac{(3+2i\sin\theta)(1+2i\sin\theta)}{1-4i^2\sin^2\theta}$

$$=\frac{(3-4\sin^2\theta)+8i\sin\theta}{1+4\sin^2\theta} \qquad (1)$$

(i) The expression in (1) is real if $\sin\theta=0$
This gives $\theta=n\pi$ where n is an integer
(ii) If the expression in (1) is purely imaginary, then

$$3-4\sin^2\theta=0 \quad i\,e \quad \sin^2\theta=\frac{3}{4}=\sin^2\frac{\pi}{3}$$

This gives $\theta=n\pi\pm\frac{\pi}{3}$, where n is an integer

21 $\quad$ (a) Let O be the origin and A_1 the vertex z_1. Let the vertex adjacent to A_1 be A_2.

Then $z_2=z_1e^{2\pi i/n}$ since $\angle A_2OA_1=2\pi/n$ (see remark (i) § 3)
Similarly if $z_3, z_4, \ldots z_n$ are the other vertices in order, then $z_3=z_1e^{4\pi i/n}$, $z_4=z_1e^{6\pi i/n}$ etc. Thus all the vertices are given by $z_k=z_1e^{2\pi ki/n}=z_1(\cos 2\pi k/n+i\sin 2\pi k/n)$

$$k=0, 1, 2, \ldots, n-1$$

(b) Let C be the centre of the polygon and A_1, A_2, A_3, the vertices represented by z_1, z_2 and z_3 respectively. Note that here C is not the origin prove that $1+\beta+\beta^2+ \quad +\beta^{n-1}=0$

59 (*a*) Find the n, nth roots of unity and prove that the sum of their pth powers vanishes unless p be a multiple of n, p being an integer, and that then the sum is n

(*b*) Find the seven seventh roots of unity and prove that sum of their nth powers always vanishes unless n be a multiple of seven, n being an integer and then the sum is seven

The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$ is

(i) -1 (ii) 0 (iii) $-i$ (iv) None (I I T 87)

60 Assume that A_i ($i=1, 2, \quad n$) are the vertices of a regular n gon inscribed in a circle of radius unity

Find (i) $|A_1A_2|^2+|A_1A_3|^2+ \quad +|A_1A_n|^2$,

(ii) $|A_1A_2| \quad |A_1A_3| \quad |A_1A_n|$

61 Prove that following inequality

$[(a_1+a_2+ \quad a_n)^2+(b_1+b_2+ \quad +b_n)^2]^{1/2}$

$< \sqrt{(a_1^2+b_1^2)}+\sqrt{(a_2^2+b_2^2)}+ \quad +\sqrt{(a_n^2+b_n^2)}$

where a_r, b_r ($r=1, 2 \quad n$) are real,

62 (*a*) Prove that

$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2$

Interpret the result geometrically and deduce that

$|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|=|\alpha+\beta|+|\alpha-\beta|$,

all numbers involved being complex

(*b*) Prove that $|z_1|+|z_2| = |\frac{1}{2}(z_1+z_2)+\sqrt{(z_1z_2)}|$
$+|\frac{1}{2}(z_1+z_2)-\sqrt{(z_1z_2)}|$

63, (i) Prove that the area of the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argand diagram is $\Sigma[(z_2-z_3)|z_1|^2/4i\,z_1]$

(ii) Show that area of the triangle on the Argand diagram formed by the complex numbers z, iz and $z+iz$ is $\frac{1}{2}|z|^2$

(I I T 1986)

64 Show that the triangle whose vertices are the points represented by the complex numbers z_1, z_2, z_3 on the Argand diagram is equilateral if and only if

$$\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2}=0$$

that is, iff $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

(ii) If $|z_1|=|z_2|=|z_3|=1$ and $z_1+z_2+z_3=0$, show that

$=(x^4-6x^2a^2+a^4)+i(4x^3a-4xa^3)$ (1)

Similarly $(x-ai)^4=(x^4-6x^2a^2+a^4)-i(4x^3a-4xa^3)$ (2)

Multiplying (1) and (2) we get

$(x+ai)^4(x-ai)^4=(x^4-6x^2a^2+a^4)^2-i^2(4x^3a-4xa^3)^2$

or $(x^2+a^2)^4=(x^4-6x^2a^2+a^4)^2+(4x^3a-4xa^3)^2$

(ii) Similar to (i) Do yourself

(i) Using the relation $1+\omega+\omega^2=0$ and $\omega^3=1$, we get

$(1-\omega+\omega^2)+(1+\omega-\omega^2)=(-2\omega)(-2\omega^2)=4\omega^3=4$

(ii) Do yourself (iii) Do yourself

(iv) $(1-\omega+\omega^2)^5+(1+\omega-\omega^2)^5=(-2\omega)^5+(-2\omega^2)^5$

$=-32(\omega^5+\omega^{10})=-32(\omega^3\omega^2+\omega^9\omega)$

$=-32(\omega^2+\omega)=-32(-1)=32$

(v) Do yourself (vi) Do yourself

(vii) L H S $=(1-\omega)(1-\omega^2)(1-\omega^3\omega)(1-\omega^3\omega^2)$

$=[(1-\omega)(1-\omega^2)]^2=(1-\omega-\omega^2+\omega^3)^2=(1+1+1)^2=9$

$[\because\ \omega+\omega^2=-1,\ \omega^3=1]$

(viii) $(1-\omega+\omega^2)(1-\omega^2+\omega^4)(1-\omega^4+\omega^8)(1-\omega^8+\omega^{16})$ to $2n$ factors

$=(1-\omega+\omega^2)(1-\omega^2+\omega)(1-\omega+\omega^2)(1-\omega^2+\omega)$ to $2n$ factors

$[\because\ \omega^4=\omega,\ \omega^8=\omega^2,\ \omega^{16}=\omega$ etc$]$

$=(-2\omega)(-2\omega^2)(-2\omega)(2\omega^2)$ to $2n$ factors

$=(2^2\omega^3)(2^2\omega^3)(2^2\omega^3)$ to n factors

$[\because\ (-2\omega)(-2\omega^2)=2^2\omega^3=2^2]$

$=(2^2)^n=2^{2n}$

(ix) Do yourself

(x) Do yourself

(xi) Expanding, we get

L H S $=x^2(1+\omega^2+\omega^4)+y^2(1+\omega^4+\omega^2)+2xy(1+\omega^3+\omega^3)$

$=x^2(1+\omega^2+\omega)+y^2(1+\omega+\omega^2)+2xy(1+1+1)$

$=6xy$ $[\because\ 1+\omega+\omega^2=0]$

(xii) $(a+b+c)(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$

$=(a+b+c)[a^2+b^2\omega^3+c^2\omega^3+ab(\omega+\omega^2)$

$+bc(\omega^2+\omega^4)+ca(\omega+\omega^2)]$

$=(a+b+c)(a^2+b^2+c^2-ab-bc-ca]$

$[\because\ \omega^4=\omega$ and $\omega+\omega^2=-1]$

$=a^3+b^3+c^3-3abc$

5 Since α β are the complex roots of unity, we may write

prove that $1+\beta+\beta^2+ \quad +\beta^{n-1}=0$

59 (a) Find the n, nth roots of unity and prove that the their pth powers vanishes unless p be a multiple of n, an integer, and that then the sum is n

(b) Find the seven seventh roots of unity and prove of their nth powers always vanishes unless n be a seven, n being an integer and then the sum is sev

The value of $\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)$ is

(i) -1 (ii) 0 (iii) $-i$ (iv) No

60 Assume that A_i ($i=1, 2, \quad n$) are the ve n-gon inscribed in a circle of radius unity

Find (i) $|A_1A_2|^2+|A_1A_3|^2+ \quad +|A_1A_n$

(ii) $|A_1A_2| \quad |A_1A_3| \quad |A_1A_n|$

61 Prove that following inequality

$[(a_1+a_2+ \quad a_n)^2+(b_1+b_2+ \quad +b_n)^2]$

$< \sqrt{(a_1^2+b_1^2)}+\sqrt{(a_2^2+b_2^2)}+ \quad +\sqrt{}$

where a_r, b_r ($r=1$ 2 n) are re

62 (a) Prove that

$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+$

Interpret the result geometric

$|\alpha+\sqrt{(\alpha^2-\beta^2)}+|\alpha-\sqrt{(\alpha^2-}$

all numbers involved being

(b) Prove that $|z_1|+ \quad |$

$+|\frac{1}{2}(z_1+z_2)-\sqrt{(z_1z_2)}|$

63 (i) Prove that the area points represented by Argand diagram is Σ

(ii) Show that are formed by the com

64 Show that the tr cated by the com gram is equilater

$\frac{1}{z_1-z_2}+\frac{1}{z_2-z_1}+$

that is, iff z_1^2+

(ii) If $|z_1|=|z_2|$

3, we have

$$n=3m+1 \quad \text{or} \quad n=3m+2$$

where m is a +ive integer

when $n=3m+1$, we have

$$1+\omega^n+\omega^{2n}=1+\omega^{3m+1}+\omega^{6m+2}=1+\omega^{3m}\,\omega+\omega^{6m}\,\omega^2$$
$$=1+\omega+\omega^2 \quad [\because \omega^{3m}=\omega^{6m}=1]$$
$$=0$$

Similarly when $n=3m+2$, we can prove the result

(ii) Since n is a multiple of 3, we have

$$\omega^n=1 \text{ and } \omega^{2n}=1$$

Hence $1+\omega^n+\omega^{2n}=1+1+1=3$

30 We have $x^3+x^2+x=x\,(x^2+x+1)=x\,(x-\omega)\,(x-\omega^2)$

Let $f(x)=(x+1)^n-x^n-1$

In order to show that x^3+x^2+x is factor of $f(x)$, we must show that $f(x)=0$ when $x=0$, $x=\omega$ and $x=\omega^2$

Now
$$f(0)=(0+1)^n-0-1=1-1=0$$
$$f(\omega)=(\omega+1)^n-\omega^n-1=(-\omega^2)^n-\omega^n-1$$
$$=(-1)^n\,\omega^{2n}-\omega^n-1=-\omega^{2n}-\omega^n-1$$
$$[\because (-1)^n=-1 \text{ since } n \text{ is odd integer}]$$
$$=-(1+\omega^n+\omega^{2n})=0 \qquad \text{[See ex 29 (i)]}$$

Similarly $f(\omega^2)=(\omega^2+1)^n-(\omega^2)^n-1=(-\omega)^n-\omega^{2n}-1$
$$=-\omega^n-\omega^{2n}-1=0$$

Hence $x\,(x-\omega)\,(x-\omega^2)$, that is, x^3+x^2+x is a factor of $f(x)$

31 We have

$xy\,(x+y)\,(x^2+xy+y^2)=xy\,(x+y)\,(y-x\omega)\,(y-x\omega^2)$

Let $f(x, y)=(x+y)^n-x^n-y^n$

Considering the given expression $f(x, y)$ as a polynomial in y, we put $y=0$

We see that at $y=0$, the polynomial $f(x, y)$ becomes $x^n-x^n=0$ i e it becomes 0 for any x. Therefore $f(x, y)$ is divisible by y. Similarly $f(x, y)$ is divisible by x. Thus $f(x, y)$ is divisible by xy

To prove that $f(x, y)$ is divisible by $x+y$, we put $y=-x$. In this case $f(x, y)$ becomes $(x-x)^n-x^n-(-x)^n$

$$=0-x^n+x^n \qquad [\because n \text{ is odd, we have } (-x)^n=-x^n]$$
$$=0$$

Consequently our polynomial is divisible by $x+y$

It remains to prove that $f(x, y)$ is divisible by

$$y-x\omega \text{ and } y-x\omega^2$$

z_1, z_2, z_3 are the vertices of ... triangle inscribed in a unit circle

$\frac{a+ib}{1+c}=\frac{1+iz}{1-iz}$

where a, b, c are real numbers and z a complex number

If A, B, C, D are four points in a plane, prove that

$$AD\ BC \leqslant BD\ CA + CD\ AB$$

(IIT 1980)

It is known that $\left|z+\frac{1}{z}\right|=a$ where z is a complex number.

What are the greatest and the least possible values of $|z|$?

74 Let z_1, z_2 be any two complex numbers and a, b two real numbers $(a^2+b^2\neq0)$. Prove the inequalities

75 Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. If z_0 is the circumcentre of the triangle then prove that $z_1^2+z_2^2+z_3^2=3z_0^2$ (IIT 1981)

76 The complex numbers z_1, z_2, z_3 are the vertices of a triangle. Find all the complex numbers z such that the triangle ... into a parallelogram

Solutions

(a) sum is real if $y_1+y_2=0$

(b) a purely imaginary number if $x_1+x_2=0$

2 We first note that the formula $\sqrt{(ab)}=\sqrt{a}\sqrt{b}$ holds if at least one of a and b is non negative

(a) Since here both a and b are negative, we cannot apply the above formula. Thus if $a=-1$ and $b=-1$, we cannot have

$$\sqrt{\{(-1)(-1)\}}=\sqrt{\{(-1)\}}\sqrt{\{(-1)\}}$$

which gives the absurd result $1=-1$

(b) The computation

$$\sqrt{(-2)}\sqrt{(-3)}=\sqrt{[(-2)(-3)]}=\sqrt{6}$$

is wrong since here both -2 and -3 are negative. The correct computation is

$$\sqrt{(-2)}\sqrt{(-3)}=i\sqrt{2}\, i\sqrt{3}=i^2\sqrt{6}=\sqrt{6}(-1)=-\sqrt{6}$$

3 $(x+iy)=\sqrt{\left\{\frac{a+ib}{c+id}\right\}}$ (1)

By the property of complex numbers we have

$$x-iy=\sqrt{\left\{\frac{a-ib}{c-id}\right\}} \quad (2)$$

Multiplying (1) and (2), we get

$$x^2+y^2=\sqrt{\left\{\frac{a+ib}{c+id}\right\}\left\{\frac{a-ib}{c-id}\right\}}$$

squaring we get

$$(x^2+y^2)^2=\frac{a+ib}{c+id}\cdot\frac{a-ib}{c-id}=\frac{a^2+b^2}{c^2+d^2}$$

And by part (i) we have

$$(a^2+b^2+c^2-bc-ca-ab)\,(x^2+y^2+z^2-yz-zx-xy)$$
$$=X^2+Y^2+Z^2-YZ-ZX-XY \qquad (3)$$

Substituting from (2) and (3) in (1), we get

$$(a^3+b^3+c^3-3abc)\,(x^3+y^3+z^3-3xyz)$$
$$=(X+Y+Z)\,(X^2+Y^2+Z^2-YZ-ZX-XY)$$
$$=X^3+Y^3+Z^3-3XYZ$$

34 We are given

$$z_1+z_2+z_3=A \qquad (1)$$
$$z_1+z_2\omega+z_3\omega^2=B \qquad (2)$$
$$z_1+z_2\omega^2+z_3\omega=C \qquad (3)$$

(i) Adding (1), (2) and (3), we get

$$3z_1+z_2\,(1+\omega+\omega^2)+z_3\,(1+\omega^2+\omega)=A+B+C$$

or $$z_1=\frac{A+B+C}{3} \qquad [\because\ 1+\omega+\omega^2=0]$$

Now multiplying (1), (2), (3) by $1, \omega^2, \omega$ respectively and adding, we get

$$z_1\,(1+\omega^2+\omega)+z_2\,(1+\omega^3+\omega^3)+z_3\,(1+\omega^4+\omega^2)$$
$$=A+B\omega^2+C\omega$$

or $$z_2=\frac{A+B\omega^2+C\omega}{3}$$

$$[\because\ 1+\omega^4+\omega^2=1+\omega+\omega^2=0 \text{ and } \omega^3=1]$$

Similarly $z_3=\dfrac{A+B\omega+C\omega^2}{3}$

(ii) We have

$$|A|^2+|B|^2+|C|^2=A\bar{A}+B\bar{B}+C\bar{C} \qquad (1)$$

But $A\bar{A}=(z_1+z_2+z_3)\,(\bar{z}_1+\bar{z}_2+\bar{z}_3)$

$$=z_1\bar{z}_1+z_2\bar{z}_2+z_3\bar{z}_3+\bar{z}_1\,(z_2+z_3)+\bar{z}_2\,(z_3+z_1)+\bar{z}_3\,(z_1+z_2)$$
$$=|z_1|^2+|z_2|^2+|z_3|^2+\bar{z}_1\,(z_2+z_3)+\bar{z}_2\,(z_3+z_1)+\bar{z}_3\,(z_1+z_2)$$

$$B\bar{B}=(z_1+z_2\omega+z_3\omega^2)\,(\bar{z}_1+\bar{z}_2\bar{\omega}+\bar{z}_3\bar{\omega}^2)$$
$$=(z_1+z_2\omega+z_3\omega^2)\,(\bar{z}_1+\bar{z}_2\omega^2+\bar{z}_3\omega)$$

$$[\because\ \bar{\omega}=\omega^2 \text{ and } (\overline{\omega^2})=\omega]$$

$$=z_1\bar{z}_1+z_2\bar{z}_2\omega^3+z_3\bar{z}_3\omega^3+\bar{z}_1\,(z_2\omega+z_3\omega^2)+\bar{z}_2\,(z_3\omega^4+z_1\omega^2)+\bar{z}_3\,(z_1\omega+z_2\omega^2)$$
$$=|z_1|^2+|z_2|^2+|z_3|^2+\bar{z}_1\,(z_2\omega+z_3\omega^2)+\bar{z}_2\,(z_3\omega+z_1\omega^2)+\bar{z}_3\,(z_1\omega+z_2\omega^2) \qquad ...(2)$$

(vi) $\frac{3-i}{2+i}+\frac{3+i}{2-i}=\frac{(3-i)(2-i)+(3+i)(2+i)}{(2+i)(2-i)}$

$=\frac{(6-5i+i^2)+(6+5i+i^2)}{2^2-i^2}=\frac{2(6-1)}{4+1}=2$

(vii) Do yourself Ans -2

(viii) Ans $\frac{1}{10}(17-9i)$

(ix) Ans i

(x) $\frac{\sqrt{(5+12i)}+\sqrt{(5-12i)}}{\sqrt{(5+12i)}-\sqrt{(5-12i)}}=\frac{[\sqrt{(5+12i)}+\sqrt{(5-12i)}]^2}{(5+12i)-(5-12i)}$

$=\frac{(5+12i)+(5-12i)+2\sqrt{[(5+12i)(5-12i)]}}{24i}$

$=\frac{10+2\sqrt{(169)}}{24i}=\frac{36}{24i}=-\frac{3}{2}i$

[Note that $1/i=i^4/i=i^3=-i$]

9 (i) $(1+i)^{-1}=\frac{1}{1+i}=\frac{1-i}{1-i^2}=\frac{1-i}{2}$

(ii) $[(\sqrt{3}+i)(\sqrt{3}-i)]^{-3/2}=(3-i^2)^{-3/2}=4^{-3/2}=\frac{1}{8}$

(iii) $[(\cos\theta+i\sin\theta)(\cos\theta-i\sin\theta)]^{-1}=(\cos^2\theta-i^2\sin^2\theta)^{-}$

$=(\cos^2\theta+\sin^2\theta)^{-1}=1$

(iv) $\frac{4(\cos 75^\circ+i\sin 75^\circ)}{0\,4(\cos 30^\circ+i\sin 30)}$

$=\frac{10(\cos 75^\circ+i\sin 75^\circ)(\cos 30^\circ-i\sin 30^\circ)}{\cos^2 30^\circ-i^2\sin^2 30^\circ}$

$=\frac{10[\cos(75^\circ-30^\circ)+i\sin(75^\circ-30^\circ)]}{\cos^2 30^\circ+\sin^2 30^\circ}$

$=10[\cos 45^\circ+i\sin 45^\circ]=\frac{10}{\sqrt{2}}(1+i)$

10 $\frac{(1+i)^2}{3-i}=\frac{1+i^2+2i}{3-i}=\frac{2i(3+i)}{3^2-i^2}$

$=\frac{2(3i+i^2)}{10}=\frac{1}{5}(3i-1)=-\frac{1}{5}+\frac{3}{5}i$

(ii) Do yourself Ans $\frac{40}{41}-\frac{9}{41}i$

(iii) Do yourself Ans $\frac{63}{25}-\frac{16}{25}i$

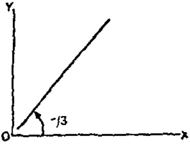
(iv) $\left(\frac{1}{1-2i}+\frac{3}{1+i}\right)\left(\frac{3+4i}{2-4i}\right)$

$=\left(\frac{1+2i}{1-4i^2}+\frac{3(1-i)}{1-i^2}\right)\left(\frac{(3+4i)(2+4i)}{(4-16i^2)}\right)$

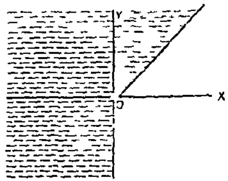
whence $x^2+y^2-1=0$, that is, $|z|=\sqrt{(x^2+y^2)}=1$. This completes the proof.

37 (i) The condition $\arg z=\pi/3$ is satisfied by all points lying on a ray emanating from the origin at an angle of $\pi/3$ to the x-axis. (See figure)

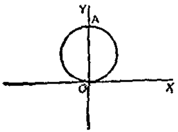
We emphasize that this condition is not satisfied by the entire straight line but only by the ray (excluding the origin). For on the other side of the line, the $\arg z$ is $-(2\pi/3)$ and not $\pi/3$. We also exclude the origin since argument of 0 is undefined.



(ii) Here the points z satisfying the condition
$$\pi/3 < \arg z \leqslant 3\pi/2$$
lie on the portion of the Argand plane containing the second and third quadrants including the entire y-axis except the origin, and a portion of the first quadrant located between rays emerging from origin at angles of $\pi/3$ and $\pi/2$ (See the figure)



(iii) Here the equation $|z-i|=1$ represents all points z on the boundary of a circle whose centre is i and radius 1. And the equation $\arg z=\pi/2$ represents all points on the positive side of y axis *excluding the origin*. *These two regions* have only one common point A whose coordinates are (0, 2)



[See the figure]

Hence the two equations $|z-i|=1$ and $\arg z=\pi/2$ represent only one point A (0, 2)

(iv) $|\pi-\arg z|<\pi/4 \Rightarrow -\pi/4<\pi-\arg z<\pi/4$
$\Rightarrow -5\pi/4<-\arg z<-3\pi/4 \Rightarrow 5\pi/4>\arg z>3\pi/4$
$\Rightarrow 3\pi/4<\arg z<5\pi/4$

Now plot the region yourself

38 (i) Let $z=x+iy$. Then $\bar{z}=x-iy$

(iii) $|-6i-3i| = |-9i| = 9$

(iv) $|(-1-i)-(2+3i)| = |-3-4i| = \sqrt{[(-3)^2+(-4)^2]} = \sqrt{(25)} = 5$

(v) $|(3-2i)-(3+5i)| = |-7i| = 7$

12 (i) Let $\sqrt{i} = x+iy$. Then $i = x^2-y^2+2ixy$

$x^2-y^2=0$ and $2xy=1$

Solving these, we get $x=y=\pm\frac{1}{\sqrt{2}}$

Hence $\sqrt{i} = \pm\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right) = \pm\frac{1}{\sqrt{2}}(1+i)$

(ii) Ans $\sqrt{-i} = \pm\frac{1}{\sqrt{2}}(1-i)$

(iii) Let $\sqrt{(5+12i)} = x+iy$

Then $5+12i = x^2-y^2+2ixy$

$x^2-y^2 = 5$ (1)

and $2xy = 12$ or $xy = 6$ (2)

Hence $(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 = 25+144 = 169$

$x^2+y^2 = 13$ (3)

Note that x^2+y^2 cannot be negative

Solving (1) and (3), we get $x=\pm 3$, $y=\pm 2$

Since product xy is +ve, we take $x=3$, $y=2$

or $x=-3$, $y=-2$

Hence $\sqrt{(5+12i)} = \pm(3+2i)$

(iv) Ans $\sqrt{(1-i)} = \pm\left[\sqrt{\left\{\frac{\sqrt{2}+1}{2}\right\}} - i\sqrt{\left\{\frac{\sqrt{2}-1}{2}\right\}}\right]$

(v) Let $\sqrt{(-8-6i)} = x+iy$

Then $-8-6i = x^2-y^2+2ixy$

$x^2-y^2 = -8$ (1)

$2xy = -6$ or $xy = -3$ (2)

$(x^2+y^2)^2 = (x^2-y^2)^2+4x^2y^2 = 64+36 = 100$

$x^2+y^2 = 10$ (3)

Solving (1) and (3), $x=\pm 1$, $y=\pm 3$

Since xy is −ve, we take opposite signs for x and y, that is we take $x=1$, $y=-3$ or $x=-1$, $y=3$

Thus $\sqrt{(-8-6i)} = \pm(1-3i)$

(vi) Ans $\pm(2+\sqrt{5}i)$

(vii) Ans $\pm(2+\sqrt{3}i)$

(viii) Ans $\pm(3-4i)$

(xi) To find the square root, we use the following device

$a^2-1+2a\sqrt{-1} = a^2+i^2+2ai = (a+i)^2$

But x and y are real. Hence the only value of z satisfying the given equation is $z=0$

(vi) Putting $z=x+iy$, the equation $z^2+|z|=0$

becomes $(x+iy)^2+|x+iy|=0$

or $(x+iy)^2+\sqrt{(x^2+y^2)}=0$

or $(x+iy)^4=x^2+y^2$

or $x^4+4x^3iy+6x^2i^2y^2+4xi^3y^3+i^4y^4=x^2+y^2$

or $(x^4-6x^2y^2+y^4)+i(4x^3y-4xy^3)=x^2+y^2$

Hence $x^4-6x^2y^2+y^4=x^2+y^2$ (1)

and $4x^3y-4xy^3=0$ i.e. $xy(x^2-y^2)=0$ (2)

From (2) $x=0$ or $y=0$ or $y=x$ or $y=-x$

When $x=0$, then (1) gives $y^4-y^2=0$ or $y=0, \pm 1$

When $y=0$, then (1) gives $x^4-x^2=0$ or $x=0, \pm 1$

When $y=\pm x$, then (1) gives $-4x^4=2x^2$ (3)

Now only real value of x from (3) is $x=0$

Thus we get the following value of $z=(x, y)$,

$z_1=(0, 0)=0,\ z_2=(0, 1)=i,\ z_3=(0, -1)=-i$

$z_4=(1, 0)=1,\ z_5=(-1, 0)=-1$

Of these z_4 and z_5 do not satisfy the given equation. Hence the required solutions are 0, i and $-i$

39 Putting $z=x+iy$, the given equations become

$$\left|\frac{x+iy-12}{x+iy-8i}\right|=\frac{5}{3} \text{ and } \left|\frac{x+iy-4}{x+iy-8}\right|=1$$

or $9|(x-12)+iy|^2=25|x+i(y-8)|^2$

and $|(x-4)+iy|^2=|(x-8)+iy|^2$

or $9[(x-12)^2+y^2]=25[x^2+(y-8)^2]$

and $(x-4)^2+y^2=(x-8)^2+y^2$

First of these equations gives

$$16x^2+16y^2+216x-400y+304=0$$

or $2x^2+2y^2+27x-50y+38=0$ (1)

And second equation gives

$8x=48$ or $x=6$

Substituting $x=6$ in (1) we get

$2\times 36+2y^2+27\times 6-50y+38=0$

or $2y^2-50y+272=0$

or $y^2-25y+136=0$

or $(y-8)(y-17)=0$

$\therefore\ y=8,\ 17$

$$=\sqrt{\left(\frac{\sqrt{2}+2}{4}\right)}=\frac{\sqrt{(2+\sqrt{2})}}{2} \quad (1)$$

And $\sin\theta=\frac{1}{\sqrt{(4+2\sqrt{2})}}=\sqrt{\left(\frac{1}{4+2\sqrt{2}}\right)}=\sqrt{\left(\frac{4-2\sqrt{2}}{16-8}\right)}$

$$=\frac{\sqrt{(2-\sqrt{2})}}{2} \quad (2)$$

The values of θ satisfying (1) and (2) is $\frac{\pi}{8}$

Hence $|1+\sqrt{2}+i|=\sqrt{(4+2\sqrt{2})}$ and $\arg(1+\sqrt{2}+i)=\frac{\pi}{8}$

14 Taking modulus of both sides, we have

$$|(1+i)\ (1+2i)\ (1+3i)\ \ (1+ni)|=|x+iy| \quad (1)$$

Since modulus of the product equals the product of moduli, we may write (1) as

$$|1+i|\,|1+2i|\quad |1+ni|=|x+iy|$$

or $\sqrt{(1+1)}\sqrt{(1+2^2)}\ (1+3^2)\ \ \sqrt{(1+n^2)}=\sqrt{(x^2+y^2)}$

Squaring we get $\quad 2\ 5\ 10\ \ (1+n^2)=x^2+y^2$

15 (i) Let $r\cos\theta=3\ \ r\sin\theta=0$

Then $r^2=3^2+0^2=9$ so that $r=3$ Also then $\cos\theta=1$, $\sin\theta=0$ so that $\theta=0$

Hence $3=r(\cos\theta+i\sin\theta)=3(\cos 0+i\sin 0)$

(ii) Let $r\cos\theta=-5$, $r\sin\theta=0$

Then $r^2=(-5)^2+0^2=25$ so that $r=5$

[Note that we always take the non negative value of r]

$\cos\theta=-1$ and $\sin\theta=0$

The principal value of θ satisfying these equations is π

Hence $-5=r(\cos\theta+i\sin\theta)=5(\cos\pi+i\sin\pi)$

(iii) Ans $6\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

(iv) Let $r\cos\theta=0$, $r\sin\theta=-2$

Then $r^2=0^2+(-2)^2=4$ so that $r=2$

$\cos\theta=0$ and $\sin\theta=-1$ which give $\theta=-\pi/2$,

Hence $-2i=2[\cos(-\pi/2)+i\sin(-\pi/2)]$

(vi) Ans $2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

(vi) $\frac{1+7i}{(2-i)^2}=\frac{1+7i}{4+i^2-4i}=\frac{1+7i}{4-1-4i}=\frac{1+7i}{3-4i}$

$$=\frac{(1+7i)\ (3+4i)}{3^2-4^2 i^2}=\frac{-25+25i}{25}=-1+i$$

Now let $\quad r\cos\theta=-1$ and $r\sin\theta=1$

these distances are equal The solution will thus be a locus of a point equidistant from two fixed points representing the numbers $-i$ and 2 *i e* from the points $(0, -1)$ and $(2, 0)$

From geometry we know that this locus is a straight line perpendicular to a line segment connecting the points $(0, -1)$ and $(2, 0)$ and passing through its mid point Hence all points z satisfying $|z+i| = |z-2|$ lie on this line

(viii) We have two equations

$$|z|-4=0 \quad \text{and} \quad |z-i|-|z+5i|=0$$

Putting $z=x+iy$, these equations become

$$|x+iy|=4 \quad i\,e \quad x^2+y^2=16 \qquad (1)$$

and $|x+iy-i| = |x+iy+5i|$

or $x^2+(y-1)^2=x^2+(y+5)^2$ *i e* $y=-2$ (2)

Putting $y=-2$ in (1), $x^2+4=16$ or $x=\pm2\sqrt{3}$

Hence the complex numbers z satisfying the given equations are $z_1=(2\sqrt{3}, -2)$ and $z_2=(-2\sqrt{3}, -2)$

that is, $z_1=2\sqrt{3}-2i$, $z_2=-2\sqrt{3}-2i$

(ix) $|z-1| = |z-3| = |z-i|$ (1)

or $|z-1|^2 = |z-3|^2 = |z-i|^2$

or $(z-1)(\bar{z}-1)=(z-3)(\bar{z}-3)=(z-i)(\bar{z}+i)$

or $z\bar{z}-(z+\bar{z})+1=z\bar{z}-3(z+\bar{z})+9=z\bar{z}+i(z-\bar{z})+1$

or $-2x+1=-6x+9=i\,2iy+1$

[$z+\bar{z}=2x$ and $z-\bar{z}=2iy$]

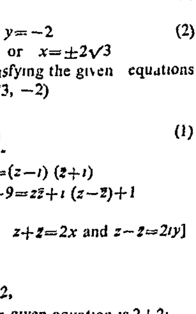
or $2x-1=6x-9=2y-1$

Now $2x-1=6x-9$ gives $x=2$,

and $2x-1=2y-1$ gives $y=x=2$,

Hence the value of z satisfying the given equation is $2+2i$

Remark Geometrically speaking the equation (1) means that z is a point equidistant from the points representing the numbers $1, 3, i$, that is equidistant from the points $A(1, 0)$, $B(3, 0)$ and $C(0, 1)$ So it is the intersection of perpendicular bisectors of segments AB and AC, that is, the intersection P of the lines represented by the equation $x=2$ and $y=x$ So we get $x=y=2$ as before (See the figure)



(x) $|z-1|^2+|z+1|^2=4$ (1)

That is why we considered two separate cases

(ix) First we note that

$2\cdot 5\,(\cos 300^\circ + i \sin 30^\circ) = (5/2)[\cos (360^\circ - 60^\circ) + i \sin 30^\circ]$

$= (5/2)[\cos (60^\circ) + i \sin 30^\circ] = (5/2)(\tfrac{1}{2} + \tfrac{1}{2}i) = (5/4)(1+i)$

Therefore we put $r \cos \theta = (5/4)$, $r \sin \theta = (5/4)$

$$\therefore \quad r = \sqrt{\{(5/4)^2 + (5/4)^2\}} = (5/4)\sqrt{2},$$

$$\cos \theta = \frac{1}{\sqrt{2}}, \ \sin \theta = \frac{1}{\sqrt{2}}, \text{ so that } \theta = \frac{\pi}{4}$$

Hence $2\cdot 5\,(\cos 300^\circ + i \sin 30^\circ)$

$$= r(\cos \theta + i \sin \theta) = (5/4)\sqrt{2}\left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}\right)$$

(x) $= -5\,(\cos 40^\circ - i \sin 40^\circ)$

$= 5\,(-\cos 40^\circ + i \sin 40^\circ)$

$= 5\,[\cos (180^\circ - 40^\circ) + i \sin (180^\circ - 40^\circ)]$

$= 5\,[\cos 140^\circ + i \sin 140^\circ]$

16 Substituting $z = \cos \phi + i \sin \phi$, we get

$$z_1 = (\cos \phi + i \sin \phi)^2 - (\cos \phi + i \sin \phi)$$

$$= \cos^2 \phi - \sin^2 \phi + 2i \cos \phi \sin \phi - \cos \phi - i \sin \phi$$

$$= \cos 2\phi + i \sin 2\phi - \cos \phi - i \sin \phi$$

$$= (\cos 2\phi - \cos \phi) + i\,(\sin 2\phi - \sin \phi)$$

$$= 2 \sin \frac{3\phi}{2} \sin\left(-\frac{\phi}{2}\right) + 2i \cos \frac{3\phi}{2} \sin \frac{\phi}{2}$$

$$= 2 \sin \frac{\phi}{2}\left[-\sin \frac{3\phi}{2} + i \cos \frac{3\phi}{2}\right]$$

$$\therefore \quad |z_1| = \sqrt{\left[4 \sin^2 \frac{\phi}{2}\left\{\sin^2 \frac{3\phi}{2} + \cos^2 \frac{3\phi}{2}\right\}\right]} = 2\left|\sin \frac{\phi}{2}\right|$$

In view of the definition of modulus, we consider three cases

Case I Let $\sin \frac{\phi}{2} = 0$, that is, $\phi = 2n\pi$, n any integer Then $|z_1| = 0$ which implies that $z_1 = 0$ Thus for $\phi = 2n\pi$ (n, any integer), arg z_1 is undefined

Case II Let $\sin\frac{\phi}{2} > 0$ which occurs when $2n\pi < \frac{1}{2}\phi < (2n+1)\pi$ that is, when $4n\pi < \phi < (4n+2)\pi$ n any integer (1)

Then $|z_1| = 2 \sin\frac{\phi}{2}$ and the trigonometrical form z_1 is as

$\Rightarrow |z|^2-|z|+1<6+3|z|$

$\Rightarrow |z|^2-4|z|-5<0$

$\Rightarrow (|z|-5)(|z|+1)<0$

$\Rightarrow |z|-5<0$ [$\because |z|+1>0$]

$\Rightarrow |z|<5$

Hence the given inequality is satisfied by complex numbers z representing points inside a circle of radius 5 with centre at the origin

44 Suppose an integer n satisfies the given equation Then

$$(1-i)^n=2^n \qquad (1)$$

We know that if two complex numbers are equal, then their moduli are also equal Hence taking moduli of both sides of (1) we get $|(1-i)^n|=|2^n|$

or $|1-i|^n=2^n$

[$\because 2^n>0$, we have $|2^n|=2^n$]

or $$(\sqrt{2})^n=2^n \qquad (2)$$

Now (2) will hold only when $n=0$ Hence the only integral solution of the given equation is $x=0$

Alternative Put $1=r\cos\theta$, $-1=r\sin\theta$ These relations give $r=\sqrt{2}$, $\theta=-\pi/4$

Then given equation takes the form

$(\sqrt{2})^n\,[\cos(-\pi/4)+i\sin(-\pi/4)]^n=2^n$

or $2^{n/2}\left[\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right]=2^n$

Equating real and imaginary parts, we get

$\cos\frac{n\pi}{4}=2^{n/2}$ and $-\sin\frac{n\pi}{4}=0$

These are satisfied only for $n=0$

Hence $n=0$ is the only solution

(ii) Ans $n=4k$ where k is an integer

Hint Proceed as in the alternative proof of part (i)

45 Let $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$ be two complex numbers

First suppose z_1, z_2 are conjugate of each other Then,

$$z_2=\bar{z}_1=x_1-iy_1$$

Hence $z_1+z_2=z_1+\bar{z}_1=2x_1$, which is real

and $z_1z_2=z_1\bar{z}_1=(x_1+iy_1)(x_1-iy_1)=x_1^2+y_1^2$, which is also real

Thus sum z_1+z_2 and product z_1z_2 are real when z_1 and z_2 are conjugate complex

Now let the sum z_1+z_2 and product z_1z_2 be real we have

(vi) $\sqrt{(x^2-2x+8)}+(x+4)\,i=y\,(2+i)$

Equating real and imaginary parts,

$$\sqrt{(x^2-2x+8)}=2y, \quad \text{or} \quad x^2-2x+8=4y^2 \qquad (1)$$

and

$$x+4=y \qquad (2)$$

Substituting for y from (2) in (1), we get

$$x^2-2x+8=4\,(x+4)^2=4x^2+32x+64$$

or $\quad 3x^2+34x+56=0$

or $\quad 3x^2+28x+6x+56=0$

or $\quad (3x+28)\,(x+2)=0$

This gives $x=-2$ or $-(28/3)$

Then $\quad y=2$ or $-(16/3)$

Hence $x=-2,\ y=2$ or $x=-(28/3),\ y=-(16/3)$.

Of these two solutions $x=-(28/3),\ y=-(16/3)$ does not satisfy given equation

Hence the only solution is $x=-2,\ y=2$

18 Since $a, b, c, \ldots k$ are the roots of the given equation, we have the identity

$$x^n+p_1x^{n-1}+p_2x^{n-2}+\ldots+p_{n-1}x+p_n \equiv (x-a)\,(x-b)\,(x-c)\ldots(x-k) \qquad (1)$$

In the identity (1) put $x=i$

Then $\quad i^n+p_1 i^{n-1}+p_2 i^{n-2}+\ldots+p_{n-1} i+p_n \equiv (i-a)\,(i-b)\,(i-c)\ldots(i-k)$

or $\quad i^n[1+p_1 i^{-1}+p_2 i^{-2}+p_3 i^{-3}+p_4 i^{-4}+\ldots+p_{n-1} i^{-(n-1)}+p_n i^{-n}] \equiv (i-a)\,(i-b)\,(i-c)\ldots(i-k)$

$$i^{-1}=\frac{1}{i}=-i,\ i^{-2}=\frac{1}{i^2}=-1,\ i^{-3}=\frac{1}{i^3}=i,\ i^{-4}=\frac{1}{i^4}=1 \text{ etc}$$

The above identity may be written as

$$i^n[(1-p_2+p_4-\ldots)-i\,(p_1-p_3+p_5\ldots)] \equiv (-1)\,(a-i)\,(b-i)\,(c-i)\ldots(k-i) \qquad (2)$$

Similarly putting $x=-i$ in (1), we shall obtain

$$(-i)^n[(1-p_2+p_4\ldots)+i\,(p_1-p_3+p_5\ldots)] \equiv (-1)^n\,(a+i)\,(b+i)\,(c+i)\ldots(k+i) \qquad (3)$$

Multiplying (2) and (3), we get

$$(-1)^n i^{2n}[(1-p_2+p_4\ldots)^2-i^2(p_1-p_3+p_5\ldots)^2] = (-1)^{2n}(a^2-i^2)\,(b^2-i^2)\,(c^2-i^2)\ldots(k^2-i^2)$$

$\because\ (-1)^n i^{2n}=(-1)^n(-1)^n=(-1)^{2n}=1$, this gives

$$(1-p_2+p_4\ldots)^2+(p_1-p_3+p_5\ldots)^2 =(a^2+1)\,(b^2+1)\,(c^2+1)\ldots(k^2+1).$$

19 The equation is

$$x^4-4x^2+8x+3=0 \qquad (1)$$

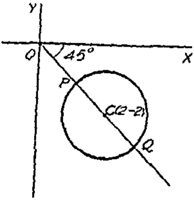
Hence the greatest value of $|z|$ is $\sqrt{5}+1$

(c) The required value of z clearly corresponds to the point P of circle $|z-2+2i|=1$ since P has the least absolute value $OP=r$, say. Clearly OC is the bisector of the fourth quadrant so that

$\angle XOP=45°$

Hence z corresponding to P is is given by $z=r\cos 45°-i\, r\sin 45°$

$=\frac{r}{\sqrt{2}}(1-i)$



To find r, we substitute the coordinates $\left(\frac{r}{\sqrt{2}}, -\frac{r}{\sqrt{2}}\right)$ of P in the equation of the circle whose cartesian form is

$$(x-2)^2+(y+2)^2=1$$

Thus $\left(\frac{r}{\sqrt{2}}-2\right)^2+\left(-\frac{r}{\sqrt{2}}+2\right)^2=1$ or $2\left(\frac{r}{\sqrt{2}}-2\right)^2=1$

or $\sqrt{2}\left(\frac{r}{\sqrt{2}}-2\right)=\pm 1$

where $r=2\sqrt{2}-1$ or $2\sqrt{2}+1$

But the value $2\sqrt{2}+1$ corresponds to the point Q

Hence $z=\frac{(2\sqrt{2}-1)}{\sqrt{2}}(1-i)$

47 Put $z=x+iy$. Then given equation takes the form

$$x^2+y^2-2i(x+iy)+2c(1+i)=0$$

or $$(x^2+y^2+2y+2c)+2i(c-x)=0$$

Equating real and imaginary parts to zero, we get

$$x^2+y^2+2y+2c=0,\ 2c-2x=0$$

These give $x=c$ and for y we have the equation

$$c^2+y^2+2y+2c=0 \text{ or } y^2+2y+c^2+2c=0 \quad (1)$$

Since we seek real values of y, the discriminant of the equation (1) must be non negative, that is,

$$\Delta=4-4(c^2+2c)>0 \text{ or } 1-c^2-2c\geqslant 0$$

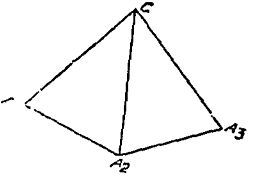
For these values of c we get

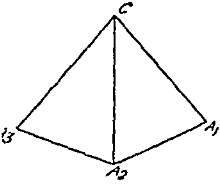
$$y=\frac{-2\pm\sqrt{\{4(1-c^2-2c)\}}}{2}=-1\pm\sqrt{(1-c^2-2c)}$$

Hence for $\Delta\geqslant 0$, the original equation has two roots

$$z_1=c+\{-1+\sqrt{(1-c^2-2c)}\}$$

Since $\angle A_1A_2A_3 = \pi - 2\pi/n$, by remark (i), § `, we have





$z_1 - z_2 = (z_3 - z_2)\, e^{i\{\pi - (2\pi/n)\}}$ from fig. 1, (1)

and $z_3 - z_2 = (z_1 - z_2)\, e^{i\{\pi - (2\pi/n)\}}$ from fig 2 (2)

Since $e^{i\pi} = \cos \pi + i \sin \pi = -1$, we obtain from (1)

$z_1 - z_2 = (z_3 - z_2)\, e^{-2\pi i/n}(-1)$ or $z_2 - z_3 = (z_1 - z_2)\, e^{2\pi i/n}$

or $z_3 = z_2 + (z_2 - z_1)\, e^{2\pi i/n}$

Similarly from (2) we get

$z_3 = z_2 + (z_2 - z_1)\, e^{-2\pi i/n}$

Hence the vertex z_3 is given by

$$z_3 = z_2 + (z_2 - z_1)\, e^{\pm 2\pi i/n} = z_2 + (z_2 - z_1)\left(\cos\frac{2\pi}{n} \pm i \sin\frac{2\pi}{n}\right)$$

22 $\frac{1-ix}{1+ix} = \frac{a-ib}{1}$

By componendo and dividendo, we have

$$\frac{(1+ix)-(1-ix)}{(1+ix)+(1-ix)} = \frac{1-(a-ib)}{1+(a-ib)}$$

or $\frac{2ix}{2} = \frac{1-a+ib}{1+a-ib}$

or $ix = \frac{(1-a+ib)(1+a+ib)}{(1+a)^2 - i^2b^2} = \frac{1-a^2-b^2+2ib}{(1+a)^2+b^2}$ (1)

If $a^2+b^2=1$ the equation (1) reduces to

$$ix = \frac{2ib}{(1+a)^2+b^2}$$

or $x = \frac{2b}{(1+a)^2+b^2}$ which is real

23 (i) We have $(x^2+a^2)^4 = (x+ai)^4\,(x-ai)^4$

Now $(x+ai)^4 = x^4 + 4x^3ai + 6x^2a^2i^2 + 4xa^3i^3 + a^4i^4$

So in this case, the roots of the original equation are given by

$$z_1, z_2 = \frac{-c^2+c\sqrt{2(2-c^2)}}{c^2-1} - i$$

(iii) If $c=\sqrt{2}$, $x = \frac{-2}{2-1} = -2$

So in this case also the original equation has only one root

$$z = -2 - i$$

(iv) If $c > \sqrt{2}$, then the original equation has no solution

Remark When $c=1$, one of the roots of equation (i) becomes infinite and so in this case we cannot apply the formula for x given in case (ii) So we first put $c=1$ in (i), and then solve it for x

49 Ans If $0 \leqslant c \leqslant \frac{1}{2}$, then the given equation has no solution if $c > \frac{1}{2}$ then the solution is

$$z = \frac{4c+\sqrt{(4c^2+3)}}{16c^2-4} + \frac{i}{4}$$

50 We shall use De Moivre's Theorem

$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

(i) $(\cos 60° + i \sin 60°)^6 = \cos (6 \times 60°) + i \sin (6 \times 60°)$

$$= \cos 360° + i \sin 360° = 1 + i0 = 1$$

(ii) $[\sqrt{2} (\cos 56° 15' + i \sin 56° 15')^8]$

$= 16 [\cos (8 \times 56° 15') + i \sin (8 \times 56° 15')]$

$= 16 [\cos 450° + i \sin 450°]$

$= 16 [\cos 90° + i \sin 90°] = 16 (0 + i) = 16i$

(iii) $(\sin \theta + i \cos \theta)^n = [\cos (\pi/2 - \theta) + i \sin (\pi/2 - \theta)]^n$

$$= \cos n (\pi/2 - \theta) + i \sin n (\pi/2 - \theta)$$

(iv) Put $1 = r \cos \theta$ $1 = r \sin \theta$

Then $r = \sqrt{2}$, $\cos \theta = 1/\sqrt{2}$, $\sin \theta = 1/\sqrt{2}$ and so $\theta = \pi/4$

Hence $\left(\frac{1+i}{\sqrt{2}}\right)^8 + \left(\frac{1-i}{\sqrt{2}}\right)^8$

$$= \frac{(r \cos \theta + ir \sin \theta)^8}{16} + \frac{(r \cos \theta - ir \sin \theta)^8}{16}$$

$$= \frac{r^8}{16} [(\cos \theta + i \sin \theta)^8 + (\cos \theta - i \sin \theta)^8]$$

$$= \frac{r^8}{16} [\cos 8\theta + i \sin 8\theta + \cos 8\theta - i \sin 8\theta]$$

$$= \frac{r^8}{16}\ 2 \cos 8\theta = 2 \times \frac{(\sqrt{2})^8}{16} \cos\left(8 \times \frac{\pi}{4}\right)$$

$$= 2 \cos (2\pi) = 2$$

$\alpha=\omega$ and $\beta=\omega^2$

Hence $\alpha^4+\beta^4+\alpha^{-1}\beta^{-1}=\omega^4+\omega^8+\omega^{-1}\omega^{-2}$

$=\omega^3\,\omega+\omega^6\,\omega^2+(\omega^3)^{-1}$

$=\omega+\omega^2+1=0$ $[\because \omega^3=1]$

26 Let $\alpha=\omega$ and $\beta=\omega^2$

Then $xyz=(a+b)\,(a\omega+b\omega^2)\,(a\omega^2+b\omega)$

a^3+b^3 from Ex 24 (ix)

27 Here $x+y+z=(a+b)+(a\omega+b\omega^2)+(a\omega^2+b\omega)$

$=a\,(1+\omega+\omega^2)+b\,(1+\omega^2+\omega)$

$=a\times 0+b\times 0=0$ (1)

Hence from the relation

$x^3+y^3+z^3-3xyz=(x+y+z)\,(x^2+y^2+z^2-yz-zx-xy)$

$=0$ by (1),

we obtain $x^3+y^3+z^3=3xyz$

But from ex 26, $xyz=a^3+b^3$

Hence $x^3+y^3+z^3=3\,(a^3+b^3)$

28 (i) First note that

$$\frac{\sqrt{3}+i}{2}=-i\left(\frac{-1+\sqrt{3}i}{2}\right)=-i\omega$$

and $\dfrac{i-\sqrt{3}}{2}=-i\left(\dfrac{-1-\sqrt{3}i}{2}\right)=-i\omega^2$

Hence $\left(\dfrac{\sqrt{3}+i}{2}\right)^6+\left(\dfrac{i-\sqrt{3}}{2}\right)^6=(-i\omega)^6+(-i\omega^2)^6$

$=i^6\,(\omega^6+\omega^{12})=-1\,(1+1)=-2$

(ii) Since $\dfrac{-1+\sqrt{3}i}{2}=\omega$ and $\dfrac{-1-\sqrt{3}i}{2}=\omega^2$

We have to prove that $\omega^n+\omega^{2n}=-1$

or $1+\omega^n+\omega^{2n}=0$

when n is not a multiple of 3

Now $1+\omega^n+\omega^{2n}=\dfrac{(1-\omega^n)\,(1+\omega^n+\omega^{2n})}{1-\omega^n}=\dfrac{1-\omega^{3n}}{1-\omega^n}$

[The formula $1-x^3=(1-x)\,(1+x+x^2)$ is used here]

$=\dfrac{1-1}{1-\omega^n}=0$

[$\because$ $1-\omega^n\neq 0$ since n is not a multiple of 3 $\therefore$ $\omega^n\neq 1$]

(iii) Hint roots of $z+1/z=1$

are $-\omega$ and $-\omega^2$

29 (i) This is the same as ex 28 (ii)

We give a second proof of this Since n is not a multiple or $\frac{x}{y}=\cos(\theta-\phi)+i\sin(\theta-\phi)$

$$\frac{y}{x}=\left(\frac{x}{y}\right)^{-1}=\cos(\theta-\phi)-i\sin(\theta-\phi)$$

$$\frac{x}{y}+\frac{y}{x}=2\cos(\theta-\phi)$$

(iii) and (iv) proceed as above

53 $(1+x)^n=p_0+p_1x+p_2x^2+p_3x^3+p_4x^4+\quad+p_nx^n$

In this put $x=i$ Then

$$(1+i)^n=p_0+p_1i-p_2-p_3i+p_4+\quad+p_ni^n \qquad (1)$$

Now put $1=r\cos\theta$, $1=r\sin\theta$, then $r=\sqrt{2}$ and $\theta=\pi/4$

Hence $(1+i)^n=(r\cos\theta+ri\sin\theta)^n=r^n(\cos\theta+i\sin\theta)^n$

$=r^n[\cos n\theta+i\sin n\theta]=(\sqrt{2})^n[\cos n\pi/4+i\sin n\pi/4]$

$=2^{n/2}[\cos n\pi/4+i\sin n\pi/4]$

Substituting in (1), we get

$$2^{n/2}[\cos n\pi/4+i\sin n\pi/4]=p_0+p_1i-p_2-p_3i+p_4+$$

Equating real and imaginary parts,

$$2^{n/2}\cos n\pi/4=p_0-p_2+p_4$$

and $2^{n/2}\sin n\pi/4=p_1-p_3+p_5$

54 Since $\cos\alpha+i\sin\alpha$ is the root of the given equation, we have $(\cos\alpha+i\sin\alpha)^n+p_1(\cos\alpha+i\sin\alpha)^{n-1}$

$$+p_2(\cos\alpha+i\sin\alpha)^{n-2}+\quad p_n=0$$

or $(\cos\alpha+i\sin\alpha)^n[1+p_1(\cos\alpha+i\sin\alpha)^{-1}$

$$+p_2(\cos\alpha+i\sin\alpha)^{-2}+p_3(\cos\alpha+i\sin\alpha)^{-3}+$$

$$+p_n(\cos\alpha+i\sin\alpha)^{-n}]=0 \qquad (1)$$

Now by De Moivre's theorem

$(\cos\alpha+i\sin\alpha)^{-1}=\cos\alpha-i\sin\alpha$,

$(\cos\alpha+i\sin\alpha)^{-2}=\cos 2\alpha-i\sin 2\alpha$ etc

Also since $\cos\alpha+i\sin\alpha\neq 0$ for any α, we can cancel the factor $(\cos\alpha+i\sin\alpha)^n$ in the equation (1) Hence (1) can be written as

$1+p_1(\cos\alpha-i\sin\alpha)+p_2(\cos 2\alpha-i\sin 2\alpha)$

$+p_3(\cos 3\alpha-i\sin 3\alpha)+\quad+p_n(\cos n\alpha-i\sin n\alpha)=0 \qquad (2)$

Equating imaginary part to 0 in (2), we get

$p_1\sin\alpha+p_2\sin 2\alpha+p_3\sin 3\alpha+\quad+p_n\sin n\alpha=0$

Remark Equating real parts in (2), we shall obtain

$$1+p_1\cos\alpha+p_2\cos 2\alpha+\quad+p_n\cos n\alpha=0$$

55 $\cos\alpha+\cos\beta+\cos\gamma=0 \qquad (1)$

$\sin\alpha+\sin\beta+\sin\gamma=0 \qquad (2)$

Multiplying (2) by i and adding to (1) we get

In $f(x,y)$, we put $y=x\omega$ It becomes

$(x+x\omega)^n-x^n-(x\omega)^n=x^n(-\omega^2)^n-x^n-x^n\omega^n$

$=x^n[(-1)^n\omega^{2n}-\omega^n-1]$

$=x^n[-\omega^{2n}-\omega^n-1]$ [n is odd]

$=x^n\times 0=0$ by ex 29 (i)

Similarly $f(x,y)$ vanishes when $y=\omega^2 x$

Hence the polynomial $f(x,y)$ is divisible by

$xy(x+y)(x^2+xy+y^2)$

32 (*a*) Let $f(x)=x^{4l}+x^{4m+1}+x^{4n+2}+x^{4p+3}$

Also $x^3+x^2+x+1=(x+1)(x^2+1)=(x+1)(x+i)(x-i)$

Then $f(-1)=(-1)^{4l}+(-1)^{4m+1}+(-1)^{4n+2}+(-1)^{4p+3}$

$=1-1+1-1=0$

Hence $x+1$ is a factor of $f(x)$

Again $f(-i)=(-i)^{4l}+(-i)^{4m+1}+(-i)^{4n+2}+(-i)^{4p+3}$

$=(-1)^{4l}i^{4l}+(-1)^{4m+1}i^{4m}i+(-1)^{4n+2}i^{4n}i^2+(-1)^{4p+3}i^{4p}i^3$

$1\cdot 1+(-1)\cdot 1\cdot i+1\cdot 1\cdot(-1)+(-1)\cdot 1\cdot(-i)$

$=1-i-1+i=0$

Hence $x+i$ is a factor of $f(x)$

Similarly $x-i$ can be proved to be a factor of $f(x)$

Thus $(x+1)(x+i)(x-i)$ that is x^3+x^2+x+1 is a factor of $f(x)$ as required

(*b*) The first equation may be written as $(z+1)(z^2+z+1)=0$, its roots are -1 ω and ω^2 The root $z=-1$ does not satisfy the second equation but $z=\omega$ and $z=\omega^2$ satisfy it Hence ω and ω^2 are the common roots

33 (i) $(a^2+b^2+c^2-bc-ca-ab)(x^2+y^2+z^2-yz-zx-xy)$

$=(a+b\omega+c\omega^2)(a+b\omega^2+c\omega)(x+y\omega+z\omega^2)(x+y\omega^2+z\omega)$

[See ex 24 (xii)]

$=[(a+b\omega+c\omega^2)(x+y\omega+z\omega^2)][(a+b\omega^2+c\omega)(x+y\omega^2+z\omega)]$

$=(ax+cy\omega^3+bz\omega^3+cx\omega^2+by\omega^2+az\omega^2+bx\omega+ay\omega+cz\omega^4)$

$\times(ax+cy\omega^3+bz\omega^3+cx\omega+by\omega^4+ay\omega$

$+bx\omega^2+ay\omega^2+cz\omega^2)$

$=[ax+cy+bz+(cx+by+az)\omega^2+(bx+ay+cz)\omega]$

$\times[(ax+cy+bz)+(cx+by+az)\omega+(bx+ay+cz)\omega^2]$

$=(X+Y\omega^2+Z\omega)(X+Y\omega+Z\omega^2)$

$=X^2+Y^2+Z^2-YZ-ZX-XY$

(ii) $(a^3+b^3+c^3-3abc)(x^3+y^3+z^3-3xyz)$

$=(a+b+c)(a^2+b^2+c^2-bc-ca-ab)$

$\times(x+y+z)(x^2+y^2+z^2-yz-zx-xy)$ (1)

Now $(a+b+c)(x+y+z)=[(ax+cy+bz)+(cx+by+az)$

$+(bx+ay+cz)]$

$=X+Y+Z$ (2)

$\cos(17\pi/12)=\cos(3\pi/2-\pi/12)=-\sin\pi/12$,

and $\sin\left(\frac{17}{12}\pi\right)=\sin\left(\frac{3\pi}{2}-\frac{\pi}{12}\right)=-\cos\frac{\pi}{12}$

Hence the roots are

$$\sqrt{2}\left(\cos\frac{\pi}{12}-i\sin\frac{\pi}{12}\right),\ -1-i,\ \sqrt{2}\left(-\sin\frac{\pi}{12}+i\cos\frac{\pi}{12}\right)$$

(ii) Ans $\pm 2^{1/4}=[\cos r\pi/12-i\sin r\pi/12]$, where $r=1$ or 7

(iii) $(-64a^4)^{1/4}=(64)^{1/4}\,a\,(-1)^{1/4}=2\sqrt{2}\,a\,[-1+i\,0]^{1/4}$

Now put $-1=r\cos\theta,\ 0=r\sin\theta$

Then $r=1$, $\cos\theta=-1$, $\sin\theta=0$ so that $\theta=\pi$

$$(-64a^4)^{1/4}=2\sqrt{2}a\,[r\cos\theta+ri\sin\theta]^{1/4}$$
$$=2\sqrt{2}a\,r^{1/4}[\cos\theta+i\sin\theta]^{1/4}$$
$$=2\sqrt{2}a\,[\cos\pi+i\sin\pi]^{1/4} \qquad [\because r=1,\ \theta=\pi]$$
$$=2\sqrt{2}a\,[\cos(2n\pi+\pi)+i\sin(2n\pi+\pi)]^{1/4}$$
$$=2\sqrt{2}a\left[\cos\frac{2n\pi+\pi}{4}+i\sin\frac{2n\pi+\pi}{4}\right],$$

where $n=0, 1, 2$ and 3

Hence the required roots are

$2\sqrt{2}a\,[\cos\pi/4+i\sin\pi/4]$, $2\sqrt{2}a\,[\cos 3\pi/4+i\sin 3\pi/4]$,
$2\sqrt{2}a\,[\cos 5\pi/4+i\sin 5\pi/4]$, $2\sqrt{2}a\,[\cos 7\pi/4+i\sin 7\pi/4]$

Now $\cos\pi/4=\cos 7\pi/4=\frac{1}{\sqrt{2}}$, $\cos 3\pi/4=\cos 5\pi/4=-\frac{1}{\sqrt{2}}$,

$\sin\pi/4=\sin 3\pi/4=\frac{1}{\sqrt{2}}$, $\sin 5\pi/4=\sin 7\pi/4=-\frac{1}{\sqrt{2}}$

Thus the roots are

$2\sqrt{2}a\,(1/\sqrt{2}+i/\sqrt{2})$, $2\sqrt{2}a\,(-1/\sqrt{2}+i/\sqrt{2})$,
$2\sqrt{2}a\,(-1/\sqrt{2}-i/\sqrt{2})$, $2\sqrt{2}a\,(1/\sqrt{2}-i/\sqrt{2})$

Hence the roots are $\pm 2a\,(1\pm i)$

57 We have to find all the values of $1^{1/n}=(1+i0)^{1/n}$

Let $1=r\cos\theta,\ 0=r\sin\theta$ so that $r=1$ and $\theta=0°$

$$(1+i0)^{1/n}=(r\cos\theta+ri\sin\theta)^{1/n}=r^{1/n}[\cos\theta+i\sin\theta]^{1/n}$$
$$=(\cos 0°+i\sin 0°)^{1/n} \qquad [\because r=1,\ \theta=0°]$$
$$=(\cos 2m\pi+i\sin 2m\pi)^{1/n}=\cos 2m\pi/n+i\sin 2m\pi/n$$

where $m=0, 1, 2, \ldots, n-1$

Hence n n^{th} roots of 1 are

1, $\cos 2\pi/n+i\sin 2\pi/n$, $\cos 4\pi/n+i\sin 4\pi/n$, ...

$$\cos\frac{2(n-1)\pi}{n}+i\sin\frac{2(n-1)}{n}$$

Similarly $C\bar{C}=|z_1|^2+|z_2|^2+|z_3|^2+z_1(\bar{z}_2\omega^2+\bar{z}_3\omega)$

$$+\bar{z}_2(z_3\omega^2+z_1\omega)+\bar{z}_3(z_1\omega^2+z_2\omega) \qquad (3)$$

Adding (1), (2) and (3), we get

$$A\bar{A}+B\bar{B}+C\bar{C}=3[|z_1|^2+|z_2|^2+|z_3|^2]$$
$$+\bar{z}_1[z_2(1+\omega+\omega^2)+z_3(1+\omega^2+\omega)]$$
$$+\bar{z}_2[z_3(1+\omega+\omega^2)+z_1(1+\omega^2+\omega)]$$
$$+\bar{z}_3[z_1(1+\omega+\omega^2)+z_2(1+\omega^2+\omega)]$$
$$=3[|z_1|^2+|z_2|^2+|z_3|^2]$$
$$[\because 1+\omega+\omega^2=0]$$

From (i) and (ii), we conclude

$$|A|^2+|B|^2+|C|^2=3[|z_1|^2+|z_2|^2+|z_3|^2]$$

35 $-3+ix^2y$ and x^2+y+4i are complex conjugates if

$$-3+ix^2y=(\overline{x^2+y+4i})=x^2+y-4i$$

Equating real and imaginary parts, we get

$$x^2+y=-3,\ x^2y=-4$$

Eliminating x^2 we get

$$(-4/y)+y=-3 \text{ or } y^2+3y-4=0 \text{ or } (y+4)(y-1)=0$$
$$y=1 \text{ or } -4$$

Then $x^2=-4/y=1$ when $y=-4$ $\quad x=\pm 1$

Hence $x=1$ $y=-4$ or $x=-1$ $y=-4$ The other value of $y=1$ gives imaginary values of x We therefore reject this case since x and y are real

Hence $x=1, y=-4$ or $x=-1$ $y=-4$

36 (a) Let $z=x+iy$ Then $|z|^2=x^2+y^2$

Therefore the condition $|z|=1$ is equivalent to

$$x^2+y^2=1 \qquad (1)$$

$$\text{Now } \frac{z-1}{z+1}=\frac{x+iy-1}{x+iy+1}=\frac{(x-1+iy)(x+1-iy)}{(x+1+iy)(x+1-iy)}$$
$$=\frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2}=\frac{2iy}{(x+1)^2+y^2} \text{ by } (1)$$

Hence $\dfrac{z-1}{z+1}$ is purely imaginary when $|z|=1$ provided

$$z\neq-1$$

When $z=1$, we have $\dfrac{z-1}{z+1}=0$

Now recall that according to the definition 2 given in § 2, 0 is a pure imaginary number

So in this case also $\dfrac{z-1}{z+1}$ is a pure imaginary number

(b) Let $\dfrac{z-1}{z+1}$ be purely imaginary Then $\dfrac{x^2+y^2-1}{(x+1)^2+y^2}=0,$

$$\alpha^{mp}=(\cos 2\pi/n+i\sin 2\pi/n)^{mp}$$

$$=\cos\frac{2\pi mp}{n}+i\sin\frac{2\pi mp}{n},\ 1\leqslant m\leqslant n-1$$

$$=1+i0=1$$

[∵ p is multiple of $n\Rightarrow mp/n$ is an integ

So in this case, each term of the series is 1 Hence in t case the sum of the series (1) is n

59 (b) Proceed as in part (a)

(c) $\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}=-i^2\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}$

$$=-i\left(\cos\frac{2\pi k}{7}+i\sin\frac{2\pi k}{7}\right)=-i\,e^{i\,2\pi k/7}$$

$$\sum_{k=1}^{6}\left(\sin\frac{2\pi k}{7}-i\cos\frac{2\pi k}{7}\right)=-i\sum_{r=1}^{6}\left(e^{i\,2\pi k/7}\right)$$

$$=-i\,[e^{i2\pi/7}+e^{i4\pi/7}+\quad 6\text{ terms}]\ \text{G P}$$

$$=-ie^{i2\pi/7}\frac{[1-e^{i12\pi/7}]}{[1-e^{i2\pi/7}]}=-ie^{i2\pi/7}\frac{[1-e^{-i2\pi/7}]}{[1-e^{i2\pi/7}]}$$

[∵ $e^{i12\pi/7}=e^{i2\pi}\quad e^{-i2\pi/7}=1\ e^{-i2\pi/7}$

$$=-i\frac{[e^{i2\pi/7}-1]}{[1-e^{i2\pi/7}]}=-i\,(-1)=i$$

Hence (iv) is correct

60 (i) With origin as the centre of the circle of radius unit let $z_1, z_2, \ldots, z_n$ represent the vertices $A_1, A_2, \ldots A_n$ of the n go Then we easily get [See Q 21 (a)]

$$z_2=z_1\,e^{2\pi i/n},\ z_3=z_1\,e^{4\pi i/n},\ \ldots,\ z_n=z_1\,e^{2(n-1)\pi i/n}$$

Now $|A_1A_r|^2=|z_1-z_r|^2=|z_1-z_1\,e^{2(r-1)\pi i/n}|^2$

$$=|z_1|^2\,|1-e^{2(r-1)\pi i/n}|^2$$

Hence the equation $z=\bar{z}$ is equivalent to $x+iy=x-iy$ which gives $2iy=0$ or $y=0$

Hence $z=x$ which means that all real numbers satisfy the given equation

(ii) Ans All pure imaginary numbers

(iii) $\bar{z}=2-z$ or $x-iy=2-x-iy$ This gives $x=1$ Hence $z=1+iy$

Hence the given equation is satisfied by all complex numbers whose real part is 1

(iv) $z^2=-\bar{z}$ or $(x+iy)^2=-(x-iy)$

or $x^2-y^2+2ixy=-x+iy$

Equating real and imaginary parts,

$$x^2-y^2=-x \tag{1}$$

and $$2xy=y \quad \text{or} \quad y(2x-1)=0 \tag{2}$$

From (2), either $y=0$ or $x=\frac{1}{2}$

When $y=0$, (1) gives $x^2=-x$ or $x(x+1)=0$

which gives $x=0$ or $x=-1$

Hence we get two sets of solutions $x=0, y=0, x=-1, y=0$

When $x=\frac{1}{2}$, (1) gives $\frac{1}{4}-y^2=-\frac{1}{2}$ or $y^2=\frac{3}{4}$ which gives $y=\pm\frac{\sqrt{3}}{2}$

Hence we obtain two more sets of solutions

$$x=\tfrac{1}{2},\ y=\frac{\sqrt{3}}{2},\ x=\tfrac{1}{2},\ y=-\frac{\sqrt{3}}{2}$$

Thus in all we get the following four solutions

$$z_1=0+i0=0,\ z_2=-1+i\,0=-1,\ z_3=\tfrac{1}{2}+i\sqrt{3}/2,$$
$$z_4=\tfrac{1}{2}-(\sqrt{3}/2)i$$

(v) $z^3=\bar{z}$ or $(x+iy)^3=x-iy$ or $x^3+3x^2iy+3xi^2y^2+i^3y^3=x-iy$

or $(x^3-3xy^2)+i(3x^2y-y^3)=x-iy$

Equating real and imaginary parts

$$x^3-3xy^2=x \tag{1}$$
$$3x^2y-y^3=-y \tag{2}$$

One obvious solution is $x=0, y=0$ i e $z=0$

Now cancelling the factor x from (1) and y from (2), we get

$x^2-3y^2=1$ or $x^2-3y^2-1=0$

and $3x^2-y^2=-1$ or $3x^2-y^2+1=0$

Solving these we get,

$$\frac{x^2}{-3-1}=\frac{y^2}{-3-1}=\frac{1}{-1+9}$$

Thus $x^2=y^2=-\frac{1}{2}$

which give imaginary values for x and y

or $1+z+z^2+\dots+z^{n-1}\equiv(z-e^{2\pi i/n})\dots(z-e^{2(n-1)\pi i/n})$

Putting $z=1$ in the above identity, we get,

$n=(1-e^{2\pi i/n})(1-e^{4\pi i/n})\dots(1-e^{2(n-1)\pi i/n})$

Hence $n=|n|=|1-e^{2\pi i/n}||1-e^{4\pi i/n}|\dots|1-e^{2(n-1)\pi i/n}|$ (2)

From (1) and (2) we get

$|A_1A_2|\,|A_1A_3|\dots|A_1A_n|=n$

61 Let $z_k=a_k+ib_k$, $k=1, 2, \dots n$

Then $z_1+z_2+\dots+z_n=(a_1+a_2+\dots+a_n)+i(b_1+b_2+\dots+b_n)$

Now $|z_1+z_2+\dots+z_n|\leqslant|z_1|+|z_2|+\dots+|z_n|$ (1)

But $|z_k|=\sqrt{(a_k^2+b_k^2)}$

and $|z_1+z_2+\dots+z_n|=\sqrt{[(a_1+a_2+\dots+a_n)^2}$ $+(b_1+b_2+\dots+b_n)^2]^{1/2}$

Hence substituting in (1), we get the required inequality

62 $|z_1+z_2|^2+|z_1-z_2|^2=$

$=(z_1+z_2)\overline{(z_1+z_2)}+(z_1-z_2)\overline{(z_1-z_2)}$

$=(z_1+z_2)(\bar z_1+\bar z_2)+(z_1-z_2)(\bar z_1-\bar z_2)$

$=2z_1\bar z_1+2z_2\bar z_2$ [*other terms cancel*]

$=2|z_1|^2+2|z_2|^2$ (1)

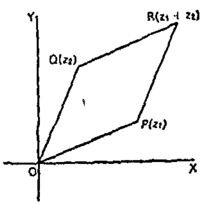
Geomterical Interpretation

Let P and Q be the points of affix z_1 and z_2 respectively

Complete the parallelogram $OPRQ$ Then R represents the point z_1+z_2 Hence

$|z_1|=OP$, $|z_2|=OQ$, $|z_1+z_2|=OR$

and $|z_1-z_2|=QP$



We know that the sum of squares of the sides of a parallelogram is equal to the sum of squares of its diagonals that is

$$2\,OP^2+2\,OQ^2=OR^2+QP^2$$

or $2|z_1|^2+2|z_2|^2=|z_1+z_2|^2+|z_1-z_2|^2$

Deduction Let $z_1=\alpha+\sqrt{(\alpha^2-\beta^2)}$ and $z_2=\alpha-\sqrt{(\alpha^2-\beta^2)}$

Then from (1), we have

$|z_1|^2+|z_2|^2=\frac{1}{2}|z_1+z_2|^2+\frac{1}{2}|z_1-z_2|^2$

$\frac{1}{2}|2\alpha|^2+\frac{1}{2}|2\sqrt{(\alpha^2-\beta^2)}|^2=2|\alpha|^2+2|\alpha^2-\beta^2|$

Hence $z_1=(6, 8)$ and $z_2=(6, 17)$
i e $z_1=6+8i$ and $z_2=6+17i$
are the two solutions

40 (i) The equation $|z|=1$ represents a circle with origin as centre and radius unity Hence all the points z satisfying the condition $|z|<1$ lie within the circle with origin as centre and radius unity

(ii) Here the points z lie outside and on the circle whose centre is origin and radius 3

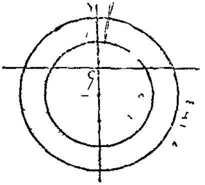
(iii) $|z-3|=1$ represents all points z on the boundary of a circle with centre 3 and radius 1

(iv) $|z-i|<1$ represents all points z inside the circle with centre i and radius 1

(v) The inequality can be written as

$$|-2|\left|z-\frac{i-1}{2}\right|>9 \text{ or } \left|z-\frac{i-1}{2}\right|>\tfrac{9}{2}$$

which represents all points z outside the circle with centre $(i-1)/2$ and radius $\frac{9}{2}$



(vi) We know that points satisfying $|z+i|\geqslant 2$ lie outside and on the circle with centre $-i$ and radius 2 Similarly the points z satisfying $|z+i|\leqslant 3$ lie inside and on the circle with centre $-i$ and radius 3 Hence the point which satisfy

$$2\leqslant|z+i|\leqslant 3$$

lie on the boundaries and inside a ring shaped region bounded by two concentric circles centred at $-i$ and having radius $r_1=2$ and $r_2=3$ (See figure above)

(vii) $|z+i|=|z-2|$ or $|z+i|^2=|z-2|^2$
or $|x+iy+i|=|x+iy-2|^2$ or $x^2+(y+1)^2=(x-2)^2+y^2$
or $4x+2y-3=0$ (1)

Hence points z satisfying the given equation lie on the straight line (1)

Alternative We know that $|z+i|=|z-(-i)|$ is the distance from the point z to the point representing the number $-i$ and $|z-2|$ is the distance from the point z to the point representing the number 2 It is required to find the points for which

$=\frac{1}{2}[(-y-x)x+y(x-y)]$
$=\frac{1}{2}(-x^2-y^2)=\frac{1}{2}(x^2+y^2)$, numerically
$=\frac{1}{2}|z|^2$

64 (i) Let the vertices A, B, C, of a $\triangle ABC$ be represented by z_1, z_2, z_3 respectively

Suppose $z_2-z_3=\alpha,\ z_3-z_1=\beta,\ z_1-z_2=\gamma$

Then $\alpha+\beta+\gamma=0$ (1)

$\overline{\alpha+\beta+\gamma}=0$ or $\bar{\alpha}+\bar{\beta}+\bar{\gamma}=0$ (2)

We first assume that $\triangle ABC$ is equilateral

Then $BC=CA=AB$

or $|z_2-z_3|=|z_3-z_1|=|z_1-z_2|$

or $|\alpha|=|\beta|=|\gamma|$ or $|\alpha|^2=|\beta|^2=|\gamma|^2$

or $\alpha\bar{\alpha}=\beta\bar{\beta}=\gamma\bar{\gamma}=k$, say (3)

From (2) and (3), we have

$k/\alpha+k/\beta+k/\gamma=0 \Rightarrow 1/\alpha+1/\beta+1/\gamma=0$

or $$\frac{1}{z_2-z_3}+\frac{1}{z_3-z_1}+\frac{1}{z_1-z_2}=0 \quad (4)$$

$\{z_1^2-z_1(z_2+z_3)+z_2z_3\}+\{\ \}+\{\ \}=0$

i e $z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1$

Conversely, assume that the condition (4) holds Then we have to prove that the $\triangle$ is equilateral,

Since $z_2-z_3=\alpha$, $z_3-z_1=\beta$ and $z_1-z_2=\gamma$, we get from (4)

$1/\alpha+1/\beta+1/\gamma=0$ or $\beta\gamma+\gamma\alpha+\alpha\beta=0$

or $\beta\gamma+(\beta+\gamma)\alpha=0$

or $\beta\gamma+(-\alpha)\alpha=0$ $[\because \alpha+\beta+\gamma=0]$

or $\alpha^2=\beta\gamma$ (5)

Then $\overline{(\alpha^2)}=\overline{\beta\gamma}$ or $(\bar{\alpha}^2)=\bar{\beta}\bar{\gamma}$ (6)

Now (5) and (6) imply that

$\alpha^2\bar{\alpha}^2=\beta\gamma\bar{\beta}\bar{\gamma} \Rightarrow (\alpha\bar{\alpha})^2=\beta\bar{\beta}\gamma\bar{\gamma} \Rightarrow (\alpha\bar{\alpha})^3=\alpha\bar{\alpha}\beta\bar{\beta}\gamma\bar{\gamma}$

Similarly $(\beta\bar{\beta})^3=\alpha\bar{\alpha}\beta\bar{\beta}\gamma\bar{\gamma}$ and $(\gamma\bar{\gamma})^3=\alpha\bar{\alpha}\beta\bar{\beta}\gamma\bar{\gamma}$

Hence $(\alpha\bar{\alpha})^3=(\beta\bar{\beta})^3=(\gamma\bar{\gamma})^3$

$\Rightarrow \alpha\bar{\alpha}=\beta\bar{\beta}=\gamma\bar{\gamma} \Rightarrow |\alpha|^2=|\beta|^2=|\gamma|^2$

$\Rightarrow |z_2-z_3|^2=|z_3-z_1|^2=|z_1-z_2|$

$\Rightarrow |z_2-z_3|=|z_3-z_1|=|z_1-z_2|$

$\Rightarrow BC=CA=AB$

Hence the $\triangle ABC$ is equilateral

(ii) $|z_1|=|z_2|=|z_3|=1$, the origin is the circumcentre of the triangle and circum radius is 1 Hence we may write $z_2=z_1e^{2\lambda i}$, or $(z-1)(\bar{z}-1)+(z+1)(\bar{z}+1)=4$

or $z\bar{z}-(z+\bar{z})+1+z\bar{z}+(z+\bar{z})+1=4$

or $2z\bar{z}=2$ or $z\bar{z}=1$ or $|z|^2=1$

Hence $|z|=1$

[Note that we cannot take $|z|=-1$ since $|z|$ is non negative]

Thus the equation (1) represents all points z on the circle with centre origin and radius unity

41 We are given $z=x+iy$ and $|z|=r=$const

(a) Let $Z=z+2$ Then $Z-2=z$ (1)

$|Z-2|=|z|=r$

(1) shows that points corresponding to $Z=z+2$ lie on a circle of radius r and centre (2, 0)

(b) Let $Z=z-1+i$ or $Z+1-i=z$ (2)

Then $|Z+1-i|=|z|=r$

(2) shows that the points representing $Z=z-1+i$ lie on a circle of radius r and having its centre at the point $-1+i$, that is at (—1, 1)

42 (a) Ans On a circle of radius 3 and centre (2, 0)

(b) Ans On a circle of radius 9 and centre (—1, 0)

43 $\log_{1/2}|z-2|>\log_{1/2}|z|$ (1)

We first observe that the left hand member of inequality (1) is meaningful for all complex numbers z, except $z=2$ And the right member is meaningful for all z except 0 So we seek the solution of the inequality (1) among complex numbers except $z=0, 2$

Since here the base of logarithms is 1/2 which is less than 1, inequality (1) implies that

$$|z-2|<|z|$$

$\Rightarrow |z-2|^2<|z|^2 \Rightarrow (z-2)(\bar{z}-2)<z\bar{z}$

$\Rightarrow z\bar{z}-2(z+\bar{z})+4<z\bar{z} \Rightarrow -4x+4<0$

$\Rightarrow -4x<-4 \Rightarrow x>1$

Hence the given inequality is satisfied by all complex numbers z representing the points on the Argand plane to the right of the line $x=1$ i e to the right of the line Re $(z)=1$ except the point (2, 0) represented by $z=2$ as remarked in the beginning

(ii) $\log_{\sqrt{3}}\dfrac{|z|^2-|z|+1}{2+|z|}<2$

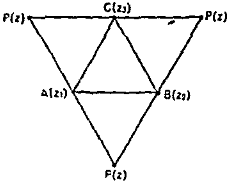
$$\Rightarrow \frac{|z|^2-|z|+1}{2+|z|}<(\sqrt{3})^2$$

Now the four points A, B, C, P form a parallelogram in the following three orders

(i) A, B, P, C (ii) B, C, P, A and (iii) C, A, P, B

(See the figure)

In case (i), the condition for A, B, P, C to form a parallelogram is



$$\overrightarrow{AB}=\overrightarrow{CP} \text{ i.e. } z_2-z_1=z-z_3$$

or $$z=z_3+z_2-z_1$$

Similarly in case (ii) and (iii), the required conditions are

$$\overrightarrow{BC}=\overrightarrow{AP} \text{ or } z_3-z_2=z-z_1 \text{ i.e. } z=z_3+z_1-z_2$$

and $$\overrightarrow{CA}=\overrightarrow{BP} \text{ or } z_1-z_3=z-z_2 \text{ i.e. } z=z_1+z_2-z_3$$

67 Let A, B, C be the points of affix z_1, z_2, z_3 and A', B', C', the points of affix z_1', z_2', z_3'. Then the $\triangle ABC$ and $\triangle A'B'C'$ are similar if

$$\overrightarrow{AB}=\lambda\,\overrightarrow{A'B'} \text{ i.e. } z_2-z_1=\lambda(z_2'-z_1')$$

and $$\overrightarrow{BC}=\lambda\,\overrightarrow{B'C'} \text{ i.e. } z_3-z_2=\lambda(z_3'-z_2')$$

$$\frac{z_2-z_1}{z_3-z_2}=\frac{z_2'-z_1'}{z_3'-z_2'},$$

or $$z_2(z_3'-z_2)-z_1(z_3'-z_2)=z_3(z_2'-z_1')-z_2(z_2'-z_1')$$

or $$z_1(z_2'-z_3')+z_2(z_3'-z_1')+z_3(z_1'-z_2')=0$$

or $$\begin{vmatrix} z_1 & z_1' & 1 \\ z_2 & z_2' & 1 \\ z_3 & z_3' & 1 \end{vmatrix}=0$$

68 Let $z_k=\cos\theta_k+i\sin\theta_k$

Then $$\frac{1}{z_k}=(\cos\theta_k+i\sin\theta_k)^{-1}=\cos\theta_k-i\sin\theta_k$$

$$z_1+z_2=(x_1+x_2)+i(y_1+y_2)$$

and $$z_1z_2=x_1x_2-y_1y_2+i(x_1y_2+x_2y_1)$$

Since z_1+z_2 and z_1z_2 are both real, we must have

$$y_1+y_2=0 \text{ and } x_1y_2+x_2y_1=0$$

These give $y_2=-y_1$ and $x_2=x_1$

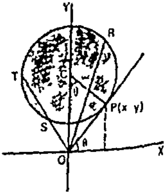
Hence $z_2=x_2+iy_2=x_1-iy_1=\bar{z}_1$, that is, z_1 and z_2 are conjugate complex

46 (a) The complex numbers z satisfying the condition

$$|z-25i|\leqslant 15 \qquad (1)$$

are represented by the points inside and on the circle of radius 15 and centre at the point C (0, 25) (See the figure below)

From the figure, it is clear that the complex number z satisfying (1) and having least positive argument corresponds to the point $P(x, y)$ which is the point of contact of a ray issuing from the origin and lying in the first quadrant to the above circle. The positive argument of all other points within and on the circle is greater than the argument of P



From the figure, we have

$$OC=25,\ CP=\text{radius}=15 \text{ and } \angle CPO=90$$

Hence $OP=\sqrt{(OC^2-CP^2)}=\sqrt{(25^2-15^2)}=20$

If $\angle PCO=\theta$, than $\angle PON=\theta$. Also $\cos\theta=\frac{PC}{OC}=\frac{15}{25}=\frac{3}{5}$

$$x=ON=OP\cos\theta=20\times(3/5)=12,$$

and $y=PN=OP\sin\theta=20\times(4/5)=16$

Hence P represents the complex number $z=x+iy=12+16i$ which is therefore the required value of z satisfying the given conditions

(b) We have $2=|z-4/z|\geqslant|z|-\left|\frac{4}{z}\right|$ or $|z|-\frac{4}{|z|}\leqslant$

or $\quad |z|^2-2|z|-4\leqslant 0$ or $(|z|-1)^2-5\leqslant 0$

or $\quad (|z|-1)^2\leqslant 5$ or $|z|-1\leqslant\sqrt{5}$ or $|z|\leqslant\sqrt{5}+1$

so that $x+iy=\dfrac{(a+i)}{a^2-i^2}=\dfrac{a^2-1+2ia}{a^2+1}$

Then $x=\dfrac{a^2-1}{a^2+1}$

$\dfrac{1+x}{1-x}=\dfrac{2a^2}{2}$ (By componendo and dividendo)

$a=\sqrt{\left(\dfrac{1+x}{1-x}\right)}=\dfrac{1+x}{\sqrt{(1-x^2)}}=\dfrac{1+x}{y}$ from (1)

Substituting $a=\dfrac{1+x}{y}$ in (2), we get

$x+iy=\dfrac{(1+x)/y+i}{(1+x)/y-i}$.

Since $y\neq 0$ $(1+x)/y$ is a real number

Hence $x+iy=\dfrac{a+i}{a-i}$ where $a=\dfrac{1+x}{y}$

71 $\dfrac{b+ic}{1+a}=z$ $\dfrac{b-ic}{1+a}=\bar{z}$ and $z\bar{z}=\dfrac{b^2+c^2}{(1+a)^2}$ (1)

Also $a^2+b^2+c^2=1$ (2)

$\dfrac{1+iz}{1-iz}=\dfrac{1+iz}{1-iz}\times\dfrac{1+i\bar{z}}{1+i\bar{z}}=\dfrac{1+i(z+\bar{z})-z\bar{z}}{1-i(z-\bar{z})+z\bar{z}}$

$=\left[1+i\dfrac{2b}{1+a}-\dfrac{(b^2+c^2)}{(1+a)^2}\right]-\left[1-\dfrac{i(2ic)}{1+a}+\dfrac{(b^2+c^2)}{(1+a)^2}\right]$

Put $b^2+c^2=1-a^2=(1-a)(1+a)$ by (2)

$\dfrac{1+iz}{1-iz}=\left[1+i\dfrac{2b}{1+a}-\dfrac{1-a}{1+a}\right]-\left[1+\dfrac{2c}{1+a}+\dfrac{1-a}{1+a}\right]$

$=\dfrac{2(a+ib)}{2(1+c)}=\dfrac{(a+ib)}{(1+c)}$

72 Let z_1, z_2, z_3, z_4 be the affixes of the points A B C, D We have the identity

$(z_1-z_4)(z_2-z_3)+(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)=0$

or $|-(z_1-z_4)(z_2-z_3)|=|(z_2-z_4)(z_3-z_1)+(z_3-z_4)(z_1-z_2)|$

or $|(z_1-z_4)(z_2-z_3)|\leqslant|(z_2-z_4)(z_3-z_1)+|(z_3-z_4)(z_1-z_2)|$

or $|(z_1-z_4)||(z_2-z_3)|\leqslant|(z_2-z_4)||(z_3-z_1)|+|(z_3-z_4)||(z_1-z_2)|$

i e $AD\ BC\leqslant BD\ CA+CD\ AB$

73 When $|z|$ assumes the maximum possible value, $|1/z|=1/|z|$ assumes the minimum value Hence it suffices to find those z whose modulus assumes the greatest value under the assumption $|z|>|1/z|$

Let $z=r(\cos\theta+i\sin\theta)$ $(0\leqslant\theta\leqslant\pi/2)$ (See the figure) and $\quad z_2 = c + \{-1 - \sqrt{(1 - c^2 - 2c)}\}\, i$

For $\Delta = 0$, we have $z_1 = z_2$ that is there is only one solution in this case For $\Delta < 0$, the equation has no roots

It remains to indicate the range of c over which there are solutions We are given $c \geqslant 0$ and in addition to this we found that c must satisfy the inequality

$$1 - c^2 - 2c \geqslant 0 \text{ or } c^2 + 2c - 1 \leqslant 0,$$

or
$$(c+1)^2 - 2 \leqslant 0 \text{ or } (c+1)^2 \leqslant 2 \text{ which implies}$$
$$-\sqrt{2} \leqslant c+1 \leqslant \sqrt{2} \text{ or } -1-\sqrt{2} \leqslant c \leqslant -1+\sqrt{2}$$

Now choosing the numbers $c \geqslant 0$ from this interval, we obtain

$$0 \leqslant c \leqslant -1+\sqrt{2}$$

Thus the final answer is as follows

(i) For $0 \leqslant c < -1+\sqrt{2}$, there are two roots given by
$$z_{1,2} = c + (-1 \pm \sqrt{(1-c^2-2c)})\, i$$

(ii) For $c = -1+\sqrt{2}$, there is only one solution given by
$$z = -1 + \sqrt{2} - i$$

(iii) For $c > -1+\sqrt{2}$, there is no solution

48 Putting $z = x + iy$ the given equation becomes
$$x + iy + c\,|\,x + iy + 1\,| + i = 0$$
or
$$x + iy + c\sqrt{[(x+1)^2 + y^2]} + i = 0$$

Equating real and imaginary parts to zero we get
$$c\sqrt{[(x+1)^2 + y^2]} + x = 0 \quad y + 1 = 0$$

whence $y = -1$ and for x, we get the equation
$$c\sqrt{[(x+1)^2 + (-1)^2]} + x = 0$$

or $\quad c^2\,[(x+1)^2 + 1] = x^2$ or $(c^2-1)\,x^2 + 2c^2\,x + 2c^2 = 0 \qquad (1)$

Since x is real, we must have
$$\Delta = 4c^4 - 4\,(c^2-1)\,2c^2 \geqslant 0 \text{ or } 2c^2 - c^4 \geqslant 0$$

or $\quad c^4 - 2c^2 \leqslant 0$ or $(c^2-1)^2 - 1 \leqslant 0$ or $(c^2-1)^2 \leqslant 1$

which implies that $-1 \leqslant (c^2-1) \leqslant 1$ or $0 \leqslant c^2 \leqslant 2$

or $\qquad 0 \leqslant c \leqslant \sqrt{2} \qquad (2)$

Since $c \geqslant 1$, from (2) we choose the interval $1 \leqslant c \leqslant \sqrt{2}$

Now we discuss four cases ,

(i) If $c = 1$, the equation (1) reduces to
$$x + 1 = 0 \text{ which gives } x = -1$$

Since $y = -1$, the original equation in this case has only one solution $\quad -1 - i$

(ii) If $1 < c < \sqrt{2}$ then (1) gives
$$x = \frac{-2c^2 \pm \sqrt{[4c^4 - 8c^2\,(c^2-1)]}}{2\,(c^2-1)} = \frac{-c \pm c\sqrt{(2-c^2)}}{c^2-1}$$

$=\sqrt{[r_1^2\cos 2\theta_1+r_2^2\cos 2\theta_2)^2+(r_1^2\sin 2\theta_1+r_2^2\sin 2\theta_2)^2]}$

$=\sqrt{[r_1^4+r_2^4+2r_1^2r_2^2\cos 2(\theta_1-\theta_2)]}$

Hence $\dfrac{2\,|\,az_1+bz_2\,|^2}{a^2+b^2}=\dfrac{2\,|\,r\cos\alpha\; r_1(\cos\theta_1+i\sin\theta_1)+r\sin\alpha\; r_2(\cos\theta_2+i\sin\theta_2)\,|^2}{r^2}$

$=2\,|\,(r_1\cos\alpha\cos\theta_1+r_2\sin\alpha\cos\theta_2)^2+i\,(r_1\cos\alpha\sin\theta_1+r_2\sin\alpha\sin\theta_2)^2\,|$

$=2\,[(r_1\cos\alpha\cos\theta_1+r_2\sin\alpha\cos\theta_2)^2+(r_1\cos\alpha\sin\theta_1+r_2\sin\alpha\sin\theta_2)^2]$

$=2\,[r_1^2\cos^2\alpha+r_2^2\sin^2\alpha+2\,r_1r_2\sin\alpha\cos\alpha\cos(\theta_1-\theta_2)]$

$=[r_1^2(1+\cos 2\alpha)+r_2^2]\,(1-\cos 2\alpha)+\,2r_1r_2\sin 2\alpha\cos(\theta_1-\theta_2)]$

$=[A+B\cos 2\alpha+C\sin 2\alpha]$

where $A=r_1^2+r_2^2$, $B=r_1^2-r_2^2$, $C=2r_1r_2\cos(\theta_1-\theta_2)$

$=[A+R\sin(2\alpha+\phi)]$ where $B=R\sin\phi$, $C=R\cos\phi$

Hence max and min values of $\dfrac{2\,|\,az_1+bz_2\,|^2}{a^2+b^2}$ are respectively $[A+R]$ and $[A-R]$ i e $[A+\sqrt{(B^2+C^2)}]$ and $[A-\sqrt{(B^2+C^2)}]$ so that

$$A-\sqrt{(B^2+C^2)}\leqslant\frac{2\,|\,az_1+bz_2\,|^2}{a^2+b^2}\leqslant A+\sqrt{(B^2+C^2)}$$

But $A=r_1^2+r_2^2=|\,z_1\,|^2+|\,z_2\,|^2$ and $B^2+C^2=|\,z_1^2+z_2^2\,|$ etc

Problem Set (B)

(Objective Questions)

$z+\bar{z}=0$ if and only if

(a) Re $(z)=0$, (b) $I_m\,(z)=0$,

(c) None of these

$z\bar{z}=0$ if and only if

(a) Re $(z)=0$, (b) $I_m\,(z)=0$,

(c) $z=0$, (d) None of these

arg bi $(b>0)$ is

(i) π, (ii) $\frac{\pi}{2}$,

(iii) $-\frac{\pi}{2}$ (iv) 0

arg a $(a<0)$ is

(i) 0, (ii) $\frac{\pi}{2}$,

(iii) π, (iv) None of these

arg z + arg $\bar{z}$ $(z\neq 0)$

(a) 0 (b) π,

(v) $\left(\frac{1+\cos\phi+i\sin\phi}{1+\cos\phi-i\sin\phi}\right)^n$

$$=\left(\frac{2\cos^2\phi/2+2i\sin\phi/2\cos\phi/2}{2\cos^2\phi/2-2i\sin\phi/2\cos\phi/2}\right)^n$$
$$=\left(\frac{\cos\phi/2+i\sin\phi/2}{\cos\phi/2-i\sin\phi/2}\right)^n$$
$$=(\cos\phi/2+i\sin\phi/2)^n(\cos\phi/2-i\sin\phi/2)^{-n}$$
$$=(\cos\phi/2+i\sin\phi/2)^n(\cos\phi/2+i\sin\phi/2)^n$$
$$=(\cos\phi/2+i\sin\phi/2)^{2n}$$
$$=\cos\{2n(\phi/2)\}+i\sin\{2n(\phi/2)\}$$
$$=\cos n\phi+i\sin n\phi$$

51 (a) $x_1\ x_2,\ x_3$

$$=(\cos\pi/2+i\sin\pi/2)(\cos\pi/2^2+i\sin\pi/2^2)(\cos\pi/2^3+i\sin\pi/2^3)$$
$$=\cos(\pi/2+\pi/2^2+\pi/2^3\ \ldots)+i\sin(\pi/2+\pi/2^2+\pi/2^3+\ \ldots)$$
$$=\cos\left(\frac{\pi/2}{1-1/2}\right)+i\sin\left(\frac{\pi/2}{1-1/2}\right)$$

(Summing the infinite G P)

$=\cos\pi+i\sin\pi=-1$ [$\cos\pi=-1$ and $\sin\pi=0$]

(b) $x^2-2x+4=0$,

$$x=\frac{2\pm\sqrt{(4-16)}}{2}=1\pm3i$$

Let $\alpha=1+\sqrt{3}i,\ \beta=1-\sqrt{3}i$

(*b*) Now put $1=r\cos\theta,\ \sqrt{3}=r\sin\theta$

Then $r=2,\ \theta=\frac{\pi}{3}$

$$\alpha^n+\beta^n=(1+\sqrt{3}i)^n+(1-\sqrt{3}i)^n$$
$$=[(r\cos\theta+ri\sin\theta)^n-(r\cos\theta-ri\sin\theta)^n]$$
$$=r^n[\cos n\theta+i\sin n\theta+\cos n\theta-i\sin n\theta]$$
$$=r^n\ 2\cos n\theta=2\ 2^n\cos(n\pi/3)$$
$$=2^{n+1}\cos(n\pi/3)$$

From the given relations

$x=\cos\theta+i\sin\theta \qquad y=\cos\phi+i\sin\phi$

(i) $xyz \qquad =\cos(\theta+\phi+\psi+\quad)+i\sin(\theta+\phi+\psi+\quad)$

$\frac{1}{xyz} \qquad =\cos(\theta+\phi+\psi+\quad)-i\sin(\theta+\phi+\psi+\quad)$

$$xyz+\frac{1}{xyz}=2\cos(\theta+\phi+\psi)$$

(ii) $\frac{x}{y}=\frac{(\cos\theta+i\sin\theta)}{(\cos\phi+i\sin\phi)}=(\cos\theta+i\sin\theta)(\cos\phi-i\sin\phi)$

17 Multiplying a complex number by i rotates the vector representing the complex number through $180°$

(a) True, (b) False

18 $\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)^{1000}=-\frac{1}{2}+\frac{\sqrt{3}}{2}i$

(a) True, (b) False

19 If the complex number z_1 and z_2 are such that the sum z_1+z_2 is a real number, then they are necessarily conjugate complexes

(a) True, (b) False

20 If the complex numbers z_1 and z_2 are such that the product $z_1 z_2$ is a real number, then they are necessarily conjugate complexes

(a) True, (b) False

21 There exist no real numbers x and y such that

$$(2-i)\,x+(1+3i)\,x+2=0$$

(a) True, (b) False

22 Let z be a complex number Then the equation $z^4+z+2=0$ cannot have a root such that $|z|<1$

(a) True, (b) False

23 If $|a_i|<1$, $\lambda_i \geqslant 0$ for $i=1, 2, \ldots n$ and $\lambda_1+\lambda_2+\ldots+\lambda_n=1$, then $|\lambda_1 a_1+\lambda_2 a_2+\ldots+\lambda_n a_n|<1$

(a) True, (b) False,

24 If $2+i\sqrt{3}$ is a root of the equation $x^2+px+q=0$ where p and q are real then $(p, q)=($ $)$ (I I T 81)

25 The greatest and least value of $|z+1|$ if $|z+4|\leqslant 3$ is and

26 If the imaginary part of $\frac{2z+1}{iz+1}$ is -2, then the locus of the point representing z in the complex plane is

(a) a circle, (b) a straight line

(c) a parabola (d) none of these

27 The complex number $z=x+iy$ which satisfy the equation $\left|\frac{z-5i}{z+5i}\right|=1$ lie on

(a) The x axis, (b) the straight line $y=5$,

(c) a circle passing through the origin, (d) none of these

(I I T 82)

28 The locus of the point z satisfying the condition $\arg\frac{z-1}{z+1}=\frac{\pi}{3}$ is

$$(\cos\alpha+i\sin\alpha)+(\cos\beta+i\sin\beta)+(\cos\gamma+i\sin\gamma)=0 \quad (3)$$

Let $z_1=\cos\alpha+i\sin\alpha$, $z_2=\cos\beta+i\sin\beta$, $z_3=\cos\gamma+i\sin\gamma$

Then (3) gives $z_1+z_2+z_3=0$

Hence $z_1^3+z_2^3+z_3^3-3z_1z_2z_3$

$=(z_1+z_2+z_3)(z_1^2+z_2^2+z_3^2-z_1z_2-z_2z_3-z_3z_1)=0$

or $z_1^3+z_2^3+z_3^3=3z_1z_2z_3$

$(\cos\alpha+i\sin\alpha)^3+(\cos\beta+i\sin\beta)^3+(\cos\gamma+i\sin\gamma)^3$

$=3(\cos\alpha+i\sin\alpha)(\cos\beta+i\sin\beta)(\cos\gamma+i\sin\gamma)$

or $(\cos 3\alpha+i\sin 3\alpha)+(\cos 3\beta+i\sin 3\beta)+(\cos 3\gamma+i\sin 3\gamma)$ (4)

$=3[\cos(\alpha+\beta+\gamma)+i\sin(\alpha+\beta+\gamma)]$

(i) Equating real parts in (4), we get

$\cos 3\alpha+\cos 3\beta+\sin 3\gamma=3\cos(\alpha+\beta+\gamma)$

(ii) Equating imaginary parts in (4) we get

$\sin 3\alpha+\sin 3\beta+\sin 3\gamma=3\sin(\alpha+\beta+\gamma)$

(iii) To prove this, we observe that

$\frac{1}{z_1}+\frac{1}{z_2}+\frac{1}{z_3}=(\cos\alpha+i\sin\alpha)^{-1}+(\cos\beta+i\sin\beta)^{-1}+(\cos\gamma+i\sin\gamma)^{-1}$

$=\cos\alpha-i\sin\alpha+\cos\beta-i\sin\beta+\cos\gamma-i\sin\gamma$

$=0$ by (1) and (2)

or $z_2z_3+z_3z_1+z_1z_2=0$ (5)

Also $z_1+z_2+z_3=0$ whence squaring, we get

$z_1^2+z_2^2+z_3^2+2z_1z_2+2z_2z_3+2z_3z_1=0$

or $z_1^2+z_2^2+z_3^2=0$ by (5)

$(\cos\alpha+i\sin\alpha)^2+(\cos\beta+i\sin\beta)^2+(\cos\gamma+i\sin\gamma)^2=0$

or $\cos 2\alpha+i\sin 2\alpha+\cos 2\beta+i\sin 2\beta+\cos 2\gamma+i\sin 2\gamma=0$

In this equating real and imaginary parts to zero, we obtain

$\cos 2\alpha+\cos 2\beta+\cos 2\gamma=\sin 2\alpha+\sin 2\beta+\sin 2\gamma=0$

56 (i) Put $2=r\cos\theta$ $2=r\sin\theta$ then $r=2\sqrt{2}$ and $\theta=\pi/4$

Hence $(2-2i)^{1/3}=(r\cos\theta-ri\sin\theta)^{1/3}$

$=r^{1/3}[\cos\theta-i\sin\theta]^{1/3}$

$=r^{1/3}[\cos(2n\pi+\theta)-i\sin(2n\pi+\theta)]^{1/3}$

$=(2\sqrt{2})^{1/3}[\cos(2n\pi+\pi/4)-i\sin(2n\pi+\pi/4)]^{1/3}$

$=\sqrt{2}[\cos\frac{1}{3}(2n\pi+\pi/4)-i\sin\frac{1}{3}(2n\pi+\pi/4)]$

Putting $n=0, 1, 2$ the required roots are

$\sqrt{2}[\cos\pi/12-i\sin\pi/12]$ $\sqrt{2}[\cos 3\pi/4-i\sin 3\pi/4]$

and $\sqrt{2}[\cos 17\pi/12-i\sin 17\pi/12]$

Now $\cos\frac{3\pi}{4}=-\frac{1}{\sqrt{2}}$, $\sin\frac{3\pi}{4}=\frac{1}{\sqrt{2}}$,

(i) interior of an ellipse, (ii) exterior of a circle,
(iii) interior and boundary of an ellipse, (iv) none of these

38 In a geometrical progression first term and common ratio are both $\frac{1}{2}(\sqrt{3}+i)$ Then the absolute value of the nth term of the progression is
(a) 2^n (b) 4^n, (c) 1, (b) none of these

39 Given that the equation $z^2+(p+iq)z+r+is=0$ where p, q r s are and non zero has a real root Then
(a) $pqr=r^2+p^2s$, (b) $prs=q^2+r^2p$
(c) $qrs=p^2+s^2q$ (d) $pqs=s^2+q^2r$

40 If the complex numbers Z_1, Z_2, Z_3 represent the vertices of an equilateral triangle such that $|Z_1|=|Z_2|=|Z_3|$
Then $Z_1+Z_2+Z_3\neq 0$
(a) True, (b) False (I I T 84)

41 The points z_1 z_2, z_3, z_4 in the complex plane are the vertices of a parallelogram taken in order if and only if
(a) $z_1+z_4=z_2+z_3$ (b) $z_1+z_3=z_2+z_4$
(c) $z_1+z_2=z_3+z_4$ (d) None of these (I I T 83)

42 The complex numbers, z_1 z_2, z_3 are the vertices A B C of a parallelogram $ABCD$, then the fourth vertex D is
(a) $\frac{1}{2}(z_1+z_2)$, (b) $\frac{1}{4}(z_1+z_2+z_3+z_4)$
(c) $\frac{1}{3}(z_1+z_2+z_3)$, (d) $z_1+z_3-z_2$

43 Among the complex numbers z satisfying the condition $|z+1-i|\leqslant 1$, the number having the least positive argument is
(a) $1-i$, (b) $-1+i$,
(c) $-i$ (d) none of these

44 If $z=x+iy$ and $\omega=\frac{1-iz}{z-i}$, then $|\omega|=1$ implies that, in the complex plane
(a) z lies on the imaginary axis
(b) z lies on the real axis
(c) z lies on the unit circle
(d) None of these (I I T 83)

45 If $z=re^{i\theta}$, then $|e^{iz}|=$

46 The vector $z=3-4i$ is turned anticlockwise through an angle of 180° and stretched 2 5 times The complex number corresponding to the newly obtained vector is

47 $\frac{(\cos\theta+i\sin\theta)^4}{(\sin\theta+i\cos\theta)^5}$ is equal to

If we put $\alpha=\cos 2\pi/n+i\sin 2\pi/n$, then

$$\cos 4\pi/n+i\sin 4\pi/n=(\cos 2\pi/n+i\sin 2\pi/n)^2=\alpha^2$$

$$\cos 6\pi/n+i\sin 6\pi/n=(\cos 2\pi/n+i\sin 2\pi/n)^3=\alpha^3$$

and so on. Hence the roots are $1, \alpha, \alpha^2, \alpha^3, \ldots \alpha^{n-1}$ where $\alpha=\cos 2\pi/n+i\sin 2\pi/n$

These roots clearly form a G P

58 As shown in ex 57, an imaginary root of the equation $z^n=1$ is of the form

$$\beta=\cos\frac{2k\pi}{n}+i\sin\frac{2k\pi}{n}, (1\leqslant k\leqslant n-1)$$

Now $1+\beta+\beta^2+\ldots+\beta^{n-1}$

$$=\frac{1(1-\beta^n)}{1-\beta}=\frac{1-(\cos 2k\pi/n+i\sin 2k\pi/n)^n}{1-(\cos 2k\pi/n+i\sin 2k\pi/n)}$$

$$=\frac{1-(\cos 2k\pi+i\sin 2k\pi)}{1-(\cos 2k\pi/n+i\sin 2k\pi/n)}$$

$$=\frac{1-1-i0}{1-\cos 2k\pi/n-i\sin 2k\pi/n}=0$$

[Note that since $1\leqslant k\leqslant n-1$, $2k\pi/n$ is not a multiple of 2π and so $1-\cos 2k\pi/n-i\sin 2k\pi/n\neq 0$]

59 (a) As in ex 57, the n, n^{th} roots of unity are $1, \alpha, \alpha^2, \ldots, \alpha^{n-1}$ where $\alpha=\cos 2\pi/n+i\sin 2\pi/n$

We have to find the sum of the p^{th} powers of these roots

$$1^p+\alpha^p+(\alpha^2)^p+(\alpha^3)^p+\ldots+(\alpha^{n-1})^p$$

$$=1+\alpha^p+\alpha^{2p}+\alpha^{3p}+\ldots+\alpha^{(n-1)p} \quad (1)$$

$$=\frac{1-(\alpha^p)^n}{1-\alpha^p}$$ [Summing the G P of common ratio α^p)

$$=\frac{1-\alpha^{pn}}{1-\alpha^p}=\frac{1-(\cos 2\pi/n+i\sin 2\pi/n)^{pn}}{1-(\cos 2\pi/n+i\sin 2\pi/n)^p}$$

$$=\frac{1-\cos 2\pi p-i\sin 2\pi p}{1-\cos 2\pi p/n-i\sin 2\pi p/n}$$

$$=\frac{1-1-i\,0}{1-\cos 2\pi p/n-i\sin 2\pi p/n}$$

$$=\frac{0}{1-\cos 2\pi p/n-i\sin 2\pi p/n}$$

$=0$ if p is not a multiple of n

If p is a multiple of n, then

Solutions

1 Ans [a]

2 Ans [c]

3 Ans (ii) Since $b > 0$, bi represents a point on the positive side of the imaginary axis on which the argument of every point is $\pi/2$

4 Ans (iii) Here a will represent a point on the negative side of real axis where the argument of each point is π

5 Ans (a),

$$\arg z + \arg \bar{z} = \arg z\bar{z} = \arg |z|^2$$

Since $|z|^2$ is a positive real number, we have $\arg |z|^2 = 0$

6 Ans (c)

$$z = \bar{z} \Rightarrow (x + iy)^2 = x - iy$$
$$\Rightarrow x^2 - y^2 - x + i(2xy + y) = 0$$
$$\Rightarrow x^2 - y^2 - x = 0 \text{ and } 2xy + y = 0$$

Now $2xy + y = 0$ gives $y = 0$ or $x = -\frac{1}{2}$

When $y = 0$, $x^2 - y^2 - x = 0$ gives $x^2 - x = 0$ or $x = 0, 1$

When $x = -\frac{1}{2}$, $x^2 - y^2 - x = 0$ gives $\frac{1}{4} - y^2 + \frac{1}{2} = 0$

or $$y = \pm\frac{\sqrt{3}}{2}$$

Hence there are four solutions

$$z_1 = 0 + i\,0 = 0,\ z_2 = 1 + i0 = 1,$$

$$z_3 = -\tfrac{1}{2} + \frac{\sqrt{3}}{2}\,i = \omega \text{ and } z_4 = -\tfrac{1}{2} - \frac{\sqrt{3}}{2}\,i = \omega^2$$

7 Ans [c]

Here $\frac{-1+\sqrt{(-3)}}{2} = \omega$ and $\frac{-1-\sqrt{(-3)}}{2} = \omega^2$,

$$\omega^{100} + (\omega^2)^{100} = \omega^{99}\,\omega + \omega^{198}\,\omega^2 = \omega + \omega^2 = -1$$

8 Ans [b]

9 Ans [d], since there exists no relation of $>$ and $<$ in complex numbers

10 Ans [d]

$$\frac{(1+i)^2}{3+i} = \frac{1+i^2+2i}{3-i} = \frac{2i(3+i)}{3^2-i^2} = \frac{6i-2}{10}$$

$$\text{Re}\left(\frac{(1+i)}{3-i}\right) = -\frac{2}{10} = -\frac{1}{5}$$

11 Ans (c)

12 Ans (iii)

$$(-64)^{1/4} = [(-64)^{1/2}]^{1/2} = [\pm 8\sqrt{(-1)}]^{1/2} = [\pm 8i]^{1/2}$$

$$= \left|1-\cos\frac{2(r-1)\pi}{n} - i\sin\frac{2(r-1)\pi}{n}\right|^2$$

[$\because$ $|z_1|$ = radius of the circle = 1]

$$= \left(1-\cos\frac{2(r-1)\pi}{n}\right)^2 + \sin^2\frac{2(r-1)\pi}{n}$$

$$= 2-2\cos\frac{2(r-1)\pi}{n}$$

Hence $\sum_{r=2}^{n} |A_1A_r|^2 = 2(n-1) - 2 \sum_{r=2}^{n}\left[\cos\frac{2(r-1)\pi}{n}\right]$

Let $S = \sum_{r=2}^{n} \cos\frac{2(r-1)\pi}{n} = \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n}$

$$+\cos\frac{6\pi}{n} + \ldots + \cos\frac{2(n-1)}{n}\pi$$

Then $2\sin\frac{\pi}{n}\ S = 2\sin\frac{\pi}{n}\cos\frac{2\pi}{n} + 2\sin\frac{\pi}{n}\cos\frac{4\pi}{n} + \ldots$

$$+ 2\sin\frac{\pi}{n}\cos\frac{(2n-1)}{n}\pi$$

$$= \left(\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}\right) + \left(\sin\frac{5\pi}{n} - \sin\frac{3\pi}{n}\right) + \ldots$$

$$+ \left(\sin\frac{2n-1}{n}\pi - \sin\frac{2n-3}{n}\pi\right)$$

$$= \sin\frac{2n-1}{n}\pi - \sin\frac{\pi}{n} = 2\cos\pi\ \sin\left(\pi - \frac{\pi}{n}\right) = -2\sin\frac{\pi}{n}$$

$$S = -1$$

Hence $\sum_{r=2}^{n} |A_1A_r|^2 = 2(n-1) - 2(-1) = 2n$

(ii) From part (i),

$$|A_1A_r| = |z_1|\ |1-e^{2(r-1)\pi i/n}|$$
$$= |1-e^{2(r-1)\pi i/n}| \qquad [\because |z_1| = 1]$$

Hence $|A_1A_2|\ |A_1A_3| \ldots |A_1A_n|$

$$= |1-e^{2\pi i/n}|\ |1-e^{4\pi i/n}| \ldots |1-e^{2(n-1)\pi i/n}| \qquad (1)$$

Since $e^{2\pi i/n}, e^{4\pi i/n}, \ldots e^{2(n-1)\pi i/n}$ are the $n-1$ imaginary, nth roots of unity, we have the identity

$$z^n - 1 \equiv (z-1)(z-e^{2\pi i/n})(z-e^{4\pi i/n}) \ldots (z-e^{2(n-1)\pi i/n})$$

or $\frac{z^n-1}{z-1} \equiv (z-e^{2\pi i/n})(z-e^{4\pi i/n}) \ldots (z-e^{2(n-1)\pi i/n})$

18 Ans (a)

$-\frac{1}{2}+\frac{\sqrt{3}}{2}i=\omega$, we have $\omega^{1000}=\omega^{999}\omega=\omega$

1 Ans (b)

For example take $z_1=2+i$ and $z_2=5-i$

20 Ans (b)

For example take $z_1=2i$ and $z_2=5i$

21 Ans (b) $x=-\frac{8}{7}, y=-\frac{4}{7}$

22. Ans (a)

From the given equation, we have

$$2=-z-z^4$$

$$|2|=|-z-z^4|=|z+z^4|\leqslant|z|+|z^4|$$

Thus $|2|\leqslant|z|+|z|^4<2$ $[\because |z|<1]$

or $2<2$ which is impossible

Hence if $|z|<1$, the equation $z^4+z+2=0$ cannot have a root

23 Ans (a)

$$|\lambda_1 a_1+\lambda_2 a_2+\ldots+\lambda_n a_n|$$
$$\leqslant|\lambda_1 a_1|+|\lambda_2 a_2|+\ldots+|\lambda_n a_n|$$
$$=|\lambda_1||a_1|+|\lambda_2||a_2|+\ldots+|\lambda_n||a_n|$$
$$=\lambda_1|a_1|+\lambda_2|a_2|+\ldots+\lambda_n|a_n| \quad [\because \lambda_i>0]$$
$$<\lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_n \quad [\because |a_i|<1]$$
$$=1$$

Thus $|\lambda_1 a_1+\lambda_2 a_2+\ldots+\lambda_n a_n|<1$

24 Ans $(-4, 7)$

Since complex roots occur in conjugate pairs, the roots of the equation are $\alpha=2+i\sqrt{3}$ and $\beta=2-i\sqrt{3}$

$$\alpha+\beta=-p \text{ and } \alpha\beta=q$$

that is $4=-p$ and $7=q$

Thus $p=-4$ and $q=7$

25 Ans 6 and 0

$$|z+1|=|z+4-3|=|(z+4)+(-3)|$$
$$\leqslant|z+4|+|-3|=|z+4|+3$$
$$\leqslant 3+3=6 \quad [\because |z+4|\leqslant 3]$$

Hence the greatest value of $|z+1|$ is 6

To find the least value, we first observe that the least value of modulus of any complex number is 0

Now $|z+1|=0$ if $z=-1$

and for this value of z, we have

$$|z+4|=|-1+4|=3$$

Then $[|z_1|+|z_2|]^2=|z_1|^2+|z_2|^2+2|z_1||z_2|$
$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|z_1z_2|$
$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\alpha^2-(\alpha^2-\beta^2)|$
$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\beta|^2$
$=2|\alpha|^2+2|\beta|^2+2|\alpha+\beta||\alpha-\beta|$
$=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha+\beta||\alpha-\beta|$ [using (1)]
$=[|\alpha+\beta|+|\alpha-\beta|]^2$

or $|z_1|+|z_2|=|\alpha+\beta|+|\alpha-\beta|$

that is, $|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|=|\alpha+\beta|+|\alpha-\beta|$

(b) Hint Using the identity (1) of part (a), we get

R H S $=\frac{1}{2}|(\sqrt{z_1}+\sqrt{z_2})^2|+\frac{1}{2}|(\sqrt{z_1}-\sqrt{z_2})^2|$
$=\frac{1}{2}|\sqrt{z_1}+\sqrt{z_2}|^2+\frac{1}{2}|\sqrt{z_1}-\sqrt{z_2}|^2$
$=\frac{1}{2}\,2|\sqrt{z_1}|^2+\frac{1}{2}\,2|\sqrt{z_2}|^2=|z_1|+|z_2|$,

63 (i) Let $z_1=x_1+iy_1=(x_1\ y_1)$

$z_2=x_2+iy_2=(x_2,y_2)$, $z_3=x_3+iy_3=(x_3,y_3)$

The required area

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}=\frac{1}{2i}\begin{vmatrix} x_1 & x_1+iy_1 & 1\\ x_2 & x_2+iy_2 & 1\\ x_3 & x_3+iy_3 & 1\end{vmatrix}$$

$$=\frac{1}{2i}\begin{vmatrix} x_1 & z_1 & 1\\ x_2 & z_2 & 1\\ x_3 & z_3 & 1\end{vmatrix}=\frac{1}{2i}\Sigma x_1(z_2-z_3)$$

$$=\frac{1}{2i}\Sigma\tfrac{1}{2}(z_1+\bar{z}_1)(z_2-z_3)$$

$$=\frac{1}{4i}\Sigma z_1(z_2-z_3)+\frac{1}{4i}\Sigma\bar{z}_1(z_2-z_3)$$

$$=\frac{1}{4i}\Sigma\frac{z_1\bar{z}_1}{z_1}(z_2-z_3)\qquad[\ \Sigma z_1(z_2-z_3)=0]$$

$$=\frac{1}{4i}\Sigma\frac{|z_1|^2}{z_1}(z_2-z_3)=\Sigma\frac{|z_1|^2(z_2-z_3)}{4iz_1}$$

(ii) We have $z=x+iy=(x,y)$,

$iz=ix-y=(-y,x)$

and $z+iz=(x-y,x+y)$

Hence area of triangle

$$=\frac{1}{2}\begin{vmatrix} x & y & 1\\ -y & x & 1\\ x-y & x+y & 1\end{vmatrix}=\frac{1}{2}\begin{vmatrix} x & y & 1\\ -y-x & x-y & 0\\ -y & x & 0\end{vmatrix}$$

18 Ans (a)

$$-\tfrac{1}{2}+\frac{\sqrt{3}}{2}i=\omega, \text{ we have } \omega^{1000}=\omega^{999}\,\omega=\omega$$

1 Ans (b)

For example take $z_1=-2+i$ and $z_2=5-i$

20 Ans (b)

For example take $z_1=2i$ and $z_2=5i$

21 Ans (b) $x=-\frac{4}{7},\ y=-\frac{2}{7}$

22. Ans (a)

From the given equation, we have

$$2=-z-z^4$$

$$|2|=|-z-z^4|=|z+z^4|\leqslant|z|+|z^4|$$

Thus $|2|\leqslant|z|+|z|^4<2$ [$|z|<1$]

or $2<2$ which is impossible

Hence if $|z|<1$, the equation $z^4+z+2=0$ cannot have a root

23 Ans (a)

$$|\lambda_1a_1+\lambda_2a_2+\ \ +\lambda_na_n|$$
$$\leqslant|\lambda_1a_1|+|\lambda_2a_2|+\ \ +|\lambda_na_n|$$
$$=|\lambda_1||a_1|+|\lambda_2||a_2|+\ \ +|\lambda_n||a_n|$$
$$=\lambda_1|a_1|+\lambda_2|a_2|+\ \ +\lambda_n|a_n| \quad [\ \lambda_i\geqslant 0]$$
$$<\lambda_1+\lambda_2+\lambda_3+\ \ +\lambda_n \quad [\ |a_i|<1]$$
$$=1$$

Thus $|\lambda_1a_1+\lambda_2a_2+\ \ +\lambda_na_n|<1$

24 Ans $(-4, 7)$

Since complex roots occur in conjugate pairs, the roots of the equation are $\alpha=2+i\sqrt{3}$ and $\beta=2-i\sqrt{3}$

$$\alpha+\beta=-p \text{ and } \alpha\beta=q$$

that is $4=-p$ and $7=q$

Thus $p=-4$ and $q=7$

25 Ans 6 and 0

$$|z+1|=|z+4-3|=|(z+4)+(-3)|$$
$$\leqslant|z+4|+|-3|=|z+4|+3$$
$$\leqslant 3+3=6 \quad [\ |z+4|\leqslant 3]$$

Hence the greatest value of $|z+1|$ is 6

To find the least value, we first observe that the least value of modulus of any complex number is 0

Now $|z+1|=0$ if $z=-1$

and for this value of z, we have

$$|z+4|=|-1+4|=3$$

and $z_3 = z_1 e^{(2A+2B)i}$ [Note that $\angle BOC = 2A$, $\angle COA = 2B$]

Hence $z_1 + z_2 + z_3 = 0 \Rightarrow z_1 + z_1 e^{2Ai} + z_1 e^{(2A+2B)i} = 0$

$\Rightarrow 1 + (\cos 2A + i \sin 2A) + \cos(2A+2B) + i \sin(2A+2B) = 0$

$\Rightarrow \cos 2A + \cos(2A+2B) = -1$

and $\sin 2A + \sin(2A+2B) = 0$

Squaring and adding $2 + 2\cos 2B = 1$

or $\cos 2B = -\frac{1}{2}$

Hence $2B = 120°$ or $B = 60°$

Now $1 + \cos 2A + \cos(2A+2B) = 0$ gives

$\cos 2A + \cos(2A+120°) = -1$ or $2\cos(2A+60°)\cos 60 = -1$

or $\cos(2A+60°) = -1$ or $2A+60° = 180°$

$A = 60°$ Hence $C = 60°$ Δ is equilateral

(iii) $BC = AC$ and $\angle C = \frac{1}{2}\pi$, we have

$$(z_2 - z_3) = (z_1 - z_3)e^{\frac{1}{2}\pi i} = i(z_1 - z_3)$$

or $(z_2 - z_3)^2 = -(z_1 - z_3)^2$

or $z_2^2 + z_3^2 - 2z_2z_3 = -z_1^2 - z_3^2 + 2z_1z_3$

or $z_1^2 + z_2^2 - 2z_1z_2 = 2z_1z_3 + 2z_2z_3 - 2z_3^2 - 2z_1z_2$

or $(z_1 - z_2)^2 = 2[(z_1z_3 - z_3^2) - (z_1z_2 - z_2z_3)]$

$= 2(z_1 - z_3)(z_3 - z_2)$

(iv) $1, w, w^2$ are the cube roots of unity and they satisfy the condition $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ for the three points z_1, z_2, z_3 to form an equilateral triangle by part (i)

65 Let the vertices of the Δ ABC be represented by z_1, z_2 and z_3. Then by our remark (i) of § 3

$$(z_3 - z_1) = (z_2 - z_1)e^{i\pi/3} \text{ and } (z_1 - z_2) = (z_3 - z_2)e^{i\pi/3}$$

whence on dividing, we get

$$\frac{z_3 - z_1}{z_1 - z_2} = \frac{z_2 - z_1}{z_3 - z_2}$$

or $(z_3 - z_1)(z_3 - z_2) = -(z_2 - z_1)^2$

or $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$ (1)

Now for an equilateral triangle, circumcentre is the same as the centroid so that

$$z_0 = (z_1 + z_2 + z_3)/3$$

or $9z_0^2 = z_1^2 + z_2^2 + z_3^2 + 2z_1z_2 + 2z_2z_3 + 2z_3z_1$

$= 3z_1^2 + 3z_2^2 + 3z_3^2$ from (1)

or $3z_0^2 = z_1^2 + z_2^2 + z_3^2$

66 Let A, B, C be the points represented by the numbers z_1, z_2, z_3 and P be the point represented by z

29 Ans (a)

$1=1-z+z$, we have

$1=|1-z+z| \leqslant |1-z|+|z|=|z-1|+|z|$

Thus $|z|+|z-1| \geqslant 1$ and so the minimum value of $|z|+|z-1|$ is 1

30 Ans (iv)

$|z-4|<|z-2| \Rightarrow |z-4|^2<|z-2|^2$

$\Rightarrow |x+iy-4|^2<|x+iy-2|^2$

$\Rightarrow (x-4)^2+y^2<(x-2)^2+y^2$

$\Rightarrow x^2-8x+16+y^2<x^2-4x+4+y^2$

$\Rightarrow -4x<-12$

$\Rightarrow x>3$

$\Rightarrow \mathrm{Re}\,(z)>3$

31. Ans (ii)

Put $\frac{\sqrt{3}}{2}=r\cos\theta,\ \frac{1}{2}=r\sin\theta$

Then $r^2=\frac{3}{4}+\frac{1}{4}=1,\ \cos\theta=\frac{\sqrt{3}}{2},\ \sin\theta=\frac{1}{2}$

$r=1$ and $\theta=\frac{\pi}{6}$

$z=(r\cos\theta+ri\sin\theta)^5+(r\cos\theta-ri\sin\theta)^5$

$=[(\cos\theta+i\sin\theta)^5+(\cos\theta-i\sin\theta)^5]$ $\quad [\ r=1]$

$=\cos 5\theta+i\sin 5\theta+\cos 5\theta-i\sin 5\theta$

By De Moivre's theorem

$=2\cos 5\theta$,

which is real Hence $I_m\,(z)=0$

32 Ans (b)

$\left|\frac{z-a}{z+\bar{a}}\right|=1 \Rightarrow |z-a|^2=|z+\bar{a}|^2$

$\Rightarrow (z-a)(\bar{z}-\bar{a})=(z+\bar{a})(\bar{z}+a)$

$\Rightarrow z\bar{z}-z\bar{a}-\bar{z}a+a\bar{a}=z\bar{z}+za+\bar{z}\bar{a}+a\bar{a}$

$\Rightarrow z\,(a+\bar{a})+\bar{z}\,(a+\bar{a})=0$

$\Rightarrow z+\bar{z}=0$ $\quad [\ a+\bar{a}=2\,\mathrm{Re}\,(a)\neq 0]$

$\Rightarrow 2x=0$

$\Rightarrow x=0$,

which is the equation of y axis

33 Ans (a)

34 Ans (a)

Solving the given equation for a, we get

$a=\frac{2\sin x\pm\sqrt{(4\sin^2 x-4)}}{2}=\sin x+i\cos x$

Then $|z_k| = \sqrt{(\cos^2 \theta_k + \sin^2 \theta_k)} = 1,\ k = 1, 2, \ldots n$

Now $z_1 + z_2 + \ldots + z_n$

$= (\cos \theta_1 + \cos \theta_2 + \ldots + \cos \theta_n) + i(\sin \theta_1 + \sin \theta_2 + \ldots + \sin \theta_n)$

And $1/z_1 + 1/z_2 + \ldots + 1/z_n$

$= (\cos \theta_1 + \cos \theta_2 + \ldots + \cos \theta_n) - i(\sin \theta_1 + \sin \theta_2 + \ldots + \sin \theta_n)$

Hence $|z_1 + z_2 + \ldots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|$

since each side is equal to

$\sqrt{[(\cos \theta_1 + \cos \theta_2 + \ldots + \cos \theta_n)^2 + (\sin \theta_1 + \sin \theta_2 + \ldots + \sin \theta_n)^2]}$

Alternative $|z_k| = 1 \Rightarrow |z_k|^2 = 1$ (1)

$\Rightarrow z_k \bar{z}_k = 1,\ k = 1, 2, \ldots, n$

Then $|z_1 + z_2 + \ldots + z_n| = |\overline{z_1 + z_2 + \ldots + z_n}|$

$= |\bar{z}_1 + \bar{z}_2 + \ldots + \bar{z}_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \ldots + \frac{1}{z_n}\right|$ from (1)

(*b*) We have

$(|z_1| + |z_2|)\left|\frac{z_1}{|z_1|} + \frac{z_2}{|z_2|}\right| \leqslant (|z_1| + |z_2|)\left[\left|\frac{z_1}{|z_1|}\right| + \left|\frac{z_2}{|z_2|}\right|\right]$

$= (|z_1| + |z_2|)\left[\frac{|z_1|}{|z_1|} + \frac{|z_2|}{|z_2|}\right]$

$= 2(|z_1| + |z_2|)$

69 $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$

$\Rightarrow |z_1 - z_2| < |1 - \bar{z}_1 z_2|$

$\Rightarrow |z_1 - z_2|^2 < |1 - \bar{z}_1 z_2|^2$

$\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2) < (1 - \bar{z}_1 z_2)(1 - z_1 \bar{z}_2)$

$\Rightarrow z_1 \bar{z}_1 - z_1 \bar{z}_2 - z_2 \bar{z}_1 + z_2 \bar{z}_2 < 1 - \bar{z}_1 z_2 - z_1 \bar{z}_2 + z_1 \bar{z}_1 z_2 \bar{z}_2$

$\Rightarrow |z_1|^2 + |z_2|^2 < 1 + |z_1|^2 |z_2|^2$

$\Rightarrow |z_1|^2 + |z_2|^2 - 1 - |z_1|^2 |z_2|^2 < 0$ (1)

$\Rightarrow (|z_1|^2 - 1)(1 - |z_2|^2) < 0$

Now the inequality (1) will hold if

$|z_1| < 1$ and $|z_2| < 1$

Hence $\left|\frac{z_1 - z_2}{1 - \bar{z}_1 z_2}\right| < 1$ if $|z_1| < 1$ and $|z_2| < 1$

70 We are given $y \neq 0$ and $|x + iy| = 1$ i.e. $x^2 + y^2 = 1$ (1)

Let $x + iy$ be represented as

$$x + iy = \frac{a + i}{a - i} \quad (2)$$

$$\Rightarrow 2(x^2+y^2)+2+2\,|(x+iy)^2-1| \leqslant 16$$
$$\Rightarrow |x^2-y^2-1+2ixy| \leqslant 7-x^2-y^2$$
$$\Rightarrow \sqrt{[(x^2-y^2-1)^2+4x^2y^2]} \leqslant 7-x^2-y^2$$
$$\Rightarrow (x^2-y^2-1)^2+4x^2y^2 \leqslant (7-x^2-y^2)^2$$
$$\Rightarrow (x^2-y^2)^2-2(x^2-y^2)+1+4x^2y^2 \leqslant 49+(x^2+y^2)^2-14(x^2+y^2)$$
$$\Rightarrow -2(x^2-y^2)+1 < 49-14(x^2+y^2)$$
$$\Rightarrow 12x^2+16y^2 \leqslant 48$$
$$\Rightarrow 3x^2+4y^2 \leqslant 12$$
$$\Rightarrow \frac{x^2}{4}+\frac{y^2}{3} \leqslant 1$$

38 Ans (c)

Here $a=\frac{1}{2}(\sqrt{3}+i)$, $r=\frac{1}{2}(\sqrt{3}+i)$

$$T_n=ar^{n-1}=[\tfrac{1}{2}(\sqrt{3}+i)]^n$$

Put $\frac{\sqrt{3}}{2}=r\cos\theta$, $\frac{1}{2}=r\sin\theta$

These give $r=1$ and $\theta=\pi/6$,

$$T_n=[r(\cos\theta+i\sin\theta)]^n$$
$$=\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)^n$$
$$=\cos\frac{n\pi}{6}+i\sin\frac{n\pi}{6},$$
$$|T_n|=\sqrt{\left(\cos^2\frac{n\pi}{6}+\sin^2\frac{n\pi}{6}\right)}=1$$

39 Ans (d)

$$z^2+(p+iq)z+r+is=0 \quad (1)$$

or $(x+iy)^2+(p+iq)(x+iy)+r+is=0$

or $(x^2-y^2+px-qy+r)+i(2xy+qx+py+s)=0$

Equating real and imaginary parts, we get

$$x^2-y^2+px-qy+r=0 \quad (2)$$

and $2xy+qx+py+s=0$ (3)

If the roots of (1) are real, then $y=0$,

(2) and (3) give

$$x^2+px+r=0 \text{ and } qx+s=0, \quad \text{or} \quad x=-s/q$$

Eliminating x, we get

$$\frac{s^2}{q^2}+p\left(-\frac{s}{q}\right)+r=0$$

or $s^2-pqs+q^2r=0$

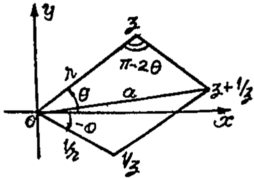
or $pqs=s^2+q^2r$,

which is the required condition

40 Ans (b)

Let Z_1, Z_2, Z_3 represent the vertices A, B, C of $\triangle ABC$ respectively and let O be the origin

Then $Z_1=\overrightarrow{OA}$, $Z_2=\overrightarrow{OB}$, $Z_3=\overrightarrow{OC}$



Since $| z+1/z | = a$, we can write this relation as

$$\left| r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta) \right| = a$$

or $\quad \left(r\cos\theta + \frac{1}{r}\cos\theta \right)^2 + \left(r\sin\theta - \frac{1}{r}\sin\theta \right)^2 = a^2$

or $\quad a^2 = r^2 + \frac{1}{r^2} + 2\cos 2\theta = r^2 + \frac{1}{r^2} - 2 + 4\cos^2\theta$

or $\quad \left(r - \frac{1}{r} \right)^2 = a^2 - 4\cos^2\theta \qquad \cos 2\theta = 2\cos^2\theta - 1$

By hypothesis, we have $r \geqslant \frac{1}{r}$, and so when r increases the difference $r - \frac{1}{r}$ increases and vice versa. Further, we have

$$\left(r - \frac{1}{r} \right)^2 = a^2 - 4\cos^2\theta \leqslant a^2$$

For $\theta = \pi/2$, we have $(r - 1/r)^2 = a^2$, and so $r - 1/r = a$, which gives $r = \{a + \sqrt{(a^2+4)}\}/2$ (rejecting $-$ive sign, why?)
It follows that the greatest value of $| z | = \{a + \sqrt{(a^2+4)}\}/2$ is attained for
$z = [\{a+\sqrt{(a^2+4)}\}/2](\cos\pi/2 + i\sin\pi/2) = i\{a+\sqrt{(a^2+4)}\}/2$.
Similarly smallest value of $| z | = r$ is given by

$$\frac{1}{r} - r = a \quad \text{at} \quad z = -ri$$

which gives $r = \{\sqrt{(a^2+4)} - a\}/2$ at $z = -i\ \{\sqrt{(a^2+4)} - a\}/2$

74 Put $\quad a = r\cos\alpha \quad b = r\sin\alpha$, $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$
and $\quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2) \qquad$ Then
$| z_1 | = r_1$, $| z_2 | = r_2$ and
$| z_1^2 + z_2^2 | = | r_1^2(\cos\theta_1 + i\sin\theta_1)^2 + r_2^2(\cos\theta_2 + i\sin\theta_2)^2 |$
$\qquad = | r_1^2(\cos 2\theta_1 + i\sin 2\theta_1) + r_2^2(\cos 2\theta_2 + i\sin 2\theta_2) |$

6 $-\frac{15}{2}+10i$

7 Ans. (d)

8 Ans (B)

Let the complex numbers $a, b, c,$ and u, v, w represent the vertices A, B, C and U, V, W, of the two Δs ABC and UVW respectively

Put $b-a=r_1e^{i\theta_1}$, $c-a=r_2e^{i\theta_2}$, $v-u=p_1e^{i\phi_1}$, $w-u=p_2e^{i\phi_2}$ and $r=\lambda e^{i\alpha}$

Substituting these values in the given relations

$c-a=r(b-a)$ and $w-u=(v-u)r$, we get

$$r_2e^{i\theta_2}=\lambda e^{i\alpha}r_1e^{i\theta_1} \quad (1)$$

and $$p_2e^{i\phi_2}=p_1e^{i\phi_1}\lambda e^{i\alpha} \quad (2)$$

whence equating moduli and arguments, in (1) we get

$r_2=\lambda r_1$, $\theta_2=\theta_1+\alpha$

i e $AC=\lambda.\ AB$ and $\angle CAB=\theta_2-\theta_1=\alpha$

Similarly from (2) we shall get

$UW=\lambda\ UV$ and $\angle\ WUV=\phi_2-\phi_1=\alpha$

Thus we get

$$\frac{AC}{UW}=\frac{AB}{UV} \text{ and } \angle\ CAB=\angle\ WUV$$

If follows that the Δs ABC and UVW are similar

9 Ans (A), (B) and (C)

We are given $|z_1|=|z_2|=1$ and $Re\,(z_1\bar{z}_2)=0$

that is $a^2+b^2=1$ (1)

$c^2+d^2=1$ (2)

and $ac+bd=0$ (3)

From (3), $\frac{a}{d}=-\frac{b}{c}=\lambda$, say

Then $a=d\lambda$ and $b=-c\lambda$

Then (1) gives $d^2\lambda^2+c^2\lambda^2=1$

or $\lambda^2(c^2+d^2)=1$ or $\lambda^2=1$ by (2)

Hence $\lambda=\pm1$, Thus $a=d$ and $b=-c$

or $a=-d$ and $b=c$

Putting $d=a$ in (1) and (2), we get

$b^2+d^2=1$ and $c^2+a^2=1$

i e $|\omega_2|=1$ and $|\omega_1|=1$

Again $a=d$ and $b=-c$ gives $ab+cd=0$

i e $Re\,(\omega_1\bar{\omega}_2)=0$

Note that $a=-d$ and $b=c$ will give the same result

(c) $\frac{\pi}{2}$, (d) None of these

6 The equation $z^2=\bar{z}$ has
(a) no solution (b) 2 solutions,
(c) four solutions, (d) an infinite number of solutions

7 $\left(\frac{1+\sqrt{(-3)}}{2}\right)^{100}+\left(\frac{-1-\sqrt{(-3)}}{2}\right)^{100}=$
(a) 2, (b) 0,
(c) -1 (d) 1

8 $(3+\omega+3\omega^2)^4$ equals
(a) 16, (b) 16ω,
(c) $16\omega^2$ (d) None of these

9 Which of the following is correct ?
(a) $2+3i > 1+4i$, (b) $6+2i > 3+3i$,
(c) $4+8i > 5+7i$ (d) None of these

10 The real part of $(1+i)^2/(3-i)$ is
(a) $\frac{1}{5}$ (b) $\frac{1}{5}$
(c) $-\frac{1}{5}$ (d) None of these

11 A square root of $3+4i$ is
(a) $\sqrt{3}+i$ (b) $2-i$,
(c) $2+i$ (d) None of these

12 $(-64)^{1/4}=$
(a) $\pm 2(1+i)$ (b) $\pm 2(1-i)$
(c) $\pm 2(1\pm i)$ (d) None of these

13 The equation $\bar{b}z+b\bar{z}=c$ where b is a non zero complex constant and c is real represents
(a) a circle (b) a straight line
(d) None of these

14 The smallest positive integer for which $\left(\frac{1+i}{1-i}\right)^n=1$ is
(a) $n=8$ (b) $n=12$
(c) $n=16$ (d) None of these (I I T 80)

15 If the cube roots of unity are 1 ω ω^2 then the roots of equation $(x-1)^3+8=0$ are
(a) -1 $1+2\omega$ $1+2\omega^2$ (b) -1 $1-2\omega$ $1-2\omega^2$
(c) -1 -1 -1 (d) None of these
(I I T 79 M N R 86)

16 $\overline{(z^{-1})}=(\bar{z})^{-1}$
(a) True (b) False

Now it is clear that $\frac{(z_1+z_2)}{(z_1-z_2)}$ is zero or pure imaginary

For example, if $z_1=1+i$ $z_2=-1-i$

then $z_1 \neq z_2$ $|z_1|=|z_2|$,

Re $(z_1)>0$ and $I_m(z_2)<0$ In this case

$I_m(z_1\bar{z}_2)=0$ since $z_1\bar{z}_2=-2$ Hence

$\frac{z_1+z_2}{z_1-z_2}=0$ in this case

Note that $a=-d$ and $b=c$ will give the same result

Again for $z_1=1+i$, $z_2=1-i$, we have

$z_1\bar{z}_2=2i$ so that $I_m(z_1\bar{z}_2)=0$

So in this case

$\frac{z_1+z_2}{z_1-z_2}$ is pure imaginary

54 (a) Hint We know that $z=z\bar{z}$ and $\overline{(z_1 \pm z_2)}=\bar{z}_1 \pm \bar{z}_2$

(b) Ans (d)

(a) a straight line, (b) a circle,
(c) a parabola, (d) none of these

29 For any complex number z the minimum value of $|z| + |z-1|$ is
(a) 1, (b) 0, (c) $\frac{1}{2}$, (d) 3/2

30 The inequality $|z-4| < |z-2|$ represents the region given by
(i) Re $(z) > 0$, (ii) Re $(z) < 0$,
(iii) Re $(z) > 2$, (iv) none of these (IIT 82)

31 If $z = \left(\frac{\sqrt{3}}{2}+\frac{i}{2}\right)^5 + \left(\frac{\sqrt{3}}{2}-\frac{i}{2}\right)^5$, then
(i) Re $(z) = 0$ (ii) $I_m(z) = 0$,
(iii) Re $(z) > 0$ $I_m(z) > 0$ (iv) Re $(z) > 0$, $I_m(z) < 0$
(IIT 82)

32 The region of the z plane for which $\left|\frac{z-a}{z+a}\right| = 1$ (Re $a \neq 0$) is
(a) x axis, (b) y axis,
(c) the straight line $x = |a|$, (d) none of these

33 The solution of the equation $|z| - z = 1 + 2i$ is
(a) $\frac{3}{2} - 2i$, (b) $\frac{3}{2} + 2i$ (c) $2 - \frac{3}{2}i$ (d) none of these

34 The equation $a^2 - 2a \sin x + 1 = 0$ has only two possible real solutions for a
(a) True (b) False

35 For complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x + iy$, we write $z_1 \cap z$ if $x_1 \leqslant x$ and $y_1 \leqslant y_2$. Then for all complex numbers z with $1 \cap z$ we have $\frac{1-z}{1+z} \cap 0$ (IIT 81)
(a) True (b) False

36 (a) The one and only case in which
$$|z_1| + |z_2| + \dots + |z_n| = |z_1 + z_2 + \dots z_n|$$
is that in which the numbers $z_1, z_2, z_3 \dots z_n$ have all the same amplitude
(a) True (b) False
(b) If z_1 and z be any two non zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$ then Arg z_1 − Arg z_2 is equal to zero
IIT 87

37 The region of Argand diagram defined by $|z-1| + |z+1| < 4$ is

quantity then the resulting series thus obtained will also be in A P

(2) Any three numbers in A P be taken as a−d, a, a+d Any four numbers in A P be taken as a 3d, a−d, a+d, a+3d

Similarly five terms in A P should be taken as $a-2d$, $a-d$, a, $a+d$, $a+2d$ and six terms as $a-5d$, $a-3d$, $a-d$ $a+d$ $a+3d$ $a+5d$ etc

(e) Two important Properties of A P

(1) In an A P the sum of terms equidistant from the beginning and end is constant and equal to the sum of first and last terms

(2) Any term of an A P (except the first) is equal to half the sum of terms which are equidistant from it

$a_n=\frac{1}{2}(a_{n-k}+a_{n+k})$ $k<n$ and for $k=1$ $a_n=\frac{1}{2}(a_{n-1}+a_{n+1})$

Problem Set (A)

1 The fifth term of an A P is 1 whereas its 31st term is −77 Find its 20th term and sum of its first fifteen terms Also find which term of the series will be −17 and sum of how many terms will be 20

2 Find the number of terms in the series

$$20,\ 19\tfrac{1}{3},\ 18\tfrac{2}{3}$$

of which the sum is 300 explain the double answer

3 (a) The nth term of a series is given to be $\frac{3+n}{4}$ find the sum of 105 terms of this series

(b) Find the sum of first 24 terms of the A P a_1 a a_3 if is it known that $a_1+a_5+a_{10}+a_{15}+a_{20}+a_{24}=225$

(c) Find $a_1+a_6+a_{11}+a_{16}$ if it is known that a_1 a_t is an A P and $a_1+a_4+a_7+$ $+a_{16}=147$

4 If the sum of first 8 and 19 terms of an A P are 64 and 361 respectively, find the common difference and sum of its n terms

5 (a) A man arranges to pay off a debt of Rs 3600 in 40 annual instalments which form an arithmetic series When 30 of the instalments are paid he dies leaving one third of the debt unpaid Find the value of the first instalment

(b) A class consists of a number of boys whose ages are in A P the common difference being four months If the youngest boy is just eight years old and if the sum of the ages is 168 years, find the number of boys

6 The interior angles of a polygon are in arithmetic progre

(a) $\cos\theta - i\sin\theta$
(b) $\cos 9\theta - i\sin 9\theta$
(c) $\sin\theta - i\cos\theta$
(d) $\sin 9\theta - i\cos 9\theta$ (M N R 85)

48 If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that

$c=(1-r)\,a+rb$ and $\omega=(1-r)\,u+rv$,

where r is a complex number, then the two triangles—

(a) have the same area (b) are similar
(c) are congruent (d) none of these

(I I T 85)

49 If $z_1=a+ib$ and $z\ =c+id$ are complex numbers such that $|z_1| = |z_*| =$ and $Re\,(z_1 z\ )=0$ then the pair of complex numbers $w_1=a+ic$ and $w\ =b+id$ satisfies—

(a) $|w_1|=l$ (b) $|w_2|=l$
(c) $Re\,(w_1\overline{w})=0$ (d) none of these (I I T 85)

50 If three complex numbers are in A P then they lie on a circle in the complex plane

(a) True (b) False (I I T 85)

51 Common roots of the equations
$z^3+2z^*+2z+1=0$ and $z^{198}+z^{100}+1=0$ are

52 The point $z=0$ lies within some polygon whose vertices are at the points

$z_k-1+z+z^2 \quad | z^{k-1}, \ |z|<1$

(a) True (b) False

53 Let z_1 and z be complex numbers such that $z_1\neq z_2$ and $|z_1| = |z_*$ If z_1 has positive real part and z_2 has negative imaginary part, then $(z_1+z_2)/(z_1-z_2)$ may be

(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) None of these (I I T 86)

54 (a) For any two complex numbers z_1, z, and any real numbers a and b

(a) $|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b\ )\,[\,|z_1|^2+|z_2|^2]$

(I I T 88)

(b) The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$ are conjugate to each other for

(a) $x=n-$ (b) $x=(n+\frac{1}{4})-$ (c) $x=0$ (d) No value of x

(I I T 88)

$$\frac{p+q}{2}\left[a+b+\frac{a-b}{p-q}\right]$$ (Roorkee 56)

17 If the pth, qth and rth terms of an A P be a, b, and c respectively, then prove that

$$a(q-r)+b(r-p)+c(p-q)=0$$

18 If the sums of p, q and r terms of an A P be a, b and c respectively then prove that

$$\frac{a}{p}(q-r)+\frac{b}{q}(r-p)+\frac{c}{r}(p-q)=0$$

19 (i) If in an A P the sum of p terms is equal to sum of q terms, then prove that the sum of $p+q$ terms is zero

(ii) In an A P, of which a is the first term if the sum of the first p terms is zero, show that the sum of the next q term is

$$-\frac{a(p+q)\,q}{p-1}$$

20 (i) The sum of first p terms of an A P is q and the sum of the first q terms is p Find the sum of the first $(p+q)$ terms (I I T 67)

(ii) Prove that the sum of the latter half of $2n$ terms of any A P is one third the sum of $3n$ terms of the same A.P

21 The sums of n terms of three arithmetical progressions are S_1, S_2 and S_3 The first term of each is unity and the common differences are 1, 2 and 3 respectively Prove that

$$S_1+S_3=2S_2$$

22 (a) The sums of n $2n$, $3n$ terms of an A P are S_1 S_2, and S_3 respectively Prove that $S_3=3(S_2-S_1)$

(b) If $S_n=n^2p$ and $S_m=m^2p, m\neq n$ in an A P prove that $S_p=p^3$

23 (a) There are n A P's whose common differences are 1, 2, 3 n respectively the first term of each being unity Prove that sum of their nth terms is $\frac{1}{2}n(n^2+1)$

(b) If there be m A P s beginning with unity whose common differences are 1, 2, 3 m respectively, show that the sum of their nth terms is

$$\tfrac{1}{2}m\,[mn-m+n+1]$$

(c) The sum of n terms of m arithmetical progressions are S_1, S_2, S_3 S_m The first term and common differences are 1, 2, 3,.. m respectively Prove that

$$S_1+S_2+S_3+\quad+S_m=\tfrac{1}{4}mn(m+1)(n+1)$$

(a) $\cos\theta - i\sin\theta$
(b) $\cos 9\theta - i\sin 9\theta$
(c) $\sin\theta - i\cos\theta$
(d) $\sin 9\theta - i\cos 9\theta$ (M N R 85)

48 If a, b, c and u, v, w are complex numbers representing the vertices of two triangles such that
$c=(1-r)a+rb$ and $\omega=(1-r)u+rv$,
where r is a complex number, then the two triangles—
(a) have the same area (b) are similar
(c) are congruent (d) none of these
(I I T 85)

49 If $z_1=a+ib$ and $z_2=c+id$ are complex numbers such that $|z_1|=|z_2|=1$ and $Re(z_1\bar{z}_2)=0$ then the pair of complex numbers $w_1=a+ic$ and $w_2=b+id$ satisfies—
(a) $|w_1|=1$ (b) $|w_2|=1$
(c) $Re(w_1\overline{w}_2)=0$ (d) none of these (I I T 85)

50 If three complex numbers are in A P, then they lie on a circle in the complex plane
(a) True (b) False (I I T 85)

51 Common roots of the equations
$z^3+2z^2+2z+1=0$ and $z^{198}+z^{100}+1=0$ are

52 The point $z=0$ lies within some polygon whose vertices are at the points
$z_k = 1+z+z^2 + \ldots + z^{k-1}$, $|z|<1$
(a) True (b) False

53 Let z_1 and z_2 be complex numbers such that
$z_1 \neq z_2$ and $|z_1|=|z_2|$. If z_1 has positive real part and z_2 has negative imaginary part then $(z_1+z_2)/(z_1-z_2)$ may be
(a) zero (b) real and positive
(c) real and negative (d) purely imaginary
(e) None of these (I I T 86)

54 (a) For any two complex numbers z_1, z_2, and any real numbers a and b
(a) $|az_1-bz_2|^2+|bz_1+az_2|^2=(a^2+b^2)[|z_1|^2+|z_2|^2]$
(I I T 88)

(b) The complex numbers $\sin x+i\cos 2x$ and $\cos x-i\sin 2x$ are conjugate to each other for
(a) $x=n\pi$ (b) $x=(n+\frac{1}{2})\pi$ (c) $x=0$ (d) No value of x
(I I T 88)

(i) $b+c, c+a, a+b,$

(ii) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (Roorkee 75)

(iii) $a^2(b+c), b^2(c+a), c^2(a+b),$

(iv) $a\left(\frac{1}{b}+\frac{1}{c}\right), b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$

(v) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

37 If a^2, b^2, c^2 are in A P, then the following are also in A P

(i) $\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}$

(ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

38 If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A P, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A P

39 If $(b-c)^2, (c-a)^2, (a-b)^2$ are in A P then prove that $\frac{1}{b-c}, \frac{1}{c-b}, \frac{1}{a-b}$ are also in A P

40 If $a_1, a_2 \ldots a_n$ are in A P where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+\ldots+\frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

(I I T 1982)

41 (a) If $\log_{10} 2$, $\log_{10}(2^x-1)$ and $\log_{10}(2^x+3)$ be three consecutive terms of an A P, then

(i) $x=0$, (ii) $x=1$,
(iii) $x=\log_2 5$, (iv) $x=\log_{10} 2$

(b) For what values of the parameter a are there values of x such that $5^{1+x}+5^{1-x}$, $a/2$, 25^x+25^{-x} are three consecutive terms of an A P

42 If the sum of m terms of an A P is equal to sum of either the next n terms or the next p terms, prove that

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

43 Show that in an arithmetical progression $a_1, a_2, a_3, \ldots$,

$$S=a_1^2-a_2^2+a_3^2-a_4^2+\ldots-a_{2k}^2=\frac{k}{2k-1}(a_1^2-a_{2k}^2)$$

$=2\sqrt{2}\sqrt{(\pm i)}$

It remains to find the value of $\sqrt{(\pm i)}$

Assume $\sqrt{i}=x+iy$ Then $i=x^2-y^2+2ixy$

$$x^2-y^2=0,\ 2xy=1 \text{ whence } x=y=\frac{1}{\sqrt{2}} \text{ or } x=y=-\frac{1}{\sqrt{2}}$$

$$\sqrt{i}=\pm\frac{1}{\sqrt{2}}(1+i)$$

Similarly $\sqrt{[(-i)]}=\pm\frac{1}{\sqrt{2}}(1-i)$

$$\sqrt{[(\pm i)]}=\pm\frac{1}{\sqrt{2}}(1\pm i)$$

Finally $(-64)^{1/4}=\pm\dfrac{2\sqrt{2}}{\sqrt{2}}(1\pm i)=\pm 2(1\pm i)$

13 Ans (*b*)

Put $z=x+iy$ $b=b_1+ib_2$ where x, y, b_1 b_2 are real

Then the equation may be written as

$$(b_1-ib_2)(x+iy)+(b_1+ib_2)(x-iy)=c$$

or $$2b_1x+2b_2y=c,$$

which is a straight line

14 Ans (d)

$$\frac{1+i}{1-i}=\frac{(1+i)^2}{1-i^2}=\frac{1+i^2+2i}{2}=\frac{1-1+2i}{2}=i,$$

we have $\left(\dfrac{1+i}{1-i}\right)^n=1 \Rightarrow i^n=1$ (1)

The smallest positive integer n satisfying (1) is 4 Hence (d) is true

15 Ans [b]

$$(x-1)^3+8=0 \text{ or } (x-1)^3=-8$$
$$x-1=(-8)^{1/3}=-2,\ -2\omega,\ -2\omega^2$$

Hence $x=-1,\ 1-2\omega,\ 1-2\omega^2$

(b) is correct answer

16 Ans (a)

Put $z=x+iy$

Then $z^{-1}=\dfrac{1}{x+iy}=\dfrac{x-iy}{x^2+y^2}$ so that $\overline{(z^{-1})}=\dfrac{x+iy}{x^2+y^2}$

Again $\bar{z}=x-iy$ and so $(\bar{z})^{-1}=\dfrac{1}{x-iy}=\dfrac{x+iy}{x^2+y^2}$

17 Ans [b] [See § 7]

(i) $b+c, c+a, a+b$,

(ii) $\frac{1}{bc}, \frac{1}{ca}, \frac{1}{ab}$ (Roorkee 75)

(iii) $a^2(b+c), b^2(c+a), c^2(a+b)$,

(iv) $a\left(\frac{1}{b}+\frac{1}{c}\right)$ $b\left(\frac{1}{c}+\frac{1}{a}\right), c\left(\frac{1}{a}+\frac{1}{b}\right)$,

(v) $\frac{1}{\sqrt{b}+\sqrt{c}}, \frac{1}{\sqrt{c}+\sqrt{a}}, \frac{1}{\sqrt{a}+\sqrt{b}}$

37 If a^2, b^2, c^2 are in A P, then the following are also in A P

(i) $\frac{1}{b+c}$ $\frac{1}{c+a}$ $\frac{1}{a+b}$

(ii) $\frac{a}{b+c}, \frac{b}{c+a}, \frac{c}{a+b}$

38 If $\frac{b+c-a}{a}, \frac{c+a-b}{b}, \frac{a+b-c}{c}$ are in A P, then $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ are also in A P

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40 If a_1, a a_n are in A P where $a_i > 0$ for all i, show that

$$\frac{1}{\sqrt{a_1}+\sqrt{a_2}}+\frac{1}{\sqrt{a_2}+\sqrt{a_3}}+ \quad + \frac{1}{\sqrt{a_{n-1}}+\sqrt{a_n}}=\frac{n-1}{\sqrt{a_1}+\sqrt{a_n}}$$

(I I T 1982)

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(iii) $x=\log_2 5$, (iv) $x=\log_{10} 2$

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42 If the sum of m terms of an A P is equal to sum of either the next n terms or the next p terms, prove that

$$(m+n)\left(\frac{1}{m}-\frac{1}{p}\right)=(m+p)\left(\frac{1}{m}-\frac{1}{n}\right)$$

43 Show that in an arithmetical progression a_1, a_2 a_3 ,

$$S=a_1^2-a_2^2+a_3^2-a_4^2+ \quad -a_{2k}^2=\frac{k}{2k-1}(a_1^2-a^2$$

Hence the condition $|z+4| \leqslant 3$ is satisfied for $z=-1$

Therefore the least value of $|z+1|$ is 0,

26 Ans (b)

$$\frac{2z+1}{iz+1}=\frac{2(x+iy)+1}{i(x+iy)+1}=\frac{2x+1+2iy}{(1-y)+ix}$$

$$=\frac{[(2x+1)+2iy][(1-y)-ix]}{(1-y)^2+x^2}$$

$$=\frac{(2x+1)(1-y)+2xy+i[-x(2x+1)+2y(1-y)]}{(1-y)^2+x^2}$$

Since $I_m\left(\frac{2z+1}{iz+1}\right)=-2$, we have

$$\frac{-x(2x+1)+2y(1-y)}{(1-y)^2-x^2}=\quad 2$$

or $\quad 2x^2-2y^2-x+2y=-2(1+y^2-2y) \quad 2x^2$

or $\quad x+2y-2=0$

which is a straight line

27 Ans [a]

$$\left|\frac{z-5i}{z+5i}\right|=1$$

or $\quad |x+iy-5i|^2=|x+iy+5i|^2$

or $\quad x^2+(y-5)^2=x^2+(y+5)^2$

or $\quad 20y=0$

or $\quad 0$

which is the equation of x axis

28 Ans (b)

Putting $z=x+iy$, we have

$$\arg\frac{z-1}{z+1}=\frac{\pi}{3} \quad \text{or} \quad \arg\frac{x+iy-1}{x+iy+1}=\frac{\pi}{3}$$

or $\quad \arg\dfrac{(x+iy-1)(x+1-iy)}{(x+1)^2+y^2}=\dfrac{\pi}{3}$

or $\quad \arg\dfrac{x^2+y^2-1+2iy}{(x+1)^2+y^2}=\dfrac{\pi}{3}$

or $\quad \tan^{-1}\dfrac{2y/[(x+1)^2+y^2]}{(x^2+y^2-1)/[(x+1)^2+y^2]}=\dfrac{\pi}{3}$

or $\quad \tan^{-1}\dfrac{2y}{x^2+y^2-1}=\dfrac{\pi}{3}$ or $\dfrac{2y}{x^2+y^2-1}=\tan\dfrac{\pi}{3}=\sqrt{3}$

or $x^2+y^2-\dfrac{2}{\sqrt{3}}y \quad 1=0,$

which is a circle

Now a will be real if $\cos x=0$ which gives $x=n\pi+\pi/2$

For these values of x, we have

$$a=\sin\left(n\pi+\frac{\pi}{2}\right)=\pm 1$$

Hence $a=1$ and $a=-1$ are only two possible real solutions

35 Ans (a)

As given for $z_1=x_1+iy_1$ and $z_2=x_2+iy_2$, the symdol $z_1 \cap z_2$ is used if $x_1 \leqslant x_2$ and $y_1 \leqslant y_2$ (1)

Therefore for $1 \cap z$ we have $1 \leqslant x$ and $0 \leqslant y$

$$\text{Now } \frac{1-z}{1+z}=\frac{1-x-iy}{1+x+iy}=\frac{(1-x-iy)(1+x-iy)}{(1+x)^2+y^2}$$

$$=\frac{1-x^2-y^2-2iy}{(1+x)^2+y^2}$$

Thus for $\frac{1-z}{1+z} \cap 0$ to be true, we must have

$$\frac{1-x^2-y^2}{(1+x)^2+y^2} \leqslant 0 \text{ and } \frac{-2y}{(1+x)^2+y^2} \leqslant 0,$$

i.e. $1-x^2-y^2 \leqslant 0$ and $y \geqslant 0$

Both these inequalities are true by virtue of (1)

36 Ans (a)

Let $z_k=r_k(\cos\theta_k+i\sin\theta_k)$, $k=1, 2, \ldots n$

Then $|z_k|=r_k$ $k=1, 2, \ldots n$

Now $|z_1+z_2|=|(r_1\cos\theta_1+r_2\cos\theta_2)+i(r_1\sin\theta_1+r_2\sin\theta_2)|$

$=\sqrt{[(r_1\cos\theta_1+r_2\cos\theta_2)+(r_1\sin\theta_1+r_2\sin\theta)^2]}$

$=\sqrt{[r_1^2+r_2^2+2r_1r_2\cos(\theta_1-\theta_2)]}$

$=\sqrt{[r_1^2+r_2^2+2r_1r_2]}$ if $\theta_1=\theta_2$

$=r_1+r_2$

Hence $|z_1+z_2|=r_1+r_2=|z_1|+|z_2|$ if $\theta_1=\theta_2$

Similary $|z_1+z_2+z_3|=[r_1^2+r_2^2+r_3^2+2r_1r_2\cos(\theta_1-\theta_2)$
$+2r_2r_3\cos(\theta_2-\theta_3)+2r_3r_1\cos(\theta_3-\theta_1)]$

Hence if $\theta_1=\theta_2=\theta_3$ we have

$|z_1+z_2+z_3|=\sqrt{(r_1^2+r_2^2+r_3^2+2r_1r_2+2r_2r_3+2r_3r_1)}$

$=r_1+r_2+r_3$

$=|z_1|+|z_2|+|z_3|$ etc

(b) It follows from part (a) where $\theta_1=\theta_2$ or $\theta_1-\theta_2=0$

37 Ans (iii)

$|z-1|+|z+1| \leqslant 4$

$\Rightarrow |z-1|^2+|z+1|^2+2|z-1||z+1| \leqslant 16$

$\Rightarrow (z-1)(\bar{z}-1)+(z+1)(\bar{z}+1)+2|z^2-1| \leqslant 16$

$\Rightarrow 2z\bar{z}+2+2|z^2-1| \leqslant 16$

Since $|Z_1|=|Z_2|=|Z_3|$, we have $OA=OB=OC$ This shows that O is the circum-centre of the $\triangle ABC$

$\angle A_1OA_2=\angle A_2OA_3=\angle A_3OA_1=\frac{2\pi}{3}$

Hence we can write,

$Z_2=Z_1e^{2\pi i/3}$ and $Z_3=Z_1e^{4\pi i/3}$ (See remark (i), § 3)

Hence $Z_1+Z_2+Z_3=Z_1(1+e^{2\pi i/3}+e^{4\pi i/3})$

$$=Z_1\left[1+\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}+\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right]$$

$$=Z_1\left(1-\tfrac{1}{2}+\frac{\sqrt{3}}{2}i-\tfrac{1}{2}-\frac{\sqrt{3}}{2}i\right)=0$$

Hence the given statement is false

41 Ans (B) Hint Equating the complex numbers corresponding to the mid points of the two diagonals we have

$$\frac{z_1+z_3}{2}=\frac{z_2+z_4}{2}$$

or $z_1+z_3=z_2+z_4$

42 Ans (D) From Ex 41, $z_4=z_1+z_3-z_2$

43 Ans (d) The required complex number is i Draw figure on the Argand plane to verify it

44 Ans (B)

$$|\omega|=1\Rightarrow\frac{|1-iz|}{|z-i|}=1\Rightarrow|1-iz|^2=|z-i|^2$$

$$\Rightarrow(1-iz)(1+i\bar{z})=(z-i)(z+i)$$

$$\Rightarrow 1-iz+i\bar{z}+z\bar{z}=z\bar{z}+iz-i\bar{z}+1$$

$$\Rightarrow 2i(z-\bar{z})=0\Rightarrow 2i(2iy)=0\Rightarrow y=0,$$

which is the equation of real axis

45 $|e^{iz}|=|e^{i\,re^{i\theta}}|$

$$=|e^{ir(\cos\theta+i\sin\theta)}|$$

$$=|e^{ir\cos\theta}\,e^{-r\sin\theta}|$$

$$=|e^{ri\cos\theta}||e^{-r\sin\theta}|$$

$$=1\,e^{-r\sin\theta}=e^{-r\sin\theta}$$

[$|e^{ri\cos\theta}|=1$ and $e^{-r\sin\theta}>0$ always so that $|e^{-r\sin\theta}|=e^{-r\sin\theta}$

50 Ans (b)

For consider, the complex numbers $e^{i\pi/3}$ $3e^{i\pi/3}$ and $5e^{i\pi/3}$ which are in A P But they do not lie on a circle In fact, they lie on the straight line $\theta = \frac{\pi}{3}$

51 Ans ω, ω^2

First equation can be written as

$$(z+1)(z^2-z+1)+2z(z+1)=0$$

or $$(z+1)(z^2+z+1)=0 \text{ whence } z=-1, \omega, \omega^2$$

$z=-1$ does not satisfy the second equation but $z=\omega, \omega^2$ satisfy it as can be easily verified

Hence the common roots are ω, ω^2

52 Ans (B)

$$z_k = \frac{1(1-z^k)}{1-z} = \frac{1-z^k}{1-z} \qquad \text{(Summing the G P)}$$

Now $$\left| z_k - \frac{1}{1-z} \right| = \left| \frac{1-z^k}{1-z} - \frac{1}{1-z} \right|$$

$$= \left| \frac{-z^k}{1-z} \right| = \frac{|z|^k}{|1-z|} < \frac{1}{|1-z|}$$

$$[\quad |z| < 1]$$

It means that all the vertices z_k lie within the circle of radius $\frac{1}{|1-z|}$ and centre $\frac{1}{1-z}$ i e within the circle

$$\left| Z - \frac{1}{1-z} \right| = \frac{1}{|1-z|}$$

Whereas the point $z=0$ lies on the boundary of this circle since

$$\left| 0 - \frac{1}{1-z} \right| = \frac{1}{|1-z|},$$

or $$\frac{1}{|1-z|} = \frac{1}{|1-z|} \text{ (an identity)}$$

Hence $z=0$ cannot lie within the polygon whose vertices are at the points

$$z_k = 1+z+z^2+ \quad +z^{k-1} \quad |z| < 1$$

53 Ans (A) and (D)

We have $$\frac{z_1+z_2}{z_1-z} = \frac{(z_1+z_2)(\bar{z}_1-\bar{z})}{(z_1-z)(\bar{z}_1-\bar{z})} = \frac{z\bar{z}_1 - z_1\bar{z}_2 + z_2\bar{z}_1 - z\bar{z}}{z\bar{z}_1 - z_1\bar{z}_2 - z\bar{z}_1 + \cdots}$$

$$= \frac{|z_1|^2 - 2i I_m(z_1\bar{z}_2) - |z_2|^2}{|z_1|^2 - 2 Re(z_1\bar{z}) + |z|^2}$$

$[\quad z_1\bar{z}_2$ and $z_2\bar{z}_1$ are conjugate

$$\frac{i I_m(z_1 z_2)}{Re(z_1\bar{z}_2) - |z_1|^2}$$

$[\because \quad |z_1| = |z_2|$

3
Progressions

§ I Arithmetical Progression

(a) Definition If certain quantities increase or decrease by the same constant then such quantities form a series which is called an arithmetical progress on This constant is called common difference For example

(i) 1, 4, 7 10 , (ii) 9 6, 3, 0, −3, (iii) a, $a+d$, $a+2d$,
The common difference in the above three series are 3, −3 and d respectively

(b) Notation The first term of the series is denoted by a, common difference by d the last terms by l, the number of term by n, sum of its n terms by S_n the nth term by T_n

Standard Results $T_n = a+(n-1)d = l$

$$S_n = \frac{n}{2}(a+l) = \frac{n}{2}[2a+(n-1)d]$$

(c) (i) Arithmetic Mean (A M)

The arithmetic mean between two given quantities a and b is x so that a, x, b are in A P i e $x-a=b-x$

or $2x=a+b$ $\quad x=\frac{a+b}{2}=A$ (Notation)

(ii) n arithmetic means between two quantities a and b

If between two given qnantities a and b we have to insert n arithmetic means x_1 x x_n then a, x_1 x_2, x_n, b will be in A P In order to find the values of these means we require the common difference The above series consists of $(n+2)$ terms and the last term is b and first term is a

$$b=T_{n+2}=a+(n+2-1))\ d \qquad d=\frac{b-a}{n+1}$$

$$x_1=T_2=a+d \quad x_2=T_3=a+2d \quad , x_n=T_{n+1}=a+nd$$

On putting the value of d we shall find the n arithmetic means

(d) An important note

(1) If each term of given arithmetical progression be increased decreased, multiplied or divided by the same non zero

ssion The smallest angle is 120° and the common difference is 5 Find the number of sides of the polygon **(I I T 1980)**

7 Find the sum of n terms of the series

$$\log a+\log\frac{a^2}{b}+\log\frac{a^3}{b^2}+\log\frac{a^4}{b^3}+\ \ n \text{ terms}$$

8 The mth term of an A P is n and its nth term is m prove that its pth term is $m+n-p$ Also show that its $(m+n)$th term is zero

9 The first and last term of an A P are a and l respectively If S be the sum of all the terms of the A P, show that the common difference is

$$\frac{l^2-a^2}{2S-(l+a)}$$

10 Show that the sum of an A P whose first term is a, second term is b and the last term is c is equal to

$$\frac{(a+c)(b+c-2a)}{2(b-a)}$$

11 The sum of n terms of a series is $3n^2+4n$ Show that the series is an A P and find the first term and common difference What will be its nth term ?

12 If the ratio of the sum of m terms and n terms of an A P be $m^2 : n^2$, prove that the ratio of its mth and nth terms will be

$$2m-1 : 2n-1$$

13 (a) The ratio between the sum of n terms of two A P s is $3n+8 : 7n+15$ Find the ratio between their 12th terms

(b) The sum of the first n terms of two A P s are as $3n+5 : 5n+9$ Prove that their 4th terms are equal

14 Find the sum of all natural numbers between 250 and 1000 which are exactly divisible by 3

15 (i) Show that the sum of all odd numbers between 2 and 1000 which are divisible by 3 is 83667

(ii) Find the sum of first n odd natural numbers

(iii) Find the sum of all odd integers between 1 and 100 divisible by 3

(iv) Sum of certain consecutive odd positive integers is 57^2-13^2 Find them

16 The pth term of an A P is a and qth term is b Prove that sum of its $(p+q)$ terms is

24 If S_1, S_2, S_3 S_m are the sums of n terms of m A P's whose first terms are 1, 2, 3, m and common differences are 1, 3, 5, , $2m-1$ respectively Show that

$$S_1+S_2+S_3+ \quad +S_m=\tfrac{1}{2}\,mn\,(mn+1)$$

25 (a) The series of natural numbers is divided into groups (1), (2, 3, 4), (5, 6, 7, 8, 9) and so on Show that the sum of the numbers in the nth group is $(n-1)^3+n^3$ (I I T 63)

(b) The series of natural numbers is divided into groups (1), (2, 3, 4), (3, 4, 5, 6, 7) (4, 5, 6, 7, 8, 9, 10) Find the sum of the numbers in nth group

26 N the set of natural numbers is partitioned into subsets $S_1=\{1\}$, $S_2=\{2\ 3\}$ $S_3=\{4, 5\ 6\}$, $S_4=\{7, 8, 9\ 10\}$ The last term of these groups is 1 1+2, 1+2+3, 1+2+3+4, so on Find the sum of the elements in the sub set S_{50} (I I T 71)

27 Show that sum of the terms in the nth bracket (1), (3 5) (7, 9, 11), is n^3

28 The sum of three numbers in A P is 15 whereas sum of their squares is 83 Find the numbers

29 The sum of three numbers in A P is 12 and the sum of their cubes is 288 Find the numbers

30 (i) Find four numbers in A P whose sum is 20 and sum of their squares is 120 (ii) sum is 32 and sum of squares is 276

31 Divide 28 into four parts in A P so that ratio of the product of first and third with the product of second and fourth is 8 15

32 Between 1 and 31 are inserted m arithmetic means so that the ratio of the 7th and $(m-1)$th means is 5 9 Find the value of m

33 Prove t[illegible] of th[illegible] means inserted between two qu[illegible] means bet-ween them

34 For wha[illegible] of a b

sum [illegible] mea

e,

in

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33 Prove that the sum of the n arithmetic means inserted between two quantities is n times the single arithmetic means between them

34 For what value of n $\dfrac{a^{n+1}+b^{n+1}}{a^n+b^n}$ is the arithmetic mean of a and b

35 The sum of two numbers is $2\frac{1}{6}$ An even number of arithmetic means are being inserted between them and their sum exceeds their number by 1 Find the number of means inserted

36 If a b, c are in A P, prove that the following are also in A.P